

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/339015833>

Meanders, zero numbers and the cell structure of Sturm global attractors

Preprint · February 2020

CITATIONS

0

READS

42

2 authors, including:



[Carlos Rocha](#)

Technical University of Lisbon

47 PUBLICATIONS 719 CITATIONS

SEE PROFILE

Meanders, zero numbers and the
cell structure of Sturm global attractors

– Dedicated to the memory of Pavol Brunovský –

Carlos Rocha
Instituto Superior Técnico
Avenida Rovisco Pais, 1049–001 Lisboa, PORTUGAL
crocha@math.ist.utl.pt
<http://www.math.ist.utl.pt/cam/>

Bernold Fiedler
Institut für Mathematik
Freie Universität Berlin
Arnimallee 7, D–14195 Berlin, GERMANY
fiedler@math.fu-berlin.de
<http://dynamics.mi.fu-berlin.de>

version of February 4, 2020

Abstract

We study global attractors $\mathcal{A} = \mathcal{A}_f$ of semiflows generated by semilinear partial parabolic differential equations of the form $u_t = u_{xx} + f(x, u, u_x)$, $0 < x < 1$, satisfying Neumann boundary conditions. The equilibria $v \in \mathcal{E} \subset \mathcal{A}$ of the semiflow are the stationary solutions of the PDE, hence they are solutions of the corresponding second order ODE boundary value problem. Assuming hyperbolicity of all equilibria, the dynamic decomposition of \mathcal{A} into unstable manifolds of equilibria provides a geometric and topological characterization of Sturm global attractors \mathcal{A} as finite regular signed CW-complexes, the Sturm complexes, with cells given by the unstable manifolds of equilibria. Concurrently, the permutation $\sigma = \sigma_f$ derived from the ODE boundary value problem by ordering the equilibria according to their values at the boundaries $x = 0, 1$, respectively, completely determines the Sturm global attractor \mathcal{A} . Equivalently, we use a planar curve, the meander $\mathcal{M} = \mathcal{M}_f$, associated to the the ODE boundary value problem by shooting.

In the previous paper [FR18d], we set up to determine the boundary neighbors of any specific unstable equilibrium \mathcal{O} , based exclusively on the information on the corresponding signed cell complex. In addition, a certain minimax property of the boundary neighbors was established. This property identifies the equilibria on the cell boundary of \mathcal{O} which are closest or most distant from \mathcal{O} at the boundaries $x = 0, 1$.

The main objective of this paper is to derive this minimax property directly from the permutation σ , the Sturm permutation, or equivalently from the meander \mathcal{M} , the Sturm meander, based on the Sturm nodal properties of the solutions of the ODE boundary value problem. This minimax result simplifies the task of identifying the equilibria on the cell boundary of each unstable equilibrium directly from the Sturm meander \mathcal{M} . It is instrumental in the identification of the geometric location, with respect to each unstable equilibrium, of all the equilibria in the Sturm global attractor \mathcal{A} . We emphasize the local aspect of this result by applying it to an example for which the identification of the equilibria in the cell boundary of \mathcal{O} is obtained from the knowledge of only a section of the Sturm meander \mathcal{M} .

1 Introduction

For quite a long time we have been pursuing the study of global attractors of scalar semilinear parabolic equations in one space dimension. Under appropriate dissipativeness conditions on the nonlinearity $f \in C^2$, equations of the form

$$(1.1) \quad u_t = u_{xx} + f(x, u, u_x), \quad 0 < x < 1,$$

with suitable boundary conditions, generate global dissipative semiflows on a Sobolev space $X \subset C^1([0, 1], \mathbb{R})$. The global attractor $\mathcal{A}_f \subset X$ of the semiflow, i.e. the maximal compact invariant subset, is called *Sturm global attractor*. In the present paper we consider the specific case of Neumann boundary conditions $u_x = 0$ at $x = 0, 1$, for which the generated semiflow is gradient-like. We refer to [He81, Pa83, Ta79] for a general background, to [Ha88, Te88, La91, BV92, Ra02] for a contextual introduction to global attractors, and to [FR14, FR15], and the citations there, for a general introduction to Sturm global attractors. See also [BF88, BF89, Fi94] for earlier results.

The stationary solutions $u(t, x) = v(x)$ of (1.1) satisfy the second order Neumann boundary value problem

$$(1.2) \quad \begin{aligned} v_{xx} + f(x, v, v_x) &= 0, \quad 0 < x < 1, \\ v_x &= 0, \quad x = 0 \text{ or } 1, \end{aligned}$$

which provides the link to the name of Sturm associated to the global attractor $\mathcal{A} = \mathcal{A}_f$. Each stationary solution of (1.1) corresponds to an equilibrium $v \in \mathcal{E}$ of the semiflow in X and is a compact invariant subset, therefore an element of the global attractor \mathcal{A} . Under the generic assumption of *hyperbolicity* of all equilibria $v \in \mathcal{E}$ the set of equilibria is finite,

$$(1.3) \quad \mathcal{E} = \{v_j : j = 1, \dots, N\},$$

with N odd due to dissipativeness. Hyperbolicity of the equilibrium $v \in \mathcal{E}$ is achieved if $\lambda = 0$ is not an eigenvalue of the linearization of (1.1) at v . Then, the gradient-like property of the semiflow provides a dynamic decomposition of the Sturm global attractor \mathcal{A} into unstable manifolds of equilibria,

$$(1.4) \quad \mathcal{A} = \bigcup_{1 \leq j \leq N} W^u(v_j).$$

This decomposition suggests a geometric and topological characterization of the Sturm global attractor \mathcal{A} as a finite regular *CW*-complex $\mathcal{C} = \mathcal{C}_f$,

$$(1.5) \quad \mathcal{C} = \bigcup_{1 \leq j \leq N} c_j,$$

with cell interiors $c_j = W^u(v_j)$. This is the regular *Thom-Smale complex* of the Sturm global attractor \mathcal{A} , or the *Sturm complex*, [FR14, FR18a].

Concurrently, the permutation $\sigma = \sigma_f \in S_N$ derived from the boundary value problem (1.2) by ordering successively the N equilibria according to their values at $x = 0$ and $x = 1$, completely determines the Sturm global attractor \mathcal{A} up to C^0 orbit equivalence, [FR00]. If we let $h_0, h_1 : \{1, \dots, N\} \rightarrow \mathcal{E}$ denote the two boundary orders of the equilibria,

$$(1.6) \quad h_\iota(1) < h_\iota(2) < \dots < h_\iota(N) \quad \text{at} \quad x = \iota \in \{0, 1\},$$

then the *Sturm permutation*

$$(1.7) \quad \sigma := h_0^{-1} \circ h_1$$

provides a combinatorial description of the Sturm global attractor \mathcal{A} determining exactly which equilibria $v_j, v_k \in \mathcal{E}$ are connected by heteroclinic connecting orbits, $v_j \rightsquigarrow v_k$. For this combinatorial description of Sturm global attractors we refer to [FR91, FR96]. Here we just mention the central role of the nodal properties of the solutions of (1.1). If $0 \leq z(\varphi) \leq \infty$ denotes the number of strict sign changes (the *zero number*) of $\varphi : [0, 1] \rightarrow \mathbb{R}$, with $\varphi \not\equiv 0$, then

$$(1.8) \quad t \longmapsto z(u^1(t, \cdot) - u^2(t, \cdot))$$

is finite and nonincreasing for $t > 0$, for any two distinct solutions u^1, u^2 of (1.1), and drops strictly whenever multiple zeros $u^1 = u^2, u_x^1 = u_x^2$ occur at any t_0, x_0 . We refer to [An88] for details and to [Ma82, FR96, FR99, FR00, Ro91, Ga04] for many aspects of nonlinear Sturm theory. For convenience, in the following we also use the signed notation

$$(1.9) \quad z(\varphi) = m_\pm$$

to indicate that φ has m strict sign changes and the index \pm to indicate that $\pm\varphi(0) > 0$.

The solution of the second order boundary value problem (1.2) by ODE shooting in phase space (v, v_x) produces a meander characterization of the Sturm permutation. Existence of solutions for $x \in [0, 1]$ is ensured here by assuming a sublinear growth of f , without loss of generality for \mathcal{A} due to its compactness. Let $v = v(x, a)$ abbreviate the solution starting from $v = a, v_x = 0$ at $x = 0$. Then, if we consider the initial set of Neumann conditions $\{(a, 0) : a \in \mathbb{R}\}$ (at $x = 0$) we obtain as image at $x = 1$ a planar regular non-selfintersecting curve $\mathcal{M} = \mathcal{M}_f := \{(v(1, a), v_x(1, a)) : a \in \mathbb{R}\}$ which transversely intersects the Neumann horizontal axis at N points corresponding to the end values of the equilibria $v_j \in \mathcal{E}$. Such a curve is called a *meander* and its existence shows that the Sturm permutation is a *meander*

permutation. We refer to [FR99] for details and to [Ka17, D&a19] for many other aspects of meander permutations. The first important outcome of the meander characterization is that the ODE Sturm permutation $\sigma = \sigma_f$ determines the PDE *Morse indices* $i(v_j) = \dim W^u(v_j)$ of the equilibria $v_j \in \mathcal{E}$, $j = 1, \dots, N$, and the number of zeros of their differences $z(v_k - v_j)$, $1 \leq j < k \leq N$. We review these results further on. For now it suffices to recall that, for odd N , a permutation $\sigma \in S_N$ is a Sturm permutation if and only if it is a *dissipative, Morse, meander permutation*, [FR99]. Dissipative here means that $\sigma(1) = 1$ and $\sigma(N) = N$ are fixed, and Morse means that the Morse indices computed from σ are all non-negative, $i(v_j) \geq 0$.

The association between Sturm complexes and Sturm permutations is crucial. The geometric characterization of Sturm global attractors involves a description of the set of signed Thom-Smale complexes for which the construction of Sturm permutations is possible. This characterization has been successfully addressed in low dimensional cases of

$$(1.10) \quad \dim \mathcal{A} := \max\{\dim W^u(v_j), 1 \leq j \leq N\},$$

for the planar case in [FR09a, FR08, FR09b], and for three-dimensional balls in [FR18a, FR18b, FR18c]. To continue this program, in [FR18d] we set up to determine the boundary neighbors (1.6) of an unstable equilibrium, based exclusively on the information of the corresponding signed cell complex. Let $\mathcal{O} \in \mathcal{E}$ denote an unstable equilibrium with $i(\mathcal{O}) = n$. The four boundary neighbors $w_{\pm}^{\iota} = w_{\pm}^{\iota}(\mathcal{O})$ of \mathcal{O} ,

$$(1.11) \quad w_{\pm}^{\iota} := h_{\iota}(h_{\iota}^{-1}(\mathcal{O}) \pm 1) \quad , \quad \iota \in \{0, 1\},$$

whenever defined, are the predecessors and successors of \mathcal{O} , along the boundary orders h_{ι} at $x = \iota = 0, 1$. Then, the identification of the boundary neighbors of $\mathcal{O} \in \mathcal{E}$ with

$$(1.12) \quad i(w_{\pm}^{\iota}) = i(\mathcal{O}) - 1 = n - 1,$$

has been obtained in terms of four sequences of equilibria of descending Morse indices, called *descendants*, defining maximal chains of heteroclinic orbit connections

$$(1.13) \quad \mathcal{O} \rightsquigarrow v^{n-1} \rightsquigarrow \dots \rightsquigarrow v^0 \quad , \quad i(v^j) = j,$$

starting from the unstable equilibrium \mathcal{O} and finishing on a stable equilibrium v^0 . Let $\mathcal{E}_{\pm}^j(\mathcal{O}) \subset \mathcal{E}$ denote the equilibrium subsets

$$(1.14) \quad \mathcal{E}_{\pm}^j(\mathcal{O}) := \{w \in \mathcal{E} : z(w - \mathcal{O}) = j_{\pm} \text{ and } \mathcal{O} \rightsquigarrow w\}.$$

The key ingredient for the identification of the boundary neighbors of $\mathcal{O} \in \mathcal{E}$ is provided by the signed zero numbers (1.9) of the differences $\varphi = w - \mathcal{O}$ for all $w \in \mathcal{E}_{\pm}^j(\mathcal{O})$. We

refer to [FR18d] for the details. Here we just remark that this identification uses the zero number decay property (1.8) and derives exclusively from the geometric structure of the signed Thom-Smale complex (1.5) of the Sturm global attractor.

To be more specific, let $\underline{v}_\pm^\iota, \bar{v}_\pm^\iota, \iota \in \{0, 1\}$, denote the equilibria

$$(1.15) \quad \underline{v}_\pm^\iota := \text{the equilibrium } v \in \mathcal{E}_\pm^{n-1}(\mathcal{O}) \text{ closest to } \mathcal{O} \text{ at } x = \iota,$$

$$(1.16) \quad \bar{v}_\pm^\iota := \text{the equilibrium } v \in \mathcal{E}_\pm^{n-1}(\mathcal{O}) \text{ most distant from } \mathcal{O} \text{ at } x = \iota.$$

Then [FR18d, Theorem 1.2] combined with [FR18d, Theorem 3.1] assert that those of the boundary neighbors w_\pm^ι of \mathcal{O} defined in (1.11) which satisfy $i(w_\pm^\iota) = n - 1$, are given by

$$(1.17) \quad w_-^0 = \underline{v}_-^0;$$

$$(1.18) \quad w_+^0 = \underline{v}_+^0;$$

$$(1.19) \quad w_-^1 = \begin{cases} \underline{v}_+^1, & \text{for even } n; \\ \underline{v}_-^1, & \text{for odd } n; \end{cases}$$

$$(1.20) \quad w_+^1 = \begin{cases} \underline{v}_-^1, & \text{for even } n; \\ \underline{v}_+^1, & \text{for odd } n. \end{cases}$$

The following additional result, also derived from the signed Thom-Smale complex of the Sturm global attractor \mathcal{A} , was obtained in Theorem 4.3 of [FR18d].

Theorem 1.1 *If any of the w_\pm^ι of the unstable equilibrium \mathcal{O} , as defined by (1.11), is more stable than \mathcal{O} , i.e. if $i(w_\pm^\iota) = n - 1$, then for the associated \underline{v}_\pm^ι of that sign \pm and that ι , in (1.15)–(1.20), we have*

$$(1.21) \quad \underline{v}_\pm^\iota = \bar{v}_\pm^{1-\iota}.$$

As pointed out in [FR18d], this additional minimax property simplifies the task of identifying the equilibria $\mathcal{E}_\pm^{n-1}(\mathcal{O})$ directly from the Sturm meander \mathcal{M} .

The main purpose of our present paper is precisely to prove Theorem 1.1 directly, based solely on the analysis of the Sturm meander \mathcal{M} . Our result gains broader relevance as a step within a much more ambitious program: the complete geometric characterization of all those signed regular cell complexes which arise as signed Thom-Smale complexes in the Sturm PDE setting. Limited to mere 3-ball global attractors \mathcal{A} , we initiated this program in [FR18a]–[FR18c]. Even with the step presented here and in the companion paper [FR18d], we are still far from that goal, in general.

The missing steps would involve the following. Of course, we first have to show that the (still elusive) characterizing geometric properties of the initial formal signed regular cell complex, are satisfied by all Sturm complexes. Given an abstract cell complex, prescribed according to the rules of the geometric characterization, we formally call the barycenters of the abstract cells “equilibria”. The cell dimensions are the “Morse indices” of the barycenter equilibria. The rules of Theorem 1.1 then formally determine predecessors and successors of all “equilibria”, recursively. Starting from the “equilibria” of maximal “Morse index”, i.e. from the barycenters of cells with maximal dimension, this recursively determines two formal total “boundary” orders h_ι of all equilibria, separately for each $\iota = 0, 1$. The two formal total orders h_ι , in turn, determine a formal permutation σ , as in (1.6), (1.7). The elusive geometric characterization would then have to guarantee, a priori, that the formal permutation σ is in fact a Sturm permutation. In a final step, it remains to show that the resulting signed Thom-Smale complex, associated to the Sturm permutation σ , coincides with the prescribed original cell complex. In particular, the barycenter “equilibria” become equilibria, their cells are their unstable manifolds, the “Morse indices” become actual Morse indices, i.e. unstable dimensions, and the formal total “boundary” orders h_ι become the orders of the equilibria at the respective boundaries $x = \iota = 0, 1$. We repeat that this program has only been completed for 3-ball Sturm attractors, so far. For further illustration in the present context we refer to the discussion section of [FR18d].

In the next section 2 we recall notation and essential results necessary to deal with the Morse indices of equilibria and the zero numbers of their differences, directly from the meander \mathcal{M} or its meander permutation. In section 3 we prove the main result, Theorem 1.1. To emphasize a local aspect of our global result, we show in section 4 an example for which the identification of the equilibria in $\mathcal{E}_+^{i(\theta)-1}(\mathcal{O})$ is obtained from the knowledge of only a section of the meander \mathcal{M} . We finish with an appendix, section 5, where we review the nonlinear Sturm-Liouville property in our meander setting.

Acknowledgments. This paper is dedicated to the memory of our longtime friend, colleague, and coauthor Palo Brunovský in deep admiration and gratitude. We owe much of our quest to his pioniering curiosity and his friendly, sharing, and noble spirit. Extended mutually delightful hospitality by the authors is gratefully acknowledged. CR expresses also gratitude to his family, friends and longtime coauthor in appreciation of their support and patience during a recent and specially hard time. This work was partially supported by DFG/Germany through SFB 910 project A4, and by FCT/Portugal through projects UID/MAT/04459/2019 and UIDB/MAT/04459/2020.

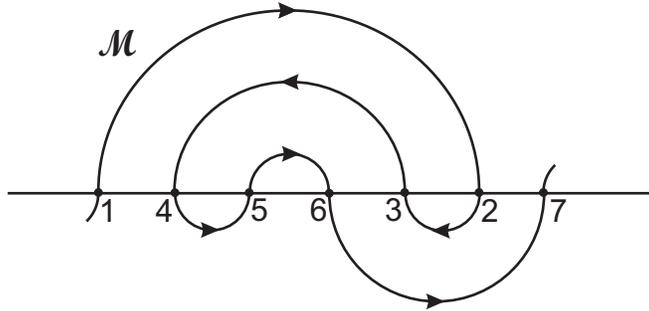


Figure 2.1: A meander \mathcal{M} in canonical form. The intersection points are labeled along the meander. Along the horizontal line the labeling corresponds to the permutation $\sigma = \{1\ 4\ 5\ 6\ 3\ 2\ 7\}$.

2 Meanders and Crossing numbers

A *meander* is a C^1 -smooth planar embedding of the (oriented) real line which crosses a positively oriented horizontal line at finitely many points with transverse intersections, [Ar88]. We consider meanders that run from Southwest to Northeast asymptotically, hence the number N of crossing points is odd. A meander for which all intersections with the base line are vertical, and all arcs joining intersection points are semicircles, is said to be in *canonical form* (see Figure 2.1 for an example).

In the following, we label the intersection points along the meander. The labeling obtained by reading along the horizontal line corresponds to a permutation $\sigma \in S_N$. Any permutation $\sigma \in S_N$ associated to a meander is then called a *meander permutation*. We recall that a permutation $\sigma \in S_N$ is called *dissipative* if it fixes the end points, i.e. $\sigma(1) = 1$ and $\sigma(N) = N$. The final ingredient has to do with the winding of the unit tangent vector of the meander. Solely based on the permutation $\sigma \in S_N$ we inductively define the Morse numbers i_j of the intersection points labeled $j = 1, \dots, N$ by

$$(2.1) \quad i_1 = 0, \quad i_{j+1} = i_j + (-1)^{j+1} \text{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)) .$$

With the meander in canonical form, these numbers count the clockwise half-windings of the unit tangent vector of the meander, from the initial intersection point $j = 1$ to the intersection point $j \in \{1, \dots, N\}$. We say that $\sigma \in S_N$ is a *Morse permutation* if all its Morse numbers are non-negative, that is, $i_j \geq 0$ for all $j = 1, \dots, N$.

A dissipative, Morse, meander permutation $\sigma \in S_N$ is called a *Sturm permutation*, and a meander with an associated Sturm permutation is called a *Sturm meander*. As it follows from [FR99, Theorem 1.2], any Sturm permutation is realizable by a boundary value problem (1.2) with dissipative nonlinearity f , and the corresponding Sturm meander is equivalent to

the associated ODE shooting meander in phase space $(u, u_x) \in \mathbb{R}^2$. Moreover, since global Sturm attractors \mathcal{A}_{f_1} and \mathcal{A}_{f_2} with the same Sturm permutation $\sigma_{f_1} = \sigma_{f_2}$ are C^0 orbit equivalent (see [FR00]), from here on we assume all meanders are in canonical form.

Let \mathcal{M} denote a Sturm meander intersecting the horizontal axis. Let the intersection points be labeled by $j = 1, \dots, N$, enumerated along the meander. For any Sturm realization (1.2) of \mathcal{M} , we label the set of equilibria $\mathcal{E} = \{v_j, j = 1, \dots, N\}$ such that $v_j = h_0(j)$. Then their Morse indices are given by the Morse numbers, $i(v_j) = i_j$. The interpretation of Figure 2.1 as a shooting diagram and the ordering (1.6) of equilibria at $x = \iota = 1$ then imply that $h_1(k)$ is the k -th equilibrium, enumerated left to right, along the horizontal axis. Suppose $h_1(k) = v_j$. Then, $h_1(k) = v_j = h_0(j)$, i.e. the positional index k of v_j along the horizontal axis is given by the inverse Sturm permutation

$$(2.2) \quad k = \sigma^{-1}(j);$$

see (1.7). Consequently, in terms of the h_1 -boundary order we obtain $v_j = h_1(\sigma^{-1}(j))$.

For any given Sturm meander \mathcal{M} in canonical form let $j < k$ and ℓ denote labels along h_0 , i.e. along the meander \mathcal{M} , of three meander intersections with the horizontal axis; see Figure 2.2. Suppose the oriented meander segment $\mathcal{M}_{j,k}$ of \mathcal{M} , from j to k , intersects the vertical line through ℓ transversely. (We ignore any crossing at ℓ itself, in case $j \leq \ell \leq k$.) We count clockwise crossings with respect to ℓ as $+1$, and counter-clockwise crossings as -1 . Then we define the *crossing number* $c_{\mathcal{M}}(j, k; \ell)$ of $\mathcal{M}_{j,k}$ with respect to ℓ as the total number of signed crossings, along the meander segment of \mathcal{M} from j to k . In other words, $c_{\mathcal{M}}(j, k; \ell)$ counts the total number of net clockwise half windings of the meander \mathcal{M} from j to k , around ℓ . For $j = k$ we naturally define $c_{\mathcal{M}}(j, j; \ell) := 0$.

For nontransverse crossings with the vertical line through ℓ , e.g. during homotopies of meanders not in canonical form, the definition can be extended by transverse approximations. For details see [FR99], but also our comment in the Appendix section 5.

For $j > k$, i.e. if we follow the meander \mathcal{M} in reverse orientation, we define

$$(2.3) \quad c_{\mathcal{M}}(j, k; \ell) := -c_{\mathcal{M}}(k, j; \ell).$$

Our definition of the half winding numbers $c_{\mathcal{M}}(j, k; \ell)$ with respect to ℓ implies the additivity property

$$(2.4) \quad c_{\mathcal{M}}(j_1, j_2; \ell) + c_{\mathcal{M}}(j_2, j_3; \ell) = c_{\mathcal{M}}(j_1, j_3; \ell),$$

for all choices of j_1, j_2, j_3 , and all ℓ .

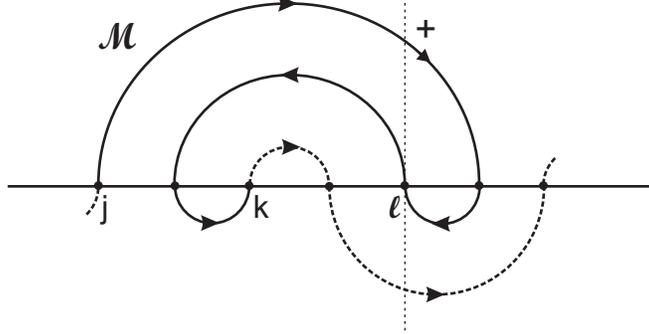


Figure 2.2: The crossing number $c_{\mathcal{M}}(j, k; \ell)$ counts the crossings between the meander segment of meander \mathcal{M} from j to k and the vertical line through ℓ , clockwise with respect to ℓ . Crossings at ℓ are not counted. Here $c_{\mathcal{M}}(j, k; \ell) = 1$. Note, $c_{\mathcal{M}}(j, k; j) = c_{\mathcal{M}}(j, k; k) = 0$.

The zero number $z(v_k - v_j)$ of the difference between any pair of distinct equilibria of any realization (1.2) of the Sturm meander \mathcal{M} can be obtained directly from the crossing numbers $c_{\mathcal{M}}(j, k; \ell)$. Indeed, from [Ro91, Proposition 3] it follows that

$$(2.5) \quad z(v_k - v_j) = \begin{cases} i(v_j) + c_{\mathcal{M}}(j, k; j), & \text{if } \mathcal{M}_{j, j+1} \text{ is in an odd quadrant with respect to } j; \\ i(v_j) - 1 + c_{\mathcal{M}}(j, k; j), & \text{if } \mathcal{M}_{j, j+1} \text{ is in an even quadrant with respect to } j. \end{cases}$$

Here $\mathcal{M}_{j, j+1}$ denotes the first arc segment of \mathcal{M} strictly between the intersection points j and $j + 1$. This is a nonlinear extension of the Sturm-Liouville property of solutions, which we review and prove in the Appendix section 5.

From (2.5) we recursively obtain the zero numbers $z_{jk} := z(v_k - v_j)$ in terms of the meander permutation σ . In fact, additivity (2.4) implies

$$(2.6) \quad z_{jk+1} - z_{jk} = c_{\mathcal{M}}(j, k + 1; j) - c_{\mathcal{M}}(j, k; j) = c_{\mathcal{M}}(k, k + 1; j),$$

for all $1 \leq j < k \leq N$, and in terms of the permutation σ we have

$$(2.7) \quad c_{\mathcal{M}}(k, k + 1; j) = \frac{1}{2}(-1)^{k+1} [\text{sign}(\sigma^{-1}(k + 1) - \sigma^{-1}(j)) - \text{sign}(\sigma^{-1}(k) - \sigma^{-1}(j))].$$

Moreover, in view of dissipativeness of f and the parabolic comparison principle, we obtain $v_1(x) < v_j(x) < v_N(x)$ for all $1 < j < N$ and all $0 \leq x \leq 1$. Hence, we define $z_{1j} = z_{jN} := 0$ for all $1 < j < N$ and the zero numbers z_{jk} for all $1 \leq j < k \leq N$ are obtained from the

decreasing recursion (see for example [FR96])

$$(2.8) \quad \begin{aligned} z_{1j} &= z_{jN} := 0 \\ z_{jk} &= z_{jk+1} + \frac{1}{2}(-1)^k [\text{sign}(\sigma^{-1}(k+1) - \sigma^{-1}(j)) - \text{sign}(\sigma^{-1}(k) - \sigma^{-1}(j))]. \end{aligned}$$

Finally we mention that Morse indices $i(v_j)$ and zero numbers z_{jk} provide all the information necessary to establish the existence of heteroclinic orbit connections between pairs of equilibria $v_j \rightsquigarrow v_k$. Such results derive from the zero number dropping argument mentioned in the Introduction, section 1, in relation with (1.8); see [FR96]. Here we recall the following adjacency notion of equilibria first introduced in [Wo02]. Two equilibria v_j and v_k are said to be *z-adjacent* if there does not exist any other equilibrium w with $w(0)$ strictly between $v_j(0)$ and $v_k(0)$ such that

$$(2.9) \quad z(w - v_j) = z(v_k - w) = z(v_k - v_j).$$

Then v_j and v_k are heteroclinically connected, $v_j \rightsquigarrow v_k$, if and only if $i(v_j) > i(v_k)$ and v_j and v_k are *z-adjacent*; see [Wo02, Theorem 2.1] and also [FR18b, Appendix 7]. If, due to an equilibrium w , the equilibria v_j and v_k are not *z-adjacent*, we say that w satisfying (2.9) *blocks* heteroclinic connections between v_j and v_k .

3 The minimax property

We now prove our main result Theorem 1.1. Let again $\mathcal{O} \in \mathcal{E}$ denote the reference unstable equilibrium with $i(\mathcal{O}) = n \geq 1$, and let $w_{\pm}^{\iota}, \iota \in \{0, 1\}$, denote its four h_{ι} -boundary neighbors given by (1.11).

Our proof proceeds along the following outline. We remark that (1.21) in Theorem 1.1 consists of four different cases according to the four choices of the sign \pm and the $\iota \in 0, 1$. We consider only the case of \underline{v}_+^1 , i.e. we chose the sign $+$ and consider the boundary $x = \iota = 1$. We assume that $i(w_{\pm}^{\iota}) = n - 1$ for the particular predecessor or successor w_{\pm}^{ι} of \mathcal{O} , associated to \underline{v}_+^1 according to (1.17)–(1.20). In other words, $i(\underline{v}_+^1) = i(w_{\pm}^{\iota}) = n - 1$. Here we need to consider the parity of n , therefore further splitting the proof into a total of eight cases. We first choose odd n . Then, by (1.15) and (1.20) we have $\underline{v}_+^1 = w_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$. It remains to show that

$$(3.1) \quad \underline{v}_+^1 = \bar{v}_+^0.$$

To prove this we use the *z-adjacency* notion (2.9) to show that the equilibrium \underline{v}_+^1 blocks any heteroclinic connections from \mathcal{O} to equilibria in $\mathcal{E}_+^{n-1}(\mathcal{O})$ further away at $x = 0$ from \mathcal{O} . By (1.16) this will prove $\underline{v}_+^1 = \bar{v}_+^0$.

For even n , (1.19) implies $\underline{v}_+^1 = w_-^1$. This case is analogous to the case of odd n , up to some obvious adaptations which we sketch in Figure 3.3 below. To help face this adaptation we recall a result appearing in [FR18b] which relates the parity of the Morse indices $i(v_j)$ and the direction of the vertical crossing of the horizontal line at j by the meander \mathcal{M} . Since it is used several times in our proof we highlight this result in Lemma 3.1 below.

We conclude this section with the remaining six cases of (1.17) in Theorem 1.1. These will follow by applications of the trivial attractor equivalences generated by the involutions

$$(3.2) \quad \tau : x \mapsto 1 - x, \quad \kappa : u \mapsto -u.$$

Lemma 3.1 *Let $j = h_0^{-1}(v_j)$ denote the h_0 -label of the j -th equilibrium $v_j \in \mathcal{E}$ along the meander \mathcal{M} in canonical form. Then, the labels $j \in \{1, \dots, N\}$ and the Morse indices $i(v_j)$ have the opposite even/odd parity. Similarly, if the Morse index $i(v_j)$ is even/odd then the meander \mathcal{M} crosses the horizontal line at j vertically in the upwards/downwards direction, respectively.*

Proof: For $j = 1$ the first Morse index $i(v_1) = 0$ is even and the first crossing of the meander \mathcal{M} of the horizontal line at 1 is vertical in the upwards direction. Afterwards, recursion (2.1) asserts that the even/odd parity of the Morse index $i(v_j) = i_j$ at the crossings j alternates, along the meander \mathcal{M} . Therefore, simple induction on j along \mathcal{M} proves the lemma. Similarly, the Jordan curve \mathcal{M} crosses the horizontal line in the upwards and downwards direction, alternatingly. \square

Proof of Theorem 1.1 in the first of eight cases: $\underline{v}_+^1 = w_+^1$, n odd.

See Figures 3.1–3.2 for illustrations of the following notation. Fix an equilibrium \mathcal{O} such that $n := i(\mathcal{O}) \geq 1$ is odd. Let $j_0 := h_0^{-1}(\mathcal{O})$ denote the h_0 -label of the equilibrium $\mathcal{O} \in \mathcal{E}$ along the meander \mathcal{M} . Consequently, the h_0 -label of the $x = 0$ boundary successor w_+^0 of \mathcal{O} , i.e. of $w_+^0 := h_0(h_0^{-1}(\mathcal{O}) + 1)$, is given by $j_0 + 1 = h_0^{-1}(w_+^0) = h_0^{-1}(\mathcal{O}) + 1$. Also, we let $j_1 := h_0^{-1}(w_+^1)$ denote the h_0 -label of the $x = 1$ boundary successor w_+^1 of \mathcal{O} , i.e. of $w_+^1 := h_1(h_1^{-1}(\mathcal{O}) + 1)$. Hence $j_1 = h_0^{-1}(w_+^1) = \sigma(h_1^{-1}(\mathcal{O}) + 1)$. By definition, the intersection points labeled j_0 and j_1 are h_1 -neighbors along the horizontal line, in this order. Since n is odd, the meander \mathcal{M} is oriented downwards at the intersection point corresponding to \mathcal{O} , by Lemma 3.1.

Our assumption $i(w_+^1) = n - 1$ implies $\underline{v}_+^1 = w_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$, by (1.15), (1.20). Therefore (1.14) implies

$$(3.3) \quad z_{j_0 j_1} := z(w_+^1 - \mathcal{O}) = z(\underline{v}_+^1 - \mathcal{O}) = n - 1,$$

which is even. Hence, $\underline{v}_+^1(x) = w_+^1(x) > \mathcal{O}(x)$ at both boundaries $x = \iota \in \{0, 1\}$. This is illustrated in Figures 3.1–3.2 by very simple examples of meanders \mathcal{M} with $\underline{v}_+^1 = w_+^1 \in$

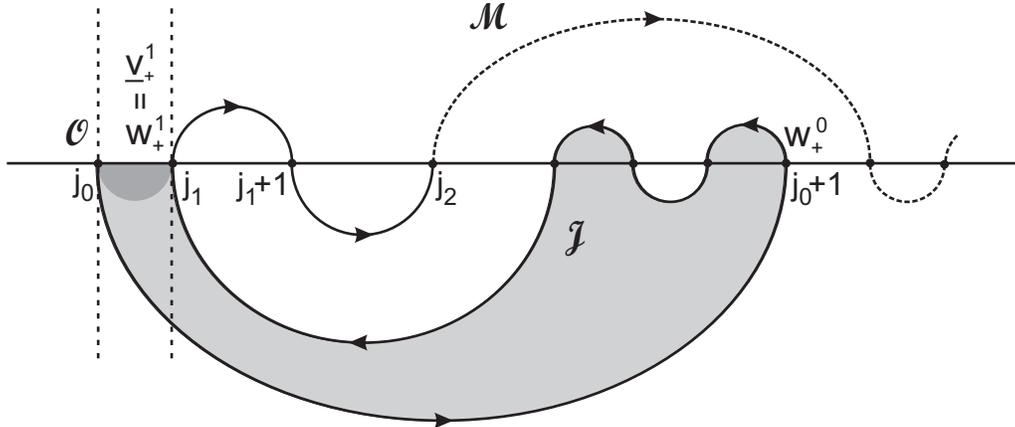


Figure 3.1: Illustration of a very simple case of a meander \mathcal{M} with $i(\mathcal{O}) = n$ odd and $\underline{v}_+^1 = w_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$. This picture assumes $i(w_+^0) = n - 1$. The case $i(w_+^0) = n + 1$ is illustrated in the next Figure. The Jordan curve \mathcal{J} is the union of the meander segment $\mathcal{M}_{j_0 j_1}$ from j_0 to j_1 with the section of the horizontal axis from j_1 back to j_0 . The dark open semicircular disk between the points labeled j_0 and j_1 is in the interior of the Jordan curve \mathcal{J} , and the unbounded meander segment of \mathcal{M} starting at the point j_1 is in the exterior of \mathcal{J} . The two vertical lines through the points j_0 and j_1 illustrate that if the meander arc segment $\mathcal{M}_{j_1 j_{1+1}}$ is to the right of j_1 then $c_{\mathcal{M}}(j_1, j_2; j_1) = c_{\mathcal{M}}(j_1, j_2; j_0)$.

$\mathcal{E}_+^{n-1}(\mathcal{O})$ and odd n . These two Figures also illustrate the alternative $i(w_+^0) = n \pm 1$ arising from $i(\mathcal{O}) = n$ and (1.11).

Let \mathcal{J} denote the union of the closed meander segment $\mathcal{M}_{j_0 j_1}$ from j_0 to j_1 with the segment of the horizontal axis from j_1 back to the adjacent intersection j_0 . Since \mathcal{M} is a meander, the closed oriented curve \mathcal{J} is a planar Jordan curve. Moreover, as the Morse indices count the clockwise half-windings of the unit tangent vector along the canonical meander \mathcal{M} (see comment to (2.1)), the assumption $i(\underline{v}_+^1) = i(w_+^1) = i(\mathcal{O}) - 1$ implies that \mathcal{J} performs a full counter-clockwise winding, i.e. \mathcal{J} is positively oriented. This shows that the (dark) open semicircular disk in the lower half-plane, with diameter given by the horizontal axis from j_0 to j_1 , is interior to \mathcal{J} . In addition, the unbounded continuing segment of the meander \mathcal{M} which starts at j_1 and runs to Northeast asymptotically (see the meander definition in the beginning of section 2), is exterior to \mathcal{J} .

We now prepare to show claim (3.1) by contradiction. Assume, indirectly, that the h_0 -maximal element \bar{v}_+^0 of $\mathcal{E}_+^{n-1}(\mathcal{O})$, most distant from \mathcal{O} at $x = 0$, differs from the h_1 -minimal element \underline{v}_+^1 in $\mathcal{E}_+^{n-1}(\mathcal{O})$, closest to \mathcal{O} at $x = 1$, i.e.

$$(3.4) \quad \bar{v}_+^0 \neq \underline{v}_+^1.$$

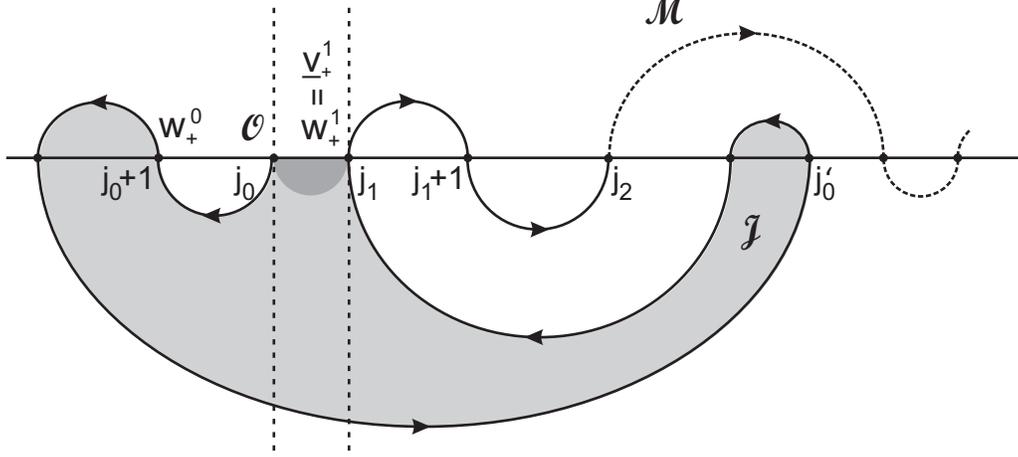


Figure 3.2: Illustration of a very simple case of a meander \mathcal{M} with $i(\mathcal{O}) = n$ odd and $\underline{v}_+^1 = \underline{w}_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$, assuming $i(\underline{w}_+^0) = n + 1$. Again the Jordan curve \mathcal{J} is the union of the meander segment $\mathcal{M}_{j_0 j_1}$ from j_0 to j_1 with the section of the horizontal axis from j_1 back to j_0 . Also, the dark open semicircular disk between the points labeled j_0 and j_1 is in the interior of the Jordan curve \mathcal{J} , and the unbounded meander segment of \mathcal{M} starting at the point j_1 is in the exterior of \mathcal{J} . As in Figure 3.1, the two vertical lines through the points j_0 and j_1 illustrate that if the meander arc segment $\mathcal{M}_{j_1 j_1^{+1}}$ is to the right of j_1 then $c_{\mathcal{M}}(j_1, j_2; j_1) = c_{\mathcal{M}}(j_1, j_2; j_0)$.

Let j_2 denote the h_0 -label of the h_0 -maximal equilibrium $\bar{v}_+^0 \in \mathcal{E}_+^{n-1}(\mathcal{O})$ most distant from \mathcal{O} at $x = 0$, i.e. $v_{j_2} := \bar{v}_+^0 = h_0(j_2)$. Since \bar{v}_+^0 is h_0 -maximal in $\mathcal{E}_+^{n-1}(\mathcal{O})$, by (1.15), and we just assumed $\bar{v}_+^0 \neq \underline{v}_+^1 = \underline{w}_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$, we have $j_2 > j_1$. See Figures 3.1–3.3. Definition (1.14) of $\mathcal{E}_+^{n-1}(\mathcal{O})$ implies

$$(3.5) \quad z_{j_0 j_2} := z(\bar{v}_+^0 - \mathcal{O}) = n - 1.$$

In the following we will also show

$$(3.6) \quad z_{j_1 j_2} := z(\bar{v}_+^0 - \underline{v}_+^1) = n - 1.$$

Then, (3.3) and (3.5) assert that $z_{j_0 j_1} = n - 1 = z_{j_0 j_2}$. Invoking (2.5) with $j = j_0$ and $k = j_1, j_2$ we obtain equality of the crossing numbers

$$(3.7) \quad c_{\mathcal{M}}(j_0, j_1; j_0) = c_{\mathcal{M}}(j_0, j_2; j_0).$$

Since $c_{\mathcal{M}}(j_0, j_2; j_0) = c_{\mathcal{M}}(j_0, j_1; j_0) + c_{\mathcal{M}}(j_1, j_2; j_0)$, by the additivity property (2.4), this shows

$$(3.8) \quad c_{\mathcal{M}}(j_1, j_2; j_0) = 0.$$

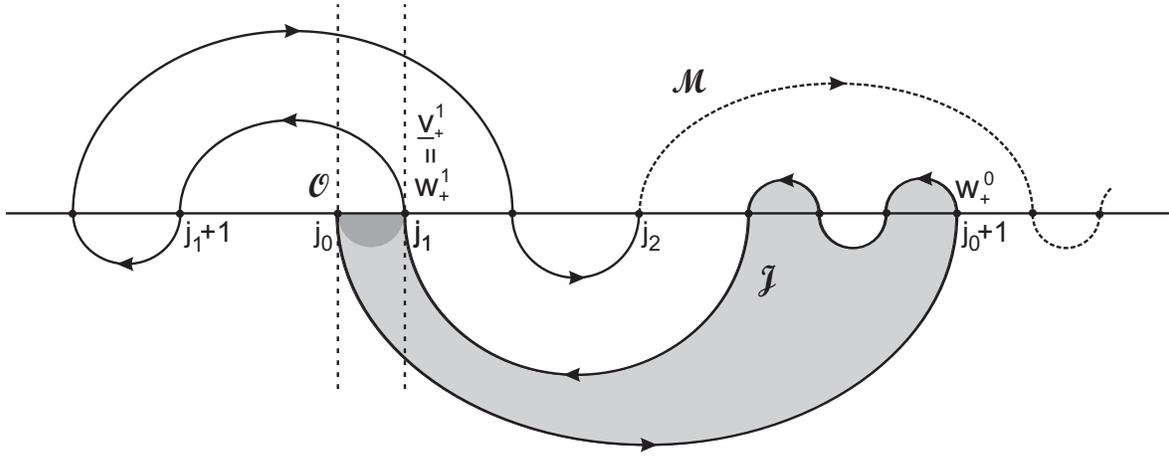


Figure 3.3: Illustration of a very simple case of meander \mathcal{M} with $i(\mathcal{O}) = n$ odd and $w_+^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$. Here the two vertical lines through the points j_0 and j_1 illustrate that if the arc segment $\mathcal{M}_{j_1 j_0^{+1}}$ is to the left of j_1 then $c_{\mathcal{M}}(j_1, j_2; j_0) = c_{\mathcal{M}}(j_1, j_2; j_1) - 1$.

Next, since the intersection points j_0 and j_1 are h_1 -adjacent neighbors along the horizontal axis, any semicircular arc segment of the meander \mathcal{M} which crosses the vertical line at j_1 also crosses the vertical line at j_0 , in the same upper or lower half-plane and in the same direction. See Figures 3.1–3.3. To take into account the arc crossing at j_1 also, we need to consider two alternative cases: either the intersection point $j_1 + 1$ is to the right of j_1 , or to the left.

If $j_1 + 1$ is to the right of j_1 , as illustrated in Figure 3.1, we conclude that the arc segment $\mathcal{M}_{j_1 j_0^{+1}}$ does not cross the vertical line at j_0 . Therefore (3.8) implies

$$(3.9) \quad c_{\mathcal{M}}(j_1, j_2; j_1) = c_{\mathcal{M}}(j_1, j_2; j_0) = 0.$$

If, on the other hand, $j_1 + 1$ is to the left of j_1 , as illustrated in Figure 3.3, the arc segment $\mathcal{M}_{j_1 j_0^{+1}}$ crosses the vertical line at j_0 , counter-clockwise with respect to j_0 . In this case we have $c_{\mathcal{M}}(j_1, j_2; j_0) = c_{\mathcal{M}}(j_1, j_2; j_1) - 1$ since the initial crossing of \mathcal{M} at the vertical line j_1 does not count. Then, (3.8) implies

$$(3.10) \quad c_{\mathcal{M}}(j_1, j_2; j_1) = c_{\mathcal{M}}(j_1, j_2; j_0) + 1 = 1.$$

We can now compute $z_{j_1 j_2}$ using (2.5) with $j = j_1$ and $k = j_2$. In the first case, Figure 3.1, where the intersection point $j_1 + 1$ is to the right of j_1 , i.e. $h_1(j_1 + 1) > h_1(j_1)$ at $x = 1$, the (first) arc segment $\mathcal{M}_{j_1 j_0^{+1}}$ belongs to an odd quadrant with respect to j_1 ;

Then, (2.5) and (3.9) imply

$$(3.11) \quad z_{j_1 j_2} = i(v_{j_1}) = i(\underline{v}_+^1).$$

In the second case, Figure 3.3, the intersection point $j_1 + 1$ is to the left of j_1 , i.e. $h_1(j_1 + 1) < h_1(j_1)$ at $x = 1$. Then the meander arc segment $\mathcal{M}_{j_1 j_1 + 1}$ belongs to an even quadrant with respect to j_1 . This time, (2.5) with $j = j_1$, $k = j_2$ and (3.10) imply (3.11), all the same. Now, since $i(\underline{v}_+^1) = i(w_+^1) = n - 1$, by assumption, we conclude that in both cases

$$(3.12) \quad z_{j_1 j_2} = n - 1.$$

This proves claim (3.6).

To reach a contradiction to our indirect assumption (3.4), we now show that the equilibrium $w := \underline{v}_+^1$ blocks the heteroclinic connection $\mathcal{O} \rightsquigarrow \bar{v}_+^0 \in \mathcal{E}_+^{n-1}(\mathcal{O})$, by blocking property (2.9) and contrary to definition (1.14) of $\mathcal{E}_+^{n-1}(\mathcal{O})$. Indeed, relations (3.3) for $w_+^1 = \underline{v}_+^1$, (3.5), and (3.6) assert

$$(3.13) \quad z(\underline{v}_+^1 - \mathcal{O}) = n - 1, \quad z(\bar{v}_+^0 - \mathcal{O}) = n - 1, \quad z(\bar{v}_+^0 - \underline{v}_+^1) = n - 1,$$

respectively. Moreover, the ordering $j_0 < j_1 < j_2$ corresponds to the ordering of the equilibria $\mathcal{O} = h_0(j_0)$, $\underline{v}_+^1 = h_0(j_1)$, $\bar{v}_+^0 = h_0(j_2)$ at the boundary $x = 0$,

$$(3.14) \quad \mathcal{O}(0) < \underline{v}_+^1(0) < \bar{v}_+^0(0).$$

Therefore, (2.9) and (3.13) show that the equilibrium \underline{v}_+^1 blocks heteroclinic connections $\mathcal{O} \rightsquigarrow \bar{v}_+^0$. This contradiction to the definition of \bar{v}_+^0 in (1.14), (1.16) proves claim (3.1), for $\underline{v}_+^1 = w_+^1$ and odd n . \square

Proof of Theorem 1.1 in the second of eight cases: $\underline{v}_+^1 = w_-^1$, n even.

To extend this proof to the case of even n , we observe that in this case the meander \mathcal{M} is oriented upwards at the intersection point $j_0 = h_0^{-1}(\mathcal{O})$, by Lemma 3.1.

We recall our assumption $i(w_-^1) = n - 1$ which by (1.15), (1.19), implies $\underline{v}_+^1 = w_-^1 \in \mathcal{E}_+^{n-1}(\mathcal{O})$. Therefore, (1.14) again implies

$$(3.15) \quad z_{j_0 j_1} := z(w_-^1 - \mathcal{O}) = z(\underline{v}_+^1 - \mathcal{O}) = n - 1,$$

which now is odd. Hence, the ordering of $w_-^1(x)$ and $\mathcal{O}(x)$ is opposite, at the respective boundaries $x = \iota \in \{0, 1\}$. This implies that $w_-^1(x) < \mathcal{O}(x)$ at $x = 1$, and $j_1 = h_0^{-1}(w_-^1)$ is the left h_1 -neighbor of $j_0 = h_0^{-1}(\mathcal{O})$.

For the equilibria in $\mathcal{E}_-^{n-1}(\mathcal{O})$, if we assume that any $i(w_-^\iota) = n - 1$, for $\iota \in \{0, 1\}$ and whichever parity of n , the remaining four cases for the sign $-$ follow immediately by the trivial equivalence $\kappa : u \mapsto -u$.

For the convenience of the reader we include a table of the action of the Klein group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ generated by the commuting involutions $\langle \tau, \kappa \rangle$ on the equilibria \underline{v}_\pm^ι .

\mathcal{O}	$\tau\mathcal{O}$	$\kappa\mathcal{O}$	$\tau\kappa\mathcal{O}$
$\underline{v}_\pm^\iota(\mathcal{O})$	$\underline{v}_\pm^{1-\iota}(\tau\mathcal{O})$	$\underline{v}_\mp^\iota(\kappa\mathcal{O})$	$\underline{v}_\mp^{1-\iota}(\tau\kappa\mathcal{O})$

4 Discussion and Example

As pointed out in the Introduction, section 1, the minimax property (1.21) simplifies the task of identifying the equilibria $\mathcal{E}_\pm^{n-1}(\mathcal{O})$ of (1.14) directly from the Sturm meander \mathcal{M} .

To emphasize the “local” aspect of our global result, we next show an example for which the identification of the equilibria in $\mathcal{E}_+^{i(\mathcal{O})-1}(\mathcal{O})$ is obtained from the knowledge of only a segment of the meander \mathcal{M} . In fact, we will only prescribe a segment of the Sturm permutation σ . We assume that our reference equilibrium $\mathcal{O} \in \mathcal{E}$ has even unstable dimension $n := i(\mathcal{O}) = 2$. As before, $j_0 = h_0^{-1}(\mathcal{O})$ denotes the h_0 -label of \mathcal{O} along the meander. We consider a Sturm permutation according to the following template

$$(4.1) \quad \sigma = \{1 \dots j_0+11 \ j_0+10 \ j_0+3 \ j_0+2 \ j_0+1 \ j_0+4 \ j_0+9 \ j_0+8 \ j_0+5 \ j_0+6 \ j_0+7 \ j_0 \dots N\}.$$

This corresponds to a meander section $\mathcal{M}_{j_0 \ j_0+11}$ which we illustrate in Figure 4.1. Note the orientation of the meander \mathcal{M} due to the assumption of even n and Lemma 3.1. Our objective is to identify the set of equilibria $\mathcal{E}_+^{n-1}(\mathcal{O}) = \mathcal{E}_+^1(\mathcal{O})$ from the partial “local” information (4.1) on σ .

By (1.11) the boundary neighbors $w_+^0 = w_+^0(\mathcal{O})$ and $w_-^1 = w_-^1(\mathcal{O})$ are given by

$$(4.2) \quad w_+^0 = h_0(h_0^{-1}(\mathcal{O}) + 1) = h_0(j_0 + 1); \quad w_-^1 = h_1(h_1^{-1}(\mathcal{O}) - 1) = h_0(j_0 + 7).$$

Indeed, $w_-^1 = h_0(j_0 + 7)$ since w_-^1 is the left h_1 -neighbor of \mathcal{O} . The recursion (2.1) for the Morse indices i_j applied to σ in (4.1), implies $i(w_-^1) = i(w_+^0) = i(\mathcal{O}) - 1 = n - 1 = 1$. Therefore, Theorem 1.1, (1.21), together with (1.18), (1.19) yields

$$(4.3) \quad w_+^0 = \underline{v}_+^0 = \bar{v}_+^1; \quad w_-^1 = \underline{v}_+^1 = \bar{v}_+^0.$$

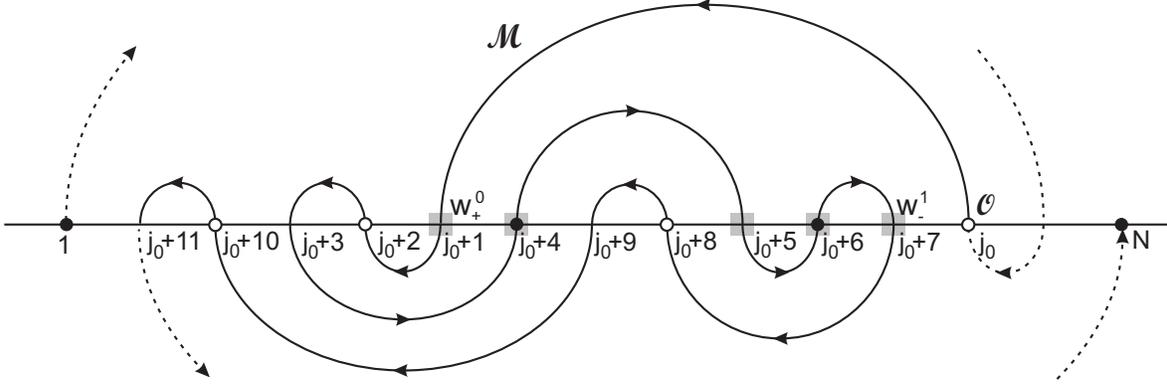


Figure 4.1: Illustration of the meander section $\mathcal{M}_{j_0, j_0+11}$ corresponding to the partial Sturm permutation $\sigma = \{1 \dots j_0 + 11 \ j_0 + 10 \ j_0 + 3 \ j_0 + 2 \ j_0 + 1 \ j_0 + 4 \ j_0 + 9 \ j_0 + 8 \ j_0 + 5 \ j_0 + 6 \ j_0 + 7 \ j_0 \dots N\}$, where the reference unstable equilibrium $\mathcal{O} = h_0(j_0)$ has even Morse index $n := i(\mathcal{O}) = 2$ and odd $i(v_+^1) = i(w_-^1) = i(w_+^0) = i(\mathcal{O}) - 1 = n - 1 = 1$. The equilibria with Morse index $i = 0$ are indicated by black dots and the equilibria with Morse index $i = 2$ are indicated by white dots. Equilibria with Morse index $i = 1$ are indicated by simple intersections. The five intersections at $j_0 + \{1, 4, 5, 6, 7\}$ in gray squares indicate the target set $\mathcal{E}_+^{n-1}(\mathcal{O})$ defined in (1.14).

Equation (4.3) implies that the set $\mathcal{E}_+^1(\mathcal{O})$ of (1.14) is strictly contained in the set of equilibria v_j with values at the boundary $x = 0$ in the interval

$$(4.4) \quad \mathcal{O}(0) < w_+^0(0) = \underline{v}_+^0(0) = v_{j_0+1}(0) \leq v_j(0) \leq v_{j_0+7}(0) = \bar{v}_+^0(0) = w_-^1(0),$$

that is, the set of all equilibria $v_j = h_0(j)$ with $j_0 + 1 \leq j \leq j_0 + 7$. Equation (4.3) also implies that $\mathcal{E}_+^1(\mathcal{O})$ is strictly contained in the set of equilibria with values at the boundary $x = 1$ in the interval

$$(4.5) \quad w_+^0(1) = \bar{v}_+^1(1) = v_{j_0+1}(1) \leq v_j(1) \leq v_{j_0+7}(1) = \underline{v}_+^1(1) = w_-^1(1) < \mathcal{O}(1).$$

We now claim that

$$(4.6) \quad \mathcal{E}_+^1(\mathcal{O}) =: \{v_{j_0+k} \mid k \in K\}, \quad \text{with } K = \{1, 4, 5, 6, 7\}.$$

Here the index set K is defined by the left equality.

Equation (4.4) implies $K \subseteq \{1, \dots, 7\}$. Equation (4.5) implies $K \subseteq \{1, 4, 9, 8, 5, 6, 7\}$. By intersection, this proves $K \subseteq \{1, 4, 5, 6, 7\}$.

We have to show, conversely, that $K \supseteq \{1, 4, 5, 6, 7\}$. The zero number formula (2.5), with $j = j_0$, $v_j = \mathcal{O}$, and v_{j_0+k} replacing v_k there, implies $z(v_{j_0+k} - \mathcal{O}) = 1_+$, for any

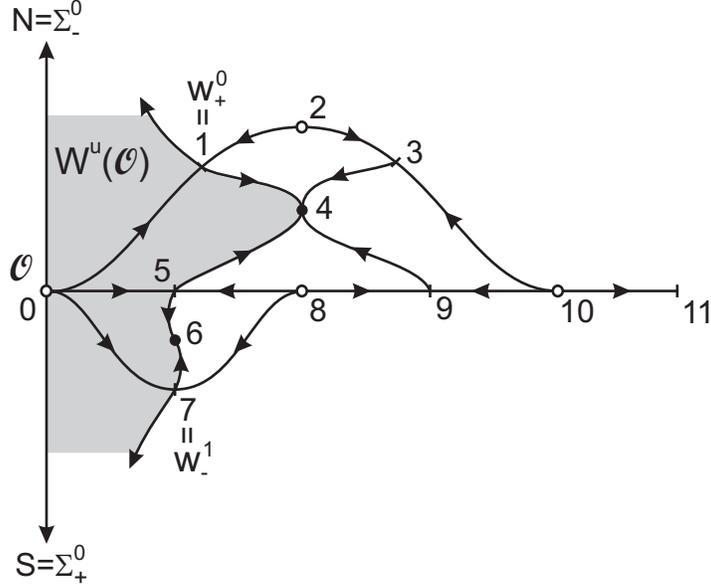


Figure 4.2: Sketch of the heteroclinic connections in the global attractor corresponding to the partial Sturm permutation σ . To abbreviate the equilibrium labels we let $j_0 = 0$. The labels corresponding to Θ , w_+^0 and w_-^1 are 0, 1 and 7, respectively. The shaded area corresponds to the subset of $W^u(\Theta)$ composed of all heteroclinic connections $\Theta \rightsquigarrow w$ with $z(w - \Theta) = 2_+$. In $W^u(\Theta)$ there are exactly two heteroclinic connections $\Theta \rightsquigarrow w$ with $z(w - \Theta) = 0_{\pm}$ which select the unique polar equilibria $\mathbf{N} = \Sigma_-^0(\Theta)$ and $\mathbf{S} = \Sigma_+^0(\Theta)$; see [FR18b]. These polar equilibria are not labeled by the partial Sturm permutation σ . According to the dashed template in Figure 4.1, one possibility is $\mathbf{N} = -2$ and $\mathbf{S} = 12$.

$k \in \{1, 4, 5, 6, 7\}$. Therefore it only remains to prove

$$(4.7) \quad k \in \{1, 4, 5, 6, 7\} \implies \Theta \rightsquigarrow v_{j_0+k}.$$

To prove claim (4.7), we recall that $v_j \rightsquigarrow v_k$ are heteroclinically connected, if and only if $i(v_j) > i(v_k)$ and v_j and v_k are z -adjacent; see (2.9). The following Morse indices $i(v_j) = i_j$ are easily determined from (2.1): we have $i(v_{j_0+k}) = 0$, for $k \in \{4, 6\}$, and $i(v_{j_0+k}) = 1$, for $k \in \{1, 5, 7\}$.

Two equilibria v_j and v_{j+1} , i.e. h_0 -boundary neighbors, are automatically z -adjacent. In fact, (2.1) implies $|i(v_j) - i(v_{j+1})| = 1$ and there is no third equilibrium w with $w(0)$ strictly between $v_j(0)$ and $v_{j+1}(0)$. In particular $\Theta = v_{j_0} \rightsquigarrow v_{j_0+1} = w_+^0$. The same assertion holds for two equilibria $v_{\sigma^{-1}(j)}$ and $v_{\sigma^{-1}(j+1)}$, i.e. h_1 -boundary neighbors at $x = 1$, by the trivial equivalence $\tau : x \mapsto 1 - x$. In particular $\Theta = v_{j_0} \rightsquigarrow v_{j_0+7} = w_-^1$. This takes care of the cases

$k \in \{1, 7\}$.

From (4.1) (see also the template Figure 4.1), we immediately verify the h_0 - boundary z -adjacency of the equilibria v_{j_0+7}, v_{j_0+6} , and the h_1 -boundary z - adjacency of the equilibria v_{j_0+1}, v_{j_0+4} . Hence, we obtain $w_-^1 = v_{j_0+7} \rightsquigarrow v_{j_0+6}$ and $w_+^0 = v_{j_0+1} \rightsquigarrow v_{j_0+4}$. By heteroclinic transitivity, this takes care of claim (4.7) for $k \in \{4, 6\}$.

For the final equilibrium v_{j_0+k} with $k = 5$, we consider all potentially $z = 1$ blocking equilibria w with $w(1)$ strictly between $\mathcal{O}(1)$ and $v_{j_0+5}(1)$. By template (4.1), there are exactly two such candidate equilibria: $w = v_{j_0+6}$ and $w = v_{j_0+7}$. Now, using the bottom row of (2.5) with $j = j_0 + 5, k = j_0 + 6, j_0 + 7$, we obtain the zero numbers

$$(4.8) \quad z_{j_0+6, j_0+5} = z(v_{j_0+5} - v_{j_0+6}) = 0, \quad z_{j_0+7, j_0+5} = z(v_{j_0+5} - v_{j_0+7}) = 0.$$

This shows that neither $w = v_{j_0+6}$, nor $w = v_{j_0+7}$, can block the $z = 1$ heteroclinic connection $\mathcal{O} \rightsquigarrow v_{j_0+5}$, at $x = 1$, and we conclude that also $v_{j_0+5} \in \mathcal{E}_+^1(\mathcal{O})$ by z -adjacency. This completes the proof of our claims (4.7) and (4.6).

In Figure 4.2 we sketch all the single-orbit heteroclinic connections obtained from the partial Sturm permutation σ . In particular, the set $\mathcal{E}_+^1(\mathcal{O})$ is illustrated with its chain of heteroclinic connections between the leading descendants w_+^0 and w_-^1 .

In Figure 4.2 we include all the single-orbit heteroclinic connections as obtained from σ , by checking z -adjacency directly from the recurrence (2.8). For completion we show next the partial zero number matrix corresponding to the Sturm permutation segment σ , completed on the diagonal by the Morse indices of the equilibria:

$$(4.9) \quad [z_{j k}]_{j,k=j_0,\dots,j_0+11} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

5 Appendix: Nonlinear Sturm-Liouville property

In this Appendix we review and prove the nonlinear Sturm-Liouville property (NSL for short) in our meander setting. Let $v_j = h_0(j), v_k = h_0(k)$ denote two equilibria with $j < k$. Then, the NSL property corresponds to the relation (2.5) between zero numbers, Morse indices and crossing numbers, which we now repeat for convenience. The claim is that the zero number $z_{jk} := z(v_k - v_j)$ is given by ([Ro91, Proposition 3])

$$(5.1) \quad z(v_k - v_j) = \begin{cases} i(v_j) + c_{\mathcal{M}}(j, k; j), & \text{if } \mathcal{M}_{jj+1} \text{ is in an odd quadrant with respect to } j; \\ i(v_j) - 1 + c_{\mathcal{M}}(j, k; j), & \text{if } \mathcal{M}_{jj+1} \text{ is in an even quadrant with respect to } j, \end{cases}$$

Here \mathcal{M}_{jj+1} denotes the first arc segment of \mathcal{M} between the intersection points j and $j + 1$. As in section 2, $c_{\mathcal{M}}(j, k; j)$ denotes the net signed clockwise crossings of the oriented meander segment \mathcal{M}_{jk} from equilibrium crossing j to k through the vertical line at j , ignoring that first crossing. See (2.3), (2.4).

Our proof of the NSL property (5.1) is based on zeros and winding numbers associated to the solutions $v = v(x, a)$ of the initial value second order ODE problem

$$(5.2) \quad 0 = v_{xx} + f(x, v, v_x) \quad , \quad v(0, a) = a, \quad v_x(0, a) = 0.$$

The equilibrium boundary value problem (1.2) is related to this ODE by the shooting condition $v_x(x, a) = 0$ at the right boundary $x = 1$. Let $a_j := v_j(0)$, $a_k := v_k(0)$ denote the initial values at $x = 0$ of the equilibria $v_j, v_k \in \mathcal{E}$, i.e.

$$(5.3) \quad v(\cdot, a_j) = v_j(\cdot), \quad v(\cdot, a_k) = v_k(\cdot).$$

Then the segment \mathcal{M}_{jk} of the meander \mathcal{M} , from the intersection point j to the intersection point k , is given by the planar curve

$$(5.4) \quad a \mapsto (v(1, a), v_x(1, a)) \in \mathbb{R}^2, \quad a_j \leq a \leq a_k$$

of boundary values at the shooting boundary $x = 1$. Note how the shooting condition $v_x(1, a) = 0$ is actually satisfied, precisely, at the equilibrium intersections of the meander \mathcal{M} with the horizontal axis $v_x = 0$ in the (v, v_x) -plane.

For $a_j < a \leq a_k$, let

$$(5.5) \quad w = w(x, a) := (v(x, a) - v_j(x))/(a - a_j)$$

denote the scaled difference between the two solutions v, v_j of (5.2). Note that $w = w(x, a)$ solves a linear second order ODE initial value problem

$$(5.6) \quad 0 = w_{xx} + q_1(x, a)w_x + q_0(x, a)w, \quad w(0, a) = 1, \quad w_x(0, a) = 0.$$

Indeed, the coefficients q_0, q_1 depend on v, v_j and are given explicitly as

$$(5.7) \quad \begin{aligned} q_0(x, a) &= \int_0^1 \partial_u f(x, r(x, a, \mu), r_x(x, a, \mu)) d\mu, \\ q_1(x, a) &= \int_0^1 \partial_p f(x, r(x, a, \mu), r_x(x, a, \mu)) d\mu, \end{aligned}$$

with $r(x, a, \mu) = \mu v(x, a) + (1 - \mu)v_j(x)$, by the Fundamental Theorem of Calculus.

To extend the above construction (5.5)–(5.7), down to $a = a_j$, let

$$(5.8) \quad q_0(x, a_j) := \partial_u f(x, v_j(x), v_{jx}(x)), \quad q_1(x, a_j) := \partial_p f(x, v_j(x), v_{jx}(x)).$$

Then $f \in C^1$ implies continuity of q_0, q_1 , and therefore continuity of $w, w_x \in C^1$, in the closed rectangle

$$(5.9) \quad (x, a) \in \mathcal{R} := [0, 1] \times [a_j, a_k].$$

The initial condition $w(0, a) = 1$ of the linear equation (5.6) implies $(w(x, a), w_x(x, a)) \neq (0, 0)$ on \mathcal{R} . We can therefore introduce (clockwise!) polar coordinates, according to the Prüfer transformation

$$(5.10) \quad w = \rho \cos \vartheta, \quad w_x = -\rho \sin \vartheta$$

and obtain

$$(5.11) \quad \begin{aligned} \rho_x &= -\rho (q_1(x, a) \sin^2 \vartheta + (1 - q_0(x, a)) \sin \vartheta \cos \vartheta) \\ \vartheta_x &= \sin^2 \vartheta - q_1(x, a) \sin \vartheta \cos \vartheta + q_0(x, a) \cos^2 \vartheta. \end{aligned}$$

Note how the initial conditions $w(0, a) = 1, w_x(0, a) = 0$ imply $\rho = 1, \vartheta = 0$ at $x = 0$. The first equation of (5.11) then implies $\rho > 0$ on the rectangle \mathcal{R} . In particular all zeros of $x \mapsto w(x, a)$ are simple, for any fixed a . The second equation implies $\vartheta_x > 0$ for $\cos \vartheta = 0$, i.e. whenever $w = 0$, alias $\vartheta = \frac{1}{2}\pi \pmod{\pi}$. Hence the total number of zeros of the solutions $x \mapsto w = w(x, a)$ relates to the winding of $x \mapsto \vartheta = \vartheta(x, a)$.

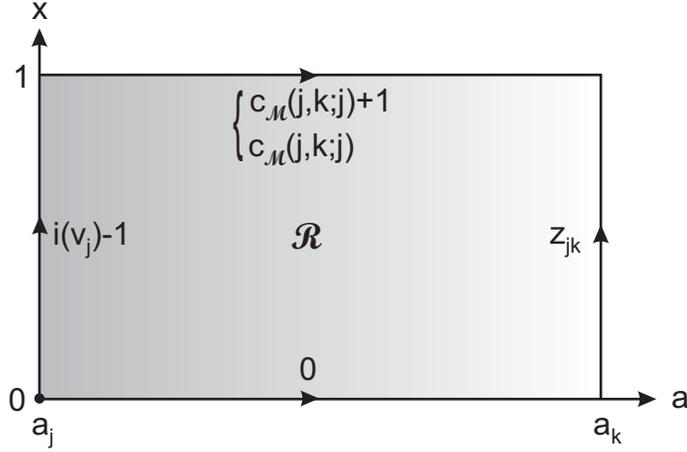


Figure 5.1: Illustration of the rectangle $\mathcal{R} := [0, 1] \times [a_j, a_k]$. On each side of \mathcal{R} is indicated the total winding of ϑ along the corresponding side. On the top side, the two values correspond to the alternative: \mathcal{M}_{j+1} is in an odd/even quadrant with respect to j .

We can now outline the remaining proof of the NSL property (5.1) as follows. We first note that the winding number of

$$(5.12) \quad \vartheta \pmod{2\pi} : \quad \partial\mathcal{R} \rightarrow S^1$$

is zero, along the boundary $\partial\mathcal{R}$ of the rectangle \mathcal{R} in (5.9). Indeed, the map extends to all of \mathcal{R} , continuously, and hence is contractible, i.e. of winding number zero. In Lemma 5.1 we relate the winding along the upper boundary $a = a_k$, $0 \leq x \leq 1$ of \mathcal{R} to the zero number $z(v_k - v_j)$ in claim (5.1). In Lemma 5.2 we relate the winding along the lower boundary $a = a_j$, $0 \leq x \leq 1$ of \mathcal{R} to the Morse index $i(v_j)$ in claim (5.1). In Lemma 5.3 we relate the winding along the right boundary $x = 1$, $a_j \leq a \leq a_k$ of \mathcal{R} to the crossing number $c_{\mathcal{M}}(j, k; j)$ in claim (5.1). Since $\vartheta = 0$ is constant along the left boundary $x = 0$, $a_j \leq a \leq a_k$, and since the total winding number is zero, this reduces the proof of the NSL property (5.1) to the three Lemmata 5.1 – 5.3.

Lemma 5.1 *The zero number $z(v_k - v_j)$ is given by*

$$(5.13) \quad z(v_k - v_j) = z(w(\cdot, a_k)) = \vartheta(1, a_k)/\pi.$$

Proof: Fix $a = a_k$. We recall that all zeros of $w(\cdot, a_k)$ are simple. They correspond to clockwise crossings of (w, w_x) through the vertical w_x -axis or, equivalently, to simple zeros of $\vartheta - \frac{1}{2}\pi \pmod{\pi}$ with positive slope $\vartheta_x > 0$. The Neumann boundary condition $v_{kx} = v_{jx} = 0$ at $x = 1$ implies $w_x = 0$ and hence $\vartheta \equiv 0 \pmod{\pi}$ there. This proves the lemma. \square

Lemma 5.2 *The Morse index $i(v_j)$, i.e. the unstable dimension of the equilibrium v_j , satisfies*

$$(5.14) \quad i(v_j) = \lfloor \vartheta(1, a_j) / \pi \rfloor + 1.$$

Here $\lfloor \cdot \rfloor$ denotes the integer valued floor function.

Proof: See for example [Ro85, Theorem 2].

Comparing (5.2) with (5.5), (5.6), (5.8), we first note that $w = v_a$ is the partial derivative of $v = v(\cdot, a)$ with respect to a , at $a = a_j$. In particular, $(w(1, a_j), w_x(1, a_j))$ is the tangent of the meander segment \mathcal{M}_{j+1} at j , at (clockwise) angle $\vartheta(1, a_j) \pmod{2\pi}$ from the horizontal line; see (5.10). Since the equilibrium v_j is assumed to be hyperbolic, $\vartheta(1, a_j) \not\equiv 0 \pmod{\pi}$. In particular the meander crosses the horizontal axis transversely at the intersection j . More precisely, the (clockwise) tangent angle $\vartheta(1, a_j)$ has to point above the horizontal axis, for odd j (alias even $i(v_j)$), and below for even j (alias odd $i(v_j)$), alternatingly:

$$(5.15) \quad \vartheta(1, a_j) \pmod{2\pi} \in \begin{cases} (0, \pi), & \text{for odd } i(v_j); \\ (\pi, 2\pi), & \text{for even } i(v_j). \end{cases}$$

Next, consider the simple eigenvalues λ_m , i.e.

$$(5.16) \quad \lambda_0 > \dots > \lambda_{i(v_j)-1} > 0 > \lambda_{i(v_j)} > \dots,$$

of the linearization

$$(5.17) \quad \lambda \tilde{v} = \tilde{v}_{xx} + q_1(x, a_j) \tilde{v}_x + q_0(x, a_j) \tilde{v}, \quad \tilde{v}(0) = 1, \quad \tilde{v}_x(0) = \tilde{v}_x(1) = 0$$

at v_j . By classical Sturm-Liouville theory, e.g. as in [CL72], the eigenfunction $\tilde{v} = \tilde{v}_m$ of $\lambda = \lambda_m$ possesses m simple zeros. Moreover, w solves (5.17) with $\lambda = 0$, but violates the Neumann boundary condition at $x = 1$. Therefore Sturm-Liouville comparison with (5.16) implies

$$(5.18) \quad i(v_j) - 1 \leq z(w) \leq i(v_j).$$

Translating zeros of w to zeros of $\vartheta - \frac{1}{2}\pi \pmod{\pi}$, as in the proof of Lemma 5.1, we obtain

$$(5.19) \quad (i(v_j) - 1)\pi < \vartheta(1, a_j) - \pi/2 < (i(v_j) + 1)\pi.$$

Combined with (5.15), this implies

$$(5.20) \quad (i(v_j) - 1)\pi < \vartheta(1, a_j) < i(v_j)\pi$$

and proves claim (5.14) of the lemma. □

We recall the definition of the signed clockwise counts $c_{\mathcal{M}}(j, k; j)$ of meander crossings, from section 2.

Lemma 5.3 *The clockwise increase*

$$(5.21) \quad \lfloor (\vartheta(1, a_k) - \vartheta(1, a_j)) / \pi \rfloor = \vartheta(1, a_k) / \pi - \lfloor \vartheta(1, a_j) / \pi \rfloor$$

of the angle $\vartheta(1, a)$ from $a = a_j$ to $a = a_k$ is given by

$$(5.22) \quad \begin{cases} c_{\mathcal{M}}(j, k; j) + 1, & \text{if } \mathcal{M}_{j, j+1} \text{ is in an odd quadrant with respect to } j; \\ c_{\mathcal{M}}(j, k; j), & \text{if } \mathcal{M}_{j, j+1} \text{ is in an even quadrant with respect to } j. \end{cases}$$

Proof: At $a = a_k$ we recall $\vartheta(1, a) / \pi \in \mathbb{N}_0$ from (5.13). This proves claim (5.21). It remains to prove claim (5.22).

From the proof of Lemma 5.2 we recall that $\vartheta(1, a_j) \pmod{2\pi}$ is the (clockwise) tangent angle of the meander segment $\mathcal{M}_{j, j+1}$ at j with the horizontal line; see (5.10). For general $a_j < a \leq a_k$, the angle $\vartheta(1, a)$ tracks the secant between j and $(w(1, a), w_x(1, a)) \in \mathcal{M}_{j, k}$.

By definition, neither the clockwise crossing count $c_{\mathcal{M}}(j, k; j)$, nor the clockwise angular increase $\lfloor (\vartheta(1, a_k) - \vartheta(1, a_j)) / \pi \rfloor$ depend on homotopies of the Jordan meander segment $\mathcal{M}_{j, k}$, as long as the initial tangent angle $\vartheta(1, a_j)$ and the final secant $\vartheta(1, a_k)$ remain fixed. Therefore we may assume all crossings of the angle $\vartheta(1, a) \pmod{\pi}$ through the levels $\pi/2$ to be transverse, and finite in number, for $a \in (a_j, a_k]$.

The two cases (5.22) at $a = a_j$ arise as follows. The even/odd parity of j at the up- or down-crossing j of the meander \mathcal{M} determines whether the initial (clockwise) tangent $\vartheta(1, a_j)$ points down or up; see (5.15). The precise direction, however, and the even/odd quadrant on that side of the horizontal line, remain undetermined. Suppose we artificially twist an initial tangent $\vartheta(1, a_j)$ from an odd quadrant at j to the even quadrant on the same side of the horizontal line. We do not require this twist to be realized by specific nonlinearities $f = f(x, v, v_x)$ in (5.2); we just compensate our twist, locally, by a homotopy of the initial meander segment $\mathcal{M}_{j, j+1}$ near j . Then our twist of the initial tangent contributes one additional clockwise crossing to the crossing count $c_{\mathcal{M}}(j, k; j)$. Since this modification leaves (5.22) invariant, we may assume the initial tangent $\vartheta(1, a_j)$ to be in an odd quadrant, without loss of generality, i.e.

$$(5.23) \quad \lfloor \vartheta(1, a_j) / \pi \rfloor + \frac{1}{2} < \vartheta(1, a_j) / \pi < \lfloor \vartheta(1, a_j) / \pi \rfloor + 1.$$

Therefore (5.21), (5.23) ensure that $\lfloor (\vartheta(1, a_k) - \vartheta(1, a_j))/\pi \rfloor - 1$ counts the net clockwise crossings of the angle $\vartheta(1, a) \pmod{\pi}$ through the levels $\pi/2$, as a increases from $a = a_j$ to $a = a_k$. This coincides with the clockwise crossing count $c_{\mathcal{M}}(j, k; j)$, by definition, and proves the lemma. \square

Contractibility of the winding map (5.12) allows us to identify the values of ϑ in lemmata 5.1 – 5.3 as real numbers, not just mod 2π . The lemmata therefore combine to show the NSL property (5.1) as follows. We first express $z_{j k} = \vartheta(1, a_k)/\pi$ in (5.13) of Lemma 5.1 via (5.21) of Lemma 5.3. We then substitute the two floor function expressions in (5.21) by (5.22) and (5.14), respectively. This proves our original claim (5.1). We may therefore evaluate $\vartheta(1, a_k)$ in (5.13) via summation of (5.11) and (5.21), to complete the proof of the NSL property (5.1).

References

- [An88] S. Angenent. The zero set of a solution of a parabolic equation. *J. Reine Angew. Math.*, **390** (1988), 79–96.
- [Ar88] V. I. Arnold. A branched covering $CP^2 \rightarrow S^4$, hyperbolicity and projective topology. *Siberian Math. J.*, **29** (1988), 717–726.
- [BV92] A. V. Babin and M. I. Vishik. *Attractors of Evolution Equations*. North Holland, Amsterdam, 1992.
- [BF88] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported* **1** (1988), 57–89.
- [BF89] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Differential Equations*, **81** (1989), 107–135.
- [CL72] E.A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw-Hill, New York 1974.
- [D&al19] V. Delecroix, É. Goujard, P. Zograf and A. Zorich. Enumeration of meanders and Masur-Veech volumes. Preprint 2019. [arXiv:1705.05190v2](https://arxiv.org/abs/1705.05190v2) [math.GT].
- [Fi94] B. Fiedler. Global attractors of one-dimensional parabolic equations: Sixteen examples. *Tatra Mountains Math. Publ.* **4** (1994), 67–92.

- [FR96] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. *J. Differential Equations*, **125** (1996), 239–281.
- [FR99] B. Fiedler and C. Rocha. Realization of meander permutations by boundary value problems. *J. Differential Equations*, **156** (1999), 282–308.
- [FR00] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. *Trans. Amer. Math. Soc.*, **352** (2000), 257–284.
- [FR08] B. Fiedler and C. Rocha. Connectivity and design of planar global attractors of Sturm type. II: Connection graphs. *J. Differential Equations*, **244** (2008), 1255–1286.
- [FR09a] B. Fiedler and C. Rocha. Connectivity and design of planar global attractors of Sturm type. I: Orientations and Hamiltonian paths. *J. Reine Angew. Math.*, **635** (2009), 71–96.
- [FR09b] B. Fiedler and C. Rocha. Connectivity and design of planar global attractors of Sturm type. III: Small and Platonic examples. *J. Dynam. Differential Equations*, **22** (2010), 121–162.
- [FR14] B. Fiedler and C. Rocha. Nonlinear Sturm global attractors: unstable manifold decompositions as regular CW-complexes. *Discrete Contin. Dyn. Syst.*, **34** (2014), no. 12, 5099–5122.
- [FR15] B. Fiedler and C. Rocha. Schoenflies spheres as boundaries of bounded unstable manifolds in gradient Sturm systems. *Discrete Contin. Dyn. Syst.*, **27** (2015), no. 3-4, 597–626.
- [FR18a] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 1: Thom-Smale complexes and meanders. *São Paulo J. Math. Sci.*, **12** (2018), 18–67.
- [FR18b] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 2: Design of Thom-Smale complexes. *J. Dynam. Differential Equations*, **31** (2019), 1549–1590.
- [FR18c] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 3: Examples of Thom-Smale complexes. *Discrete Contin. Dyn. Syst.*, **38** (2018), no. 7, 3479–3545.
- [FR18d] B. Fiedler and C. Rocha. Boundary orders and geometry of the signed Thom-Smale complex for Sturm global attractors. Preprint 2018. [arXiv:1811.04206v2](https://arxiv.org/abs/1811.04206v2) [math.DS], to appear in *J. Dynam. Differential Equations*.

- [FR91] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Differential Equations*, **91** (1991), 75–94.
- [Ga04] V. A. Galaktionov. *Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications*. Chapman & Hall, Boca Raton, 2004.
- [Ha88] J. K. Hale. *Asymptotic Behavior of Dissipative Systems*. Math. Surv. **25**. AMS Publications, Providence, 1988.
- [He81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Lect. Notes Math. **804**, Springer-Verlag, New York, 1981.
- [Ka17] A. Karnauhova. *Meanders – Sturm Global Attractors, Seaweed Lie Algebras and Classical Yang-Baxter Equation*. De Gruyter, Berlin, 2017.
- [La91] O. A. Ladyzhenskaya. *Attractors for Semigroups and Evolution Equations*. Cambridge University Press, 1991.
- [Ma82] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sec. IA.*, **29** (1982), 401–441.
- [Pa83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.
- [Ra02] G. Raugel. Global attractors in partial differential equations. *Handbook of Dynamical Systems*, **2** (2002), 885–982.
- [Ro85] C. Rocha. Generic properties of equilibria of reaction-diffusion equations with variable diffusion. *Proc. Roy. Soc. Edinburgh Sect. A*, **101** (1985), 385–405.
- [Ro91] C. Rocha. Properties of the attractor of a scalar parabolic PDE. *J. Dynam. Differential Equations*, **3** (1991), 575–591.
- [Ta79] H. Tanabe. *Equations of Evolution*. Pitman, Boston, 1979.
- [Te88] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Springer-Verlag, New York, 1988.
- [Wo02] M. Wolfrum. A sequence of order relations: encoding heteroclinic connections in scalar parabolic PDE. *J. Differential Equations* **183** (2002), no. 1, 56–78.