Stable, Oscillatory Viscous Profiles of Weak, non-Lax Shocks in Systems of Stiff Balance Laws

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1 Introduction

This thesis is devoted to a phenomenon in hyperbolic balance laws, first described by Fiedler and Liebscher [FL98], which is similar in spirit to the Turing instability [Tur52]. The combination of two individually stabilising effects can lead to quite rich dynamical behaviour, like instabilities, oscillations, or pattern formation.

Our problem is composed of two ingredients. As a first part we have a strictly hyperbolic conservation law which has rarefaction waves and shocks with monotone viscous profiles as elementary solutions. The second part is a source term which, alone, would describe a simple, stable kinetic behaviour: all trajectories end by converging monotonically to an equilibrium. The balance law, constructed of these two parts, however, can support viscous shock profiles with oscillatory tails. They emerge from a Hopf-like bifurcation point that belongs to a curve of equilibria of the associated travelling-wave system. The linearised flow at the rest points along this curve possesses a pair of conjugate complex eigenvalues which crosses the imaginary axis at the Hopf-point.

The nature of the oscillatory shocks as well as their stability properties are the subjects of this thesis. Our main result, theorem 1.5, establishes convective stability of oscillatory viscous profiles to weak shocks with extreme speed: if the speed of the wave exceeds any characteristic speed, then the profile is linearly stable in a suitable exponentially weighted space. For intermediate speeds, the profiles are absolutely unstable.

Methods of two areas, hyperbolic conservation laws and dynamical systems, are combined in this work. This makes it necessary to give a brief review of basic facts and methods of both fields in sections 2 and 5.

1.1 Travelling waves

Searching for viscous shock profiles of the Riemann problem, we consider systems of hyperbolic balance laws of the form

$$u_t + f(u)_x \;=\; \varepsilon^{-1} g(u) + \varepsilon\delta u_{xx}, \tag{1.1}$$

with $u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n$, $f \in C^6$, $g \in C^5$, $\delta > 0$ fixed, and with real time t and space x. The balance law is assumed to be hyperbolic, i.e. $f'(u)$ has only real eigenvalues.

The case of conservation laws, $g \equiv 0$, has been studied extensively. Some fundamental methods and results will be reviewed in section 2. Here, we consider stiff source terms $\varepsilon^{-1}g$, $\varepsilon > 0$ small. Stiff sources arise frequently in models of chemical reactions and combustion processes and in relaxation systems.

We are looking for travelling-wave solutions with wave speed s of the form

$$u = u\left(\frac{x - st}{\varepsilon}\right). \tag{1.2}$$

To solve (1.1) they must satisfy

$$-s\dot{u} + A(u)\dot{u} = g(u) + \delta\ddot{u}. \tag{1.3}$$

Here $A(u) = f'(u)$ denotes the Jacobian of the flux and the dot indicates the differentiation with respect to the comoving, rescaled coordinate $\xi = (x - st)/\varepsilon$. Note that (1.3) is independent of ε.

Viscous profiles are travelling waves (1.2) which connect two asymptotic states $u^{\pm}$. They are solutions of (1.3) for which asymptotic states

$$\lim_{\xi \to \pm\infty} u(\xi) = u^{\pm} \tag{1.4}$$

exist. In the limit $\varepsilon \searrow 0$ they give rise to a solution of the Riemann problem of (1.1) with values $u = u^{\pm}$ connected by a shock front travelling with shock speed s.

The second-order travelling-wave equation (1.3) can be rewritten as a first-order system

$$\begin{aligned} \dot{u} &= v, \\ \delta\dot{v} &= -g(u) + (A(u) - s)v. \end{aligned} \tag{1.5}$$

Viscous profiles are now heteroclinic orbits connecting equilibria ($u = u^{\pm}, v = 0$) of (1.5). Note that any such equilibrium is also a zero of the source term,

$$g(u^{\pm}) = 0. \tag{1.6}$$

For pure conservation laws, $g \equiv 0$, this last condition does not restrict the values $u^{\pm}$. We can, in this case, integrate the viscous-profile equation (1.3) once and analyse the resulting first-order dynamical system as described in section 2.5.

In general, the source term $g(u)$ may depend on some but not all components of u. For instance a chemical reaction depends on concentrations and temperature, but not on velocities.

In this work, we assume that the u_0-component does not contribute to the reaction terms and that the origin is an equilibrium state:

$$\begin{aligned} g &= g(u_1, \ldots, u_{n-1}) = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}, \\ g(0) &= 0. \end{aligned} \tag{1.7}$$

This gives rise to a line of equilibria

$$u_0 \in \mathbb{R}, \qquad u_1 = \cdots = u_{n-1} = 0, \qquad v = 0, \tag{1.8}$$

of our viscous-profile system (1.5).

The asymptotic behaviour of viscous profiles $u(\xi)$ for $\xi \to \pm\infty$ depends on the linearisation L of (1.5) at $u = u_\pm$, $v = 0$. In block-matrix notation corresponding to coordinates (u, v) we have

$$L = \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1} g' & \delta^{-1}(A - s) \end{pmatrix}. \tag{1.9}$$

Here $A = A(u)$ and $g' = g'(u)$ describe the JACOBI matrices of the flux f and reaction term g at $u = u^\pm$. To abbreviate (1.9), we wrote s rather than $s \cdot \mathrm{id}$.

In the case $g \equiv 0$ of pure conservation laws, the linearisation L possesses an n-dimensional kernel that corresponds to the then arbitrary choice of the equilibrium $u \in \mathbb{R}^n$, $v = 0$. Normal hyperbolicity of this family of equilibria, in the sense of dynamical systems [HPS77], [Fen77], [Wig94], is ensured for wave speeds s that are not in the spectrum of the strictly hyperbolic Jacobian $A(u)$:

$$s \notin \operatorname{spec} A(u) = \{\sigma_0, \ldots, \sigma_{n-1}\}. \tag{1.10}$$

Indeed, under this condition the only zeros that arise in the real spectrum

$$\operatorname{spec} L = \{0\} \cup \delta^{-1} \operatorname{spec}(A(u) - s) \tag{1.11}$$

are the trivial ones. Varying the parameter s, bifurcations occur at values $s \in \operatorname{spec} A(u)$. Travelling waves emerge that correspond to the usual LAX shocks, see section 2.5.

We now investigate the failure of normal hyperbolicity of L along the line of equilibria $u = (u_0, 0, \ldots, 0)$, $v = 0$. Although our method applies in complete generality, we just present the analysis for systems of the smallest possible dimension, $n = 3$, for which purely imaginary eigenvalues of L arise when $\delta > 0$ is fixed small enough.

1.2 Bifurcation from lines of equilibria

In this section, we investigate the three-dimensional case, $n = 3$, of the viscous-profile system (1.5, 1.7). We restrict[1] ourselves to the case

$$\begin{aligned} A(u_0,0,0) &= A_0 + u_0 \cdot A_1 + \mathcal{O}(u_0^2), \\ A_0 &= \begin{pmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \alpha_2 \end{pmatrix}, \qquad A_1 = (\beta_{ij})_{0\le i,j\le 2} \quad \text{symmetric}, \\ g'(0) &= \begin{pmatrix} 0 & & \\ & \gamma_{11} & \gamma_{12} \\ & \gamma_{21} & \gamma_{22} \end{pmatrix}, \qquad \gamma_{ij} \neq 0, \end{aligned} \tag{1.12}$$

with omitted entries being zero. Note that these data can arise from flux functions f which are gradient vector fields. In corollary 1.4 we present a specific choice of the parameters where the reaction terms $\dot{u} = g(u)$ *alone*, likewise, do not support oscillatory behaviour. All trajectories of the pure kinetics end by converging monotonically to an equilibrium. The interaction of flux and reaction, in contrast, is able to produce purely imaginary eigenvalues of the linearisation L, as described in proposition 1.1 below.

In [FL98] an special example of the above type was investigated and parameter values were given, such that purely imaginary eigenvalues exist, too. Unfortunately this example had several drawbacks. The main problem is, that the source of the example in [FL98] has unstable zeros. Therefore every travelling wave of the form (1.2, 1.4) has left and right asymptotic states which are unstable with respect to homogeneous perturbations. So, any viscous profile will be sensitive to perturbations with long wavelength.

Here, in contrast, we shall exactly compute the range of parameters where Hopf points occur in the more general setup (1.12). We shall identify parameter regions with stable asymptotic states and investigate the linear stability of the resulting shocks.

Proposition 1.1 *Consider the linearisation $L = L(u_0)$, see (1.9), along the line $u_0 \in \mathbb{R}, u_1 = u_2 = 0$ of equilibria of the viscous-profile system (1.5) in $\mathbb{R}^3$. Assume (1.12) holds.*

For $s \notin \{\alpha_0, \alpha_1, \alpha_2\}$, small $|u_0|$, and $\delta \searrow 0$ the spectrum of L then decouples into two parts:

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[1] The main restriction here is the fact that the u_0-direction, the equilibrium line, coincides with an eigendirection of A_0. This renders the following calculation a bit easier to acquire. We want to emphasise again, that our aim is to find oscillations in an example which is as simple as possible.

(i) an unboundedly growing part $\operatorname{spec}_\infty L \;=\; \delta^{-1}\operatorname{spec}(A-s)+\mathcal{O}(1)$

(ii) a bounded part $\operatorname{spec}_{bd} L \;=\; \operatorname{spec}((A-s)^{-1}g')+\mathcal{O}(\delta)$

Here A and g are evaluated at $u=(u_0,0,0)$. For $\alpha_1\neq\alpha_2$, $\gamma_{11}+\gamma_{22}\neq 0$, $\delta\searrow 0$, the bounded part, $\operatorname{spec}_{bd} L$, at $u_0=0$ limits for values

$$\begin{aligned} s \;=\; s^{\mathrm{crit}} \;&=\; \frac{\alpha_1\gamma_{22}+\alpha_2\gamma_{11}}{\gamma_{11}+\gamma_{22}}, \\ 1 \;&<\; \frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} \end{aligned} \tag{1.13}$$

onto simple eigenvalues

$$\begin{aligned} \mu_0 \;&\in\; \{0,\pm i\omega_0\}, \\ \omega_0 \;&=\; \frac{\gamma_{11}+\gamma_{22}}{\alpha_1-\alpha_2}\sqrt{\frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}}-1}, \end{aligned} \tag{1.14}$$

with eigenvectors $\binom{\tilde u}{\tilde v}$ given by $\tilde v=\mu_0\tilde u$ and

$$\begin{aligned} \tilde u \;&=\; \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} && \textit{for } \mu_0=0, \\ \tilde u \;&=\; \begin{pmatrix} 0 \\ 1 \\ -\frac{\gamma_{11}}{\gamma_{12}}\pm i\frac{\gamma_{11}}{\gamma_{12}}\sqrt{\frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}}-1} \end{pmatrix} && \textit{for } \mu_0=\pm i\omega_0. \end{aligned} \tag{1.15}$$

The proof will be postponed to section 4.1.

Proposition 1.1 describes a situation which looks like a Hopf bifurcation.[2] But, of course, no parameter is involved. Unusually, there is no foliation of the vector field near the critical point, transversal to the line of equilibria. (In a parameter dependent system, we could regard the parameter as an additional variable in an extended phase space. Then any equilibrium would become a curve with a transversal foliation due to the parameter.)

Bifurcations from lines of equilibria in absence of parameters have been investigated in [Lie97], [FLA98a] from a theoretical point of view. For a related application to binary oscillators in discretised systems of hyperbolic balance laws see [FLA98b]. We recall the result of [FLA98a].

[2]For background information about Hopf bifurcation, please see [MM76].

Consider C^5 vector fields

$$\dot{\mathbf{u}} \; = \; F(\mathbf{u}) \tag{1.16}$$

with $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^N$. We assume that the u_0 axis consists of equilibria

$$0 \; = \; F(u_0, 0, \ldots, 0). \tag{1.17}$$

At the origin the JACOBI matrix $F'(u_0, 0, \ldots, 0)$ has a trivial kernel vector along the u_0-axis. We suppose that, in addition, there exists a pair of simple, complex conjugate, nonzero eigenvalues $\mu(u_0), \overline{\mu(u_0)}$ crossing the imaginary axis transversely as u_0 increases through the origin $u_0 = 0$:

$$\begin{aligned} \mu(0) \; &= \; i\omega(0), \qquad \omega(0) \; > \; 0, \\ \Re\mathrm{e}\, \mu'(0) \; &\neq \; 0. \end{aligned} \tag{1.18}$$

In the complementary directions $F'(u_0, 0, \ldots, 0)$ is assumed to be hyperbolic.

Let Z be the two-dimensional real eigenspace of $F'(0)$ associated to $\pm i\omega(0)$. By Δ_Z we denote the Laplacian with respect to variations of u in the eigenspace Z. Coordinates in Z are chosen as coefficients of the real and imaginary parts of the complex eigenvector associated with $i\omega(0)$. Note that the linearisation acts as a rotation with respect to these, not necessarily orthogonal, coordinates. Let P_0 be the one-dimensional eigenprojection onto the trivial kernel along the u_0-axis. Finally, we impose a nondegeneracy assumption which reads

$$\Delta_Z P_0 F(0) \; \neq \; 0. \tag{1.19}$$

This assumption can be seen as an exclusion of foliations transversal to the u_0-axis near the origin. If (1.19) is satisfied (and this is decided by second order terms) then no foliation transversal to the line of equilibria can exist.[3]

After fixing the orientation along the positive u_0-axis, we can consider $\Delta_Z P_0 F(0)$ as a real number. Depending on the sign

$$\eta \; := \; \mathrm{sign}(\Re\mathrm{e}\, \mu'(0)) \; \cdot \; \mathrm{sign}(\Delta_Z P_0 F(0)), \tag{1.20}$$

we call the “bifurcation” point $u_0 = 0$ *elliptic*, if $\eta = -1$, and *hyperbolic*, for $\eta = +1$.

The following result from [FLA98a] concerns the qualitative behaviour of solutions in a normally hyperbolic[4] three-dimensional centre manifold to $\mathbf{u} = 0$.

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[3]See section 3 for more details.

[4]Please note that we have now introduced three different meanings of the term “hyperbolic” which

Theorem 1.2 (Fiedler, Liebscher, Alexander [FLA98a]) *Let the assumptions (1.16–1.19) be satisfied for the C^5-vector field $\dot{\mathbf{u}} = F(\mathbf{u})$. Then the following holds true in a neighbourhood U of $\mathbf{u} = 0$ within a three-dimensional centre manifold to $\mathbf{u} = 0$.*

In the hyperbolic case, $\eta = +1$, all non-equilibrium trajectories leave the neighbourhood U in positive or negative time direction, or both. The stable and unstable sets of $\mathbf{u} = 0$, respectively, form cones around the positive/negative u_0-axis, with asymptotically elliptic cross section near their tips at $\mathbf{u} = 0$. These cones separate regions with different convergence behaviour. See figure 1.1a.

In the elliptic case all non-equilibrium trajectories starting in U are heteroclinic between equilibria $\mathbf{u}^\pm = (u_0^\pm, 0, \dots, 0)$ on opposite sides of $u_0 = 0$. If $F(\mathbf{u})$ is real analytic near $\mathbf{u} = 0$, then the two-dimensional strong stable and strong unstable manifolds, $W^s(u^+)$ and $W^u(u^-)$, of the asymptotic states within the centre manifold intersect at an angle that possesses an exponentially small upper bound in terms of $|\mathbf{u}^\pm|$. See figure 1.1b.

The results for the hyperbolic case, $\eta = +1$, are based on normal-form theory and a spherical blow-up construction inside the centre manifold. The elliptic case, $\eta = -1$, is based on Neishtadt's theorem on exponential elimination of rapidly rotating phases [Nei84]. This will be described in section 3.

1.3 Shock profiles

Our next task is to apply theorem 1.2 to the problem of viscous profiles of systems of hyperbolic balance laws near Hopf points of the kind observed in proposition 1.1.

Theorem 1.3 *Consider the problem (1.4, 1.5) of finding viscous profiles with shock speed s to the system of viscous hyperbolic balance laws (1.1). Let assumptions (1.7, 1.12, 1.13) hold, so that a pair of purely imaginary simple eigenvalues occurs at the origin for the linearisation L, in the limit $\delta \to 0$, for a fixed speed $s = s^{\mathrm{crit}}$.*

Then there exist nonlinearities $A(u) = f'(u)$ and $g(u)$, compatible with the above assumptions, such that the assumptions and conclusions of theorem 1.2 are valid for the viscous-profile system (1.5) for small, positive δ at a point $(u_0^\delta, 0, \dots, 0)$, with $\lim_{\delta \searrow 0} u_0^\delta = 0$. Both

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should not be mixed up. First, the hyperbolicity of a conservation law restricts the eigenvalues of the derivative of the flux to real numbers. In contrast to this, normal hyperbolicity of manifolds in dynamical systems describes the existence of transversally intersecting stable and unstable manifolds with uniformly bounded contraction and expansion rates, see [Wig94]. Finally, the classification of the two bifurcation types in (1.20) is motivated by the structure of the normal form, see section 3.

the elliptic and the hyperbolic case occur; see figure 1.1.

Since both conditions define open regions, the results persist for shock speeds s in a small open interval around s^{crit}, even when f, g remain fixed. The Hopf *bifurcation will then occur at a point $(\breve{u}_0^\delta, 0, \ldots, 0)$, with $\breve{u}_0^\delta$ converging to a point near the origin, for $\delta \searrow 0$.*

In particular, the assumptions of theorem 1.2 are satisfied if, and only if,

$$\begin{array}{lrcl} (i) & \alpha_1 & \neq & \alpha_2, \\ (ii) & 0 & \neq & \gamma_{11} + \gamma_{22}, \\ (iii) & 1 & < & \frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}}, \\ (iv) & \alpha_0 & \neq & s^{\text{crit}} = \frac{\alpha_1\gamma_{22}+\alpha_2\gamma_{11}}{\gamma_{11}+\gamma_{22}}, \\ (v) & 0 & \neq & \beta_{11}\gamma_{22} + \beta_{22}\gamma_{11} - \beta_{12}\gamma_{21} - \beta_{21}\gamma_{12}, \\ (vi) & 0 & \neq & g_0''(0)[\tilde{u}, \overline{\tilde{u}}]. \end{array} \tag{1.21}$$

The first four conditions are these of proposition 1.1. Assumption (v) represents the transversality condition (1.18). The last requirement matches the nondegeneracy condition (1.19), where $\tilde{u}$ is the first part of the Hopf*-eigenvector (1.15) and $\overline{\tilde{u}}$ its complex conjugate.*

The type of bifurcation (1.20) is determined by

$$\begin{aligned} \eta \;=\; & \operatorname{sign}(\beta_{12}\gamma_{21} + \beta_{21}\gamma_{12} - \beta_{11}\gamma_{22} - \beta_{22}\gamma_{11}) \operatorname{sign}(\gamma_{11}\gamma_{22}) \cdot \\ & \cdot \operatorname{sign}(\alpha_0 - s^{\text{crit}}) \operatorname{sign}(g_0''(0)[\tilde{u}, \overline{\tilde{u}}]). \end{aligned} \tag{1.22}$$

Specific choices of flux and reaction terms are presented in corollary 1.4; see (1.23).

In the elliptic case, $\eta = -1$, we observe (at least) pairs of weak shocks with oscillatory tails, connecting u^- and u^+. In the hyperbolic case, $\eta = +1$, viscous profiles leave the neighbourhood U and thus, possibly, represent large shocks. At $u_0 = 0$, their profiles change discontinuously and the role of the equilibrium on the u_0-axis switches from left to right asymptotic state with oscillatory tail.

We shall focus on the elliptic case, $\eta = -1$. The heteroclinic connections, figure 1.2a, which fill an entire neighbourhood of the Hopf point in the centre manifold then lead to travelling waves of the balance law (1.1, 1.2). In figure 1.2b, such a wave is shown, and a generic projection of the n-dimensional space of u-values onto the real line was used. The oscillations imposed by the purely imaginary eigenvalues now look like a Gibbs phenomenon. But here, they are an intrinsic property of the analytically derived solution.

To prove theorem 1.3, we first check the transversality condition (1.18) for the purely imaginary eigenvalues. Then we compute an expansion in terms of δ for the eigenprojection P_0 onto the trivial kernel along the u_0-axis. Finally, we check nondegeneracy condition (1.19) for $\Delta_Z P_0 F(0)$, in the limit $\delta \searrow 0$, completing the proof by reduction to theorem 1.2. This is done in section 4.

Corollary 1.4 *Theorems 1.2, 1.3 hold true for $\eta = \pm 1$ with the following specific choices of a gradient flux term $f(u) = \nabla\Phi(u)$ and a reaction term $g(u)$:*

$$\begin{aligned} \Phi(u) &= -\tfrac{1}{2}u_0^2 + u_1^2 + \tfrac{1}{2}u_2^2 - \tfrac{1}{2}u_0 u_1^2, \\ g(u) &= \begin{pmatrix} -\eta u_1^2 & - & \eta u_2^2 \\ \tfrac{1}{4}u_1 & + & \tfrac{3}{2}u_2 \\ -\tfrac{3}{2}u_1 & - & 4u_2 \end{pmatrix}. \end{aligned} \tag{1.23}$$

These choices correspond to values $\alpha_0 = -1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_{11} = -1$, all other $\beta_{ij} = 0$, $\gamma_{11} = 1/4$, $\gamma_{12} = 3/2$, $\gamma_{21} = -3/2$, $\gamma_{22} = -4$. In addition, g_0'' is positive ($\eta = -1$) or negative ($\eta = 1$) definite. The critical shock speed is $s^{\text{crit}} = \frac{31}{15}$.

In the above example the flux function is a gradient. For the conservation-law part without the source g, the travelling-wave equation (1.3) would be represented by a gradient flow

$$\delta\dot{u} \;=\; \nabla\left(\Phi(u) - \frac{s}{2}\langle u, u\rangle - \langle u, \Phi(u^-) - su^-\rangle\right). \tag{1.24}$$

without any possible oscillatory behaviour. The pure kinetics, $\dot{u} = g(u)$, too, does not support oscillations. On the u_1, u_2 subspace, the flow $g'(u)$ is linear with real and negative eigenvalues. All trajectories end by converging monotonically to the equilibrium line. Anyhow, the interaction of the conservation law with the source produces oscillatory viscous profiles. These profiles can be even stable, in some sense, as we shall see later on.

A very similar phenomenon is known as TURING-instability in reaction-diffusion equations. Reaction and diffusion, which separately stabilise constant, homogeneous solutions, can form instabilities and spatial patterns [Tur52]. In our case, we have to deal with the interaction of reaction and transport.

1.4 Stability, main result

Our next aim is to find stability of the shock profiles (1.2) with respect to perturbations in the parabolic equation (1.1).

The first requirement that we want to impose is the stability of the right and left asymptotic states $u^{\pm}$ with respect to homogeneous perturbations. The points $u^{\pm}$ have to be stable equilibria of

$$u_t \;=\; \varepsilon^{-1} g(u). \tag{1.25}$$

Otherwise, any shock connecting u^- and u^+ would be very sensitive to long wave perturbations. Therefore all eigenvalues of $g'(0) = g'(u_0, 0, \ldots, 0)$ should be contained in the closed left half plane.

Secondly, we want to insist on the possibility of oscillatory shocks for source terms that *resist* oscillations. Especially, g' should have real eigenvalues.

Collecting all these requirements and (1.13) from proposition 1.1, we impose the following restrictions of the parameters (1.12) of flux and source terms:

(i) real eigenvalues of g'

$$(\gamma_{11} - \gamma_{22})^2 + 4\gamma_{12}\gamma_{21} \;>\; 0, \tag{1.26}$$

(ii) two negative eigenvalues of g' in addition to the trivial one

$$\gamma_{11} + \gamma_{22} \;<\; 0 \qquad \text{and} \qquad \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \;>\; 0, \tag{1.27}$$

(iii) HOPF-point

$$\alpha_1 \;\neq\; \alpha_2 \qquad \text{and} \qquad 1 \;<\; \frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}}. \tag{1.28}$$

We want to simplify this a bit. Because of (ii) and (iii), $\gamma_{11}\gamma_{22}$ has to be negative. Without loss of generality, we can choose $\gamma_{11} > 0 > \gamma_{22}$. Then, (ii) implies that $\gamma_{12}\gamma_{21}$ must be negative, too, and of absolute value grater than $|\gamma_{11}\gamma_{22}|$. Taking (1.26) into account we arrive at

$$0 \;<\; \gamma_{11} \;<\; -\gamma_{22}, \qquad -\gamma_{11}\gamma_{22} \;<\; -\gamma_{12}\gamma_{21} \;<\; \frac{1}{4}(\gamma_{11} - \gamma_{22})^2, \qquad \alpha_1 \;\neq\; \alpha_2. \tag{1.29}$$

One easily checks, that — up to the choice of the signs of γ_{11}, γ_{22} — (1.29) is equivalent to (1.26 – 1.28).[5] Note that the example given in corollary 1.4 fulfils this condition.

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[5]The structure of source terms g that satisfy this condition becomes even more apparent with the

Now we can investigate the linear stability of the travelling waves of (1.1, 1.2, 1.4) observed in theorem 1.3 in the elliptic case.

To achieve this, we have to study the spectrum of the linearisation of the p.d.e. operator associated to (1.1). The spectrum is a disjoint union of the *point spectrum*[6] and the *essential spectrum*[6]. If the entire spectrum is contained in the left half plain, we speak of spectral stability, which can be extended to linear stability by establishing suitable estimates of the resolvent. Parts of the essential spectrum in the right half plane typically correspond to continua of unstable modes. In some cases the growth of these modes may be dominated by their convection to either $-\infty$ or $+\infty$. Perturbations may grow but travel away. Pointwise, the perturbations decay to zero. We call this situation *convective stability*[6] in contrast to *absolute instability*[6], where growing modes exist which do not travel away.

Convective stability can be investigated through the introduction of exponentially weighted norms $\|\cdot\|_\nu$,

$$\|u\|_\nu^2 \;=\; \int_{-\infty}^{\infty} |e^{\nu x} u(x)|^2 \,\mathrm{d}x, \tag{1.30}$$

instead of the usual L^2 norm. Choose, for instance, $\nu > 0$. Then, perturbations of fixed L^2 norm near $+\infty$ will be very large in the weighted norm. Near $-\infty$, however, a perturbation of the same L^2 norm will be very small due to a small weight. Modes that travel towards $-\infty$ are multiplied by $e^{\nu x}$ which gets smaller as x goes to $-\infty$. In fact, growing modes might even become decaying. Parts of the essential spectrum that corresponds to convective modes can be shifted if the spectrum is calculated in a norm with exponential weights. Absolute instabilities are, in contrast, not affected by supplementary weights. Therefore, convective stability is defined as spectral stability with respect to a

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substitutions

$$\gamma_{11} \;=\; g\,\frac{1}{\theta}, \qquad \gamma_{22} \;=\; -g\,\theta, \qquad \gamma_{12} \;=\; g\,h\,\rho, \qquad \gamma_{21} \;=\; -g\,h\,\frac{1}{\rho}.$$

Condition (1.29) now reads

$$0 \;<\; g, \qquad 1 \;<\; \theta, \qquad 0 \;\neq\; \rho, \qquad 1 \;<\; h \;<\; \frac{1}{2}\left(\theta + \frac{1}{\theta}\right), \qquad \alpha_1 \;\neq\; \alpha_2.$$

Both conditions cover the same parameter region. The critical shock speed as well as the Hopf eigenvalues that were calculated in proposition 1.1 (1.13 – 1.14) now have the form

$$s^{\mathrm{crit}} \;=\; \frac{\theta^2\alpha_1 - \alpha_2}{\theta^2 - 1}, \qquad \mu_0 \;=\; \pm i\omega_0 \;=\; \mp i\frac{g\sqrt{h^2-1}}{\alpha_1 - \alpha_2}\frac{\theta^2 - 1}{\theta}$$

[6]See section 5 for definitions.

appropriate exponentially weighted norm. A precise definition as well as a review about basic concepts and results of spectral theory of travelling waves will be given in section 5.

Then, in section 6, the following theorem will be proven.

Theorem 1.5 *Consider the viscous balance law (1.1) under the conditions (1.7, 1.12, 1.21(iv-vi), 1.29). Then theorem 1.3 applies; suppose the elliptic case holds.*[7] *Then for small δ, $0 < \delta < \delta_0$, there exist viscous profiles (1.2, 1.4) to weak shocks, see figure 1.2. These profiles travel with speed s near s^{crit}, as defined in theorem 1.3, and connect asymptotic states $u^\pm$ of small distance, $|u^+ - u^-| < \epsilon$.*

Suppose further, that the considered profiles have extreme speed s that exceeds all characteristic speeds, in particular

$$s^{\text{crit}} \; > \; \alpha_0, \alpha_1, \alpha_2 \qquad \text{or} \qquad s^{\text{crit}} \; < \; \alpha_0, \alpha_1, \alpha_2. \tag{1.31}$$

We call the corresponding shocks hypersonic.

Then, possibly after further reduction of δ_0 and ϵ, all those travelling waves are convectively stable. The weight of convective stability, as introduced above, must possess a positive exponent for $s^{\text{crit}} > \alpha_i$ and a negative exponent for $s^{\text{crit}} < \alpha_i$.

For intermediate wave speeds, within the range of characteristic speeds, the constructed profiles cannot be stabilised by any exponential weight.[8] *They are absolutely unstable.*

Note that if we fix the signs of γ_{11} and γ_{22} as in (1.29) then we obtain the following equivalences:

$$\begin{array}{rclcrcl} s & > & \alpha_1, \alpha_2 & \Longleftrightarrow & \alpha_1 & > & \alpha_2, \\ s & < & \alpha_1, \alpha_2 & \Longleftrightarrow & \alpha_1 & < & \alpha_2. \end{array} \tag{1.32}$$

For example, under the assumption $\alpha_1 > \alpha_0, \alpha_2$ theorem 1.5 establishes convective stability of all the constructed oscillatory profiles. This result holds regardless of the source g, as long as g satisfies the assumptions of the Hopf-bifurcation theorem 1.3 to give rise to any profiles at all.

Note how the oscillatory profiles contradict the common expectations that arise from the study of hyperbolic conservation laws. Viscous profiles to weak shocks, especially stable ones, should respect the Lax criterion. The profiles which we have constructed are

[7] This is indeed possible, see for instance Corollary 1.4

[8] This is true, even if one takes different weights for positive and negative values x, in the attempt to fine-tune the concept of exponential weight.

of non-Lax nature. In particular, they are stable only for hypersonic speed (i.e. a speed that exceeds all characteristic speeds).

In the limit $\varepsilon \searrow 0$ the observed oscillations manifest themselves as overshooting at the shock layer. If such a phenomenon occurs in numerical simulations then, typically, an incapable numerical scheme is blamed for it. Numerical viscosities are used to smooth out unwanted oscillations. However, the oscillations in our example are intrinsic properties of the solution and should be represented by a numerical scheme.

1.5 The hyperbolic limit

Our problem (1.1) depends on two small parameters, ε, δ. For small values of δ our analysis holds, see theorems 1.3 and 1.5, and for small ε the constructed travelling waves becomes steep, see figure 1.2b.

The Hopf bifurcation depends on the bounded part of the spectrum of the linearised viscous-profile equation, see proposition 1.1. We could even fix ε and set $\delta = 0$. Then, the pure balance law without viscosity,

$$u_t + f(u)_x \;=\; \varepsilon^{-1} g(u), \tag{1.33}$$

still permits travelling-wave solutions with oscillatory tails. The bifurcation is indeed imposed by the interaction of flux and source. The viscosity acts only as a regularisation.

1.6 Acknowledgements

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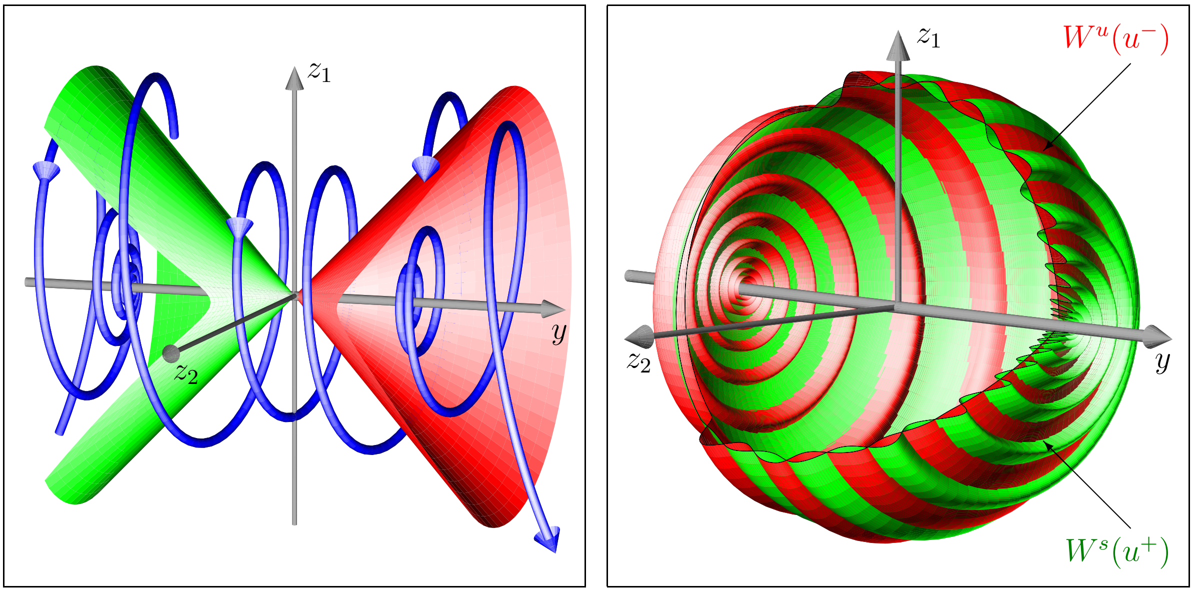
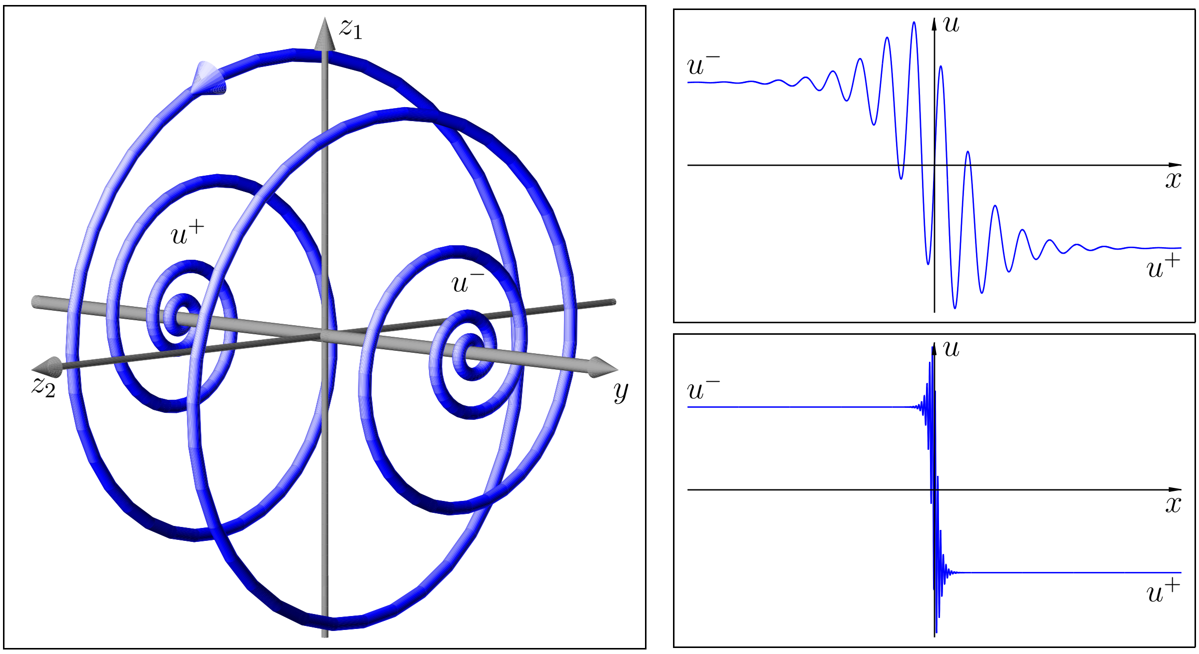


Case a) hyperbolic, $\eta = +1$.   Case b) elliptic, $\eta = -1$.

Figure 1.1: Dynamics near HOPF bifurcation from lines of equilibria.



a) heteroclinic orbit of the travelling-wave o.d.e (1.5) near the HOPF point.   b) oscillating viscous profile in the singular limit $\varepsilon \searrow 0$.

Figure 1.2: Viscous profiles emerging from the elliptic case of example (1.23).

2 Hyperbolic Conservation Laws

Conservation laws

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{m} \frac{\partial}{\partial x_i} F_i(u) \;=\; 0, \qquad x \in \mathbb{R}^m, \quad u \in \mathbb{R}^n, \tag{2.1}$$

arise in various physical models including fluid dynamics [LL59, Jos90], magneto-hydrodynamics [FS95], elasticity [KK80], multiphase flow in oil recovery [SS87, MPS97], cosmology [ST95], and many more.

The prototype of a conservation law is the one-dimensional, scalar BURGERS equation

$$\begin{aligned} u_t + (\tfrac{1}{2}u^2)_x &= 0 \qquad \text{or, alternatively,} \\ u_t + u\,u_x &= 0, \qquad x \in \mathbb{R}, u \in \mathbb{R}. \end{aligned} \tag{2.2}$$

It was introduced in [Bur40] as a model of turbulence.

In this section, we want to summarise the fundamental problems and basic methods of the theory of hyperbolic conservation laws. We shall focus on the case of one-dimensional systems, $m = 1$,

$$u_t + F(u)_x \;=\; 0, \qquad x \in \mathbb{R}, \quad u \in \mathbb{R}^n. \tag{2.3}$$

A typical requirement is the hyperbolicity of the system of conservation laws. All eigenvalues of F' should be real numbers. This guarantees the well-posedness of the associated initial-value problem.

For a more detailed introduction into the topic, see [Smo94].

2.1 Characteristics and the formation of discontinuities

Assuming the existence of a smooth solution $u(x,t)$ to the CAUCHY problem of a one-dimensional, scalar conservation law

$$\begin{aligned} u_t + F(u)_x &= 0, \qquad x \in \mathbb{R}, t \in \mathbb{R}^+, u \in \mathbb{R}, \\ u(x, t = 0) &= u_0(x), \end{aligned} \tag{2.4}$$

one easily identifies characteristic curves: Along

$$x \;=\; \phi(t), \quad \text{where} \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \;=\; F'(u(\phi(t), t)), \tag{2.5}$$

the solution $u(x,t)$ of (2.4) has to be constant. Therefore, the characteristic curves $(\phi(t),t)$ are, in fact, straight lines.

Because of the nonlinearity of the flux F, the slope of the characteristic lines depends on the initial condition u_0. Typically, there are intersections of different characteristics after finite time, such that any smooth solution breaks down at these intersections. Discontinuities, called shocks, arise, see figure 2.2. For the BURGERS equation (2.2), globally defined solutions in the class C^1 only exist if the initial function u_0 is monotonically increasing.

In systems of conservation laws, there are several characteristic fields, one for each eigendirection of F'. A typical, simplifying assumption is the *strict* hyperbolicity: all eigenvalues of F' ought to be real, simple, and *distinct*.

The information about the initial condition is transported along the characteristics. Conservation laws permit a finite speed of propagation of information.

For scalar equations, the notion of characteristics can be extended to cover intersections at discontinuities. These generalised characteristic curves prove to be a strikingly useful tool, if seen from a slightly different point of view as backward characteristics [Daf77]. From any arbitrarily chosen point (x,t) one can follow the maximal/minimal backward characteristic to find the domain that influences the value at (x,t). This property can be used to obtain a very deep insight into the structure of solutions. For scalar conservation laws *with source*, for instance, backward characteristics are an important ingredient to understand the structure of the attractor [Här99].

2.2 Weak solutions

Due to the nonlinearity, discontinuities can arise after finite time even for smooth initial data. Hyperbolic conservation laws, in general, can be solved only in a weak sense. The weak formulation of (2.1) has the form

$$\int\limits_{t\geq 0} \left(u \, \frac{\partial}{\partial t}\psi + F(u)\, \frac{\partial}{\partial x}\psi \right) \mathrm{d}x\mathrm{d}t \; + \int\limits_{t=0} u_0 \psi \, \mathrm{d}x \;=\; 0, \qquad \forall \psi \in C_0^1(\mathbb{R}\times\mathbb{R}, \mathbb{R}^n). \tag{2.6}$$

This equation imposes constraints on discontinuities, the so called RANKINE-HUGONIOT jump conditions, as we shall see now. Let u be a weak solution. Consider a point (x_0,t_0) on a smooth curve Γ of discontinuity. Assume u to be smooth near (x_0,t_0) outside Γ with well-defined limits $u^{\pm}$ on both Sides of Γ. Let s be the speed of the discontinuity, i.e. s^{-1} is the slope of Γ at u, see figure 2.1. Integrating u on a small ball around (x,t) and using

For all test functions ψ with $\operatorname{supp}\psi \subset D$, the divergence theorem gives:

$$
\begin{aligned}
0 &= \int_D u\psi_t + F(u)\psi_x \,\mathrm{d}x\mathrm{d}t \\
&= \int_{D^-} (u\psi)_t + (F(u)\psi)_x \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{D^+} (u\psi)_t + (F(u)\psi)_x \,\mathrm{d}x\mathrm{d}t \\
&= \oint_{\partial D^-} (-u\,\mathrm{d}x + F(u)\,\mathrm{d}t)\psi \\
&\quad + \oint_{\partial D^+} (-u\,\mathrm{d}x + F(u)\,\mathrm{d}t)\psi \\
&= \int_X^Y \Big((F(u^-) - F(u^+))\,\mathrm{d}t - (u^- - u^+)\,\mathrm{d}x\Big)\psi.
\end{aligned}
$$

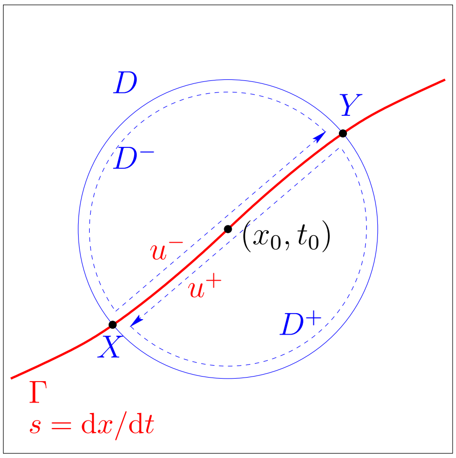


Figure 2.1: Jump condition for a shock wave

the divergence theorem gives the jump conditions

$$F(u^+) - F(u^-) \;=\; s(u^+ - u^-). \tag{2.7}$$

Unfortunately, by only considering the weak formulation of the problem we lose the uniqueness of the solution, see figure 2.3. One must impose additional constraints, so called *entropy conditions*, to identify the "physically relevant" solutions. These conditions are motivated by the second law of thermodynamics in gas-dynamical systems. Along discontinuities, the entropy increases and information gets lost. Discontinuities account for the irreversibility of the evolution induced by the conservation law. We shall discuss entropy conditions in the next section.

A second problem arises. Nonlinear transformations of the hyperbolic conservation law can change the set of solutions. Smooth solutions, of course, are not affected by transformations. In contrast, the weak formulation and the Rankine-Hugoniot condition change.

Example 2.1 *We consider the following two formulations of the same conservation law, namely the* Burgers *equation (2.2):*

$$u_t + \left(\frac{1}{2}u^2\right)_x \;=\; 0 \tag{2.8}$$

and

$$v_t + \left(\frac{2}{3}v^{3/2}\right)_x \;=\; 0. \tag{2.9}$$

They are connected by the transformation $v = u^2$. (We consider only positive solutions.)

The jump conditions (2.7) now read

$$
\begin{array}{llllll}
(2.8) & \Rightarrow & s = \dfrac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u_+ - u_-} & & = \dfrac{1}{2}(u_+ + u_-), \\
(2.9) & \Rightarrow & s = \dfrac{\frac{2}{3}v_+^{3/2} - \frac{2}{3}v_-^{3/2}}{v_+ - v_-} & = \dfrac{2}{3}\dfrac{u_+^3 - u_-^3}{u_+^2 - u_-^2} & = \dfrac{2}{3}\dfrac{u_+^2 + u_+u_- + u_-^2}{u_+ + u_-}.
\end{array}
\tag{2.10}
$$

This leads to different shock speeds. The RANKINE-HUGONIOT *conditions are* not *invariant under nonlinear transformations. We obtain different sets of weak solutions.*

However, in applications, the physical model typically has the form of an integral equation and provides the correct weak formulation along with the conservation law.

2.3 Entropy and viscosity

To ensure uniqueness of weak solutions to hyperbolic conservation laws one has to impose additional constraints. Motivated by the second law of thermodynamics, entropy conditions are a very popular class of properties. In principle, they ensure that information can only get lost in discontinuities.

Before we give a precise definition of entropy, we investigate an alternative approach to the problem of uniqueness. It is rooted in the observation that conservation laws are idealised physical models. Frequently, effects due to dissipation (like viscosity, heat conduction, and so on) and dispersion are neglected. We shall consider one of these: viscous effects. Then, one should regard the conservation law (2.3) as a singular limit $\varepsilon \searrow 0$ of a parabolic[9] equation

$$u_t + F(u)_x \;=\; \varepsilon(Bu_x)_x. \tag{2.11}$$

This approach is often called the "viscosity method". Unfortunately, for general systems the singular limit is very difficult to handle. One reason is the violation of the scaling invariance of the conservation law by its parabolic regularisation. In fact, with the scaling $(\tilde{x}, \tilde{t}) = (\varepsilon^{-1}x, \varepsilon^{-1}t)$, the limit $\varepsilon \searrow 0$ can be regarded as the long time behaviour of the parabolic equation for fixed ε.

[9]The viscous effects are expected to regularise the problem. In particular we want to obtain an equation with a well-posed CAUCHY problem. The matrix B should have eigenvalues with positive real parts. Zero eigenvalues may be allowed. Frequently, the case $B = \mathrm{id}$ is considered. However, the identity matrix, typically, does not correctly represent the physical properties of the problem.

With an artificial regularisation, $\varepsilon t(Bu_x)_x$, that respects the scaling properties, results about convergence of the singular limit were obtained by several authors [Daf74, Fan92, Tza96].

Nevertheless, viscosity approximations can be used to find appropriate admissibility criteria for shock waves.

Definition 2.2 *For systems of conservation laws (2.3), a discontinuity with speed s, connecting u^- on the left and u^+ on the right, is admissible, if there exists a sequence of travelling-wave solutions $u_\varepsilon(x,t) = u((x-st)/\varepsilon)$ to the parabolic regularisation (2.11), that converges, for $\varepsilon \searrow 0$, to the piecewise constant solution*

$$u(x,t) \;=\; \begin{cases} u^- & x < st \\ u^+ & x > st \end{cases} . \tag{2.12}$$

The parabolic regularisation (2.11) also motivates the abstract definitions of the entropy criteria.

Definition 2.3 *An entropy, entropy-flux pair of the system of conservation laws (2.3) is a pair of continuously differentiable, real-valued functions $(U = U(u), G = G(u))$ that satisfies the relation*

$$U'(u) \cdot F'(u) \;=\; G'(u). \tag{2.13}$$

As long as the solution of the system of conservation laws (2.3) remains smooth, the entropy is conserved. Indeed, we obtain an additional conservation law

$$U(u)_t + G(u)_x \;=\; U'(u)u_t + G'(u)u_x \;=\; -U'(u)F'(u)u_x + G'(u)u_x \;=\; 0, \tag{2.14}$$

that is satisfied for smooth solutions but *not* across discontinuities. Across discontinuities we require the entropy to decrease.[10]

Definition 2.4 *An entropy solution to (2.3), with respect to the entropy, entropy-flux pair (U,G), is a weak solution that, additionally, satisfies the entropy inequality*

$$U(u)_t + G(u)_x \;\le\; 0, \tag{2.15}$$

in the weak sense, i.e..

$$\int\limits_{t\ge 0} \left(U(u)\,\frac{\partial}{\partial t}\psi + G(u)\,\frac{\partial}{\partial x}\psi \right) \mathrm{d}x\mathrm{d}t \;+\; \int\limits_{t=0} U(u_0)\psi\,\mathrm{d}x \;\ge\; 0, \qquad \forall \psi \in C_0^1(\mathbb{R}\times\mathbb{R}, \mathbb{R}^+). \tag{2.16}$$

[10] This definition seems to contradict the physical notion of entropy. We can think of $-U$ as an increasing function. The point here is the monotonicity of U in t.

Adding a small viscosity term, we obtain for a solution of (2.11):

$$U(u)_t + G(u)_x \;=\; \varepsilon U'(u)(Bu_x)_x \;=\; \varepsilon (U'(u)Bu_x)_x - \varepsilon U''(u)[u_x, Bu_x]. \tag{2.17}$$

If B is the identity matrix and $U(u)$ is a *convex* function, we can conclude

$$U(u)_t + G(u)_x \;\leq\; \varepsilon U(u)_{xx}. \tag{2.18}$$

This means that the entropy condition (2.15) with a convex entropy $U(u)$ and the viscosity approximation with an identity viscosity matrix are consistent. So, one mainly is interested in convex entropies.

Definition 2.5 *A weak solution to (2.3) is called entropy-admissible, if it is an entropy solution to any possible convex entropy U and corresponding entropy flux G.*

Unfortunately, the condition (2.13) consists of n first-order equations for the two scalar unknowns U and G. If $n \geq 3$, then this system is overdetermined. In general, we expect to find nontrivial entropies only in dimensions $n \leq 2$. Nevertheless, there are important examples of larger systems that admit nontrivial entropy functions, for example systems with gradient or anti-gradient flux.

For scalar, one-dimensional conservation laws one can show that a shock connecting u^- and u^+ is entropy-admissible if, and only if,

$$\frac{F(u^+) - F(u)}{u^+ - u} \;\leq\; \frac{F(u^-) - F(u)}{u^- - u} \tag{2.19}$$

for every intermediate value $u = \alpha u^+ + (1-\alpha)u^-,\ 0 < \alpha < 1$. The slope of the secant to the graph of F through the points at u^-,u should be greater that the slope of the secant through the points at u^+,u. In fact, the shock speed s will have a value between these slopes.

If, in addition, the flux F is a convex function, as for instance in the Burgers equation (2.2), then the entropy condition simply requires that along shocks the solution only jumps downward, $u^- > u^+$.

A third admissibility criterion was introduced by Lax, [Lax57]. It has intuitive geometrical meaning.

Definition 2.6 *Consider a strictly hyperbolic system (2.3). Let $\lambda_1(u) < \cdots < \lambda_n(u)$ denote the real, distinct eigenvalues of $F'(u)$. A shock that connects u^- with u^+ and travels with speed s is* Lax*-admissible if the inequalities*

$$\lambda_{i-1}(u^-) \;<\; s \;\leq\; \lambda_i(u^-) \qquad \text{and} \qquad \lambda_i(u^+) \;\leq\; s \;<\; \lambda_{i+1}(u^+) \tag{2.20}$$

are satisfied for a suitable index i.

The LAX condition requires, that the characteristics of the i-th field are absorbed in the shock at both sides. All other characteristics cross the shock. In particular, no field of characteristics may be emitted at once from both sides of the shock. Therefore, figure 2.3 shows non-admissible shocks.

For scalar, one-dimensional conservation laws with convex flux, all these admissibility criteria are equivalent. The entropy-solution is then globally defined and unique.

Already for nonconvex fluxes, the LAX condition produces different (non-unique) results. However, the criterion can be improved [Ole59, Liu76a].

Two-dimensional systems have been investigated to some extent even in the case of non-strict hyperbolicity [Liu76b, ČP95, SS87, SPM97]. For larger systems, only solutions to initial data with small total variation are well understood [Gli65, DiP76, DiP79, BL97]. With the appropriate choice of the viscosity matrix B, equivalence of the viscosity criterion to the entropy condition for weak shocks was proven [MP85].

2.4 The Riemann problem

Before we further investigate the properties of shock waves, we want to give an overview about the elementary solution of conservation laws. As already noted before, conservation laws are invariant under the scaling $(\tilde{x}, \tilde{t}) = (cx, ct)$. It is natural to look for solutions that have this symmetry too, i.e. which remain fixed under this scaling. Eminently, the initial values $u_0(x)$ of such solutions must be constant for $x < 0$ and for $x > 0$,

$$u_0(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases} . \tag{2.21}$$

This type of CAUCHY-problem was investigated by RIEMANN [Rie92] to describe gas tube experiments. It is now called the RIEMANN problem. Its solution is used as a building block of approximate solutions to general initial data in various existence and uniqueness proofs [Gli65, GL70, Liu77] and numerical schemes, like GODUNOV and GLIMM schemes [Krö97].

In the simplest case of a scalar conservation law with convex flux function F, for instance the BURGERS equation, there are two possible types of entropy solutions to the RIEMANN problem.

(i) For $u^- > u^+$, a *shock wave*, see figure 2.4a,

$$u(x,t) \;=\; \begin{cases} u^- & x < st \\ u^+ & x > st \end{cases}, \qquad s \;=\; \frac{F(u^-) - F(u^+)}{u^- - u^+}. \tag{2.22}$$

The shock speed s is determined by the jump condition (2.7). There is only one characteristic field, with characteristic speed $F'(u)$. On both sides of the shock, characteristics must get absorbed in the discontinuity. To the left of an admissible shock with speed s characteristics must be faster, $F'(u^-) > s$, and to the right they have to be slower, $F'(u^+) < s$. The convexity of the flux, i.e. the monotone increase of F', now restricts admissible shocks to the case $u^- > u^+$.

(ii) For $u^- < u^+$, a *rarefaction wave*, see figure 2.4b,

$$u(x,t) \;=\; \begin{cases} u^- & x < s^-t \\ u(\frac{x}{t}) & s^- \le \frac{x}{t} \le s^+ \\ u^+ & x > s^+t \end{cases}, \qquad F'(u(\tfrac{x}{t})) \;=\; \tfrac{x}{t}, \quad s^\pm \;=\; F'(u^\pm). \tag{2.23}$$

The above equation is obtained easily by the ansatz $u(x,t) = u(x/t)$ for a scale-invariant solution. The monotonicity of F' guarantees the existence of a unique solution $u(x/t)$ for $u^- < u^+$ and $s^- < s^+$.

Shocks and rarefaction waves are called *elementary waves*.

For non-convex fluxes, the solution of the Riemann problem can be constructed as a sequence of elementary waves. These waves can be obtained by examining the convex hull of the graph of the flux F on the interval between the asymptotic states u^- and u^+. Subintervals, where the convex hull coincides with the graph of f, correspond to rarefaction waves. Straight lines indicate shocks, see figure 2.5. Due to the entropy condition (2.19), the hull has to be taken above the graph of F for $u^- > u^+$ and below for $u^- < u^+$.

Systems of n conservation laws (in one space dimensions) cause some additional difficulties. There are now elementary waves associated to every characteristic field:

(i) Shock waves

$$s(u^- - u^+) \;=\; F(u^-) - F(u^+) \;=\; F'(u^-)(u^- - u^+) + \mathcal{O}\left((u^- - u^+)^2\right). \tag{2.24}$$

At least for weak shocks, $|u^- - u^+| \ll 1$, the difference of the asymptotic states, $u^- - u^+$, must be close to an eigenvector of F'. The speed s is then close to the corresponding eigenvalue, i.e. the characteristic speed. In fact, for fixed u^-, the shocks can be seen as emerging from bifurcations at speeds s, for which $(F'(u^-)-s)$ becomes singular.

(ii) Rarefaction fans

$$F'(u(\tfrac{x}{t}))u'(\tfrac{x}{t}) \;=\; \tfrac{x}{t}u'(\tfrac{x}{t}). \tag{2.25}$$

As long as $F'(u(s = x/t))$ has a unique right i-th eigenspace, $\operatorname{span} r_i(u(s))$, that is not orthogonal to the gradient of the eigenvalue $\lambda_i(u(s))$, we obtain for the i-th family the evolution

$$u'(s) \;=\; \frac{r_i(u(s))}{\langle \lambda_i'(u(s)), r_i(u(s))\rangle}. \tag{2.26}$$

Given u^-, this equation can be solved for $s^- < s < s^+$ with initial values $s^- = \lambda_i(u^-)$, $u(s^-) = u^-$.

For *strictly* hyperbolic systems, the wave speeds of different characteristic fields are ordered according to the distinct eigenvalues of F'. Notably, there are always unique eigenspaces, $\operatorname{span} r_i(u)$, that depend continuously on u.

However, the above constructions not necessarily result in well-defined curves that are parametrised by the speed s. We need an additional assumption:

Definition 2.7 *The i-th characteristic field is called* genuinely nonlinear*, if the gradient of the i-th eigenvalue is nowhere orthogonal to the right eigenvector,*

$$\forall u: \;\; \langle \lambda_i'(u), r_i(u)\rangle \;\neq\; 0. \qquad \textit{(GNL)}$$

On the contrary, the field is called linearly degenerate (LD)*, if the gradient of the eigenvalue and the matching eigenvalue are everywhere orthogonal,*

$$\forall u: \;\; \langle \lambda_i'(u), r_i(u)\rangle \;=\; 0. \qquad \textit{(LD)}$$

Genuine nonlinearity is a generalisation of the convexity of the flux in scalar conservation laws. If the i-th field is genuinely nonlinear, then the jump condition (2.24) and the o.d.e. (2.26) define curves in the space $\mathbb{R}^n$ of values of u. Points on these curves can be connected by a single elementary wave. The curves are parametrised by s. Both curves, $R_i(u^-)$, $S_i(u^-)$, that belong to a fixed value u^- and family i, intersect at u^- with a quadratic tangency, see figure 2.6. The admissibility of connections on these curves is controlled by the sign of $\langle \lambda_i', r_i\rangle$, for rarefaction fans, and by the entropy condition or the viscosity criterion, for shocks.

Linear degeneracy, though it is a very non-generic assumption, is always satisfied for the second field in the important EULER equations of gas dynamics. It produces so called *contact discontinuities*. The wave curve that is imposed by a linearly degenerate field is a straight line in direction of the eigenspace, span r_i. Any two points, u^- and u^+, on this line are connected by a shock-like discontinuity that travels at speed $s = \lambda_i$, independent of u^-, u^+. For a linearly degenerate field i the rarefaction curve R_i and the shock curve S_i are not parametrised by s anymore and coincide identically. The characteristics on both sides of a contact discontinuity are parallel to it.

The fundamental idea in now to construct a solution to the RIEMANN problem by concatenating elementary waves[11] of the various characteristic fields. To accomplish this, one has to find a path

$$u^- = u^{[1]}, u^{[2]}, \dots, u^{[n+1]} = u^+, \tag{2.27}$$

where $u^{[i]}, u^{[i+1]}$ belong to a common wave curve (shock or rarefaction) to the i-th family in an admissible direction, see figure 2.6.

For weak initial discontinuities, $|u^+ - u^-|$ small, this is possible, provided that the system is strictly hyperbolic and all characteristic fields are either genuinely nonlinear or linearly degenerate. Indeed, the implicit function theorem provides us with the sequence (2.27). In this case, from any point in $\mathbb{R}^n$ any other point nearby can be reached by a path of up to n pieces of elementary wave curves. The strict hyperbolicity ensures that the speeds of the resulting elementary waves are strictly ordered according to their wave numbers. (Remember that $|u^+ - u^-|$ is small. Therefore the wave speeds can only slightly differ from the eigenvalues at u^-.)

Regrettably, for general initial conditions this is not always possible. From a starting point u^- not all points in $\mathbb{R}^n$ may be reachable by a path of pieces of elementary wave curves. Even if there exists a path, the wave speeds may no longer be ordered. For example, a rarefaction fan may become broad enough to intersect with an adjacent shock. The concatenation of the constructed elementary waves may not result in a well defined solution.

In nonstrictly hyperbolic systems, it is even for weak initial discontinuities sometimes necessary to introduce additional elementary waves, see, for example, [IMP90, SS91, SPM97].

[11] A possible generalisation would be the use of composite wave as described before in the case of a scalar conservation law with non-convex flux.

2.5 Viscous profiles of shock waves

We conclude this review with some remarks about the viscosity criterion for shock waves, see definition 2.2.

We look for shock solutions

$$u(x,t) \;=\; \begin{cases} u^- & x < st \\ u^+ & x > st \end{cases} \tag{2.28}$$

that admit a viscous profile. A viscous profile is a solution of the form

$$u \;=\; u(\xi) \;=\; u\left(\frac{x-st}{\varepsilon}\right) \tag{2.29}$$

to the parabolic regularisation (2.11). With this ansatz, we obtain an ordinary differential equation in the free variable ξ:

$$-s\frac{\mathrm{d}}{\mathrm{d}\xi}u(\xi) + \frac{\mathrm{d}}{\mathrm{d}\xi}F(u(\xi)) \;=\; \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\xi). \tag{2.30}$$

The scaling of ξ renders equation (2.30) independent of ε. Equation (2.30) can be integrated once. We obtain the first-order equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u(x) \;=\; -su(\xi) + F(u(\xi)) + C, \tag{2.31}$$

with some constant C.

In the limit $\varepsilon \searrow 0$, the profile $u(\cdot)$ converges to the shock solution (2.28) if, and only if,

$$\lim_{\xi\to\pm\infty} u(\xi) \;=\; u^{\pm}. \tag{2.32}$$

Limit sets of a trajectory of a dynamical system are flow-invariant. Here, the limit sets are isolated points. These points must be equilibria of (2.31). We obtain the conditions

$$\begin{aligned} C \;&=\; su^- - F(u^-) \qquad \text{and} \\ C \;&=\; su^+ - F(u^+). \end{aligned} \tag{2.33}$$

Note that (2.33) must be satisfied *simultaneously.* We have arrived at the Rankine-Hugoniot condition (2.7). This shows again the consistency of the weak formulation (2.6) and the viscosity method.

Consider asymptotic states $u^\pm$ such that the Rankine-Hugoniot condition (2.7) holds. Then, the shock solution (2.28) admits a viscous profile if, and only if, the dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}u(x) \;=\; F(u(\xi)) - su(\xi) - \Big(F(u^-) - su^-\Big) \tag{2.34}$$

contains a heteroclinic orbit that connects u^-, for $\xi \to -\infty$, with u^+, for $\xi \to +\infty$. The unstable manifold of u^- and the stable manifold of u^+ must intersect.

The question about viscous profiles can be regarded as an global bifurcation problem that depends on the parameters s, F, u^- [Sch92, FS95].

Consider for instance the linearisation of (2.34) at the asymptotic fixed points:

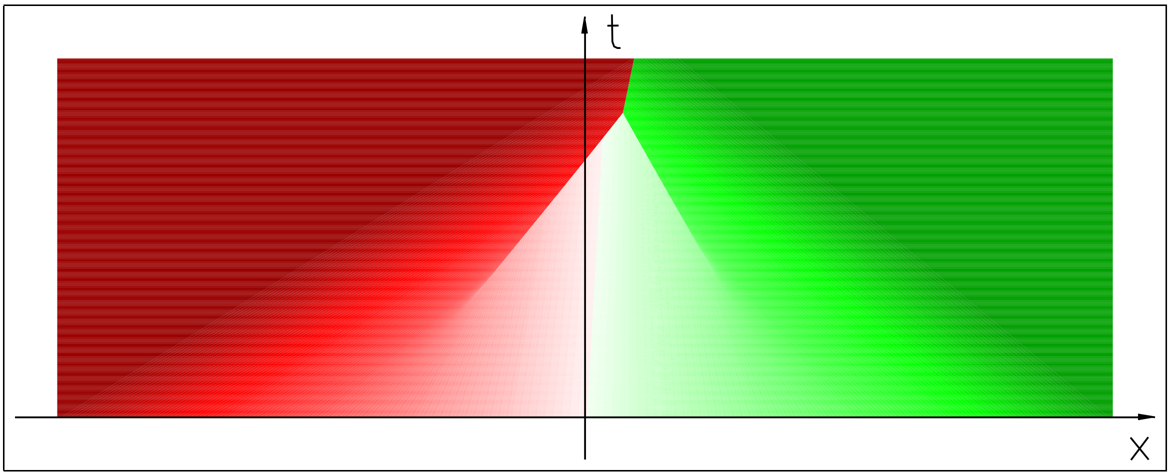
$$F'(u^\pm) - s. \tag{2.35}$$

The LAX admissibility criterion (2.20) requires that u^- has an unstable eigenspace of dimension $n - i + 1$ and u^+ has an stable eigenspace of dimension i. Generically, any intersection of the $n - i + 1$ dimensional unstable manifold of u^- and the i dimensional stable manifold of u^+ persist under small variations of the parameters s, F, u^-. The LAX criterion is also a structural stability condition.

Fix u^-. The linearisation $F'(u^-) - s$ becomes singular, if s is an eigenvalue of F'. For strictly hyperbolic systems, a single real eigenvalue λ_i becomes zero. A bifurcation of equilibria occurs. Generically[12], for s near λ_i a family of equilibria $u^+(s)$ near u^- exists. The bifurcating equilibrium has an unstable dimension that differ by one from the unstable dimension of the original equilibrium. For any s near the bifurcation point a connecting heteroclinic orbit exists that correspond to a LAX shocks. Weak LAX shocks generically admit viscous profiles.

Viscous profiles to strong shocks are very difficult to obtain. Topological methods, for example the CONLEY index may help [CS71, CS72, CS73]. In [FS95] a bifurcation analysis for magnetohydrodynamic shock waves was carried out.

[12] The i-th field must be genuinely nonlinear near u^-.



The first picture shows the solution to the initial data in the second picture. There is a locally smooth solution. After finite time, two shocks are formed by intersecting characteristics. These discontinuities then interact and form a single shock curve. The u-values are represented by the colour. Characteristics are observable as lines of constant colour.

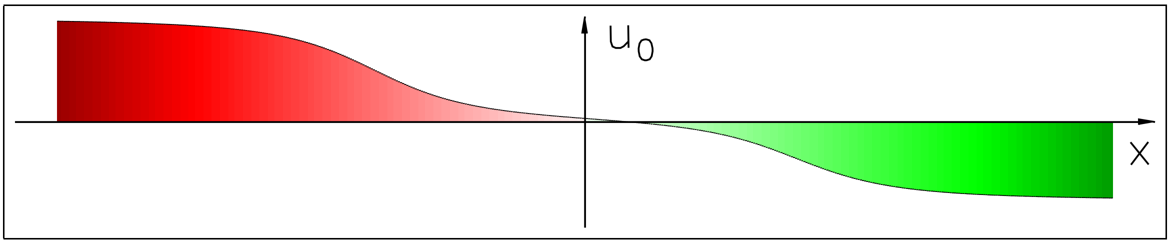
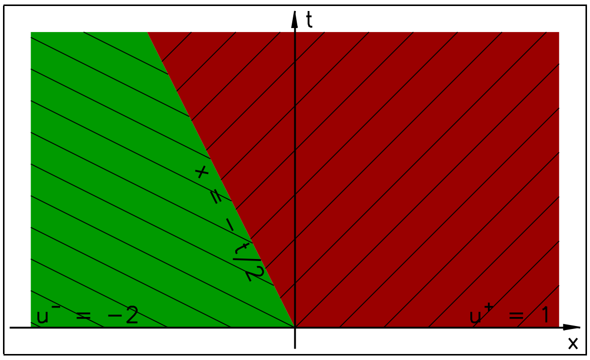


Figure 2.2: Shock formation for the BURGERS equation (2.2) and smooth initial data

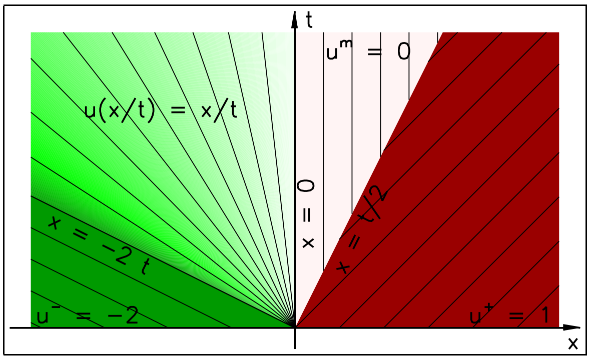
Here, two weak solutions (2.6) of the BURGERS equation (2.2) are shown. The two solutions have the same initial data,

$$u_0(x) = \begin{cases} -2 & x < 0 \\ 1 & x > 0 \end{cases}$$

of a RIEMANN problem, see (2.21). Without an additional constraint, the solution cannot be unique.
In fact, the entropy condition (2.19) will reject both of the sketched solutions. Only the pure rarefaction wave, figure 2.4b, will remain for the above initial values.

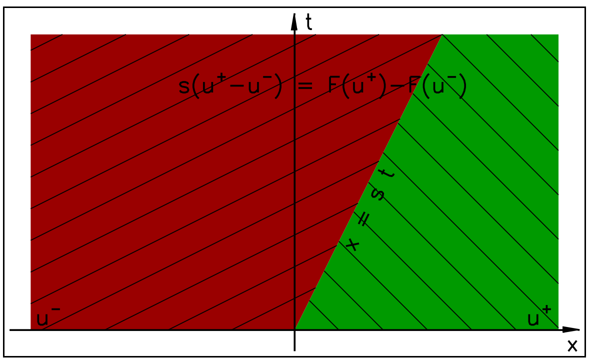


a) a shock wave

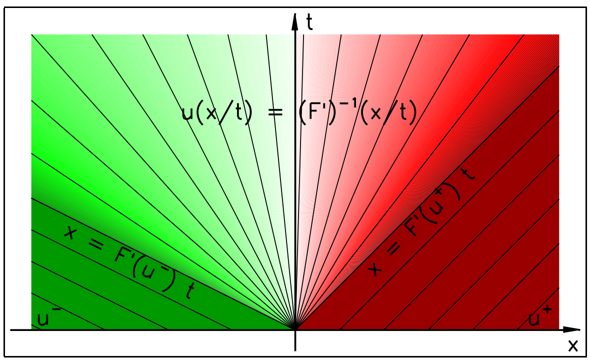


b) a rarefaction fan and a shock

Figure 2.3: Non-unique weak solutions to the BURGERS equation

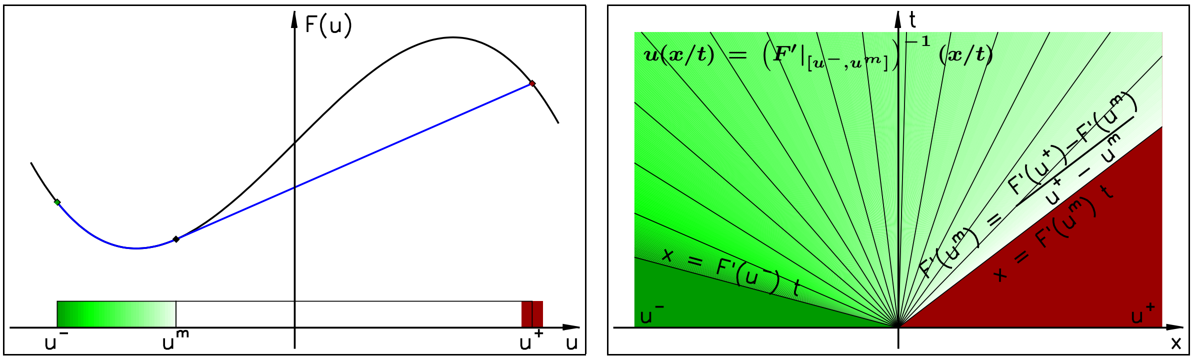


a) shock wave



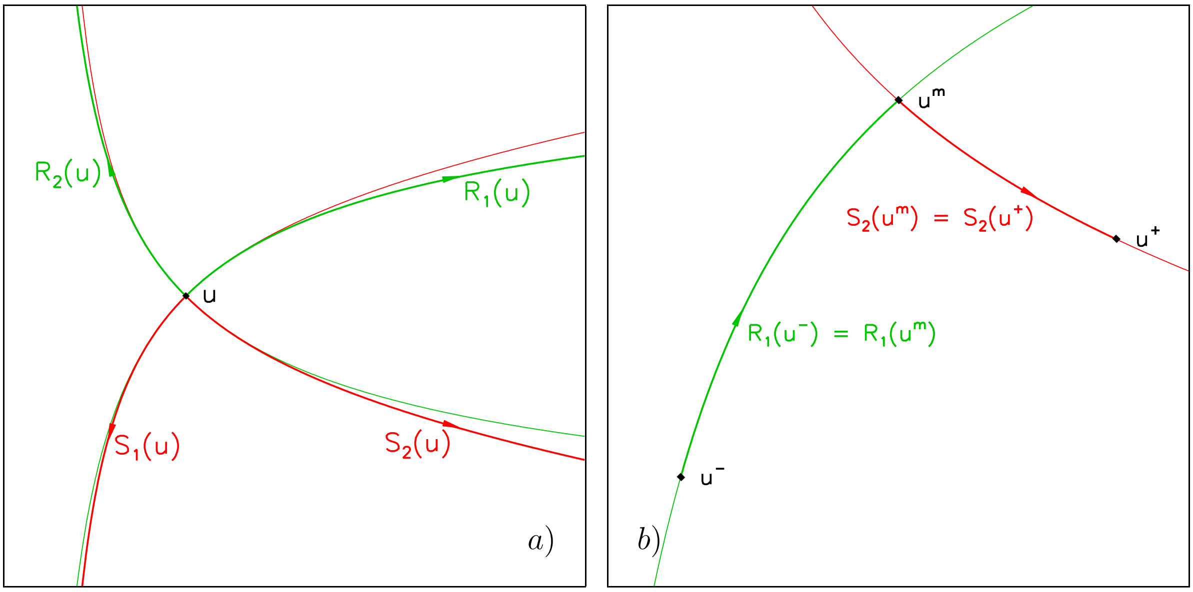
b) rarefaction fan

Figure 2.4: Elementary waves for a scalar conservation law



convex hull of the graph of the flux   corresponding solution

Figure 2.5: Solution of the RIEMANN problem for a single, scalar conservation law



In the figure a) the RANKINE-HUGONIOT curves, $S_i(u)$, and the rarefaction curves, $R_i(u)$, of both characteristic families to a given value $u \in \mathbb{R}^2$ are sketched.
The connection from u^- along $R_1(u^-)$ to u^m and along $S_2(u^m)$ to u^+ in figure b) leads to the solution shown in figure c).

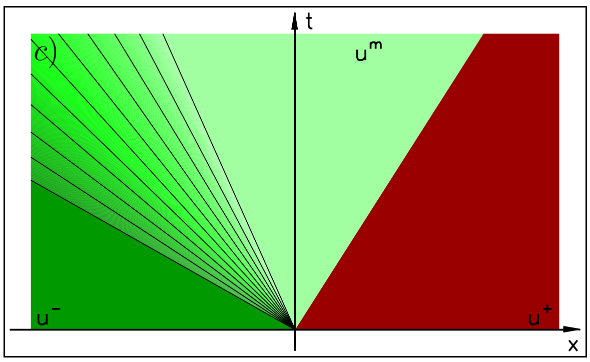


Figure 2.6: RIEMANN solution as a concatenation of elementary waves of different fields

3 Bifurcation From Lines Of Equilibria

Here it is necessary to discuss a bit more the properties of Hopf-bifurcations from lines of equilibria. Also, we shall give an outline of the proof of theorem 1.2. Please refer to [FLA98a] for further details.

As in the introduction, we consider C^5 vector fields

$$\dot{\mathbf{u}} \;=\; F(\mathbf{u}), \qquad \mathbf{u} \in \mathbb{R}^N, \tag{3.1}$$

with a line of equilibria that coincides with the u_0-axis, see (1.16, 1.17). We further assume that $F'(0)$ possesses a pair of Hopf-eigenvalues that satisfy the transversality condition (1.18).

We want to investigate the dynamics near the Hopf-point. Therefore, we can restrict our attention to the three-dimensional centre manifold at the origin.[13] This manifold is normally hyperbolic, all bounded solution near the critical point are contained in the centre manifold. To avoid unnecessary notation, we assume that F is already the reduced flow on the centre manifold, in particular $N = 3$. The basis of $\mathbb{R}^3$ should be aligned with the eigenvectors of $F'(y, 0, 0)$ along the y-axis of equilibria, at least near the origin.

3.1 Normal form

As a first step, we calculate a normal form of F. We use the coordinates

$$\mathbf{u} \;=\; (y, z) \in \mathbb{R} \times \mathbb{C}, \tag{3.2}$$

where the line of equilibria coincides with the y-axis. The coordinate $z \in \mathbb{C} = \mathbb{R}^2$, which we shall also write as $re^{i\varphi}$ in polar coordinates, parametrises the generalised eigenspace to the Hopf-eigenvalues $\pm i\omega$.

The linearisation of the flow at the origin has the form

$$L \;=\; F'(0) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & i\omega \end{pmatrix}. \tag{3.3}$$

It is equivariant with respect to the S^1 action of $\exp(A\varphi)$ which is a rotation around the y-axis imposed by the Hopf-eigenvalues. The line of equilibria remains fixed under this action.

---

[13] For details about centre manifolds, please refer to [GH82, Van89].

The remaining terms can be simplified by a normal-form algorithm. We use normal-form theory as presented, for example, in [Van89]. Due to the semisimple spectrum of L, in each normal-form step, the remaining part is given simply by the average

$$\tilde{h}(\mathbf{u}) \;:=\; \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \exp(-L\varphi)h(\exp(L\varphi)\mathbf{u})\mathrm{d}\varphi \tag{3.4}$$

along a period of the rotation around the y-axis. Here h represents n-th order terms before normal-form transformation, which become $\tilde{h}$, in normal form. The necessarily associated transformation in $\mathbf{u}$, however, modifies higher order terms in each step. Therefore, h does not coincide with F in general, except for terms of second order.

The aforementioned normal-form procedure results in a flow

$$\dot{\mathbf{u}} \;=\; \tilde{F}(\mathbf{u}) + \vartheta(u). \tag{3.5}$$

The error terms ϑ can be pushed to any finite order $\mathcal{O}(|\mathbf{u}|^K)$, as long as F is smooth enough.

The flow $\tilde{F}$, that is given by the truncated normal form, obtains some structural properties from the averaging procedure (3.4). Mainly, it is equivariant with respect to the S^1-action

$$\tilde{F}(\exp(L\varphi)\mathbf{u}) \;=\; \exp(L\varphi)\tilde{F}(\mathbf{u}). \tag{3.6}$$

In particular, the line of equilibria and the critical point at the origin are preserved:

$$\tilde{F}(y, z=0) \equiv 0, \qquad \tilde{F}'(0,0) = 0. \tag{3.7}$$

In polar coordinates, $z = e^{i\varphi}$, the truncated normal form $\tilde{F}$ is independent of φ:

$$\tilde{F}(\mathbf{u}) \;=\; \tilde{F}(y, r^2). \tag{3.8}$$

Together with (3.7), we can now specify the normal form of the original flow (3.1):

$$\begin{aligned} \dot{y} &= r^2\,\tilde{F}^y(y,r^2) &+& \quad\vartheta^y(y, re^{i\varphi}),\\ \dot{r} &= ry\,\tilde{F}^r(y,r^2) &+& \quad\vartheta^r(y, re^{i\varphi}),\\ \dot{\varphi} &= \tilde{F}^\varphi(y,r^2) &+& \; r^{-1}\vartheta^\varphi(y, re^{i\varphi}). \end{aligned} \tag{3.9}$$

Please note that the constant terms of the first two equations vanish because the y-axis consists of equilibria. The factors of $\tilde{F}^{y,r}$ are imposed by the equivariance relation (3.6). The error terms $\vartheta^{y,r,\varphi}$ are of order K and vanish along the line of equilibria:

$$\vartheta \;=\; \mathcal{O}((|y|+|r|)^K), \qquad \vartheta^y(y,0) \;=\; \vartheta^r(y,0) \;=\; r^{-1}\vartheta^\varphi(y,0) \;=\; 0. \tag{3.10}$$

At the origin, the normal form (3.9) has the same linear part as the original vector field. Along the line of equilibria, the linear parts also match because we have aligned the coordinates with the eigenvectors. Therefore, $F'(y,0)$ commutes with $F'(0,0) = L$ and the averaging (3.4) leaves the linear part, $F'(y,0)$, untouched. We conclude that

$$\begin{array}{rclcl} \tilde{F}^r(0,0) & = & \partial_y \Re\mathrm{e}\,\mu(0) & = & \Re\mathrm{e}\,\mu'(0) \\ \tilde{F}^\varphi(0,0) & = & \Im\mathrm{m}\,\mu(0) & = & \omega. \end{array} \tag{3.11}$$

Finally, we calculate the first averaging step at the origin and obtain the value of $\tilde{F}^y(0,0)$:

$$\begin{array}{rcl} \tilde{F}^y(0,0) & = & \frac{1}{2}\partial_r^2 \Big(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} F^y(y, r\cos(\omega\varphi), r\sin(\omega\varphi))\mathrm{d}\varphi \Big)\Big|_{(y=0,r=0)} \\ & = & \frac{1}{4}\Delta_z F^y(0,0). \end{array} \tag{3.12}$$

The requirement that $\tilde{F}^y(0,0)$ does not vanish exactly matches the nondegeneracy condition (1.19).

We see, that the transversality condition (1.18) and the nondegeneracy condition (1.19) are generically fulfilled.[14] They require, that in the normal form the lowest order terms that could be nonzero are nonzero indeed.

Consider, again, the case of the usual type of Hopf-bifurcation that is described for example in [MM76]. Then y is a parameter, i.e. $F^y \equiv 0$. The nondegeneracy condition (1.19) fails. The phase space is foliated in planes $y =$ constant which are invariant under the flow F. The same holds true for the vector field $\tilde{F}$ in normal form. A one-dimensional family of periodic orbits emerges from the bifurcation point.

Even in the general case, any foliation that is transversal to the line of equilibria would be, after normal-form transformation, equivariant with respect to the rotation $\exp(L\varphi)$. This means that the planes $y =$ constant would be invariant under the truncated normal form, i.e. $\tilde{F}^y \equiv 0$. Therefore, the nondegeneracy condition (1.19) excludes the case of the usual Hopf-bifurcation. It requires that any foliation fails already in the second order approximation.

---

[14] The term "generic" means the following. Let X be the set of all vector fields F with a line of equilibria through the origin and a pair of purely imaginary eigenvalues of F' at the origin. Let $Y \subset X$ be set of all vector fields that, additionally, satisfy the conditions (1.18) and (1.19). Then Y is of the second Baire category with respect to X. Y is the intersection of countably many open and dense subsets of X. Topologically, Y is *thick*. In particular, Y is dense in X.

3.2 Simplified normal form

Under the assumption of transversality (1.18) and nondegeneracy (1.19), $\tilde{F}^y, \tilde{F}^r, \tilde{F}^\varphi$ are nonzero at the origin. We can find a C^1 diffeomorphism of the phase space, such that in a suitably rescaled time the normal form (3.9) is simplified to

$$\begin{array}{rcrcl} \dot{y} &=& \frac{1}{2}\eta r^2 &+& \theta^y(y,r,\varphi), \\ \dot{r} &=& ry &+& \theta^r(y,r,\varphi), \\ \dot{\varphi} &=& \omega, && \end{array} \tag{3.13}$$

with new error terms $\theta = \mathcal{O}((|y|+|r|)^K)$ that vanish on the y-axis. See [FLA98a] for a proof. The parameter η can be scaled to ± 1 and has the sign

$$\begin{array}{rcl} \eta &=& \operatorname{sign} \tilde{F}^r(0,0) \;\cdot\; \operatorname{sign} \tilde{F}^y(0,0) \\ &=& \operatorname{sign}(\Re\mathrm{e}\, \mu'(0)) \;\cdot\; \operatorname{sign}(\Delta_Z F^y(0,0)), \end{array} \tag{3.14}$$

that coincides with our previous definition (1.20).

The truncated normal form in independent of φ and can be regarded as a flow on the (y,r)-plane:

$$\begin{array}{rcl} \dot{y} &=& \frac{1}{2}\eta r^2, \\ \dot{r} &=& ry. \end{array} \tag{3.15}$$

If we transform this planar vector field with an Euler multiplier r^{-1}, we obtain a linear system

$$\begin{array}{rcl} \dot{y} &=& \frac{1}{2}\eta r, \\ \dot{r} &=& y. \end{array} \tag{3.16}$$

The linear equations have the same trajectories as (3.15), while the direction is reversed for $r < 0$. In the reduced system, the origin will be a hyperbolic equilibrium for $\eta = +1$ and an elliptic equilibrium for $\eta = -1$. This motivates the names for the two types of the Hopf-bifurcation in theorem 1.2. In figure 3.1, the planar phase portraits of the truncated normal form are shown. They are superimposed by the rotation $\dot{\varphi} = \omega$.

3.3 Higher order terms

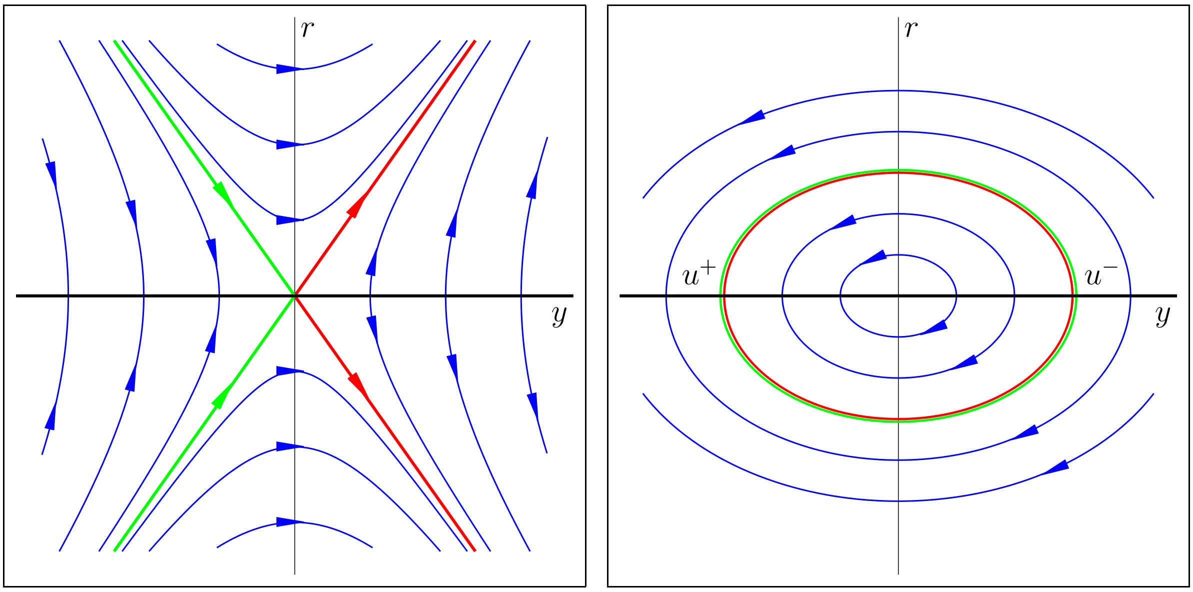
The control of the error terms in (3.13) is now the main issue in the proof of theorem 1.2. To accomplish this, a spherical blow-up of the origin can be used. Details are made explicit in [FLA98a]. A general approach to blow-up constructions can be found in [DR98]. Here, we only want to remark about exponentially small splittings in the elliptic case, $\eta = -1$, in which we are interested here.

The time-ω map of the planar vector field (3.15) coincides with the POINCARÉ map of the truncated normal form (3.13 without the error terms). In figure 3.1b the heteroclinic orbits connecting u^- and u^+, for the planar vector field, now must be regarded as coinciding stable and unstable manifolds of $u^\pm$, for the POINCARÉ map.

For a *map*, the coincidence of stable and unstable manifolds is a highly non-generic property. Typical error terms will lead to a splitting of the manifolds in the POINCARÉ map of the complete system (3.13 with the error terms). The stable and unstable manifold then look like in figure 3.2. Any non-constant trajectory of the POINCARÉ map consists of isolated points on the intersection of the unstable manifold of an unstable equilibrium u^-, shown in red, and the stable manifold of a stable equilibrium u^+, shown in green. The corresponding three-dimensional picture is displayed in figure 1.1b.

This splitting leads to some bizarre peculiarities of the oscillatory viscous profiles that we have constructed in theorem 1.3. Any heteroclinic orbit in the travelling-wave equation near the HOPF point gives rise to a viscous profile of the original balance law. In contrast to the common expectation, for fixed speed s and fixed asymptotic state u^- on one side, there are connections to an entire interval $(u^+_{\min}, u^+_{\max})$ on the other side. The decision of which asymptotic state is chosen by a particular profile depends on the phase shift φ of the superimposed rotation. In the limit of small ε, the oscillations of the profile (that provide the information about the phase shift) are hidden in the very thin shock layer, see figure 1.2b. In an experiment or a numerical calculation, one may be unable to resolve this shock layer. The mutual independence of the asymptotic states would be a very unsuspected observation.

Nevertheless, with NEISHTADT's theorem on exponential elimination of rapidly rotating phases, [Nei84], one can establish an upper bound on the size of the above interval $(u^+_{\min}, u^+_{\max})$. In fact, it is exponentially small in terms of $|u^- - u^+|$, provided f, g are real analytic functions. The described effect may be invisible for profiles of weak shocks. Profiles with larger asymptotic states are beyond the local analysis of this paper. However, they might still exist.



Case a) hyperbolic, $\eta = +1$. Case b) elliptic, $\eta = -1$.

Figure 3.1: Normal-form flow (3.15) near a HOPF bifurcation from a line of equilibria.

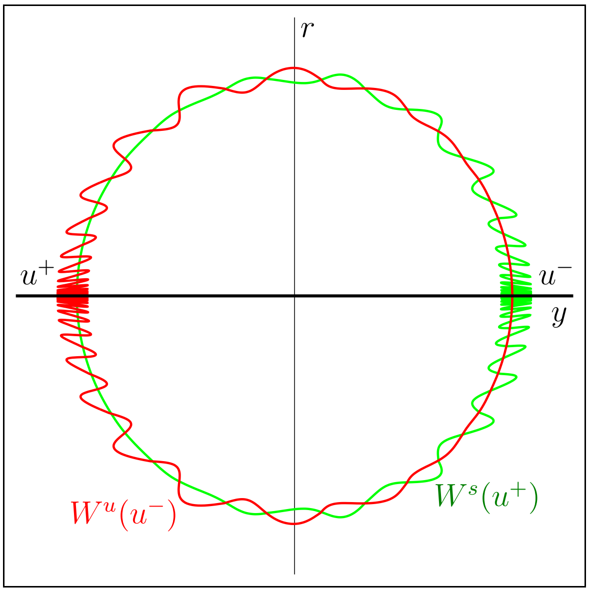


Figure 3.2: Transverse splitting due to higher order terms in (3.13).

4 Oscillatory Viscous Shocks

In this section, we prove the theorems mentioned in the introduction that establish the Hopf bifurcation along the line of equilibria. The considerations are similar to those in [FL98].

4.1 Hopf point

Proof of proposition 1.1: Regular perturbation theory, see for example [Kat66], applies to the scaled block matrix

$$\delta L \;=\; \begin{pmatrix} 0 & 0 \\ -g' & A-s \end{pmatrix} + \delta \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

which becomes a lower triangular matrix for $\delta = 0$. This provides us with the unbounded part $\mathrm{spec}_\infty L$ of the spectrum, generated in v-space alone (with $u = 0$).

Moreover δL possesses a three-dimensional kernel, at $\delta = 0$, given by

$$g'u \;=\; (A-s)v. \tag{4.2}$$

On this kernel, the eigenvalue problem for L reduces to

$$\mu_0 u \;=\; v \;=\; (A-s)^{-1} g' u. \tag{4.3}$$

We insert the assumption (1.12). Then, the characteristic polynomial of (4.3) at $u_0 = 0$ is given by

$$p_0(\mu) \;=\; \mu\left(\mu^2 - \frac{(\gamma_{11}+\gamma_{22})s - \alpha_1\gamma_{22} - \alpha_2\gamma_{11}}{(s-\alpha_1)(s-\alpha_2)}\mu + \frac{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}{(s-\alpha_1)(s-\alpha_2)}\right), \tag{4.4}$$

which for

$$s \;=\; s^{\mathrm{crit}} \;=\; \frac{\alpha_1\gamma_{22} + \alpha_2\gamma_{11}}{\gamma_{11}+\gamma_{22}}$$

becomes

$$\begin{aligned} p_0^{\mathrm{crit}}(\mu) \;&=\; \mu\left(\mu^2 + \frac{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}{(s^{\mathrm{crit}}-\alpha_1)(s^{\mathrm{crit}}-\alpha_2)}\right) \\ &=\; \mu\left(\mu^2 + \frac{(\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21})(\gamma_{11}+\gamma_{22})^2}{(\alpha_2-\alpha_1)\gamma_{11}(\alpha_1-\alpha_2)\gamma_{22}}\right) \\ &=\; \mu\left(\mu^2 + \frac{(\gamma_{11}+\gamma_{22})^2}{(\alpha_2-\alpha_1)^2}\left(\frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} - 1\right)\right). \end{aligned} \tag{4.5}$$

Under the hypothesis $1 < \gamma_{12}\gamma_{21}/\gamma_{11}\gamma_{22}$, we obtain two purely imaginary eigenvalues as claimed in the proposition. Note that $s^{\text{crit}} \neq \alpha_1, \alpha_2$, because $\alpha_1 \neq \alpha_2$ and $\gamma_{11}\gamma_{22} \neq 0$.

The eigenvectors can be obtained by direct calculation. We shall not use the explicit form of the eigenvectors in the following considerations and skip this part, here. ∎

4.2 Linearisation and transverse eigenvalue crossing

Here we continue our linear analysis of the linearisation

$$L^\delta(u_0) \;=\; \begin{pmatrix} 0 & \text{id} \\ -\delta^{-1}g' & \delta^{-1}(A - s^{\text{crit}}), \end{pmatrix} \tag{4.6}$$

with $A = A(u)$ and $g' = g'(u)$ evaluated along the line of equilibria $u = (u_0, 0, 0)$, see (1.9) and proposition 1.1. In the limit $\delta \searrow 0$, we address the issue of transverse crossing of purely imaginary eigenvalues in lemma 4.1. In lemma 4.2, we explicitly compute the one-dimensional eigenprojection P_0^δ onto the trivial kernel of $L^\delta(0)$.

By proposition 1.1, purely imaginary eigenvalues of $L^\delta(u_0)$ arise from an $\mathcal{O}(\delta)$ perturbation of the matrix

$$(A(u_0,0,0) - s^{\text{crit}})^{-1}g'(u_0,0,0) \;=\; \Big(A_0 - s^{\text{crit}} + u_0A_1 + \mathcal{O}(|u_0^2|)\Big)^{-1} g'(0) \tag{4.7}$$

with spectrum $\text{spec}_{bd}\, L$. Note that g is assumed to be independent of u_0, see (1.7)

Let $\mu(u_0), \bar\mu(u_0)$ denote the continuation of the pair of simple, purely imaginary eigenvalues

$$\mu(0) \;=\; i\omega_0, \quad \bar\mu(0) \;=\; -i\omega_0, \tag{4.8}$$

with u_0-derivatives $\mu'(u_0)$, $\bar\mu'(u_0)$.

Lemma 4.1 *In the above setting and notation, the derivative of the real part of the* Hopf *eigenvalue is given by*

$$\Re\text{e}\,\mu'(0) \;=\; \frac{1}{2}\frac{(\gamma_{11}+\gamma_{22})^2}{(\alpha_2-\alpha_1)^2}\frac{\beta_{12}\gamma_{21}+\beta_{21}\gamma_{11}-\beta_{11}\gamma_{22}-\beta_{22}\gamma_{11}}{\gamma_{11}\gamma_{22}}. \tag{4.9}$$

Proof: Since the unit vector e_0 in u_0-direction is a trivial kernel vector of $g'(0)$ and since the the remaining eigenvalues of $(A - s^{\text{crit}})^{-1}g'$ remain conjugate complex for small $|u_0|$, we can start with the expression

$$\Re\text{e}\,\mu(u_0) \;=\; \frac{1}{2}\,\text{trace}\left((A - s^{\text{crit}})^{-1}g'\right). \tag{4.10}$$

In particular, $\mathrm{trace}((A_0 - s^{\mathrm{crit}})^{-1} g') = 0$. With the u_0-expansion

$$\begin{aligned} (A-s)^{-1} &= \left(A_0 - s + u_0 A_1 + \mathcal{O}(|u_0|^2)\right)^{-1} \\ &= (A_0-s)^{-1} - u_0 (A_0-s)^{-1} A_1 (A_0-s)^{-1} + \mathcal{O}(|u_0|^2) \end{aligned} \tag{4.11}$$

we immediately obtain

$$\Re\mathrm{e}\, \mu'(u_0) \;=\; -\frac{1}{2}\, \mathrm{trace}\left((A_0 - s^{\mathrm{crit}})^{-1} A_1 (A_0 - s^{\mathrm{crit}})^{-1} g'\right). \tag{4.12}$$

Inserting A_0, A_1, g', and s^{crit}, see (1.12, 1.13), yields

$$\begin{aligned} &(A_0 - s^{\mathrm{crit}})^{-1} A_1 (A_0 - s^{\mathrm{crit}})^{-1} g' \\ &= \frac{(\gamma_{11}+\gamma_{22})^2}{(\alpha_1-\alpha_2)^2} \begin{pmatrix} * & & \\ & \frac{1}{\gamma_{11}} & \\ & & -\frac{1}{\gamma_{22}} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & \beta_{11} & \beta_{12} \\ * & \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} * & & \\ & \frac{1}{\gamma_{11}} & \\ & & -\frac{1}{\gamma_{22}} \end{pmatrix} \begin{pmatrix} 0 & & \\ & \gamma_{11} & \gamma_{12} \\ & \gamma_{21} & \gamma_{22} \end{pmatrix} \\ &= \frac{(\gamma_{11}+\gamma_{22})^2}{(\alpha_1-\alpha_2)^2} \begin{pmatrix} * & * & * \\ * & \frac{\beta_{11}}{\gamma_{11}} & \frac{\beta_{12}}{\gamma_{11}} \\ * & -\frac{\beta_{21}}{\gamma_{22}} & -\frac{\beta_{22}}{\gamma_{22}} \end{pmatrix} \begin{pmatrix} 0 & & \\ & 1 & \frac{\gamma_{12}}{\gamma_{11}} \\ & -\frac{\gamma_{21}}{\gamma_{22}} & -1 \end{pmatrix} \\ &= \frac{(\gamma_{11}+\gamma_{22})^2}{(\alpha_1-\alpha_2)^2} \begin{pmatrix} 0 & * & * \\ * & \frac{\beta_{11}}{\gamma_{11}} - \frac{\beta_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} & * \\ * & * & \frac{\beta_{22}}{\gamma_{22}} - \frac{\beta_{21}\gamma_{12}}{\gamma_{11}\gamma_{22}} \end{pmatrix}. \end{aligned} \tag{4.13}$$

The stars indicate arbitrary values that do not matter in the final result. Omitted entries are zero. The combination of (4.12) and (4.13) proves the lemma. ∎

By regular perturbation of $\mathrm{spec}_{bd}\, L$, any result $\Re\mathrm{e}\,\mu'(0) \neq 0$ of lemma 4.1 extends to small positive δ.

We now turn to an expansion for the eigenprojection P_0^δ onto the one-dimensional kernel of the 6×6-matrix $L^\delta(u_0)$ at $u_0 = 0$; see (4.6), (1.12). Aligning the notations of proposition 1.2 and of theorem 1.3, we decompose

$$\mathbf{u} \;=\; (u, v) \;\in\; \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3. \tag{4.14}$$

Again, $e_0^T = (1,0,0)$ denotes the first unit vector in $\mathbb{R}^3$ and $\mathbf{e}_0^T = (e_0^T, 0)$ the first unit vector in $\mathbb{R}^6$.

Lemma 4.2 *In the above setting and notation, we have*

$$\begin{aligned} P_0^\delta &= \mathbf{e}_0 \cdot \mathbf{e}_\delta^T, \qquad \textit{with} \\ \mathbf{e}_\delta^T &= \left(1 + (\tfrac{\delta}{\alpha_0 - s^{\mathrm{crit}}})^2\right)^{-1/2} (e_0^T, -\tfrac{\delta}{\alpha_0 - s^{\mathrm{crit}}} e_0^T) \end{aligned} \tag{4.15}$$

Proof: Kernel and co-kernel of $L^\delta(u_0)$ are one-dimensional and correspond to the simple zero eigenvalue of $L^\delta(u_0)$. Obviously

$$\ker L^\delta(u_0) = \mathbf{e}_0, \tag{4.16}$$

because $g'(u_0, 0, 0)e_0 = 0$. At $u_0 = 0$, the co-kernel of $L^\delta(u_0)$ is given by

$$0 \;=\; \mathbf{e}_\delta^T \cdot \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1} g' & \delta^{-1}(A_0 - s^{\mathrm{crit}}) \end{pmatrix} \tag{4.17}$$

The structure of A_0 and g', see (1.12), has the following implications: The lower, right submatrix of

$$g' \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{21} & \gamma_{22} \end{pmatrix}$$

is regular, due to assumption (1.14). Therefore, the last two components of $\mathbf{e}_\delta$ must be zero. Then, the second and third components vanish too, because A_0 is diagonal. The two remaining components are proportional:

$$\frac{(\mathbf{e}_\delta)_0}{(\mathbf{e}_\delta)_3} \;=\; -\delta^{-1}(\alpha_0 - s^{\mathrm{crit}}) \tag{4.18}$$

The normalisation of the resulting vector proves the lemma. ∎

4.3 Nondegeneracy

To finish the proof of theorem 1.3 it remains to check the nondegeneracy assumption $\Delta_Z P_0 F \neq 0$, see (1.19). Here F denotes the flow of the viscous-profile system (1.5):

$$F(\mathbf{u}) \;=\; F(u, v) \;=\; \begin{pmatrix} v \\ -\delta^{-1} g(u) + \delta^{-1}(A(u) - s^{\mathrm{crit}})v \end{pmatrix}. \tag{4.19}$$

In lemma 4.3, we check this assumption in the limit $\delta \searrow 0$. In corollary 4.4, we provide explicit expressions for the type determining sign $\eta = \pm 1$ defined in (1.20). In particular, corollary 1.4 shows that both the hyperbolic case $\eta = +1$ and the elliptic case $\eta = -1$ can be realized by our nonlinear hyperbolic balance laws, even with gradient flux terms. This completes the proof of theorem 1.3.

To check the nondegeneracy condition (1.19) on $\Delta_Z P_0 F$ in the limit $\delta \searrow 0$, we use the following notation. By transverse eigenvalue crossing at $\delta = 0$, lemma 4.1, we obtain purely imaginary eigenvalues $\pm i\omega^\delta$ at equilibria $\mathbf{u}^\delta = (u_0^\delta, 0, \dots, 0) = (u^\delta, 0)$ on the u_0-axis, for small $\delta > 0$. Let Z^δ denote the corresponding eigenspace. See (4.15) for the expression of the eigenprojection P_0^δ onto the trivial kernel. Note that $\mathbf{u}^\delta$, ω^δ, P_0^δ, and Z^δ vary differentiably with δ.

Lemma 4.3 *In the above setting and notation, we have*

$$\Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) \;=\; \left(1 + \Big(\frac{\delta}{\alpha_0 - s^{\text{crit}}}\Big)^2\right)^{-1/2} \frac{1}{\alpha_0 - s^{\text{crit}}}\, g_0''(u^\delta)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right] \mathbf{e}_0 \tag{4.20}$$

at the HOPF *point* $\mathbf{u}^\delta = (u^\delta, 0)$ *with complex eigenvector* $(\tilde{u}^\delta, \tilde{v}^\delta)$ *of* $i\omega^\delta$.

Consider, for instance, quadratic forms $g_0''(0)$*, which are strictly positive/negative definite*[15] *on* (u_1, u_2)*-space, with* $\Gamma = \pm 1$ *indicating the sign of definiteness. Then*

$$\operatorname{sign} \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) \;=\; \Gamma \cdot \operatorname{sign}(\alpha_0 - s^{\text{crit}}), \tag{4.21}$$

for all small $\delta > 0$.

Proof: By lemma 4.2 we have

$$\left(1 + \Big(\frac{\delta}{\alpha_0 - s^{\text{crit}}}\Big)^2\right)^{1/2} P_0^\delta \;=\; \mathbf{e}_0 \cdot \mathbf{e}_0^T - \frac{\delta}{\alpha_0 - s^{\text{crit}}}\, \mathbf{e}_0 \cdot (0, e_0^T) \tag{4.22}$$

The explicit form (4.19) of the nonlinearity F implies

$$\Delta_{Z^\delta} P_0^0 F(\mathbf{u}^\delta) \;=\; \mathbf{e}_0 \Delta_{Z^\delta} v_0 \;=\; 0 \tag{4.23}$$

on any subspace Z^δ and for any $\mathbf{u}^\delta$, simply because the u-component of F is linear. With P_0^δ instead of P_0^0, and with the abbreviation $\mathcal{A}$ instead of $A - s^{\text{crit}}$, we obtain the more general formula

$$\begin{aligned} &\left(1 + \Big(\tfrac{\delta}{\alpha_0 - s^{\text{crit}}}\Big)^2\right)^{1/2} \mathbf{e}_0^T\, \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}) \\ &\quad= -\Delta_{Z^\delta}\left(\Big(0, -\tfrac{\delta}{\alpha_0 - s^{\text{crit}}}\, e_0^T\Big)\begin{pmatrix} v \\ \delta^{-1}(-g(u) + \mathcal{A}(u)v) \end{pmatrix}\right) \\ &\quad= -\tfrac{1}{\alpha_0 - s^{\text{crit}}} \Delta_{Z^\delta}\Big(-g_0(u) + (\mathcal{A}(u)v)_0\Big). \end{aligned} \tag{4.24}$$

[15] Here, a cautioning remark is necessary. While the nondegeneracy condition, $\Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) \neq 0$, is generically fulfilled, the positive/negative definiteness of $g_0''(0)$ is *not* a generic property. Typical systems satisfy the nondegeneracy assumption although $g_0''(0)$ is indefinite. However, if, fortunately, $g_0''(0)$ is positive or negative definite, then the test of the nondegeneracy condition becomes very easy. We do not even need to compute the HOPF eigenvalues. In particular, our example in corollary 1.4 has a positive/negative definite $g_0''(0)$, depending on the parameter η.

Here $(\mathcal{A}(u)v)_0$ denotes the zero-component of $\mathcal{A}(u)v$. We treat this term first, using the notation

$$\tilde{\mathbf{u}}^\delta \;=\; \begin{pmatrix} \tilde{u}^\delta \\ \tilde{v}^\delta \end{pmatrix} \tag{4.25}$$

for the complex eigenvector of the purely imaginary Hopf eigenvalue $\mu^\delta = i\omega^\delta$ at $u = u^\delta, v = 0$. Then,

$$Z^\delta \;=\; \mathrm{span}\{\Re\mathrm{e}\, \tilde{\mathbf{u}}^\delta, \Im\mathrm{m}\, \tilde{\mathbf{u}}^\delta\}. \tag{4.26}$$

We denote the standard Laplacian by $\Delta_\beta = \partial^2_{\beta_1} + \partial^2_{\beta_2}$, evaluated at $\beta = 0$. The insertion $\tilde{v}^\delta = \mu^\delta \tilde{u}^\delta$ yields

$$\begin{aligned}
&\Delta_{Z^\delta} (\mathcal{A}(u)v)_0\Big|_{u=u^\delta, v=0} \\
&\quad= \Delta_\beta \left(\mathcal{A}(u^\delta + \beta_1 \Re\mathrm{e}\, \tilde{u}^\delta + \beta_2 \Im\mathrm{m}\, \tilde{u}^\delta)\ (\beta_1 \Re\mathrm{e}\, \tilde{v}^\delta + \beta_2 \Im\mathrm{m}\, \tilde{v}^\delta)\right)_0 \\
&\quad= 2\left((A'(u^\delta)\, \Re\mathrm{e}\, \tilde{u}^\delta)\, \Re\mathrm{e}\, \tilde{v}^\delta + (A'(u^\delta)\, \Im\mathrm{m}\, \tilde{u}^\delta)\, \Im\mathrm{m}\, \tilde{v}^\delta\right)_0 \\
&\quad= 2\ \Re\mathrm{e}\left((A'(u^\delta)\tilde{u}^\delta)\overline{\tilde{v}^\delta}\right)_0 \\
&\quad= 2\ \Re\mathrm{e}\left(\overline{\mu^\delta}\right)\ \left(f''(u^\delta)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right]\right)_0 \\
&\quad= 2\ \Re\mathrm{e}(\mu^\delta)\ f_0''(u^\delta)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right] \\
&\quad= 0
\end{aligned} \tag{4.27}$$

all along the Hopf curve $u = u^\delta$, $v = 0$. Here we have used $A(u) = f'(u)$ for the flux function and the fact that the Hessian matrix $f_0''(0)$ is symmetric.

Therefore, we can conclude from (4.24), (4.27) that

$$\begin{aligned}
&\left(1 + \left(\tfrac{\delta}{\alpha_0 - s^{\mathrm{crit}}}\right)^2\right)^{1/2} \mathbf{e}_0^T\ \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) \\
&\quad= \tfrac{1}{\alpha_0 - s^{\mathrm{crit}}}\ \Delta_{Z^\delta} g_0(u)\Big|_{u=u^\delta} \\
&\quad= \tfrac{1}{\alpha_0 - s^{\mathrm{crit}}}\ g_0''(u^\delta)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right] \\
&\quad= \tfrac{1}{\alpha_0 - s^{\mathrm{crit}}}\ g_0''(0)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right].
\end{aligned} \tag{4.28}$$

This proves (4.20) and the lemma. ∎

Corollary 4.4 *Combining lemmata 4.1 and 4.3, the* sign $\eta = \pm 1$ *distinguishing elliptic from hyperbolic* Hopf *bifurcation along our line of equilibria is, for* $\delta > 0$ *small enough, given explicitly by*

$$\begin{aligned}
\eta \;=\;& \mathrm{sign}(\beta_{12}\gamma_{21} + \beta_{21}\gamma_{12} - \beta_{11}\gamma_{22} - \beta_{22}\gamma_{11})\ \mathrm{sign}(\gamma_{11}\gamma_{22}) \cdot \\
&\cdot\ \mathrm{sign}\left(g_0''(0)\left[\tilde{u}^\delta, \overline{\tilde{u}^\delta}\right]\right)\ \mathrm{sign}(\alpha_0 - s^{\mathrm{crit}}).
\end{aligned} \tag{4.29}$$

If $g_0''(0)$ is positive or negative definite, then we can write

$$\begin{aligned}\eta \;=\;& \operatorname{sign}(\beta_{12}\gamma_{21}+\beta_{21}\gamma_{12}-\beta_{11}\gamma_{22}-\beta_{22}\gamma_{11})\operatorname{sign}(\gamma_{11}\gamma_{22})\,\cdot\\ &\cdot\,\Gamma\;\operatorname{sign}(\alpha_0-s^{\mathrm{crit}}).\end{aligned} \tag{4.30}$$

Here derivatives are evaluated at $u=0$, and are assumed to be chosen such that $\eta \neq 0$. The parameter $\Gamma = \pm 1$ indicates positive/negative definiteness of $g_0''(0)$ on (u_1, u_2)-space. Obviously, both signs of η can be realized.

5 Spectra Of Travelling Waves

In this section we review and adapt some definitions, concepts and results for our spectral stability considerations later on. Refer to [Sat76, Cop78, Hen81, AGJ90, SS00] for further details.[16]

A crucial point of our investigation is the reformulation of the eigenvalue problem to the elliptic p.d.e. on the real line as a dynamical system with the spatial coordinate as the time-like variable. This idea of investigating the spatial dynamics and of applying results from the theory of dynamical systems goes back to KIRCHGÄSSNER [Kir82].

We consider the viscous conservation law (1.1). In comoving, rescaled coordinates $\xi = (x - st)/\varepsilon$ and rescaled time $\tau = t/\varepsilon$, this equation has the form

$$u_\tau = \delta u_{\xi\xi} + s u_\xi - A(u) u_\xi + g(u). \tag{5.1}$$

Travelling waves $u(x, t) = u(\xi)$ become stationary, i.e. time independent, solutions of (5.1). Their linear stability is determined by the linearisation

$$\tilde{u}_\tau \;=\; \mathcal{M}\,\tilde{u} \;=\; \delta\tilde{u}_{\xi\xi} + s\tilde{u}_\xi - A'(u(\xi))[\tilde{u}, u'(\xi)] - A(u(\xi))\tilde{u}_\xi + g'(u(\xi))\tilde{u} \tag{5.2}$$

of (5.1). We arrive at the associated eigenvalue problem

$$0 \;=\; (\mathcal{M} - \lambda)\,\tilde{u} \;=\; \delta\tilde{u}_{\xi\xi} + s\tilde{u}_\xi - A'(u(\xi))[\tilde{u}, u'(\xi)] - A(u(\xi))\tilde{u}_\xi + g'(u(\xi))\tilde{u} - \lambda\tilde{u}. \tag{5.3}$$

This linear, second order, non-autonomous o.d.e. can be rewritten as the first-order system

$$\begin{aligned} \begin{pmatrix} \tilde{u}_\xi \\ \tilde{v}_\xi \end{pmatrix} &= \begin{pmatrix} 0 & \mathrm{id} \\ \delta^{-1}\big(\lambda + A'(u(\xi))u'(\xi) - g'(u(\xi))\big) & \delta^{-1}\big(A(u(\xi)) - s\big) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \\ &= M(\xi, \lambda) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \end{aligned} \tag{5.4}$$

that is associated to the operator

$$\mathcal{T}(\lambda) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - M(\cdot, \lambda) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}. \tag{5.5}$$

The eigenvalue problem for travelling waves can be interpreted as a dynamical system in the spatial variable ξ. Of course, the spectrum depends on the considered space.

[16] Unfortunately, there seems to be no comprehensive review of this approach in the literature.

In fact, the choice of an appropriate space will be one of the main issues in the following analysis. For the moment, we can think of $\mathcal{M} - \lambda$ respectively $\mathcal{T}(\lambda)$ acting on $L^2(\mathbb{R}, \mathbb{C}^n)$ and $H^1(\mathbb{R}, \mathbb{C}^n) \times L^2(\mathbb{R}, \mathbb{C}^n)$. These operators are densely defined with domains $H^2(\mathbb{R}, \mathbb{C}^n)$ and $H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n)$, where H^l denotes the usual Sobolev space of L^2-functions with l weak derivatives in L^2. On the real line, H^1 is embedded in the space of bounded continuous functions [Ada75]. The spectrum can be investigated by using two basic concepts: Fredholm properties and exponential dichotomies.

5.1 Fredholm properties

The Fredholm properties of the operator $\mathcal{M} - \lambda$ that act on the space $L^2(\mathbb{R}, \mathbb{R}^n)$ qualitatively characterise different points in the spectrum of $\mathcal{M}$.

Definition 5.1 *The* essential spectrum*,* $\operatorname{spec}_{\text{ess}} \mathcal{M}$*, is the set of all complex numbers* λ *such that the operator* $\mathcal{M} - \lambda$ *is either not* Fredholm *or of a nonzero* Fredholm *index. The* point spectrum*,* $\operatorname{spec}_{\text{pt}} \mathcal{M}$*, is the set of all* λ *such that* $\mathcal{M} - \lambda$ *is* Fredholm *with index zero and has a nontrivial kernel. The complement,* $\mathbb{C} \setminus (\operatorname{spec}_{\text{ess}} \cup \operatorname{spec}_{\text{pt}})$*, is called the* resolvent set*.* $\mathcal{M} - \lambda$ *is bounded invertible for all* λ *belonging to this set.*

Note that $(\mathcal{M} - \lambda)$ and $\mathcal{T}(\lambda)$ have the same Fredholm properties. In fact, the dimensions of their nullspaces are equal, as well as the dimensions of the nullspaces of the adjoint operators.

Example 5.2 *Let us consider the linear convection-diffusion equation*

$$u_t \;=\; \mathcal{M}u \;=\; u_{\xi\xi} + a u_\xi + b u \tag{5.6}$$

on the space $L^2(\mathbb{R}, \mathbb{R})$ *with real constants* a*,* b*.* Fourier *transformation yields*

$$((\widehat{\mathcal{M} - \lambda})\hat{u})(k) \;=\; (-k^2 + iak + b - \lambda)\, \hat{u}(k). \tag{5.7}$$

Bounded invertibility of $\mathcal{M} - \lambda$ *on* L^2 *fails if, and only if, there exists a real value of* k *that satisfies*

$$0 \;=\; -k^2 + iak + b - \lambda. \tag{5.8}$$

Therefore

$$\operatorname{spec}_{\text{ess}} \mathcal{M} \;=\; \{-k^2 + iak + b \mid k \in \mathbb{R}\}. \tag{5.9}$$

(There is no point spectrum. The eigenmodes to any $\lambda \in \operatorname{spec} \mathcal{M}$ *are periodic waves, that have an unbounded* L^2*-norm.) For positive parameters* b*, the trivial solution of (5.2) is spectrally unstable.*

The transformation $v(\xi) = \exp(-a\xi/2)u(\xi)$ *eliminates the transport term* au_ξ *of (5.6). We obtain the equation*

$$v_t \;=\; v_{\xi\xi} + (b - \tfrac{1}{4}a^2)v. \tag{5.10}$$

For $b < a^2/4$ *all solutions of (5.10) to bounded initial data decay to zero.*

We conclude that for $0 < b < a^2/4$ *the original equation (5.6) is pointwise stable, although it is spectrally unstable. In particular, all solutions to initial data with compact support decay pointwise to zero.*

5.2 Exponential dichotomies

Any bounded solution of (5.4) corresponds to an eigenfunction of the operator $\mathcal{M}$. Exponential dichotomies characterise operators for which any bounded solution decays in fact exponentially fast to zero as ξ goes to $\pm\infty$. Details and extended results can be found in [Cop78, Hen81, PSS97].

To make a precise definition, let $M(\xi,\lambda) \in \mathbb{C}^{N\times N}$ be a general function of $(\xi,\lambda) \in \mathbb{R}\times\mathbb{C}$ of the form

$$M(\xi,\lambda) \;=\; \tilde{M}(\xi) + \lambda\tilde{N}(\xi) \tag{5.11}$$

which is smooth in $\xi \in \mathbb{R}$ and analytic in $\lambda \in \mathbb{C}$.

We assume that our problem has asymptotically constant coefficients That means that there are positive constants K, κ, independent of ξ and λ, and matrices $M^\pm$, analytically dependent on λ, such that

$$\|M(\xi,\lambda) - M^\pm(\lambda)\| \;\le\; Ke^{-\kappa|\xi|}, \qquad \text{as } \xi \to \pm\infty. \tag{5.12}$$

Lemma 5.3 *The eigenvalue problem (5.4) of viscous shock profiles that arise from our viscous balance law (1.1) has asymptotically constant coefficients.*

Proof: The asymptotic matrices $M^\pm(\lambda)$ are given by $M(\pm\infty,\lambda)$, i.e. by the matrices evaluated at $u^\pm = u(\pm\infty)$, see (5.4). The heteroclinic orbit $u(\xi)$ belongs to the intersection of the stable manifold of u^- and the unstable manifold of u^+ in the viscous-profile o.d.e (1.5). Let us consider the convergence to u^- for $t \to -\infty$ in the unstable manifold of u^-. The rate of convergence is determined by the eigenvalue of the linearisation (1.9) with the smallest positive real part. For small $|u^-|$ this the HOPF eigenvalue, see proposition 1.1. Its real part has the value $u^- \,\Re\mathrm{e}\,\mu'(0) + \mathcal{O}(|u^-|^2)$. We take $\kappa = \frac{1}{2}u^- \,\Re\mathrm{e}\,\mu'(0)$ and obtain exponential convergence of the trajectory $u(\xi)$ to u^- for $\xi \to -\infty$ with the rate κ. The linearisation at $u(\xi)$ converges with the same rate. The additional eigenvalue parameter

λ in (5.4) does not change these arguments. The limit $\xi \to +\infty$ can be investigated analogously. ∎

We now consider the linear operator $\mathcal{T}(\lambda) : H^1(\mathbb{R}, \mathbb{C}^N) \to L^2(\mathbb{R}, \mathbb{C}^N)$,

$$\mathcal{T}(\lambda)\mathbf{u} \;=\; \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi} - M(\cdot, \lambda)\mathbf{u} \tag{5.13}$$

and the associated o.d.e.

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u} \;=\; M(\cdot, \lambda)\mathbf{u}, \qquad \mathbf{u} \in \mathbb{C}^N. \tag{5.14}$$

Definition 5.4 (Exponential dichotomies) *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$, or $\mathbb{R}$. Fix $\lambda \in \mathbb{C}$. We say that equation (5.14) has an exponential dichotomy on I if there exist positive constants K, κ^s, κ^u and a family of projections $P(\xi, \lambda)$, $\xi \in I$, such that the following properties hold.*

(i) Stability. *For any fixed $\xi^* \in I$ and $u^* \in \mathbb{C}^N$, a solution $\phi^s(\xi, \xi^*)u^*$ of (5.14) with initial value $\phi^s(\xi^*, \xi^*)u^* = P(\xi^*)u^*$ exists. Furthermore, the following estimate of exponential decay holds:*

$$\forall \xi, \xi^* \in I, \xi \geq \xi^* : \quad |\phi^s(\xi, \xi^*)| \;\leq\; K \exp(-\kappa^s |\xi - \xi^*|). \tag{5.15}$$

(ii) Instability. *For any fixed $\xi^* \in I$ and $u^* \in \mathbb{C}^N$, a solution $\phi^u(\xi, \xi^*)u^*$ of (5.14) with initial value $\phi^u(\xi^*, \xi^*)u^* = (1 - P(\xi^*))u^*$ exists. Furthermore, the following estimate of exponential decay in the reverse direction holds:*

$$\forall \xi, \xi^* \in I, \xi \leq \xi^* : \quad |\phi^u(\xi, \xi^*)| \;\leq\; K \exp(-\kappa^u |\xi - \xi^*|). \tag{5.16}$$

(iii) Invariance. *The solution operators, ϕ^s and ϕ^u, satisfy*

$$\begin{aligned} &\forall u^* \in \mathbb{C}^N, \xi, \xi^* \in I, \xi \geq \xi^* : \qquad \phi^s(\xi, \xi^*)u^* \in \operatorname{im} P(\xi), \\ &\forall u^* \in \mathbb{C}^N, \xi, \xi^* \in I, \xi \leq \xi^* : \qquad \phi^u(\xi, \xi^*)u^* \in \ker P(\xi). \end{aligned} \tag{5.17}$$

The dimension of $\ker P(\xi)$ *is independent of ξ. It is called the* Morse *index, $i(\lambda)$, of the dichotomy. When we consider different dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, we use the notation i^+, i^- to distinguish both cases.*

Exponential dichotomies are a powerful tool to prove the existence and stability of travelling waves and other patterns in one spatial coordinate, see for instance [PSS97]. The dynamical point of view gives geometric insight into the problem and therefore improves the understanding of intrinsic properties and coherences.

Under our assumption of asymptotically constant coefficients (5.12), the existence of exponential dichotomies on $\mathbb{R}^\pm$ is determined by the asymptotic matrices $M^\pm(\lambda)$:

Theorem 5.5 (Coppel [Cop78], ch. 6) *For a fixed λ, equation (5.14) has an exponential dichotomy on $\mathbb{R}^-$ if, and only if, the matrix $M^-(\lambda)$ is hyperbolic. In this case, the Morse index i^- is given by the dimension of the generalised unstable eigenspace of $M^-(\lambda)$. The same statement holds on $\mathbb{R}^+$ with respect to $M^+(\lambda)$.*

Furthermore, the equation has an exponential dichotomy on the entire real line $\mathbb{R}$ if, and only if, there exists exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ with projections $P^\pm(\xi)$ that satisfy $\ker P^-(0) \oplus \operatorname{im} P^+(0) = \mathbb{C}^N$. Especially, the Morse *indices must match, $i^+(\lambda) = i^-(\lambda)$.*

The exponents κ^u, κ^s of the dichotomy, see definition 5.4, depend only on the eigenvalues of the asymptotic coefficients $M^\pm$. In particular, κ^u can be any value less than the smallest positive real part of the eigenvalues of M^-. Analogously, κ^s is a bound on the stable rates of M^+.

Exponential dichotomies are robust under small perturbation of the operator:

Theorem 5.6 ([Cop78], ch. 4) *Suppose that equation (5.14) has an exponential dichotomy on $I = \mathbb{R}^+$, $\mathbb{R}^-$, or $\mathbb{R}$ with constants $K > 1, \kappa > 0$ as in definition 5.4. Let $N(\xi)$ be a bounded, continuous matrix-function, with*

$$\varepsilon \;:=\; \sup_{\xi \in I} \|N(\xi)\| \;<\; \frac{\kappa}{4K^2}. \tag{5.18}$$

Then the perturbed equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{u} \;=\; \Big(M(\cdot) + N(\cdot)\Big)\mathbf{u} \tag{5.19}$$

has an exponential dichotomy on I with constants $\tilde{K} = 5K^2/2$ and $\tilde{\kappa} = \kappa - 2K\varepsilon$.

If the perturbation depends continuously on a parameter, the projections of the dichotomy can be chosen to also depend continuously on that parameter. Even higher regularity can be achieved.

5.3 Relations between both concepts

The following fundamental result of Palmer relates the dichotomies of equation (5.14) and the Fredholm properties of the associated operator $\mathcal{T}(\lambda)$.

Theorem 5.7 (Palmer [Pal84, Pal88]) *For a fixed λ, the operator $\mathcal{T}(\lambda)$ is* Fredholm *if, and only if, the equation (5.14) possesses exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. If this is the case, then the* Fredholm*-index is equal to the difference of the associated* Morse*-indices $i^-(\lambda) - i^+(\lambda)$.*

Invertibility of $\mathcal{T}(\lambda)$ is satisfied if, and only if, there exists an exponential dichotomy of the equation (5.14) on the entire real line $\mathbb{R}$.

In connection with theorem 5.5, this relation implies that the essential spectrum is determined by the asymptotic matrices $M^{\pm}(\lambda)$, alone. It remains to compute the point spectrum. One has to determine the kernel of $T(\lambda)$, i.e. one has to find nonzero bounded solutions of the eigenvalue problem. Alternatively, one can investigate the transversality of $\ker P^-(0)$ and $\operatorname{im} P^+(0)$[17], see figure 5.1. Any bounded solution $u(\cdot)$ of the eigenvalue problem (5.13) must belong to the stable evolution ϕ^s on $\mathbb{R}^+$ and to the unstable evolution ϕ^u on $\mathbb{R}^-$. Therefore $u(0) \in \ker P^-(0) \cap \operatorname{im} P^+(0)$. Outside the essential spectrum, the spaces $\ker P^-(0)$ and $\operatorname{im} P^+(0)$ depend analytically on λ. The transversality of their intersection can be measured by their wedge product that is also called the Evans function [AGJ90].

Note that, even with theorem 5.7 in mind, the robustness of exponential dichotomies, theorem 5.6, is a stronger statement than the robustness of Fredholm operators under small perturbations. The exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ are robust separately, the corresponding operator need not to be Fredholm. In addition, we have explicit bounds on the size of the allowed perturbation.

Example 5.8 (5.2, continued) *We can now verify our previous calculation of the spectrum of (5.6). The associated first-order system has the form*

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda - b & -a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{5.20}$$

with constant coefficients. Both asymptotic matrices are, therefore, equal and the essential

[17]Note that dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ exist for λ outside the essential spectrum.

spectrum consists of all values $\lambda \in \mathbb{C}$ where

$$M(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - b & -a \end{pmatrix} \tag{5.21}$$

is non-hyperbolic. We obtain the eigenvalues of $M(\lambda)$ as

$$-\tfrac{1}{2}a \pm \sqrt{\tfrac{1}{4}a^2 - b + \lambda}. \tag{5.22}$$

The asymptotic matrices are non-hyperbolic, if they have a purely imaginary eigenvalue $ik, k \in \mathbb{R}$:

$$\begin{aligned} \pm\sqrt{\tfrac{1}{4}a^2 - b + \lambda} &= ik + \tfrac{1}{2}a, \qquad k \in \mathbb{R} \\ \lambda &= -k^2 + iak + b. \end{aligned} \tag{5.23}$$

This result coincides with (5.9).

5.4 Exponential weights — convective stability

Roughly knowing how to compute the spectrum of the linearisation, it is now time to give a first definition of stability.

Definition 5.9 *An equilibrium solution of the p.d.e. (5.1) is called* spectrally stable *(in L^2) if the spectrum of the linearised operator $\mathcal{M}$, see (5.2), is entirely contained in the open left half plane, i.e. if $\mathcal{M} - \lambda$ is bounded invertible (in L^2) for all λ with $\Re\mathrm{e}\,\lambda \geq 0$.*[18]

On unbounded domains spectral stability might be a too restrive requirement if it is considered with respect to a translationally invariant norm, e.g. L^2. For instance, in the example 5.2 the trivial solution is spectrally unstable, but all perturbations decay pointwise to zero.

Parts of the essential spectrum with positive real part typically correspond to continua of unstable modes. Perturbations that excite these modes grow in time, with respect to the considered norm. In an unbounded, translationally invariant space growing perturbations may at the same time travel away to plus/minus infinity. Due to their convective nature, on any bounded subdomain these perturbations could nevertheless decay to zero.

To investigate this kind of behaviour, we introduce exponentially weighted norms, that break the translational symmetry.

[18]Typically, the definition of spectral stability has to be adjusted for different problems. Mainly, the case of spectrum on the imaginary axis has to be investigated carefully depending on the problem.

Definition 5.10 *For any pair $\nu = (\nu^-, \nu^+) \in \mathbb{R}^2$ we define the space*

$$L_\nu \;=\; \{u \in L^2_{\text{loc}} \mid \|u\|_\nu < \infty\} \tag{5.24}$$

with the norm

$$\|u\|_\nu \;=\; \int_{-\infty}^{0} e^{\nu^-\xi}|u(\xi)|^2 \mathrm{d}\xi + \int_{0}^{+\infty} e^{\nu^+\xi}|u(\xi)|^2 \mathrm{d}\xi. \tag{5.25}$$

Weighted Sobolev *spaces can be defined analogously.*

$$H^l_\nu \;=\; \left\{ u \in H^l_{\text{loc}} \;\middle|\; \sum_{k=0}^{l} \left\| \frac{\mathrm{d}^k}{\mathrm{d}\xi^k} u \right\|_\nu \;<\; \infty \right\}. \tag{5.26}$$

Take, for instance, $\nu^- > 0$. Then the growth rate of any perturbation that is convected to the left is reduced due to the negative exponent of the weight for negative ξ. If a perturbation travels faster than it grows, with respect to the weight, it decays in the weighted space. So, convective instabilities can be stabilised by the introduction of appropriate weighted norms.

Definition 5.11 *An equilibrium solution of the p.d.e. (5.1) is called* convectively stable *if there exists a weight $\nu = (\nu^-, \nu^+) \in \mathbb{R}^2$, such that it is spectrally stable with respect to the weighted space L_ν. If there does not exist a stabilising weight, then the solution is called* absolutely unstable.

Here, some remarks are necessary. The above definition suits our purposes, although more refined (and less restrictive) definitions of various convective stability properties are possible.

First, using our definition of convective stability we consider the eigenvalue problem (5.4) with respect to a fixed weight ν for all λ. One could admit λ-dependent weights $\nu(\lambda)$. However, then one does not obtain spectral stability in any weighted space, as in definition 5.11. Despite this fact, pointwise stability can still be proven in many cases.

Secondly, it is not always possible to stabilise a pointwise stable wave in an exponentially weighted space. Perturbations may split and travel to $\pm\infty$ at the same time [SS00]. In [SS00], Sandstede and Scheel distinguish between convective, transient, remnant and absolute instabilities.

The spaces L^2, H^l and L_ν, H^l_ν are connected by the isomorphisms

$$\begin{aligned} J_\nu \;&:\; L_\nu \longrightarrow L^2,\;\; H^l_\nu \longrightarrow H^l, \\ (J_\nu u)(\xi) \;&=\; e^{\nu(\xi)\xi}u(\xi), \qquad \nu(\xi) \;=\; \begin{cases} \nu^- & \text{for } \xi < 0 \\ \nu^+ & \text{for } \xi > 0 \end{cases}. \end{aligned} \tag{5.27}$$

On L_ν this operator is isometric, on H^l_ν the induced norm is equivalent to the usual Sobolev norm on H^l.

The isomorphism J_ν allows us to write the linearised operator $\mathcal{M} : H^2_\nu \to L_\nu$ in the weighted space again as a modified operator on L^2:

$$\begin{aligned} \mathcal{M}_\nu &: H^2 \longrightarrow L^2, \\ \mathcal{M}_\nu &= J_\nu \mathcal{M} J_\nu^{-1} = J_\nu \mathcal{M} J_{-\nu}. \end{aligned} \tag{5.28}$$

We now must determine the spectrum of $\mathcal{M}_\nu$ in the space L^2. As before, we could again write the eigenvalue problem $0 = (\mathcal{M}_\nu - \lambda)u$ as a first-order dynamical system. Alternatively, we can just impose the exponential weights onto the family $\mathcal{T}(\lambda)$ of the associated first-order system and use the modified family

$$\begin{aligned} \mathcal{T}_\nu(\lambda) &: H^1 \longrightarrow L^2, \\ \mathcal{T}_\nu(\lambda) &= J_\nu \mathcal{T}(\lambda)\Big|_{L_\nu} J_\nu^{-1}. \end{aligned} \tag{5.29}$$

Both variants are equivalent due to the fact that

$$\frac{\mathrm{d}}{\mathrm{d}\xi} J_\nu = J_\nu \left(\nu + \frac{\mathrm{d}}{\mathrm{d}\xi}\right), \tag{5.30}$$

see example 5.12.

The family of first-order operators with weights has the form

$$\mathcal{T}_\nu(\lambda)u = \frac{\mathrm{d}}{\mathrm{d}\xi}u - M_\nu(\cdot,\lambda)u = \frac{\mathrm{d}}{\mathrm{d}\xi}u - M(\cdot,\lambda)u - \nu(\cdot)u. \tag{5.31}$$

The weight just adds a scalar multiple of u to the operator. In particular, the asymptotic matrices $M^\pm(\lambda)$ and their spectrum are shifted by the amount $\nu^\pm$:

$$M^\pm_\nu(\lambda) = M^\pm(\lambda) + \nu^\pm \cdot \mathrm{id}\,. \tag{5.32}$$

Example 5.12 (5.2, continued) *With exponential weights we have*

$$\begin{aligned} \mathcal{M}_\nu u &= u_{\xi\xi} - 2\nu u_\xi + \nu^2 u + a u \xi - a\nu u + bu \\ &= u_{\xi\xi} + (a - 2\nu)u_\xi + (b - a\nu + \nu^2)u \end{aligned} \tag{5.33}$$

For simplicity, we only consider the case $\nu^+ = \nu^- = \nu$. Then this equation has the same structure as in the case without weights. We obtain

$$\operatorname{spec}_{\mathrm{ess}} \mathcal{M}_\nu = \{-k^2 + ik(a - 2\nu) + (b - a\nu + \nu^2) \mid k \in \mathbb{R}\}. \tag{5.34}$$

The spectrum lies in the left half plane, i.e. we obtain convective stability, if we find a weight ν, such that

$$0 \;>\; b - a\nu + \nu^2 \;=\; (\nu - \tfrac{1}{2}a)^2 + b - \tfrac{1}{4}a^2 \tag{5.35}$$

holds. This is possible if, and only if, $b < \frac{1}{4}a^2$.

We can also calculate the essential spectrum with the help of the asymptotic matrices. Writing $(\mathcal{M}_\nu - \lambda)u = 0$ as a first-order system, we get

$$\begin{aligned}
\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \lambda - (b - a\nu + \nu^2) & -(a - 2\nu) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \begin{pmatrix} \nu & 1 \\ \lambda - b & \nu - a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\nu & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} M_\nu(\lambda) \begin{pmatrix} 1 & 0 \\ -\nu & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
\end{aligned} \tag{5.36}$$

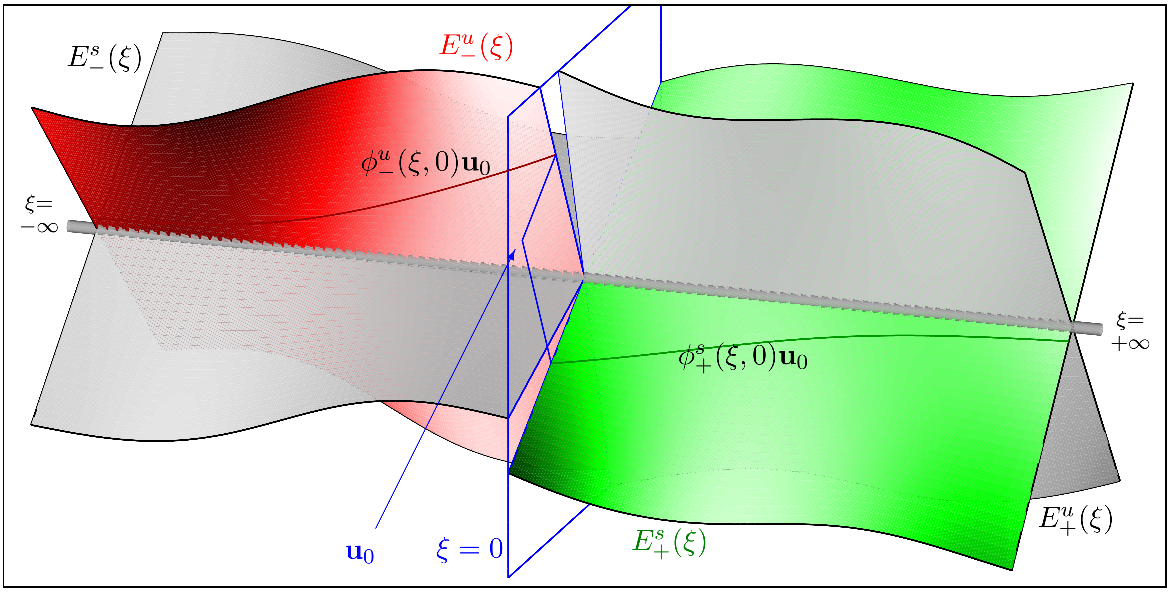
where $M_\nu(\lambda)$ is the Matrix $M(\lambda)$ from the unweighted case shifted by ν. The eigenvalues of $M_\nu(\lambda)$ are

$$-\tfrac{1}{2}a \pm \sqrt{\tfrac{1}{4}a^2 - b + \lambda} + \nu. \tag{5.37}$$

Purely imaginary eigenvalues $ik, k \in \mathbb{R}$, i.e. non-hyperbolic asymptotic matrices, occur for

$$\begin{aligned}
\pm\sqrt{\tfrac{1}{4}a^2 - b + \lambda} &= ik + \tfrac{1}{2}a - \nu \\
\lambda &= -k^2 + ik(a - 2\nu) + b - a\nu + \nu^2.
\end{aligned} \tag{5.38}$$

We obtain, of course, the same essential spectrum as before in (5.34). The trivial solution of (5.6) is convective stable if, and only if, $b < \frac{1}{4}a^2$.



The subspaces $E^s_\pm(\xi)$ are the images of the projections $P^\pm(\xi)$, whereas $E^u_\pm(\xi)$ are the null-spaces of $P^\pm(\xi)$. In $E^s_+(\xi)$ solutions $u(\xi)$ decay exponentially fast for $\xi \to +\infty$, in $E^u_-(\xi)$ they decay for $\xi \to -\infty$. In the complementary spaces the solution grows. Any solution u that is bounded on the entire real line must satisfy $\mathbf{u}_0 = \mathbf{u}(0) \in E^u_-(0) \cap E^s_+(0)$. The kernel of the associated operator is, therefore, given by the intersection $E^u_-(0) \cap E^s_+(0)$. The point spectrum is determined by a nontrivial intersection for dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$ with coinciding index.

Figure 5.1: Exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$.

6 Linear Stability Of Oscillatory Viscous Shocks

In this section we prove theorem 1.5. For small parameters $\delta > 0$ in the three-dimensional case (1.12) all heteroclinic orbits in a small neighbourhood of the HOPF point give rise to convectively stable travelling waves, provided they have extreme speeds. In fact, they are linearly stable, all small perturbation decay to zero in the linearised equation (5.2) with respect to a suitably chosen weighted space L_ν. This is done in four steps.

First, we investigate the asymptotic matrices $M_\nu^\pm(\lambda) = \lim_{\xi\to\infty} M_\nu(\xi,\lambda)$ of the associated linearised first-order system (6.2), for $\lambda = 0$ and for $\Re\mathrm{e}\,\lambda \gg 1$. A necessary condition for stability is the hyperbolicity of these matrices with the same unstable dimension $i^\pm(\lambda = 0) = i^\pm(\lambda|_{\Re\mathrm{e}\,\lambda\gg 1})$. Otherwise, on any path in the complex plane from the origin to a point with large real part there would be a value $\tilde\lambda$ with non-hyperbolic matrix $M_\nu^-(\tilde\lambda)$ or $M_\nu^+(\tilde\lambda)$. This violates the condition of theorem 5.7. The value $\tilde\lambda$ would be contained in the essential spectrum.

Because our problem origintes from a second order equation we then can conclude, that for large real parts of λ we always have hyperbolic asymptotic matrices with equal stable and unstable dimension. Therefore, the above condition can be simplified to the requirement that at the origin we have the same index $i^\pm(0) = n = 3$. This implies that we must find vertical lines $-\nu^\pm + i\mathbb{R}$ that divide the spectra of $M_\nu^\pm(0)$ into sets each of three eigenvalues.

In the second part, we shall estimate the matrices $M_\nu^\pm(\lambda)$ for intermediate values λ and prove their hyperbolicity for a suitable parameter region. This will show that the essential spectrum of the linearisation $\mathcal{M}_\nu$, see (6.1), is entirely bounded to the left of the imaginary axis.

After that, in the third part, we exclude the possibility of point eigenvalues for travelling waves with small amplitudes. We regard the linearisation around these travelling waves as a small perturbation of the linearisation around the homogeneous solution $u(x,t) = 0$ and use the structural stability of invertible operators, theorem 5.6. This will prove the convective stability of our oscillatory waves.

Finally, we shall establish linear stability.

6.1 Essential spectrum

Let us start with the linearisation (5.2) of our hyperbolic viscous balance law (5.1) around an oscillatory travelling wave $u(\xi)$, see theorem 1.3, in an exponentially weighted space L_ν. In this section, we assume that we have a viscous hyperbolic balance law that admits a Hopf bifurcation of elliptic type near the origin, for small δ. More precisely, we assume (1.7, 1.12, 1.21(iv-vi), 1.29) to be satisfied.

According to (5.27, 5.28) the linearisation can be regarded as a modified operator on L^2:

$$\begin{aligned}\tilde{u}_\tau &= \mathcal{M}_\nu \tilde{u} \\ &= \delta\tilde{u}_{\xi\xi} + (s - A'(u)u' - A(u) - 2\delta\nu)\tilde{u}_\xi + (g'(u) - s\nu + \nu A'(u)u' + \nu A(u) + \delta\nu^2)\tilde{u}\end{aligned} \tag{6.1}$$

We can write the eigenvalue problem problem $0 = (\mathcal{M}_\nu - \lambda)\tilde{u}$ as a first-order system. The associated operator is equivalent to

$$\begin{aligned}\mathcal{T}_\nu(\lambda)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} &= \tfrac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} - M_\nu(\cdot,\lambda)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} - \Big(M(\cdot,\lambda) + \nu(\cdot)\Big)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}, \\ M(\cdot,\lambda) &= \begin{pmatrix}0 & \mathrm{id}\\ \delta^{-1}\big(\lambda + A'(u)u' - g'(u)\big) & \delta^{-1}\big(A(u) - s\big)\end{pmatrix},\end{aligned} \tag{6.2}$$

which is just the operator $T(\lambda)$ for the unweighted case (5.4, 5.5) with the additional term due to the weight, see (5.29–5.31).

Lemma 6.1 *For any weight ν there exists a positive constant $C \gg 1 + (\nu^\pm)^2$ such that for all λ with large real Part, $\Re\mathrm{e}\,\lambda > C$, the spectra of the asymptotic matrices*

$$M_\nu^\pm(\lambda) = \lim_{\xi\to\pm\infty} M_\nu(\xi,\lambda) = \begin{pmatrix}0 & \mathrm{id}\\ \delta^{-1}\big(\lambda - g'(0)\big) & \delta^{-1}\big(A(u^\pm) - s\big)\end{pmatrix} + \nu^\pm, \tag{6.3}$$

see (6.2), consist of three eigenvalues with positive real part

$$+\sqrt{\frac{\lambda}{\delta}} + \mathcal{O}(1) + \nu^\pm \tag{6.4}$$

and three eigenvalues with negative real part

$$-\sqrt{\frac{\lambda}{\delta}} + \mathcal{O}(1) + \nu^\pm. \tag{6.5}$$

Here $\sqrt{\frac{\lambda}{\delta}}$ denotes the root with positive real part, $\Re\mathrm{e}\sqrt{\frac{\lambda}{\delta}} \geq \sqrt{\frac{\Re\mathrm{e}\,\lambda}{\delta}} \geq \sqrt{\frac{C}{\delta}}$, and the eigenvalues are counted with their algebraic multiplicity.

Proof: The asymptotic matrices $M_\nu^\pm(\lambda)$ can be diagonalised. For $\Re\mathrm{e}\,\lambda \longrightarrow \infty$ we have in blockmatrix notation, omitting the factor $\cdot\,\mathrm{id}$ in all 3×3 blocks,

$$\begin{aligned} M_\nu^\pm(\lambda) &= \begin{pmatrix} 0 & 1 \\ \delta^{-1}\lambda + \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} + \nu^\pm \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{\lambda}{\delta}} & \sqrt{\frac{\lambda}{\delta}} \end{pmatrix}\left[\begin{pmatrix} -\sqrt{\frac{\lambda}{\delta}} & 0 \\ 0 & \sqrt{\frac{\lambda}{\delta}} \end{pmatrix} + \mathcal{O}(1)\right]\begin{pmatrix} 1 & -\sqrt{\frac{\delta}{\lambda}} \\ 1 & \sqrt{\frac{\delta}{\lambda}} \end{pmatrix} + \nu^\pm. \end{aligned} \tag{6.6}$$

This immediately proves the lemma. ∎

At $\lambda = 0$ the asymptotic matrices have the form of the linearisation (1.9) of the travelling-wave o.d.e.,

$$M_\nu^\pm(0) = \begin{pmatrix} 0 & \mathrm{id} \\ \delta^{-1}g'(u^\pm) & \delta^{-1}\Big(A(u^\pm) - s\Big) \end{pmatrix} + \nu^\pm = L(u^\pm) + \nu^\pm. \tag{6.7}$$

In proposition 1.1 we have already calculated the spectrum of L. We obtain:

Lemma 6.2 *For $s \notin \{\alpha_0, \alpha_1, \alpha_2\}$ and small $\delta, |u^\pm|$ the spectrum of $M_\nu^\pm(0)$ consists of three large eigenvalues*

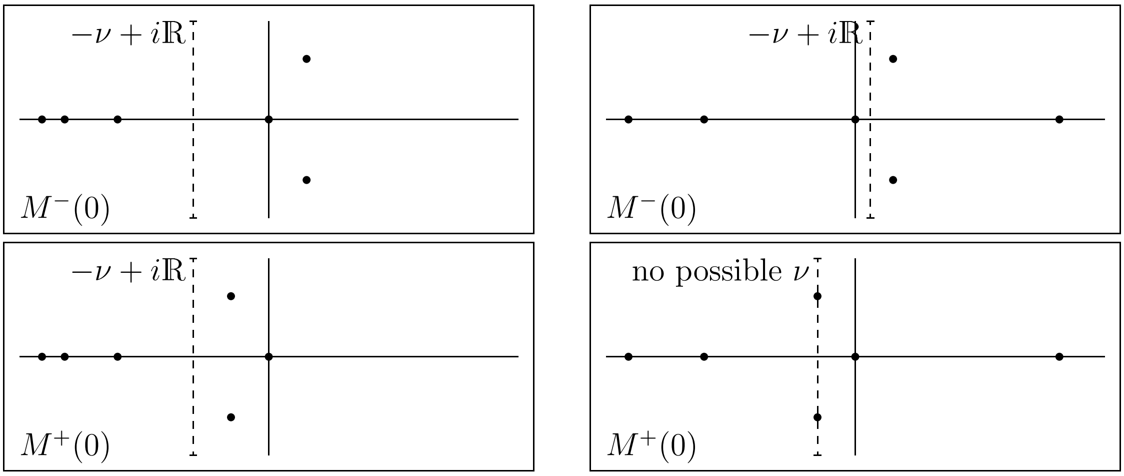
$$\nu^\pm + \delta^{-1}(\alpha_i - s) + \mathcal{O}(1) \tag{6.8}$$

and three small eigenvalues

$$\nu^\pm, \quad \nu^\pm + i\omega + u_0^\pm \mu'(0) + \mathcal{O}(\delta, |u^\pm|^2), \quad \nu^\pm - i\omega + u_0^\pm \mu'(0) + \mathcal{O}(\delta, |u^\pm|^2). \tag{6.9}$$

Lemma 6.3 *A necessary condition for convective stability of a travelling wave solution to our balance law is the existence of weights $\nu^\pm$ such that the spectra of $M_\nu^\pm(0)$ consist of three eigenvalues with positive and three eigenvalues with negative real part. In the complex plane, vertical lines $-\nu^\pm + i\mathbb{R}$ must exist that divide the spectrum of $L(u^\pm)$ into three points on each side, counted with algebraic multiplicity.*

Proof: Assume convective stability with a weight $\nu = (\nu^-, \nu^+)$ satisfying the condition of definition 5.11. Suppose $M_\nu^\pm(0)$ to have $i_\nu^+(0) = i_\nu^-(0) \neq 3$ eigenvalues with positive real part. (The indices i_ν^+, i_ν^- must be equal due to the stability assumption.) On any path in the complex right half plane from the origin, $\lambda = 0$, to large real parts, $\Re\mathrm{e}\,\lambda \gg 1$, the indices $i_\nu^\pm$ must finally change to a value of $i_\nu^\pm(\lambda, \Re\mathrm{e}\,\lambda \gg 1) = 3$, see lemma 6.1. Hyperbolicity, say of $M_\nu^+(\lambda)$, with a fixed unstable dimension i_ν^+, is an open property,



a) extreme wave speed $s > \alpha_0, \alpha_1, \alpha_2$. b) intermediate speed $\alpha_0 > s > \alpha_1, \alpha_2$.

Figure 6.1: Eigenvalues of the asymptotic matrices $M_\nu^\pm(\lambda = 0)$.

in the parameter λ. Therefore, there exists a point $\tilde{\lambda}$ with $\Re\mathrm{e}\,\tilde{\lambda} > 0$, on any of the above mentioned paths, where the hyperbolicity of $M_\nu^+(\tilde{\lambda})$ fails, i.e. $M_\nu^+(\tilde{\lambda})$ has a purely imaginary eigenvalue. We have $\tilde{\lambda} \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{M}|_{L_\nu})$, see theorem 5.7. This contradiction to the stability assumption proves the lemma. ∎

The main restriction is now imposed by the two Hopf eigenvalues, that have the same real part, $u_0^\pm\,\Re\mathrm{e}\,\mu'(0) + \mathcal{O}(\delta, |u^\pm|^2)$, and, therefore, cannot be divided by a vertical line. In addition, the Hopf eigenvalues are on different sides of the trivial eigenvalue at the asymptotic states $u^\pm$, see figure 6.1.

If the three large eigenvalues, see lemma 6.2, have the same sign, i.e. $s > \alpha_0, \alpha_1, \alpha_2$ or $s < \alpha_0, \alpha_1, \alpha_2$, then the only possible dividing line $-\nu^\pm + i\mathbb{R}$ has the three large eigenvalues on one side and the three small eigenvalues on the other side, for both asymptotic states. If, on the other hand, one large eigenvalue has a different sign than the remaining two, then for one of the asymptotic states the Hopf eigenvalues are accompanied by two large eigenvalues on one side and the third large eigenvalue together with the trivial one on the other side. No partition of three to three eigenvalues is possible, in this case. We arrive at the following conclusion.

Corollary 6.4 *In the three dimensional case, convective stability of the oscillatory profiles near the* Hopf *point implies that the speed of the travelling wave is extreme. Precisely, one of the cases*

(i) $s^{\mathrm{crit}} > \alpha_0, \alpha_1, \alpha_2$

(ii) $s^{\text{crit}} < \alpha_0, \alpha_1, \alpha_2$

must hold. The two weight exponents $\nu^\pm$ *must be positive in case (i) and negative in case (ii), in order for convective stability to hold.*

The next task is to show that the above condition is also sufficient to control at least the essential spectrum of the linearised operator (6.1).

Lemma 6.5 *Suppose*

$$\sigma \ := \ s - \max_{0 \le i \le 2} \alpha_i \ > \ 0. \tag{6.10}$$

Then there exist a weight exponent $\nu = (\nu^-, \nu^+)$ *and constants* δ_0, ε*, with* $\nu^- = \nu^+ > 0$, $\delta_0 > 0$, $\varepsilon > 0$*, such that the following statement holds.*

For all δ *with* $0 < \delta < \delta_0$*, for all asymptotic states near the origin,* $|u_0^\pm| < \varepsilon$*, and for any* λ *with* $\Re\mathrm{e}\, \lambda > -\nu^\pm \sigma/4$*, the asymptotic matrices (6.3) are hyperbolic with three stable and three unstable eigenvalues.*

Notably, the essential spectrum of $\mathcal{M}_\nu$*, see (6.1), is bounded to the left of the imaginary axis with gap* $\nu^\pm\sigma/4$ *for small* δ *and oscillatory viscous profiles, given by theorem 1.3, in a small enough neighbourhood of the* Hopf *point.*[19]

Due to symmetry, the same statement hold for $s < \alpha_0, \alpha_1, \alpha_2$ *and for a negative weight-exponent.*

Proof: In lemma 6.1, we have already fixed the unstable dimension, which cannot change as long as the matrices remain hyperbolic. Therefore, it is sufficient to prove hyperbolicity.

We shall first formulate equivalent statements to the failure of hyperbolicity for some

[19]More precisely, we have to take δ small enough, such that the Hopf-point for this positive δ is located inside the ε-neighbourhood of the origin. Then we can consider heteroclinic orbits around this Hopf-point with asymptotic states $u^\pm$ inside this ε-ball centred at the origin.

number $\lambda \in \mathbb{C}$:

$$\begin{aligned}
& M^\pm(\lambda) + \nu^\pm \quad \text{is not hyperbolic} \\
\iff\ & \exists \chi \in \mathbb{R} \\
& i\chi - \nu^\pm \ \in \ \operatorname{spec} M^\pm(\lambda) \\
\iff\ & \exists \chi \in \mathbb{R}, \tilde{u}, \tilde{v} \in \mathbb{C}^n : \\
& (i\chi - \nu^\pm) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \ = \ \begin{pmatrix} 0 & \mathrm{id} \\ \delta^{-1}\big(\lambda - g'(0)\big) & \delta^{-1}\big(A(u^\pm) - s\big) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \\
\iff\ & \exists \chi \in \mathbb{R}, \tilde{u} \in \mathbb{C}^n : \\
& (i\chi - \nu^\pm)^2 \tilde{u} \ = \ \delta^{-1}\big(\lambda - g'(0)\big)\tilde{u} + \delta^{-1}\big(A(u^\pm) - s\big)(i\chi - \nu^\pm)\tilde{u} \\
\iff\ & \exists \chi \in \mathbb{R} : \\
& \lambda - \delta(\nu^\pm - i\chi)^2 \ \in \ \operatorname{spec}\Big((\nu^\pm - i\chi)(A(u^\pm) - s) + g'(0)\Big)
\end{aligned} \tag{6.11}$$

Now, the matrix $A(u) = f'(u)$ has only real eigenvalues, due to the hyperbolicity of our conservation law. It has the form $A(u^\pm) = A_0 + u_0^\pm A_1 + \mathcal{O}((u_0^\pm)^2)$, where A_0 is diagonal with entries α_i, see (1.12). For small $|u^\pm|$, the matrix A is a small perturbation of A_0. Therefore, we can find an $\varepsilon > 0$ such that for all $|u_0^\pm| < \varepsilon$

$$\operatorname{spec}(A(u^\pm) - s) \ \subset \ (-\infty, -\frac{3}{4}\sigma), \tag{6.12}$$

which implies, for positive $\nu^\pm$,

$$\operatorname{spec}\Big((1 + i\frac{\chi}{\nu^\pm})(A(u^\pm) - s)\Big) \ \subset \ (-\infty, -\frac{3}{4}\sigma) + i\mathbb{R}. \tag{6.13}$$

Again, for large $\nu^\pm$, the norm of $(\nu^\pm)^{-1}g'(0)$ is arbitrarily small with respect to the spectral gap $3\sigma/4$ of (6.13). There exists a value $\nu^\pm > 0$, independent of χ, such that

$$\operatorname{spec}\Big((\nu^\pm - i\chi)(A(u^\pm) - s) + g'(0)\Big) \ \subset \ (-\infty, -\frac{1}{2}\nu^\pm\sigma) + i\mathbb{R}. \tag{6.14}$$

Let us fix

$$\delta_0 \ := \ \frac{\sigma}{4\nu^\pm} \ > \ 0. \tag{6.15}$$

This enables us to estimate for all $\Re\mathrm{e}\,\lambda > -\nu^\pm\sigma/4$ and $0 < \delta < \delta_0$:

$$\Re\mathrm{e}(\lambda - \delta(\nu^\pm - i\chi)^2) \ = \ \Re\mathrm{e}\,\lambda - \delta(\nu^\pm)^2 + \delta\chi^2 \ > \ \Re\mathrm{e}\,\lambda - \delta(\nu^\pm)^2 \ > \ -\tfrac{1}{2}\nu^\pm\sigma. \tag{6.16}$$

Combining (6.14) and (6.16) with (6.11) we have proven the lemma. ∎

6.2 Resolvent set

With the essential spectrum under control, we now investigate the point spectrum.

Consider the linearisation around the trivial solution $u(\xi) \equiv 0$,

$$\tilde{u}_\tau \;=\; \mathcal{M}^0 \tilde{u} \;=\; \delta \tilde{u}_{\xi\xi} + s\tilde{u}_\xi - A_0 \tilde{u}_\xi + g'(0)\tilde{u}. \tag{6.17}$$

The eigenvalue problem $0 = (\mathcal{M}^0_\nu - \lambda)\tilde{u}$ has an associated family of operators

$$\mathcal{T}^0_\nu(\lambda) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - M^0_\nu(\lambda) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \tag{6.18}$$

with *constant coefficients*. The matrix M^0_ν is independent of ξ. (The structure is the same as before in (6.2), if $u(\xi)$ replaced by 0.) In lemma 6.5, we have already proven that M^0_ν is hyperbolic for all λ with $\Re\mathrm{e}\,\lambda > -\nu^\pm\sigma/4$. Therefore, for these λ, the equation (6.18) has an exponential dichotomy on the entire real line, given by the eigenprojections of M^0_ν. Theorem 5.7 now states that all λ with $\Re\mathrm{e}\,\lambda > -\nu^\pm\sigma/4$ belong to the resolvent set of $\mathcal{M}^0$. In particular all point eigenvalues have real part less than $-\nu^\pm\sigma/4$.

The robustness of exponential dichotomies, theorem 5.6, and the pre-compactness of bounded sets in $\mathbb{C}$ lead to the following conclusion:

Corollary 6.6 *Suppose* $\sigma = s - \max_{0\le i\le 2}\alpha_i > 0$. *Choose a suitable weight-exponent* $\nu = (\nu^-, \nu^+)$ *according to lemma 6.5. Then for any constant* $C > 0$ *there exist positive constants* δ_0, ε*, possibly smaller than those of lemma 6.5, such that the following statement holds.*

Take any oscillatory viscous profiles, given by theorem 1.3, for parameters δ *with* $0 < \delta < \delta_0$ *and asymptotic states* $u^\pm = (u^\pm_0, 0, \ldots, 0)$ *with* $|u^\pm_0| < \varepsilon$*. Then all* λ *with* $\Re\mathrm{e}\,\lambda > -\nu^\pm\sigma/4$ *and* $|\lambda| < C$ *belong to the resolvent set of the weighted, linearised operator* $\mathcal{M}_\nu$*, given in (6.1).*

6.3 Stability

The last part of our proof of stability is standard theory of elliptic and parabolic operators, see for instance [Hen81].

The linear operator $-\mathcal{M}_\nu$, see (6.1), is sectorial, because the second order term is the Laplace operator on the real line with diagonal coefficient matrix and the remaining terms are of lower order and have bounded coefficients. In particular, there exist positive constants a, b ($b < \pi/2$) such that the entire sector

$$\{\lambda \in \mathbb{C}| - \tfrac{\pi}{2} - b < \arg(\lambda - a) < \tfrac{\pi}{2} + b\} \tag{6.19}$$

is part of the resolvent set of $\mathcal{M}_\nu$.

The sector (6.19) includes the entire complex half plane of nonnegative real parts with the exception of a bounded domain around the origin. Corollary 6.6 covers this bounded domain. In fact, we can take $C > \max\{a, a\cot(b)\}$ in corollary 6.6. This confirms that all λ with $\Re\mathrm{e}\,\lambda \geq 0$ belong to the resolvent set of $\mathcal{M}_\nu$, provided the assumptions of corollary 6.6 are met. The convective stability of the viscous profiles, theorem 1.5, is proven.

We can even show that all λ with $\Re\mathrm{e}\,\lambda \geq -\nu^\pm\sigma/4$ are contained in the resolvent set of $\mathcal{M}_\nu$. We must choose C in corollary 6.6 slightly larger than before, e.g. $C^2 > (a + \nu^\pm\sigma/4)^2 \cot(b)^2 + (a + \nu^\pm\sigma/4)^2$. A sectorial operator with a bound on the real part of its spectrum provides an estimate on the generated semigroup. We obtain

$$\| \exp(\mathcal{M}_\nu)\|_{\mathcal{L}(L^2,L^2)} \;\leq\; c\, e^{-\nu^\pm\sigma/4}, \tag{6.20}$$

for some constant c. Any solution of (6.1) decays exponentially fast to zero in L^2. This result finally establishes the linear stability of the viscous profiles under the assumptions of theorem 1.5 in the exponentially weighted space.

7 Discussion

We have constructed viscous profiles with oscillatory tails in a three-dimensional system of hyperbolic conservation laws with stiff source terms. They arise near a Hopf-like bifurcation point along a line of equilibria in the associated travelling-wave equation.

The oscillations arise despite the fact that our problem is composed of two ingredients which, if considered separately, resist oscillations. The conservation-law part is strictly hyperbolic, the flux can even be a gradient. Without the source, the conservation law gives rise to monotone viscous profiles of weak shocks. The source, taken alone, stabilises the dynamics: all trajectories of the pure kinetics end by converging monotonically to equilibria. The unsuspected existence of oscillations due to the combination of nonoscillatory ingredients is similar to the Turing instability observed in reaction-diffusion equations.

In numerical approximations of conservation laws with sources, oscillations need not to be numerical artifacts. They may be intrinsic properties of the solution. The oscillations manifest themselves as overshooting near the shock front.

The observed waves can be constructed for any wave speed, regardless of the characteristic speeds of the conservation law. In three dimensional systems, they are convectively stable only for hypersonic shocks, i.e. for profiles with extreme speeds larger or smaller than all characteristic speeds. For intermediate speeds the waves are absolutely unstable. These results still hold for general three-dimensional systems as long as the profiles emerge from a Hopf bifurcation of the kind described in theorem 1.2. The special structure (1.12) that we used to construct the oscillatory profiles is not needed in the proof of stability.

The exponential weights which we used to prove convective stability are strong in relation to the rates at which the profiles converge to their asymptotic states. This has the following reason. Because we investigate the stability of a particular wave, the considered norm must separate it from the continuum of profiles that emerge from the Hopf point. We recall that, for fixed speed, an entire neighbourhood of the bifurcation point within its centre manifold is filled with heteroclinic orbits. For nearby wave speeds, a Hopf bifurcation occurs at a slightly shifted critical point. In addition, the travelling-wave equation is equivariant with respect to translations. Therefore, we obtain at least a four-parameter family of oscillatory profiles. The exponential weights centred at an oscillatory travelling wave push any other profile out of the examined space.

For example, consider an oscillatory profile $u(\xi)$. Then the linearisation (5.2) of the balance law around $u(\cdot)$ has a zero eigenvalue with the eigenvector $u_\xi(\cdot)$, due to the translational symmetry. This eigenvector does not belong to the exponentially weighted space that we used to prove convective stability, and the trivial eigenvalue has been eliminated. Perturbations that would lead to a shift of the wave are likewise eliminated from our stability analysis.

So, what does convective stability mean? Take any small perturbation with bounded support of a convectively stable oscillatory profile, for example. Then, in the linearised equation, this perturbation will pointwise decay to zero. On any bounded subinterval it will decay uniformly.

We can think of an alternative approach to the question of stability. One could consider the entire four-parameter family of oscillatory profiles and the possible convergence of nearby initial data to this entire family with respect to a suitable metric. Unfortunately, the structure of this family is quite involved, especially near the Hopf point, as we explained above. Such an investigation is therefore beyond the scope of the present thesis.

We have proven linear stability in the exponentially weighted space. Nonlinear stability is a much more difficult problem due to the exponential weights. Perturbations that are convected to $-\infty$ may increase[20] exponentially in the L^2 or supremum norm. If they become large enough, the linearisation of the balance law is no longer a sufficient approximation of the whole system. Nonlinear effects become dominant. The nonlinearity could accelerate the growth of the perturbation. The growth need not to be dominated by the convection to $-\infty$, anymore. The perturbation may even spread out in the opposite direction and destroy the very wave structure itself. Completely different patterns may evolve. See, for example, [CC97].

A bridge from the linear stability, as we proved here, to nonlinear stability on bounded domains is provided by the results of Sandstede and Scheel [SS00]. Indeed, with suitable separated boundary conditions[21], the spectrum of the linearisation at an oscillatory profile is bounded to the left of the imaginary axis with positive gap. This is sufficient to ensure even nonlinear stability on any fixed bounded subinterval. Perturbations can grow only slightly and then disappear through the boundary. Actually, on bounded subintervals, all weighted norms are equivalent to the L^2-norm.

[20] for weights with positive exponents.

[21] In fact, the boundary conditions must not introduce additional unstable point eigenvalues. Appropriate conditions are given in [SS00]

In numerical simulations, the calculations must be carried out on bounded domains. The described effects should be nonlinearly stable phenomena. In fact, our example could become a test-case for numerical schemes that are designed for systems with stiff source terms. Further work has to be done is this direction.

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