Bifurcations from homoclinic orbits to non-hyperbolic equilibria in reversible lattice differential equations

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Abstract

The study of traveling waves in a lattice differential equation (LDE) leads naturally to a forward-backward-delay equation. Structures which are present in the LDE typically are inherited by the traveling wave equation. In this article we are interested in a situation, where the traveling wave equation is reversible and possesses a symmetric (not necessarily small) homoclinic solution. Moreover, we consider the case that the asymptotic steady state possesses exactly two purely imaginary eigenvalues $\pm i\omega$, $\omega \neq 0$. As a consequence, a family of small periodic solutions exists near the steady state.

It is the aim of this work to analyse the interaction of the homoclinic solution with these periodic solutions by exploiting the underlying reversibility of the equation. As one of the main results we find all homoclinic orbits to the center manifold, which stay close to the primary homoclinic solution and which approach a periodic orbit or the steady state in forward and backward time.

1 Introduction

We are interested in one-dimensional lattice differential equations of the form

$$\partial_t u^i(t) = F(u^{i-M}(t), \ldots, u^i(t), \ldots, u^{i+M}(t)), \tag{1}$$

where $F : \mathbb{R}^{(2M+1)N} \to \mathbb{R}^N$ is a smooth map. Hence, the dynamic of the i-th particle only depends on its M nearest neighbors. Looking for traveling wave solutions $u^i(t) = \psi(i - ct)$, $c \neq 0$, leads to the traveling wave equation

$$-c\psi'(\xi) = F(\psi(\xi - M), \ldots, \psi(\xi), \ldots, \psi(\xi + M)), \tag{2}$$

where we have set $\xi = i - ct$. The right hand side of (2) defines a map from $C^0([-M, M], \mathbb{R}^N)$ to $\mathbb{R}^N$; hence equation (2) is a nontrivial functional differential equation of mixed type. General properties of these type of equations have first been investigated in [26] and afterward studied by many others

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[19, 20, 21, 27, 6, 7, 8, 9]. A special feature of these equations is the fact that they do not generate a flow. This is in sharp contrast to delay differential equations, where the existence of a semiflow is well known, see [18].

The task of finding homoclinic or heteroclinic solutions of (2), which then corresponds to the existence of pulses or fronts in the original equation (1), is therefore a difficult task. However, there has been some recent progress. Let us mention the important work of Friesecke and Wattis [5], where the existence of solitary waves of one-dimensional lattices with Hamiltonian structure has been proved using variational methods. Mallet-Paret investigated the existence of traveling waves in a quite general class of lattice differential equations [22]. Iooss et al. restricted their attention to the Fermi-Pasta-Ulam lattice and the Klein-Gordon equation (3), see [11, 12]. They could prove the existence of small localized solutions using a center manifold reduction. In particular, they could analyse the essential dynamics of the traveling wave equation (2) near a steady state.

An essential feature of many traveling wave equations (2) associated to a certain lattice differential equations (1) is a *time-reversibility*, which leads to a special property of the traveling equation. More precisely, this means that whenever $\psi(\xi)$ is a solution of (2) also $R\psi(-\xi)$ is a solution. Here, $R \in L(\mathbb{R}^N)$ is a bounded linear map with the property $R^2 = id$. However, since the initial value problem with respect to equation (2) is not well-posed, we usually prefer to define reversibility differently (see (12) and (14)). As we show in section 3, the traveling wave equation associated to the Fermi-Pasta-Ulam lattice has this property. There are other important examples, i.e. the traveling wave equation of the Klein-Gordon equation, [1, 11],

$$\begin{aligned} \dot{u}_n &= w_n \\ \dot{w}_n &= u_{n+1} - 2u_n + u_{n-1} + V'(u_n), \qquad n \in \mathbb{Z}, \end{aligned} \tag{3}$$

also has this property, see section 3.

So far, a lot of *local* bifurcations near steady states have been analysed in lattice differential equations, see [11, 12]. But what can be said about more general bifurcation scenarios, which are not restricted to a small neighborhood of a steady state and therefore cannot be investigated by a center manifold reduction? And how can we faithfully exploit the underlying reversibility of the traveling wave equation (2)?

A first step has been performed in [8], where a bifurcation of a not necessarily small homoclinic orbit to a nonhyperbolic equilibrium in a general functional differential equation has been investigated. However, no symmetry considerations have been taken into account there. The aim of the present work is to show that the approach of casting equation (2) in a specific abstract setting, which has been initiated in [7, 8, 9, 27], is strong enough to analyse homoclinic bifurcations in reversible systems similar to the ODE-case. Moreover, we are interested in the case of a reversible forward-backward delay equation (2) (the term "reversible" in this context will be specified below), which does not necessarily possess any additional structure. However, since a lot of well-known lattice differential equations, like the Klein-Gordon lattice, do have a Hamil-

tonian structure, we will also point out some consequences which are imposed by such a structure.
In order to motivate our results, let us first restrict our attention to an ordinary differential equation $\dot{x} = \mathcal{F}(x), x \in \mathbb{R}^{2N}$, and recall why homoclinic orbits are so typical in reversible systems. We assume that the homoclinic solution h under consideration satisfies

$$\lim_{|\xi|\to\infty} h(\xi) = 0,$$

where zero is a hyperbolic steady state and (6) is satisfied for a linear operator R with $R^2 = id$. Of particular importance is now the case of a *symmetric* homoclinic orbit, i.e. the orbit intersects the fixed point space $\text{Fix}(R) = \{x \in \mathbb{R}^{2N} : Rx = x\}$. Let us restrict to the situation that $\dim(\text{Fix}(R)) = N$, which will be similar to our assumptions concerning the more general equation (2). The question of persistence of a symmetric homoclinic orbit under variation of a parameter in a reversible setting now reduces to a question regarding the relative position of $\text{Fix}(R)$ and the stable manifold $W^{s,+}(0)$ of the steady state zero. More precisely, if

$$\text{Fix}(R) + T_{h(0)}W^{s,+}(0) = \mathbb{R}^{2N} \tag{4}$$

then the homoclinic orbit will persist upon slight variation of a parameter. Indeed, the stable manifold will still intersect $\text{Fix}(R)$ and induce a homoclinic orbit via reversibility. Let us note that (4) is possible, since $\dim W^{s,+}(0) = N$ and $\text{Fix}(R) = N$; hence, (4) is equivalent to the trivial intersection of the two spaces. These arguments show that the persistence of (symmetric) homoclinic orbits is related to the dimension of the sum of stable manifold and $\text{Fix}(R)$.
Coming back to the case of general forward-backward-delay equations, we may ask whether such a scenario has an analogon in this more general framework. Of course, the setting is now infinite dimensional and due to the non-existence of flows, concepts such as Poincaré-maps do not exist. However, it has been shown in a series of recent papers [6, 7, 8, 9] that locally invariant manifolds near the homoclinic orbit do exist. Here, we additionally want to exploit the underlying reversibility and show that bifurcations near the homoclinic orbit could be analysed similar to the corresponding ODE scenario.

The main scenario in the ODE-case

For the sake of illustration let us first explain the main scenario in the framework of an ordinary differential equation

$$\dot{x}(t) = \mathcal{F}(x(t), \lambda), \qquad \lambda \in \mathbb{R}^2, \quad x \in \mathbb{R}^{2N+2}, \tag{5}$$

where the vector field satisfies

$$\mathcal{F} \circ R = -R\mathcal{F}, \tag{6}$$

for a linear map $R \in L(\mathbb{R}^N)$ such that $\dim(\text{Fix}(R)) = N + 1$. We are interested in a situation where the asymptotic steady state is *non*-hyperbolic and consider the case that that the linearization of $\mathcal{F}$ at zero possesses two simple eigenvalues $\pm i\omega$, $\omega \neq 0$, on the imaginary axis. The Lyapunov center theorem (see [3]) then implies that the center manifold $\mathcal{M}_\lambda$ consists entirely of periodic orbits for all $\lambda \sim 0$. We remind that the center manifold is given as a graph over the generalized center eigenspace and contains all global solutions of (5) which stay in a sufficiently small neighborhood of zero. Let us additionally assume the existence of a symmetric homoclinic orbit h of (5) for $\lambda = 0$, which approaches the trivial steady state and

$$h(0) \in \mathrm{Fix}(R). \tag{7}$$

Due to the existence of periodic orbits near the steady state, h has to approach the steady state along the strong stable manifold as $t \to \infty$. In particular, $h(t)$ approaches zero in forward time with exponential rate. By reversibility, h also converges to zero in backward time with exponential rate. We will consider the generic case that the center stable manifold $W^{cs,+}(0)$ and the strong unstable manifold $W^{u,-}(0)$ intersect only along the homoclinic orbit for $\lambda = 0$. Hence

$$T_{h(0)} W^{cs,+}_{\lambda=0}(0) \cap T_{h(0)} W^{u,-}_{\lambda=0}(0) = \ \mathrm{span}\{\partial_t h(0)\}. \tag{8}$$

In order to ensure genericity of the parameter, we assume that

$$\{U \in W^{s,+}_\lambda(0) : \lambda \sim 0\} \ \text{ is transverse to } \ \mathrm{Fix}(R). \tag{9}$$

This last assumption assures the existence of homoclinic orbits to the steady state to be a codimension-one-phenomenon. Moreover, counting dimensions we see that the center stable manifold $W^{cs,+}_\lambda(0)$ intersects the space $\mathrm{Fix}(R)$ transversely at $h(0)$. Let us focus on the generic case that

$$\dim(\mathrm{Fix}(R) \cap T_{h(0)} W^{cs,+}_0(0)) = 1, \tag{10}$$

and we refer to [14] for the scenario in case of a two-dimensional intersection. We now discuss the resulting bifurcation scenario. More precisely, we are interested in the existence of solutions to the center manifold. At least in the ODE-case this has already been achieved in [14, 15]. In particular, one has to distinguish the cases

$$\begin{aligned} &a) \quad \dim\,[T_{h(0)} W^{cs,+}_{\lambda=0}(0) \cap T_{h(0)} W^{cu,-}_{\lambda=0}(0)] = 2, \\ &b) \quad \dim\,[T_{h(0)} W^{cs,+}_{\lambda=0}(0) \cap T_{h(0)} W^{cu,-}_{\lambda=0}(0)] = 3. \end{aligned} \tag{11}$$

Case a) is the generic case, which implies that center stable and center unstable manifold intersect transversely along h. In this case the bifurcation scenario is easy to analyse. Indeed, on account of hypothesis (9), we expect the existence of a symmetric homoclinic solution h^λ to the steady state for λ lying on a suitable continuous curve in the two-dimensional parameter plane through the origin. Taking into account (10) there exists a one-parameter family of symmetric homoclinic solutions $\tilde{h}^{sym,\kappa}$ for all $\lambda \approx 0$, which are parametrized over $\kappa \sim 0$ and which approach a periodic orbit in forward and backward time. In particular, for each fixed κ the solution $\tilde{h}^{sym,\kappa}$ selects a unique periodic

orbit, that is approached in forward and backward time. The solutions $\tilde{h}^{sym,\kappa}$ account for the intersection of the center stable manifold and the space $\mathrm{Fix}(R)$. Finally, we have to consider additional intersections of $W^{cs,+}(0)$ and $W^{cu,-}(0)$, which may induce unsymmetric global solutions. But these do not exist, if the intersection of their tangent spaces satisfies a).
However, if the ordinary differential equation also possesses a Hamiltonian structure, case a) cannot occur (see also section 7). More interesting is therefore case b), which is compatible with a Hamiltonian structure. We then additionally conclude the existence of two one-parameter families of unsymmetric homoclinic solutions to the center manifolds, where one family is induced by the other via reversibility.

The main result

We can now state our main result, which addresses this bifurcation scenario in the more general reversible equation (2), see also theorem 6 and 7 for a statement of the results that lists all relevant hypotheses explicitly. Let us point out again that we mainly want to understand *how* a reversible structure of the forward-backward alias traveling wave equation (2) facilitates the bifurcation analysis analogous to the ODE-scenario. The case that the lattice equation (1) possesses an additional Hamiltonian structure will be addressed in section 7.
For the statement of the main result we introduce the following notation. Differentiating (2) at 0 and $\lambda = 0$ we can write for any $\phi \in C^0([-M, M], \mathbb{R}^N)$

$$D\mathcal{F}(0,0)[\phi] =: \sum_{j=-M}^{M} A_j^+ \phi(j),$$

where $A_j^+ \in L(\mathbb{R}^N, \mathbb{R}^N)$ are matrices for any $j \in \{-M, \dots, M\}$. The next theorem is the main result of this work and we refer to [14, 15] for a statement in terms of ordinary differential equations.

Theorem 1

Let us consider equation (2), where $F \in C^2(\mathbb{R}^{(2M+1)N} \times \mathbb{R}, \mathbb{R}^N)$ and $F(0, \lambda) = 0$ for all λ. Moreover, assume that (2) is reversible, that is for any $\phi \in C^0([-M, M], \mathbb{R}^N)$ it is true that

$$F((R\phi(-M), \dots, R\phi(M)), \lambda) = -RF((\phi(-M), \dots, \phi(M)), \lambda), \tag{12}$$

where $R \in L(\mathbb{R}^N)$ satisfies $R^2 = id$ and (15) below. Let us assume that there is a homoclinic solution h of (2) for $\lambda = 0$, which satisfies $Rh(0) = h(0)$ and approaches zero in forward and backward time with exponential rate. Moreover, we suppose that the equation

$$det \left[\lambda \cdot id - \sum_{j=-M}^{M} A_j^+ (e^{\lambda j} \cdot id) \right] = 0$$

possesses exactly two simple zeros $\lambda = \pm i\omega$, $\omega \neq 0$, on the imaginary axis and the real part of all other zeros with positive real part is larger than zero.

Then, under generic assumptions (see the hypotheses in the statement of theorem 7), there exists a continuous curve Hom in the two-dimensional parameter space, such that exactly for parameter values λ on Hom the equation (2) possesses a homoclinic solution h^λ, which approaches the steady state in forward and backward time. Moreover, exactly one of the following cases occurs:

i) There exists a one-parameter family $\tilde{h}^{sym,\kappa}$ for each fixed $\lambda \sim 0$, such that every $\tilde{h}^{sym,\kappa}$ is a solution of (2) defined on $\mathbb{R}$. Moreover, $\tilde{h}^{sym,\kappa}(0) \in Fix(R)$ and the solution approaches a nontrivial periodic orbit in forward and backward time. These are the only homoclinic orbits to the center manifold for fixed $\lambda \neq 0$ and $\lambda \approx 0$.

ii) Additionally to the family $\tilde{h}^{sym,\kappa}$ (as described in i)) there exist two one-parameter families of non-symmetric homoclinic orbits to the center manifold. These approach the steady state or a periodic orbit in forward and backward time.

Remark
Let us note that this result covers *all* scenarios which may arise in the case of a reversible equation if the generic assumptions (8) and (10) are satisfied, since assumption (9) only states that we consider a generic parameter. The hypotheses listed above do not make sense for the more general equation (2) in the moment, but will later be formulated in a way that they make sense in the general setting, see theorem 7. So we can give a complete picture of the resulting bifurcation scenario of (2), as long as only homoclinic solutions to the center manifold are addressed; see [14, 10] for the existence of two-homoclinic solutions and more complicated behavior in the framework of ordinary differential equations.

Remark
As we will argue in section 7, we expect only case ii) in the statement of the above theorem to occur if the original lattice differential equation (1) has a Hamiltonian structure. Indeed, we will show that in the case of the Klein-Gordon lattice, for example, this additional structure induces a first integral for the traveling wave equation (2). This fact then typically prevents a transverse intersection of center stable and center unstable manifold in the traveling wave equation (2) and excludes possibility i) in theorem 1; see section 7 for details.

Let us comment on some difficulties which have to be addressed in the sequel. In the course of proving theorem 1 we will derive a bifurcation function, whose zeros correspond to homoclinic solutions to the center manifold. This function measures the distance between initial values in the center stable and center unstable manifold in an appropriate complement of $T_{H(0)}W^{cs,+}(0)+T_{H(0)}W^{cu,-}(0)$ within a Poincaré section. This method is known as Lin's method, see [28, 17]. We are therefore introducing this concept to forward-backward-delay equations (even in the case of nonhyperbolic steady states) as long as we are only interested in one-homoclinics. In particular, the notion of codimension of homoclinic orbits in equation (2) becomes rigorous.

By adapting Lin's method to our situation we in particular have to deal with the construction of appropriate (R-invariant) complements of $T_{H(0)}W^{s,+}(0) + T_{H(0)}W^{u,-}(0)$. Once we can construct suitable complements and determine its dimensions the situation becomes analogous to the ODE-case. Of course we are now working in an infinite dimensional setting, where the construction of closed R-invariant complements and the counting of dimensions are non-trivial tasks. This is even getting more complicated as the center stable and center unstable manifold are constructed as submanifolds of a Banach space which is not an Hilbert space, see [8, 9] and section 4. However, the approach taken in this work is strong enough to reproduce the global picture of the bifurcation scenario analogous to the ODE-case. In particular, we could now aim at analysing quite general bifurcation scenarios of traveling waves in time-reversible lattices (1) in the future.
We proceed as follows. After introducing some notation in the next section we discuss the set up in section 3. There we also will make precise the notion of reversibility (see section 3.1) and study properties of the linear system, which arises after linearizing (2) along the homoclinic solution. The existence of invariant manifolds as well as the local dynamics near the steady state is addressed in section 4. In 5 we finally prove our main theorem by adapting Lin's method to the case of forward-backward-delay equations. The work is concluded by a discussion section (see section 7) where we want to discuss the relevance of our results for the Klein-Gordon and Fermi-Pasta-Ulam lattice differential equation.

2 Notation and Definitions

Throughout this paper we will denote by $BC^{\eta}(J,E)$ for some interval $J \subset \mathbb{R}$ and some Banach space E and some $\eta \in \mathbb{R}$ the space of all continuous functions $w : J \to E$, which are bounded with respect to the norm

$$\|w\|_{\eta} := \sup_{t\in J} e^{\eta|t|}|w(t)|_E.$$

Moreover, the following spaces will be used throughout this paper:

$$\begin{aligned} Y &:= \mathbb{R}^N \times L^2([-M,M],\mathbb{R}^N), \\ \tilde{Z} &:= \mathbb{R}^N \times L^\infty([-M,M],\mathbb{R}^N) \\ X &:= \{(\xi,\varphi) \in Y \mid \varphi \in H^1([-M,M],\mathbb{R}^N) \text{ and } \varphi(0) = \xi\} \\ \tilde{X} &:= \{(\xi,\phi) \in \mathbb{R}^N \times C^0([-M,M],\mathbb{R}^N) : \phi(0) = \xi\} \end{aligned}$$

As a convention, if subspaces are furnished with an additional $\tilde{}$, they are regarded as subspaces of $\tilde{X}$ with the induced norm. If they are furnished with an additional $\hat{}$, they are viewed as subspaces of X.
In the following we will use the notation $x_t \in L^2([-M,M],\mathbb{R}^N)$ for an integrable function $x : \mathbb{R} \to \mathbb{R}^N$. This function is defined by $x_t(\theta) := x(t+\theta)$ for any $t \in \mathbb{R}$.

3 The setting

Instead of studying the traveling wave equation (2) directly, we prefer to work with the abstract equation

$$\begin{aligned} \dot{U}(t) &= \mathcal{F}((\xi(t), \phi(t,\cdot)), \lambda) \\ &= \begin{pmatrix} F(\phi(t,-M), \ldots, \phi(t,0), \ldots, \phi(t,M)), \lambda) \\ \partial_\theta \phi(t,\theta) \end{pmatrix}, \end{aligned} \tag{13}$$

where $F \in BC^2(\mathbb{R}^{N(2M+1)} \times \mathbb{R}; \mathbb{R}^N)$ and $F(0,\lambda) = 0$ for all λ. This approach has first been used in [11, 12] although with a slightly different choice of state spaces X, Y. Let us now define what a strong solution of (13) is.

Definition 1
We call a continuous function $U(t) : [t_1, t_2) \to Y$ a solution of (13) on (t_1, t_2), where $-\infty < t_1 < t_2 \leqslant \infty$, if $t \to U(t)$ is continuous regarded as a map on (t_1, t_2) with values in X, if $t \to U(t)$ is differentiable regarded as a map on (t_1, t_2) with values in Y and (13) is satisfied on (t_1, t_2).
We call a differentiable function $U(t) : (-\infty, t_2) \to Y$ a solution of (13) on $(-\infty, t_2)$ and $t_2 \in \mathbb{R}$, if $t \to U(t)$ is continuous regarded as a map on $(-\infty, t_2)$ with values in X and (13) is satisfied on $(-\infty, t_2)$.

The next lemma clarifies the connection between solutions of (13) and our original equation (2). The proof can be found in [6, 7, 8].

Lemma 1
Let

$$U(t) = \begin{pmatrix} \xi(t) \\ \varphi(t)(\cdot) \end{pmatrix}$$

be a solution of (13) on $(t_1 - M, t_2 + M)$. Then $\varphi(t)(\theta) = \xi(t + \theta)$ for all $t \in (t_1 - M, M + t_2)$ and $\theta \in [-M, M]$ with $t + \theta \in (t_1 - M, t_2 + M)$. Furthermore $\xi(t)$ solves (2) on the interval (t_1, t_2).

3.1 Reversibility

We introduce the notion of reversibility of our abstract equation (13) in this section. More precisely, we want to assume that

$$\mathcal{R}\mathcal{F}(U, \lambda) = -\mathcal{F}(\mathcal{R}U, \lambda) \tag{14}$$

for any $U = (\xi, \phi) \in X$, where the linear map $\mathcal{R} : Y \to Y$ is defined by

$$\mathcal{R}(\xi, \phi(\theta)) := (R\xi, R[\mathcal{S}\phi(\cdot)]) = (R\xi, R\phi(-\theta))$$

and $(\mathcal{S}\phi)(\theta) := \phi(-\theta)$ for any $\phi \in C^0([-M, M], \mathbb{R}^N)$. Moreover, we only consider $R \in L(\mathbb{R}^N)$ that can be represented in the form

$$R = P_{i_1} \circ P_{i_2} \circ \ldots \circ P_{i_n}, \tag{15}$$

where the reflection P_i, $1 \leqslant i \leqslant N$, is defined by

$$P_i(x^1, \dots, x^N) \mapsto (x^1, \dots, x^{i-1}, -x^i, x^{i+1}, \dots, x^N).$$

The next to examples provide two well-known lattice differential equations, where the corresponding abstract equation satisfies (14).

Examples
a) Let us consider the Klein-Gordon equation (3). A travelling wave ansatz leads to the abstract equation

$$\begin{pmatrix} \partial_t x(t) \\ \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} \xi(t) \\ \phi^1(t,1) + \phi^1(t,-1) - 2\phi^1(t,0) + V'(\phi^1(t,0)) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix},$$

where ϕ^1 denotes the first component of $\phi = (\phi^1, \phi^2) : [-1,1] \to \mathbb{R}^2$ and $(x, \xi, \phi) \in X = \{(x, \xi, (\phi^1, \phi^2)) \in \mathbb{R}^2 \times H^1([-1,1], \mathbb{R}^2) : \phi(0) = (x, \xi)\}$. Then $\mathcal{R}$ is given by

$$\mathcal{R}(x, \xi, \phi(\theta)) \mapsto (x, -\xi, \phi^1(-\theta), -\phi^2(-\theta)), \tag{16}$$

which has the upper form (14).
b) Another important example is given by the Fermi-Pasta-Ulam model, which has the form

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} + V'(u_{n+1} - u_n) + V'(u_{n-1} - u_n), \qquad n \in \mathbb{Z}.$$

Making a travelling wave ansatz and casting the equation in abstract form then leads to the equation

$$\begin{aligned} \begin{pmatrix} \partial_t x(t) \\ \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} &= \begin{pmatrix} \xi(t) \\ \frac{-1}{c}(\phi^1(t,1) + \phi^1(t,-1) - 2\phi^1(t,0)) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \frac{-1}{c}(V'(\phi^1(t,1) - \phi^1(t,0)) + V'(\phi^1(t,-1) - \phi^1(t,0))) \\ 0 \end{pmatrix} \end{aligned} \tag{17}$$

with $(x, \xi, \phi) \in X$, where X is defined as in example a) and where $c \neq 0$ denotes the travelling wave speed. Again, this equation is reversible with respect to $\mathcal{R}$ as given in (16).

Coming back to equation (2), we want to assume the existence of a symmetric homoclinic orbit.

Hypothesis 1
Equation (2) possesses a homoclinic solution h for $\lambda = 0$, which satisfies $h(t) = Rh(-t)$ for all $t \geqslant 0$ and

$$|h(t)| \leqslant Me^{-\alpha t}$$

for some $M, \alpha > 0$ and $t \geqslant 0$.

Note that h induces a homoclinic solution H for the abstract equation via $H(t) = (h(t), h_t)$, which satisfies $\mathcal{R}H(0) = H(0)$; that is, H is symmetric. More generally, we call a solution U of (13) *symmetric* if its orbit intersects $\mathrm{Fix}(\mathcal{R})$ in a point different from a steady state.

3.2 Solution operators for the non-autonomous linear equation

In this chapter we want to review some known facts about linear functional differential equations of mixed type which we will use in the sequel, see also [20, 27]. We investigate the linear equation

$$\dot{y}(t) = D_1 F(h(t-M), \ldots, h(t), \ldots, h(t+M), 0) y_t =: L(t) y_t, \qquad (18)$$

where we recall that $y_t(\theta) := y(t+\theta)$ for any $\theta \in [-M, M]$.

Definition 2
We call a function $x \in L^2([-M,\tau), \mathbb{C}^N)$ a solution of (18) for some $M < \tau \leqslant \infty$ and some initial condition $\phi \in L^2([-M,M], \mathbb{C}^N)$, if $x \in H^1_{loc}([0,\tau), \mathbb{C})$, $x_0 = \phi$ and (18) is satisfied for almost every $t \in [0,\tau)$.

Note that in any case $L(t)\phi$ for fixed t and $\phi \in C^0([-M,M], \mathbb{R}^N)$ has the form

$$L(t)\phi = \sum_{j=-M}^{M} A_j(t)\phi(j) \qquad (19)$$

for some $A_j(\cdot) \in BC^0(\mathbb{R}, L(\mathbb{R}^N, \mathbb{R}^N))$. Let us now state a hypothesis which implies that two solutions $\tilde{y}, y \in H^1(\mathbb{R}, \mathbb{R}^N)$ of (18) are identical provided they coincide on some interval of length $2M$, see [6, 27].

Hypothesis 2
$\det(A_{-M}(\cdot))$ and $\det(A_M(\cdot))$ do not vanish identically on any nontrivial interval of $\mathbb{R}$.

As in the nonlinear case we can relate equation (18) to the abstract equation

$$\partial_t V(t) = \mathcal{A}(t) V(t), \qquad (20)$$

where the linear operator $\mathcal{A}(t) : X \subset Y \to Y$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} \xi \\ \varphi \end{pmatrix} = \begin{pmatrix} L(t)\varphi \\ \partial_\theta \varphi \end{pmatrix}$$

for $(\xi, \varphi) \in X$. Let us set $\mathcal{A}_+ := \lim_{t\to\infty} \mathcal{A}(t)$ (i.e. where $L(t)$ in the definition of $\mathcal{A}(t)$ is replaced by $L_+ := \lim_{t\to\infty} L(t)$). Then it is known that the spectrum of the densely defined operator $\mathcal{A}_+ : X \subset Y \to Y$ only consists of eigenvalues of finite multiplicity. Moreover, an element $\lambda_* \in \mathbb{C}$ is in $\mathrm{spec}(\mathcal{A}_+)$, if the characteristic function vanishes at λ_*, that is, if

$$\det(\triangle(\lambda)) := \det \left[\lambda \cdot id - \sum_{j=-M}^{M} A_j^+ (e^{+j\lambda} \cdot id) \right] = 0 \qquad (21)$$

for $\lambda = \lambda_*$, where $A_j^+ := \lim_{t\to\infty} A_j(t)$. Furthermore, the algebraic multiplicity of λ_* as an eigenvalue of $\mathcal{A}_+$ (which is the dimension of its generalized eigenspace) coincides with the order of λ_* as a zero of $\det\triangle(\cdot)$; we refer to [6, 7, 27] for proofs of these statements.
The following result implies that on suitable subspaces the abstract equation (20) can be solved in forward- and backward time, respectively. The proof can again be found in [6, 7, 27].

Theorem 2 (Center-dichotomy on $\mathbb{R}_+$)
(20) possesses a center-dichotomy on $\mathbb{R}_+$. That is, for any $\delta > 0$ there exist constants $K, \alpha > 0$ and a family of strongly continuous projections $P(t) : Y \to Y$, $t \geqslant 0$, with the following properties. For $U \in Y$ and $t_0 \geqslant 0$

- *there exists a continuous function $\Phi_+^{cs}(\cdot,\cdot)U : \{(t,t_0) : t \geqslant t_0; t, t_0 \geqslant 0\} \to Y$, such that $\Phi_+^{cs}(t_0,t_0)U = P(t_0)U$. Moreover, $\Phi^{cs}(t,t_0)U \in Rg(P(t))$ and $|\Phi_+^{cs}(t,t_0)U|_Y \leqslant Ke^{\delta|t-t_0|}|U|_Y$ for all $t \geqslant t_0 \geqslant 0$.*
- *There exists a continuous function $\Phi_+^{u}(\cdot,\cdot)U : \{(t,t_0) : t \leqslant t_0; t, t_0 \geqslant 0\} \to Y$, such that $\Phi_+^{u}(t_0,t_0)U = (id - P(t_0))U$. Moreover, $\Phi_+^{u}(t,t_0) \in ker(P(t))$ and $|\Phi_+^{u}(t,t_0)U|_Y \leqslant Ke^{-\alpha|t-t_0|}|U|_Y$ for all $t_0 \geqslant t \geqslant 0$.*

In the special case $U \in X$ the functions $t \mapsto \Phi^{cs}(t,t_0)U$ and $t \mapsto \Phi_+^{u}(t,t_0)U$ define classical solutions of (20) on their domain of definition. In any case, if $U \in Rg(P(t_0))$ with $U = (\zeta, \phi(\cdot))$ the map $\Phi_+^{cs}(t,t_0)U$ is of the form $(x(t), x_t)$ for $t > t_0$, $\Phi_+^{cs}(t_0,t_0)U = U$ and $x(\cdot)$ defines a solution of (18) with $x_0 = \phi$. An analogous statement holds for $\Phi^u(t,t_0)U$.

Definition
We often call the existence of such a projection $P(t)$ and such solution operators a *center dichotomy* on $\mathbb{R}_+$. The difference to an *exponential dichotomy* is the fact that the norm of the solution operator $\Phi_+^{cs}(t,t_0)$ does not decay exponentially in the difference $|t - t_0|$.

Let us stress the fact that for *any* point U in the center stable subspace $\mathrm{Rg}(P(t_0))$ there exists a function $\Phi^{cs}(t,t_0)U = (x(t), x_t)$, such that $x(t)$ is a solution of our original equation (18) for $t > t_0$. Thus, we need not exclusively focus on initial values $U \in X$ (although these induce classical solutions of the abstract equation (20)), since we are actually interested in solving the equation $\dot{x}(t) = L(t)x_t$.
Alternatively, there exist continuous solution operators $\Phi_+^s(t,t_0)$, $\Phi_+^{cu}(t,t_0)$ for $t \geqslant t_0 \geqslant 0$ and $t_0 \geqslant t \geqslant 0$, respectively, which define strong solutions for initial values $U \in X$ and satisfy the estimates

$$\|\Phi_+^s(t,t_0)\|_{L(Y,Y)} \leqslant Ke^{-\alpha|t-t_0|}, \qquad \|\Phi_+^{cu}(t,t_0)\|_{L(Y,Y)} \leqslant Ke^{\delta|t-t_0|}.$$

Finally, we can also prove the existence of a center dichotomy on $\mathbb{R}_-$.
Let us now consider the case that the equation $\dot{x}(t) = L(t)x_t$ possesses a center dichotomy on $\mathbb{R}_+$ with associated solution operators Φ_+^{cs}, Φ_+^{u} and Φ_+^{s}, Φ_+^{cu}. The next lemma states that the projections $\Phi^{cs}(t_0,t_0)$ approach the projection π_{cs} associated to the center stable eigenspace of the autonomous linear equation $\dot{x}(t) = D_1F(0,0)x_t =: L_+x_t$.

Lemma 2
Consider the equation $\dot{x}(t) = L_+ x_t + \tilde{L}(t)x_t$, *where* $\tilde{L}(t) := L(t) - L_+$, $\tilde{L}(\cdot) \in BC^0(\mathbb{R}, L(C^0, \mathbb{R}^N))$ *and*

$$|\tilde{L}(t)|_{L(C^0,\mathbb{R}^N)} \leqslant Me^{-\gamma|t|}$$

for $t \to \infty$. *In case that* $\dot{x}(t) = L_+ x_t$ *is hyperbolic (not hyperbolic) the projection* $\Phi^s_+(t,t)$ $(\Phi^{cs}(t,t))$ *approaches* π_s (π_{cs}) *for* $t \to \infty$ *with respect to the* $L(Y,Y)$*-norm, the* $L(\tilde{X},\tilde{X})$*-norm or the* $L(X,X)$*-norm.*

Proof
We can assume without loss of generality that $\dot{x}(t) = L(t)x_t$ is hyperbolic as $t \to \pm\infty$. Otherwise we consider the translated equation

$$\dot{x}(t) = -\sigma x(t) + L(t)[e^{\sigma \bullet} x_t(\bullet)]$$

for small $\sigma > 0$, which is asymptotically hyperbolic for $t \to \infty$. Note that any solution $x(t)$ on $\mathbb{R}_+$ of the original equation $\dot{x}(t) = L(t)x_t$ induces a solution $y(t)$ of the translated equation via $y(t) = e^{-\sigma t}x(t)$. Hence, let us consider the abstract equation

$$\begin{aligned} \begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} &= \begin{pmatrix} L_+[\phi(t,\cdot)] \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} + \begin{pmatrix} \tilde{L}(t)[\phi(t,\cdot)] \\ 0 \end{pmatrix} \\ &=: A_+ \begin{pmatrix} \xi(t) \\ \phi(t,\cdot) \end{pmatrix} + B(t) \begin{pmatrix} \xi(t) \\ \phi(t,\cdot) \end{pmatrix} \end{aligned} \tag{22}$$

We denote the solution operators associated to the exponential dichotomy of the equation $\dot{U} = A_+ U$ by $e^{A^s_+ t}$ for $t \geqslant 0$ and $e^{A^u_+ t}$ for $t \leqslant 0$, where $|e^{A^s_+ t}U|_Y \leqslant Me^{-\beta|t|}|U|_Y$ for $t \geqslant 0$ and $|e^{A^u_+ t}U|_Y \leqslant Me^{-\iota|t|}|U|_Y$ for $t \leqslant 0$ and some $0 < \beta < \iota$.
Via integration (see [29] for example) we can see that the operator $\Phi^s_+(t,s)$ associated to an exponential dichotomy of (22) on $\mathbb{R}_+$ solves the equation

$$\begin{aligned} e^{A^s_+(t-s)}W_* &= \Phi^s_+(t,s)W_* + e^{A^s_+ t}\Phi^u_+(0,s)W_* \\ &+ \int_t^\infty e^{A^u_+(t-\tau)}B(\tau)\Phi^s_+(\tau,s)W_* d\tau - \int_s^t e^{A^s_+(t-\tau)}B(\tau)\Phi^s_+(\tau,s)W_* d\tau \\ &+ \int_0^s e^{A^s_+(t-\tau)}B(\tau)\Phi^u_+(\tau,s)W_* d\tau \end{aligned} \tag{23}$$

for $t \geqslant s \geqslant 0$ and $W_* \in X$. Here, the equation can be considered in Y and all integrals can be regarded as Riemann integrals. Note that the integrands are in fact well-defined: Since for $W_* \in X$ it turns out that $\Phi^s_+(\tau,s)W_* = (x(\tau), x(\tau+\cdot))$ for some continuous function $x(\cdot)$, the value $B(\tau)\Phi^s_+(\tau,s)W_*$ is well-defined. Setting $t = s = t_0$ gives

$$\begin{aligned} \pi_s W_* &= \Phi^s_+(t_0,t_0)W_* + \int_{t_0}^\infty e^{A^u_+(t_0-\tau)}B(\tau)\Phi^s_+(\tau,t_0)W_* d\tau \\ &\quad + \int_0^{t_0} e^{A^s_+(t_0-\tau)}B(\tau)\Phi^u_+(\tau,t_0)W_* d\tau + e^{A^s_+ t_0}\Phi^u_+(0,t_0)W_*. \end{aligned} \tag{24}$$

Now all integrals can be estimated in the Y-norm by $M(t_0)|W_*|_Y$, where $M(t_0) \to 0$ as $t_0 \to \infty$. In fact the bound $M(t_0)$ is independent in $W_* \in X$. Note that for the estimate of the second integral appearing on the right hand side of (24) we actually make use of the the assumption that $|B(\tau)|_{L(Y,Y)}$ converges to zero as $\tau \to \infty$ with exponential rate.
Let us now discuss the situation in the case of the space X next. Now, $W_* \in X$ and we have to consider the integrals in (24) as weak* integrals (as explained in the appendix). Still, the value of the sum of the two integrals appearing in equation (24) is in X for each $t_0 > 0$ large enough, see the appendices of [7, 8, 9]. Moreover, one can still estimate the second integral in (24) with respect to the X-norm and show that it converges to zero, see again the appendix of [8] for details. □

4 Invariant manifolds

Since we want to study bifurcating homoclinic solutions of the abstract equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \mathcal{F}((\xi(t), \phi(t, \cdot)), \lambda) \tag{25}$$

near H, the existence of parameter dependent invariant manifolds of the steady state becomes important. In this section we review and state existence results of center manifolds, center (un-)stable and strong (un-)stable manifolds of the equilibrium, which have been proved in [8, 9]. Let us start with the center manifold, which helps us to clarify the set of bounded solutions near zero.

4.1 Local dynamics near the steady state

Let us make the following assumption, which states that the steady state zero possesses exactly two simple purely imaginary eigenvalues. We remind that the characteristic equation $\det\triangle(\cdot)$ has been defined in (21).

Hypothesis 3
$\det\triangle(\cdot)$ possesses exactly two simple, purely imaginary zeros $\pm i\omega$, $\omega \neq 0$. Hence, all other zeros of the characteristic equation have real part bigger than zero.

The last hypothesis implies that the center eigenspace E_c, defined as the generalized eigenspace corresponding to all purely imaginary eigenvalues of the operator $\mathcal{A}_+ : Y \to Y$ is two-dimensional and consists of the linear span of the corresponding two eigenvectors associated to $\pm i\omega$. Note that E_c can be defined as the range of the spectral projection $P_c : Y \to Y$, where for $\Phi \in Y$

$$P_c\Phi := -\frac{1}{2\pi i} \int_\gamma (\mathcal{A}_+ - \lambda)^{-1} \Phi d\lambda,$$

and γ denotes a positively oriented closed curve enclosing $\pm i\omega$, that stays sufficiently close to the imaginary axis; see [6, 7, 13]. Let us set $\tilde{E}_h := \mathrm{Rg}(id -$

$P_c)\big|_{\tilde{X}}$, equipped with the $\tilde{X}$-topology. The following theorem has been proved in [6].

Theorem 3 (Center manifold)
(25) possesses a two-dimensional, local invariant manifold $\mathcal{M} = \mathcal{M}_\lambda \subset \tilde{X}$, which is tangent to E_c at $0 \in \mathcal{M}$ for $\lambda = \lambda_0$. The vector field $\mathcal{F}\big|_{\mathcal{M}}$ on $\mathcal{M}$ is reversible with respect to $\mathcal{R}$, that is

$$\mathcal{F}(\mathcal{R}U, \lambda) = -\mathcal{R}\mathcal{F}(U, \lambda)$$

for all $U \in \mathcal{M}_\lambda$, as long as U is close enough to zero. Moreover, $\mathcal{M}$ depends two times differentiable on μ. That is, $\mathcal{M}$ can locally be represented as a graph of a function $\Psi^\lambda : E_c \cap \mathcal{U} \to \tilde{E}_h$ for $\lambda = 0$, such that Ψ^λ is two times differentiable with respect to λ and where $\mathcal{U}$ denotes a suitable small neighborhood of zero.

Together with hypothesis 3 we have the following corollary.

Corollary 1
If hypothesis 3 is satisfied, the two-dimensional center manifold consists entirely of periodic orbits for every $\lambda \approx 0$. That is, every initial value on the center (except zero) lies on a nontrivial periodic orbit.

Proof
The proof of this claim follows by the Lyapunov-Center theorem. Let us indicate the proof. First of all note that the eigenspace corresponding to $\pm i\omega$ is given by the linear span of $e_1 = (1, \cos(\omega\theta))$ and $e_2 = (0, \sin(\omega\theta))$. Moreover, $e_1 \in \mathrm{Fix}(\mathcal{R})$ and $e_2 \in \mathrm{Fix}(-\mathcal{R})$. If we consider the reduced dynamic on the center eigenspace, the linear flow is just given by a rotation. In particular, every solution of the linear flow is periodic and hits the e_1-axis transversely. Hence, by reversibility, the flow of the nonlinear flow also consists of periodic orbits, since near zero the nonlinear flow is just a small perturbation of the linear flow. This picture remains true for small $\lambda \neq 0$. $\square$

4.2 Invariant manifolds near the homoclinic orbit

In this section we state a result, which assures the existence of a center stable manifold of zero near any point on the homoclinic orbit. For the precise statement of the theorem we need to introduce some notation.

Notation
Since the linear equation (20) possesses a center dichotomy on $\mathbb{R}_+$, there exist solution operators Φ^{cs}_+, Φ^u_+, satisfying the properties of theorem 2. Let us set $E^{cs}_+(t_0) := \mathrm{Rg}[\Phi^{cs}_+(t_0, t_0)]$ and $E^u_+(t_0) := \mathrm{Rg}[\Phi^u_+(t_0, t_0)]$ for any $t_0 \geqslant 0$. Choosing $\delta > 0$ there also exists a center dichotomy on $\mathbb{R}_-$ with solution operators Φ^u_-, Φ^{cs}_- satisfying

$$\|\Phi^u_-(t,s)\|_{L(Y,Y)} \leqslant Me^{-\alpha|t-s|}, \qquad \|\Phi^{cs}_-(s,t)\|_{L(Y,Y)} \leqslant Me^{\delta|t-s|} \tag{26}$$

for $t \leqslant s \leqslant 0$, some $\alpha, M > 0$, and $M = M(\delta)$. We define $E^u_-(t_0) := \mathrm{Rg}[\Phi^u_-(t_0, t_0)]$ and $E^{cs}_-(t_0) := \mathrm{Rg}[\Phi^{cs}_-(t_0, t_0)]$ for any $t_0 \leqslant 0$.

Theorem 4 (The center stable manifold)
Equation (13) possesses a C^2-manifold $W^{cs,+}(0) = W^{cs,+}_\lambda(0) \subset \tilde{X}$ near $H(0)$ with the following properties:

a) *The tangent space of $W^{cs,+}_0(0)$ at $H(0) \in W^{cs,+}_0(0)$ is $\tilde{E}^{cs}_+(0) := \tilde{X} \cap E^{cs}_+(0)$.*

b) *If $W_+ \in W^{cs,+}_\lambda(0)$ and $W_+ \approx H(0)$ with respect to the $\tilde{X}$-norm, then there exists a continuous function $W(t) : [0,\infty) \to \tilde{X}$, such that $W(t) \in W^{cs,+}_\lambda(0)$ for some small time interval $[0,t_*)$, $W(0) = W_+$ and $W(t) = (\xi(t), \xi_t)$, where $\xi(t) : [-M, t_*) \to \mathbb{R}^N$ is a solution of (2) on $(0,t_*)$. Moreover, if $\lambda \sim 0$ is sufficiently small, then $\xi(t)$ is defined on $[-M,\infty)$ and approaches a periodic orbit or the steady state zero in forward time $t \to \infty$.*

c) *If there is a solution $W(t)$ of the abstract equation (13) for $\lambda = 0$, such that $|W(t) - H(t)|_{\tilde{X}}$ is sufficiently small for all $t \geqslant 0$, then $W(0) \in W^{cs,+}_0(0)$.*

d) *$W^{cs,+}_\lambda(0)$ is two times differentiable with respect to λ: If we supply equation (13) with $\dot{\lambda} = 0$ then the extended system possesses a local invariant C^2-manifold $W^{cs,+}_{ex}(0) \subset \tilde{X} \times \mathbb{R}^2$ and $W^{cs}_{ex}(0) \cap \left(\tilde{X} \times \{\lambda\}\right)$ satisfies the properties a), b), c).*

Sketch of the proof
Let us make some comments concerning the proof of this theorem and refer to [8, 9] for a complete proof. The first step is to parametrize solutions $U(t)$ near $H(t)$ via $U(t) = V(t) + H(t)$. Making this ansatz one can show that $V(t)$ solves an equation of the form

$$\dot{V}(t) = \mathcal{A}(t)V(t) + \mathcal{G}(t, V(t), \lambda), \tag{27}$$

where $\mathcal{A}(t)$ denotes, as before, the linearization of (13) along the homoclinic orbit H, see (20). Moreover, $\mathcal{G}$ is given by $\mathcal{G}(t, V, \lambda) := \mathcal{F}(H(t) + V, \lambda) - \mathcal{A}(t)V - \mathcal{F}(H(t), 0)$. We therefore try to find the function V as a fixed point of the integral equation

$$\begin{aligned} V(t) &= \Phi^{cs}_+(t,0)V^s_0 + \int_0^t \Phi^{cs}_+(t,s)\mathcal{G}_{mod}(s, V(s), \lambda)ds \\ &+ \int_\infty^t \Phi^{u}_+(t,s)\mathcal{G}_{mod}(s, V(s), \lambda)ds, \end{aligned} \tag{28}$$

where $V^s_0 \in \tilde{E}^{cs}_+ := E^{cs}_+ \cap \tilde{X}$ and $\mathcal{G}_{mod}$ denotes a modified nonlinearity, such that the Lipschitz constant remains globally small. We want to look for fixed points V in the space $BC^{2\delta} := BC^{2\delta}([0,\infty), \tilde{X})$, where functions remain bounded with respect to t in the $\tilde{X}$-norm when multiplied by $e^{-2\delta|t|}$ (where δ has been defined in (26)). But we have to give a meaning to the integral terms appearing in (28), since the solution operators Φ^{cs}_+, Φ^u_+ do not map $\tilde{X}$ to $\tilde{X}$. Moreover, the map $t \mapsto \Phi^{cs}_+(t,s)U$ with values in $\tilde{Z} = \mathbb{R}^N \times L^\infty([-M,M], \mathbb{R}^N)$ is not

even integrable. However, the integrals are well defined as *weak* integrals*, see the appendix or [8, 9]. One can now show that there exists a unique solution $V \in BC^\delta$ of the integral equation (28), see [8, 9].
Finally, let us make a remark concerning the smoothness of $W^{cs,+}(0)$ as a submanifold of $\tilde{X}$. Here, we make use of the fact that solutions starting near $H(0)$ in $W^{cs,+}(0)$ approach an orbit on the center manifold for $t \to \infty$, as has been proved in [9]. Hence, solutions starting near $H(0)$ stay uniformly close to H for all times. Under these assumptions smoothness of $W^{cs,+}(0)$ has been proved in [9]. □

We also need the existence of an strong unstable manifold $W^{u,-}(0)$ near $H(0)$.

Theorem 5 (The strong unstable manifold)
Equation (13) possesses a C^2-manifold $W^{u,-}(0) = W^{u,-}_\lambda(0) \subset \tilde{X}$ near $H(0)$, with the following properties:

a) *The tangent space of $W^{u,-}_0(0)$ at $H(0) \in W^{u,-}_0(0)$ is $\tilde{E}^u_-(0) := \tilde{X} \cap E^u_-(0)$.*

b) *If $W_- \in W^{u,-}_\lambda(0)$ and $W_+ \approx H(0)$ with respect to the $\tilde{X}$-norm, then there exists a continuous function $W(t) : (-\infty, 0] \to \tilde{X}$, such that $W(t) \in W^{u,-}_\lambda(0)$ for some small time interval $(-t_*, 0]$, $W(0) = W_-$ and $W(t) = (\xi(t), \xi_t)$, where $\xi(t) : (-\infty, M] \to \mathbb{R}^N$ is a solution of (2) on $(-\infty, 0)$ which converges to 0 with exponential rate $\kappa > 0$ as $t \to -\infty$.*

c) *If there is a solution $W(t)$ of the abstract equation (13), such that $W(t) \to 0 \in \tilde{X}$ with exponential rate κ for $t \to -\infty$ and if $W(0)$ is close enough to $H(0)$ with respect to the $\tilde{X}$-norm, then $W(0) \in W^{u,-}_\lambda(0)$.*

d) *$W^{u,-}_\lambda(0)$ is two times continuously differentiable with respect to λ: We can represent $W^{u,-}_\lambda(0)$ in the form $W^{u,-}_\lambda(0) = H(0) + graph(\Psi(\cdot, \lambda))$ and $\Psi(\cdot, \lambda) : (\tilde{E}^u_-(0) \times \mathbb{R}^2) \cap U \to \tilde{E}^{cs}_-(0)$ is two times continuously differentiable with respect to the parameter λ, if U denotes a small neighborhood of zero.*

Similarly, one can prove the existence of a strong stable manifold $W^{s,+}_\lambda(0)$ and a center unstable manifold $W^{cu,-}_\lambda(0)$ near $H(0)$.

5 Properties of reversible equations

Before we state the main hypotheses we will collect some basic properties of reversible forward-backward delay equations. We start with the following lemma.

Lemma 3
If $U(t) = (\xi(t), \xi_t) \in X$ is a solution of (13) on $[0, \infty)$, such that $\xi : [-M, \infty) \to \mathbb{R}^N$ solves (2), then $V(t) = (\eta(t), \eta_t)$ is a solution of (13) on $(-\infty, 0)$, where $\eta : (-\infty, M] \to \mathbb{R}^N$ is defined by

$$\eta(t) := R\xi(-t).$$

Proof
Let us first remark that instead of viewing F as a function from $\mathbb{R}^{(2M+1)N}$ to $\mathbb{R}^N$ we can also regard F as map from $C^0 := C^0([-M,M],\mathbb{R}^N)$ to $\mathbb{R}^N$. Now note that $\eta(t)$ is well defined on $t \leqslant 0$, since $\xi(t)$ is defined on $t \geqslant 0$. By definition of $\mathcal{R}$ and reversibility we have $f(R[\mathcal{S}\phi]) = -Rf(\phi)$ for any $\phi \in C^0$. Thus,

$$\begin{aligned} \partial_t \eta(t) &= -R\partial_t \xi(-t) = -RF(\xi_{-t}(\cdot)) \\ &= -RF(R[\mathcal{S}\eta_t]) = f(\eta_t). \end{aligned}$$

This shows that $\eta(t)$ is a solution on $\mathbb{R}_-$. Moreover, we have $\partial_t \eta_t = \partial_\theta \eta_t$ on $(-\infty, 0)$ which proves the claim. □

The next lemma implies that the tangent spaces of stable and unstable manifold are related to each other via the involution $\mathcal{R}$.

Lemma 4
$\mathcal{R}\left[T_{H(0)}W_0^{s,+}(0)\right] = T_{H(0)}W_0^{u,-}(0)$.

Proof
Let us show that $\mathcal{R}[W_0^{s,+}(0)] = W_0^{u,-}(0)$ and consider a point $U = (\xi,\psi) \in W_0^{s,+}(0)$; then there exists a solution $U(t) = V(t) + H(t) = (\xi(t), \xi_t)$, where $V(t)$ solves the integral equation

$$\begin{aligned} V(t) &= \Phi_+^s(t,0)V_0^s + \int_0^t \Phi_+^s(t,s)\mathcal{G}_{mod}(s,V(s),\lambda)ds \\ &\quad + \int_\infty^t \Phi_+^{cu}(t,s)\mathcal{G}_{mod}(s,V(s),\lambda)ds \end{aligned} \tag{29}$$

for some appropriate $V_0^s \in \tilde{E}_+^s(0)$. If $V(t)$ takes values in X for all $t \geqslant 0$ then the proof is trivial. Indeed, considering $U(t) = H(t) + V(t)$ and using lemma 3 we get a (classical) solution $W(t) = H(t) + \tilde{V}(t)$ on $\mathbb{R}_-$, where $\tilde{V}(t)$ solves the integral equation corresponding to $W^{u,-}(0)$. Since the claim is therefore true on a dense set and both manifolds are C^2, the proof is completed. □

6 The main scenario

In this section we finally want to analyse the bifurcation scenario, which we explained in the introduction by means of an ordinary differential equation. We start by making some reasonable assumptions which we state in the next two sections.

6.1 Relative Positions

6.1.1 The relative position of center stable and strong unstable manifold

First of all we want to assume that center stable and strong unstable manifold intersect only along the homoclinic orbit H, which is the generic case.

Hypothesis 4
$T_{H(0)}W_0^{cs,+}(0) \cap T_{H(0)}W_0^{u,-}(0) = span\,\langle \mathcal{F}(H(0),0)\rangle$.

On account of theorem 2 the linearization of (13) along $H(t)$ for $\lambda = 0$ is of the form (20) and possesses a center-dichotomy on $\mathbb{R}_+$ with solution operators $\Phi_+^{cs}(t,s)$, $\Phi_+^u(s,t)$ for $t \geqslant s \geqslant 0$. We set

$$E_+^{cs}(t_0) := \mathrm{Rg}(\Phi_+^{cs}(t_0,t_0)\big|_Y), \qquad \tilde{E}_+^{cs}(t_0) := \mathrm{Rg}(\Phi_+^{cs}(t_0,t_0)\big|_{\tilde{X}}).$$

Moreover, the linearization of (13) along $H(t)$ for $\lambda = 0$ possesses a center-dichotomy on $\mathbb{R}_-$ with solution operators $\Phi_-^{cs}(t,s)$, $\Phi_-^u(s,t)$ for $s \leqslant t \leqslant 0$ and we set $E_-^{cs}(t_0) := \mathrm{Rg}(\Phi_-^{cs}(t_0,t_0)\big|_Y)$, $\tilde{E}_-^{cs}(t_0) := \mathrm{Rg}(\Phi_-^{cs}(t_0,t_0)\big|_{\tilde{X}})$. Similarly, the spaces $\tilde{E}_-^u(t_0)$, $E_-^u(t_0)$, $E_+^s(t_0)$, $\tilde{E}_+^s(t_0)$ are defined.

Remark
Let us point out that in the case of a center-dichotomy on $\mathbb{R}_+$ only the space $E_+^{cs}(0)$ is uniquely defined. We are therefore free to choose a closed complement E of $E_+^{cs}(0)$ in Y, which then plays the role of the space $E_+^u(0) = \mathrm{Rg}(\Phi_+^u(0,0))$; we refer to [29] for a proof of this fact. In particular, on account of hypothesis 4 we can make the choice

$$E := \mathcal{E}_-^u(0) + \mathrm{span}\,\langle \Psi_* \rangle\,,$$

where $\mathcal{E}_-^u(0) \subset E_-^u(0)$ denotes a closed complement of $\mathcal{F}(H(0),0)$ and Ψ_* is a nonzero vector which spans a complement of the sum $E_-^u(0) + E_+^{cs}(0)$. With this choice E is closed, has trivial intersection with $E_+^{cs}(0)$ and defines a complement. Hence, there exists a center dichotomy on $\mathbb{R}_+$ with solution operators Φ_+^{cs}, Φ_+^u and $\mathrm{Rg}(\Phi_+^u(0,0)) = E$, see [29]. Similar considerations apply to a center dichotomy on $\mathbb{R}_-$.

6.1.2 The relative position of center stable manifold and center unstable manifold

Let us now make some general observations concerning the intersection of $W^{cs,+}(0)$ and $W^{cu,-}(0)$. By assumption, the codimension of

$$T_{H(0)}W^{cs,+}(0) + T_{H(0)}W^{u,-}(0)$$

in $\tilde{X}$ is one. Since the codimension of $T_{H(0)}W^{u,-}(0) = \tilde{E}_-^u(0)$ in $T_{H(0)}W^{cu,-}(0) = \tilde{E}_-^{cu}(0)$ is two (see the proof of lemma 5 below), we conclude that

$$2 \leqslant \dim(T_{H(0)}W^{cu,-}(0) \cap T_{H(0)}W^{cs,+}(0)) \leqslant 3.$$

In fact, this intersection cannot be higher dimensional since otherwise the intersection of the tangent spaces of $W^{s,+}(0)$ and $W^{cu,-}(0)$ would at least contain a two-dimensional linear subspace.
Let us now focus on the more interesting case that center stable and center unstable manifold intersect non transversely along H; in particular we only address case ii) of theorem 1 (or equivalently, case b) of theorem 7). So let us assume

Hypothesis 5
The manifolds $W^{cs,+}(0)$ and $W^{cu,-}(0)$ intersect non transversely along H, that is

$$dim\left[T_{H(0)}W^{cs,+}(0)\cap T_{H(0)}W^{cu,-}(0)\right]=3.$$

We remind the reader that this case is of particular interest, since the travelling wave equations (2) of the most important lattice differential equations (like the Klein-Gordon lattice) prevent a two-dimensional intersection of $T_{H(0)}W^{cu,-}(0)$ and $T_{H(0)}W^{cs,+}(0)$, see the discussion in section 7.1.

6.1.3 The relative position of center stable manifold and Fix($\mathcal{R}$)

Since $\mathcal{R}W^{cs,+}(0) = W^{cu,-}(0)$ we conclude that $T_{H(0)}W^{cs,+}(0)\cap \text{Fix}(\mathcal{R}) \subset T_{H(0)}W^{cs,+}(0)\cap T_{H(0)}W^{cu,-}(0)$. Taking into account that $\mathcal{F}(H(0))$ is contained in $T_{H(0)}W^{cs,+}(0)\cap T_{H(0)}W^{cu,-}(0)\cap \text{Fix}(-\mathcal{R})$ we get

$$\dim(T_{H(0)}W^{cs,+}(0)\cap \text{Fix}(\mathcal{R}))+1 \leqslant \dim(T_{H(0)}W^{cs,+}(0)\cap T_{H(0)}W^{cu,-}(0)) \leqslant 3.$$

We make the following assumption.

Hypothesis 6
$T_{H(0)}W^{cs,+}(0)\cap Fix(\mathcal{R}) = 1$.

This is the generic case in the framework of ordinary differential equations. Indeed, if the space is $(2N+2)$-dimensional we conclude $\dim((\text{Fix})\mathcal{R}) = N+1$ and $\dim(T_{H(0)}W^{cs,+}(0)) = N+2$, which generically leads to a one-dimensional intersection of these two spaces.

6.2 Counting codimensions

Looking for homoclinic solutions to the steady state, we have to analyse how stable and unstable manifold split when varying the parameter λ. It is therefore important to determine the co-dimension of the sum of the corresponding tangent spaces of stable and unstable manifold in $\tilde{X}$, since we want to measure the distance of points in these manifolds within this complement.
It is the aim of this chapter to determine this codimension, which we expect to be three by comparing the situation to an ordinary differential equation. However, we work in the Banach space $\tilde{X}$ where we cannot count dimensions as in the ODE-case. Let us now show that there exists a three-dimensional subspace $\tilde{K}$ of $\tilde{X}$ which satisfies

$$\tilde{K}\oplus\left[\tilde{E}^s_+(0)+\tilde{E}^u_-(0)\right]=\tilde{X}. \tag{30}$$

Note that the dimension of $\tilde{K}$ in a finite dimensional setting (in the case of an ODE) is indeed three: We assumed that the linearization of $\mathcal{F}$ in 0 possesses exactly two eigenvalues on the imaginary axis; hence, stable and unstable eigenspaces possess a two-dimensional complement. Now note that along the homoclinic orbit $H(t)$ stable and unstable manifold intersect which increases the dimension of the complement by one, hence $\dim(\tilde{K}) = 3$.

Lemma 5
The space

$$\tilde{\Xi} := Rg(\Phi_+^s(0,0)\big|_{\tilde{X}}) + Rg(\Phi_-^u(0,0)\big|_{\tilde{X}}) = \tilde{E}_+^s(0) + \tilde{E}_-^u(0)$$

has codimension 3 *in the space* $\tilde{X}$.

Proof
In order to prove the claim we will construct a particular complement $\tilde{K}$ of $\tilde{\Xi}$ in $\tilde{X}$ and determine its codimension. Let us remind that for every $U \in E_+^{cs}(0) \subset Y$ there exists a continuous function $V(t) = \Phi_+^{cs}(t,0)U = (\eta(t), \eta_t) : [0,\infty) \to Y$, such that

$$\|V(t)\|_Y \leqslant Me^{\delta|t|}\|U\|_Y, \qquad t \geqslant 0, \tag{31}$$

where $\delta > 0$ can be chosen small enough and $\eta(t)$ solves the equation $\dot{\eta}(t) = D_1 f(h_t, 0)\eta_t$ on $\mathbb{R}_+$. Moreover, the space $E_+^{cs}(0)$ is maximal with this property; that is, every solution $\eta(t)$ on $\mathbb{R}_+$ which satisfies (31) also satisfies $(\eta(0), \eta_0) \in E_+^{cs}(0)$, see [7, 27]. Instead of constructing a center-dichotomy on $\mathbb{R}_+$ with solution operators Φ_+^{cs}, Φ_+^u we can alternatively consider the solution operators Φ_+^s, Φ_+^{cu} as explained after theorem 2, where $\|\Phi_+^s(t,s)\|_{L(Y,Y)} \leqslant Me^{-\alpha|t-s|}$ for some $\alpha > 0$ and $t \geqslant s \geqslant 0$. Of course,

$$E_+^s(0) = \mathrm{Rg}(\Phi_+^s(0,0)\big|_Y) \subset E_+^{cs}(0) \tag{32}$$

by maximality of $E_+^{cs}(0)$. We now choose a complement $Y^c \subset Y$ of $E_+^s(0)$ in $E_+^{cs}(0)$. Let us show that Y^c is 2-dimensional, which would then imply that the codimension of $E_+^s(0) + E_-^u(0)$ in Y is three, since $E_+^{cs}(0) + E_-^u(0)$ has a one-dimensional complement in Y by hypothesis 4. First of all note that $\Phi_+^{cs}(t_0, t_0)$ approaches π_{cs} with respect to the $L(Y,Y)$ norm for $t_0 \to \infty$ as we have shown in lemma 2. Similarly, $\Phi^s(t_0, t_0)$ approaches π_s with respect to the $L(Y,Y)$-norm for $t_0 \to \infty$. This shows that $E_+^s(t_0)$ possesses a two-dimensional complement in $E_+^{cs}(t_0)$; at least if $t_0 >> 0$ is large enough: Indeed, note that $\mathrm{Rg}(\pi_s)$ has codimension two in $\mathrm{Rg}(\pi_{cs})$ by hypothesis 3. We claim that this implies that Y^c is two-dimensional. In order to see this, we consider the translated homoclinic orbit $H^{t_0}(t) := H(t + t_0)$. The linearization along H^{t_0} possesses a center dichotomy on $\mathbb{R}_\pm$ with solution operators $\Phi_\pm^{cs,t_0}, \Phi_\pm^{u,t_0}$ and $\Phi_\pm^{s,t_0}, \Phi_\pm^{cu,t_0}$. If we now choose $V \in \mathrm{Rg}(\Phi_+^{cs,t_0}(0,0)) \cap X$ there is a classical solution $V(t) = (\xi(t), \phi(t,\cdot))$ of the equation

$$\dot{V}(t) = \mathcal{A}_{t_0}(t)V(t) := \begin{pmatrix} D_1 f(h_{t+t_0}, 0)\phi(t,\cdot) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix},$$

for $t \geqslant 0$ and $V(0) = V$. By defining $W(t) = V(t - t_0)$ for $t \geqslant t_0$, we observe that $V = W(t_0) \in E_+^{cs}(t_0)$, which shows that $\mathrm{Rg}(\Phi_+^{cs,t_0}(0,0)) \subset E_+^{cs}(t_0)$. Similar, one can now show that the other direction holds, that is $\mathrm{Rg}(\Phi_+^{cs,t_0}(0,0)) \supset E_+^{cs}(t_0)$ and therefore we conclude that the two spaces coincide. We infer from this that the codimension of $\mathrm{Rg}(\Phi_+^{s,t_0}(0,0))$ in $\mathrm{Rg}(\Phi_+^{cs,t_0}(0,0))$ with respect to Y is two, if $t_0 > 0$ is sufficiently large. Moreover, the complement of $\mathrm{Rg}(\Phi_+^{cs,t_0}(0,0)) + \mathrm{Rg}(\Phi_-^{u,t_0}(0,0))$ is one-dimensional by hypothesis 4. Hence, the codimension of $\mathrm{Rg}(\Phi_+^{s,t_0}(0,0)) + \mathrm{Rg}(\Phi_-^{u,t_0}(0,0))$ in Y is three. Now consider the operator

$$\begin{aligned} \iota_{t_0} &: Y \times Y \to Y \\ \iota_{t_0} &: (U,V) \mapsto \Phi_+^{s,t_0}(0,0)U - \Phi_-^{u,t_0}(0,0)V. \end{aligned}$$

Note that $\mathrm{Rg}(\iota_{t_0}) = \mathrm{Rg}(\Phi_+^{s,t_0}(0,0)) + \mathrm{Rg}(\Phi_-^{u,t_0}(0,0))$. A short computation of the adjoint operator $\iota_{t_0}^*$ reveals that any element in the kernel of $\iota_{t_0}^*$ (and hence in a complement of $\mathrm{Rg}(\iota_{t_0})$) defines a solution of the adjoint equation

$$\dot{V}(t) = -(\mathcal{A}_{t_0}(t))^* V(t) \tag{33}$$

where $U(t) = (\xi(t), \phi(t,\cdot))$ and the adjoint operator has to be understood with respect to the Y-scalar product. This shows that the adjoint equation (33) possesses three linear independent solutions on $\mathbb{R}$, which may grow asymptotically with algebraic rate (since all purely imaginary eigenvalues of the linearization at zero are simple, the solutions are actually bounded on $\mathbb{R}$ as has been shown in [6]. However, we won't make use of this fact). Therefore, equation (33) also possesses three linear independent solutions on $\mathbb{R}$ for $t_0 = 0$ by translation. Now any solution $W(t)$ of (33) for $t_0 = 0$ admits the representation

$$W(t) = \begin{cases} (\Phi_+^{cu}(0,t))^* U_+ & : \quad t \geqslant 0 \\ (\Phi_-^{cs}(0,t))^* U_+ & : \quad t \leqslant 0 \end{cases}$$

for some suitable U_+ in $\mathrm{Rg}(\Phi_-^{cs}(0,0)^*) \cap \mathrm{Rg}(\Phi_+^{cu}(0,0)^*)$, where the latter space defines a complement to $E_+^s(0) + E_-^u(0)$ in Y. This observation together with the uniqueness hypothesis 2 shows that $E_+^s(0) + E_-^u(0)$ has codimension three in Y. We can therefore choose a three-dimensional orthogonal complement of $E_+^s(0) + E_-^u(0)$ in Y, which is spanned by the vectors e_1, e_2, e_3, say. Since X is dense in Y, we can find vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \in X \subset \tilde{X}$ sufficiently close to e_1, e_2, e_3, such that $\mathrm{span}\langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$ is still a complement of $E_+^s(0) + E_-^u(0)$ in Y. But since $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ are now elements of $\tilde{X}$ this shows that $\tilde{E}_+^s(0) + \tilde{E}_-^u(0)$ has codimension three in $\tilde{X}$. $\square$

Remark

We have factored out a 2-dimensional subspace of $E_+^{cs}(0)$ such that $\Phi_+^{cs}(t,0)$ decays with exponential rate on the complement $E_+^s(0)$ with respect to t. This shows that we in fact have obtained a *trichotomy* with solution operators $\Phi_+^s, \Phi_+^u, \Phi_+^c$. Here, Φ_+^s, Φ_+^u decay exponentially in forward and backward time, respectively, and Φ_+^c "captures" the center part, that is, Φ_+^c is defined on a two-dimensional space and may grow with algebraic rate.

Let us now collect more informations regarding a suitable complement $\tilde{K}$ of $\tilde{\Xi}$. In particular, we want to assure that we can choose an $\mathcal{R}$-invariant subspace $\tilde{K}$. The next lemma is a step toward this direction.

Lemma 6
There exists a two-dimensional subspace $\tilde{Y}^c \subset (T_{H(0)}W_0^{cs,+}(0) \cap T_{H(0)}W_0^{cu,-}(0))$, *which has the property* $\tilde{Y}^c \cap \mathit{span}\,\langle \mathcal{F}(H(0),0)\rangle = \{0\}$ *and*

$$\mathit{dim}(\tilde{Y}^c \cap \mathit{Fix}(\mathcal{R})) = 1, \qquad \mathit{dim}(\tilde{Y}^c \cap \mathit{Fix}(-\mathcal{R})) = 1.$$

Proof
Observe that the intersection of the tangent spaces is $\mathcal{R}$-invariant: Any vector $V \in T_{H(0)}W_0^{cs,+}(0)$ is mapped to $T_{H(0)}W_0^{cu,-}(0)$ and vice versa. On the other hand, the intersection can only contain a one-dimensional subspace of $\mathrm{Fix}(\mathcal{R})$ by hypothesis 6. Taking into account hypothesis 5 we can therefore find a two-dimensional subspace $\tilde{Y}^c$ with the desired properties. □

We now make the generic assumption that $\mathrm{Fix}(\mathcal{R})$ is transverse to $T_{H(0)}W_0^{cs,+}(0)+T_{H(0)}W_0^{cu,-}(0)$:

Hypothesis 7
There exists a vector $b_0 \in \mathit{Fix}(\mathcal{R})$, *such that*

$$\mathit{span}\,\langle b_0\rangle + T_{H(0)}W_0^{cs,+}(0) + T_{H(0)}W_0^{cu,-}(0) = \tilde{X}.$$

Let C denote a one-dimensional, $\mathcal{R}$-invariant complement of $\tilde{E}_+^{cs}(0) + \tilde{E}_-^{u}(0)$, such that $C \subset \mathrm{Fix}(\mathcal{R})$. Note that such a complement exists on account of hypothesis 7. We can therefore define a complement $\tilde{K}$ of $\tilde{\Xi}$ in the form $\tilde{K} = \tilde{Y}^c \oplus C$; $\tilde{K}$ is then $\mathcal{R}$-invariant and

$$\dim(\tilde{K} \cap \mathrm{Fix}(\mathcal{R})) = 2, \qquad \dim(\tilde{K} \cap \mathrm{Fix}(-\mathcal{R})) = 1.$$

In particular, there exist basis vectors b_0, b_1 in $\mathrm{Fix}(\mathcal{R})$ and b_2 in $\mathrm{Fix}(-\mathcal{R})$ which span the complement $\tilde{K}$ of $\tilde{\Xi}$. This is a generic assumption, since we expect that neither $\mathrm{Fix}(\mathcal{R})$ nor $\mathrm{Fix}(-\mathcal{R})$ is completely contained in $\tilde{\Xi}$.
But why can't we simply choose all vectors b_0, b_1, b_2 in $\mathrm{Fix}(\mathcal{R})$ then with the same reasoning? This is not possible if we restrict our attention to the ODE case, however. Indeed, let us consider the case of an ODE in $\mathbb{R}^{2N+2}$; then $\dim\mathrm{Fix}(\mathcal{R}) = \dim(\mathrm{Fix})(-\mathcal{R}) = N+1$ and moreover $\dim T_{H(0)}W^{s,+}(0) = \dim T_{H(0)}W^{u,-}(0) = N$. Taking into account that $\tilde{\Xi}$ is $\mathcal{R}$-invariant a straightforward computation leads to the fact that only a N-dimensional subspace of $\mathrm{Fix}(-\mathcal{R})$ and a $(N-1)$-dimensional subspace of $\mathrm{Fix}(\mathcal{R})$ can be contained in $\tilde{\Xi}$. Hence, at least one basis vector b_i has to be chosen in $\mathrm{Fix}(-\mathcal{R})$, if we want to obtain an $\mathcal{R}$-invariant complement.

6.3 One-homoclinic orbits to the steady state

In this section we want to obtain the existence of (symmetric) homoclinic solutions to the steady state (which are contained in the intersection of strong stable and strong unstable manifold). Later on, we are also interested in the existence of homoclinic orbits to the center manifold which approach a periodic orbit in forward and backward time. These solutions lie in the intersection of center stable and center unstable manifold. However, in order to be able to distinguish these solutions from homoclinic orbits to the steady state we will start by studying the intersection of the strong stable manifold $W^{s,+}(0)$ and the strong unstable manifold $W^{u,-}(0)$ of zero. We begin by defining a Poincaré section at $H(0)$ which will be helpful in the subsequent analysis. The following lemma will be needed for that purpose.

Lemma 7
There exist complements $\tilde{\mathcal{E}}^s_+(0) \subset \tilde{E}^s_+(0)$ and $\tilde{\mathcal{E}}^u_-(0) \subset \tilde{E}^u_-(0)$ of span$\langle \mathcal{F}(H(0)) \rangle$ which are closed with respect to the $\tilde{X}$-norm. Furthermore, the space $\tilde{\mathcal{E}}^s_+(0) + \tilde{\mathcal{E}}^u_+(0)$ is closed in $\tilde{X}$.

Proof
Let us first prove the claim concerning the existence of $\tilde{\mathcal{E}}^s_+(0)$. We start by defining a complement $\mathcal{C}$ of span$\langle \mathcal{F}(H(0)) \rangle$ in X and set $\mathcal{C} := \mathrm{Rg}(\Phi^s_+(0,0)\big|_{\hat{E}})$, where $\hat{E}$ denotes a closed subspace of the Hilbert space X of codimension one, such that

$$\mathrm{Rg}(\Phi^s_+(0,0)\big|_{\hat{E}}) \oplus \mathrm{span}\,\langle \mathcal{F}(H(0)) \rangle = E^s_+(0).$$

The closure $\mathcal{E}^s_+(0)$ of the space $\mathrm{Rg}(\Phi^s_+(0,0)\big|_{\hat{E}}) \subset Y$ in the Hilbert space Y provides a closed complement of span$\langle \mathcal{F}(H(0)) \rangle$ in $E^s_+(0)$. Indeed, the intersection of the closure of this space with span$\langle \mathcal{F}(H(0)) \rangle$ remains trivial; otherwise $\mathcal{E}^s_+(0)$ would coincide with the whole space $E^s_+(0)$. This is impossible, however, since any vector in the orthogonal complement of $\mathrm{Rg}(\Phi^s_+(0,0)\big|_{\hat{E}})$ with respect to the Y-norm cannot be contained in the closure. This argument shows that

$$\mathcal{C} \subset \mathcal{E}^s_+(0) \tag{34}$$

and the right hand side of (34) defines a complement of span$\langle \mathcal{F}(H(0)) \rangle$ in $E^s_+(0)$ (in particular, with trivial intersection).
Let us now consider the closure of $\mathcal{C}$ in $\tilde{X}$ and denote this closure by $\tilde{\mathcal{E}}^s_+(0)$, that is $\tilde{\mathcal{E}}^s_+(0) := \bar{\mathcal{C}}$ where the closure is taken with respect to the $\tilde{X}$-norm. This defines a closed complement of span$\langle \mathcal{F}(H(0)) \rangle$ in $\tilde{E}^s_+(0)$: Note first that $\tilde{\mathcal{E}}^s_+(0)$ has only trivial intersection with span$\langle \mathcal{F}(H(0)) \rangle$ on account of (34) (indeed, the closure of $\mathcal{C}$ with respect to the $\tilde{X}$-norm is still contained in the right hand side of (34)). By construction of $\tilde{\mathcal{E}}^s_+(0)$ the space span$\langle \mathcal{F}(H(0)) \rangle + \tilde{\mathcal{E}}^s_+(0)$ is dense in $\tilde{E}^s_+(0)$. Analogously to the proof of lemma 6.2 in [8] one can now show that $\tilde{\mathcal{E}}^s_+(0) + \mathrm{span}\,\langle \mathcal{F}(H(0)) \rangle$ is closed in $\tilde{X}$, which proves that the sum coincides with $\tilde{E}^s_+(0)$. The space $\tilde{\mathcal{E}}^u_-(0)$ can now be defined via

$$\tilde{\mathcal{E}}^u_-(0) := \mathcal{R}[\tilde{\mathcal{E}}^s_+(0)].$$

Then $\tilde{\mathcal{E}}^u_-(0) + \tilde{\mathcal{E}}^s_+(0)$ is $\mathcal{R}$-invariant and $\tilde{\mathcal{E}}^u_-(0) \subset \tilde{E}^u_-(0)$ is a closed complement of span $\langle \mathcal{F}(H(0), 0) \rangle$. The proof that $\tilde{\mathcal{E}}^u_-(0) + \tilde{\mathcal{E}}^s_+(0)$ is closed follows once more from the proof of lemma 6.2 in [8]. □

Remark
The last proof uses the fact that we have shown the existence of the solution

operator Φ_+^s in the Hilbert space Y: We defined the desired complements as closures of appropriate X-subspaces in the $\tilde{X}$-norm. The relation $X \subset \tilde{X} \subset Y$ allowed us to control these closures, making use of the fact that X, Y are Hilbert spaces. This procedure reminds at the sun-star theory for pure delay differential equations [18], where one always uses advantages of the "bigger" and "smaller" space (here Y and $\tilde{X}$) to end up with a powerful machinery.

Using this last lemma we can therefore define by

$$\tilde{\Sigma} := \tilde{\mathcal{E}}_+^s(0) \oplus \tilde{\mathcal{E}}_-^u(0) \oplus \tilde{K} \subset \tilde{X}$$

a closed, $\mathcal{R}$-invariant Poincaré section to the homoclinic orbit at $H(0)$. The proof that $\tilde{\Sigma}$ is closed follows from the proof of lemma 6.2 in [8].

Definition 3
A tuple (γ^+, γ^-) is called Lin solution, if the following conditions are satisfied.

a) $\gamma^{+/-}(\cdot)$ are solutions of (13) on $\mathbb{R}_+$, $\mathbb{R}_-$, respectively, whose orbits are close to the orbit of the homoclinic solution H.

b) $\gamma^+(0), \gamma^-(0) \in \tilde{\Sigma} + H(0)$

c) $\gamma^+(0) - \gamma^-(0) \in \tilde{K}$

d) $\gamma^+(0) \in W^{s,+}(0)$ and $\gamma^-(0) \in W^{u,-}(0)$.

Choices
In the next section we want to construct Lin-solutions and derive a bifurcation function. It will be very helpful during the analysis to make a special choice for the subspaces $E_+^{cu}(0), E_-^{cs}(0) \subset Y$ with respect to a center dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Note that we already pointed out in the remark after hypothesis 4 that these subspaces can be chosen; at least as long as they provide closed complements of $E_+^s(0)$ and $E_-^u(0)$, respectively. We now make the specific choice

$$E_-^{cs}(0) := \mathcal{E}_+^s(0) + \tilde{K}, \qquad E_+^{cu}(0) := \mathcal{E}_-^u(0) + \tilde{K}, \tag{35}$$

where $\mathcal{E}_+^s(0) \subset Y$ denotes a closed complement of $\mathcal{F}(H(0), 0)$ in $E_+^s(0)$, which satisfies $\tilde{\mathcal{E}}_+^s(0) \subset \mathcal{E}_+^s(0)$, see (34). Similarly, $\mathcal{E}_-^u(0) \subset Y$ denotes a closed complement of $\mathcal{F}(H(0), 0)$ in $E_-^u(0)$, which satisfies the property $\tilde{\mathcal{E}}_-^u(0) \subset \mathcal{E}_-^u(0)$.

6.3.1 Construction of Lin-solutions

Let us now construct Lin solutions $(\gamma^+(0), \gamma^-(0))$. Since $\gamma^+(0) \in W^{s,+}(0)$, we can look for $\gamma^+(\cdot)$ in the form $\gamma^+(t) = H(t) + v^s(t)$, where $v^s(0) = v^s(0, V^s, \lambda)$ is defined for any $V^s \in \tilde{\mathcal{E}}_+^s(0)$ by

$$\begin{aligned} v^s(0, V^s, \lambda) \;\; &: \;\; \left(\tilde{\mathcal{E}}_+^s(0) \times \mathbb{R}\right) \cap B_\varepsilon(0) \to \tilde{\Sigma} \\ v^s(0, \cdot, \cdot) \;\; &: \;\; (V^s, \lambda) \mapsto V^s + \int_\infty^0 \Phi_+^{cu}(0, s) \mathcal{G}_{mod}(s, V_*^s(s), \lambda) ds, \end{aligned}$$

where $V^s_*(\cdot) = V^*(\cdot, V^s, \lambda) \in BC^{-\gamma}(\mathbb{R}_+, \tilde{X})$ is the unique solution of the fixed point equation with respect to $W^{s,+}(0)$ corresponding to $V^s \in \tilde{E}^s_+(0)$. $\mathcal{G}$ has been defined in (27). Similarly, for any $V^u \in \tilde{\mathcal{E}}^u_-(0)$ we define

$$\begin{aligned} v^u(0, V^u, \lambda) &: \left(\tilde{\mathcal{E}}^u_-(0) \times \mathbb{R}\right) \cap B_\varepsilon(0) \to \tilde{\Sigma} \\ v^u(0, \cdot, \cdot) &: (V^u, \lambda) \mapsto V^u + \int_{-\infty}^0 \Phi^{cs}_+(0, s)\mathcal{G}_{mod}(s, V^u_*(s), \lambda)ds, \end{aligned}$$

where $V^u_*(\cdot) = V^u_*(\cdot, V^u, \lambda) \in BC^{-\gamma}(\mathbb{R}_-, \tilde{X})$ is the unique solution of the fixed point equation with respect to $W^{u,-}(0)$ corresponding to $V^u \in \tilde{E}^u_-(0)$. In particular, we can write

$$\begin{aligned} v^s(0, V^s, \lambda) &= V^s + y^u(V^s, \lambda) + k^s(V^s, \lambda) \in \tilde{\mathcal{E}}^s_+(0) \oplus \tilde{\mathcal{E}}^u_-(0) \oplus \tilde{K}, \\ v^u(0, V^u, \lambda) &= V^u + y^s(V^u, \lambda) + k^u(V^u, \lambda) \in \tilde{\mathcal{E}}^u_-(0) \oplus \tilde{\mathcal{E}}^s_+(0) \oplus \tilde{K} \quad (36) \end{aligned}$$

due to our special choice for the spaces $E^{cu}_+(0), E^{cs}_-(0)$ in (35). On account of b) in definition 3 we want to have

$$V^s = y^s(V^u, \lambda), \qquad V^u = y^u(V^s, \lambda) \tag{37}$$

for all $\lambda \sim 0$. (37) can be solved by the implicit function theorem: Consider

$$\begin{pmatrix} V^s - y^s(V^u, \lambda) \\ V^u - y^u(V^s, \lambda) \end{pmatrix} = 0 \tag{38}$$

for $(V^s, V^u, \lambda) \in \left(\tilde{\mathcal{E}}^s_+(0) \times \tilde{\mathcal{E}}^u_-(0) \times \mathbb{R}\right) \cap B_\varepsilon(0)$. Differentiating the left hand side with respect to (V^s, V^u) at $(V^s, V^u) = 0$ at $\lambda = 0$ induces an invertible map due to $\partial_{V^u} y^s(0, 0) = 0$ and $\partial_{V^s} y^u(0, 0) = 0$. Hence, (38) holds for a curve $(V^s, V^u) = (V^s(\lambda), V^u(\lambda))$, $\lambda \sim 0$. Let us summarize our results in the next lemma.

Lemma 8
For each sufficiently small $|\lambda|$ there is a unique Lin orbit (γ^+, γ^-) tending to the equilibrium zero.

Proof
We define

$$\begin{aligned} \gamma^+(t) = \gamma^{+,\lambda}(t) &:= H(t) + v^s(V^s(\lambda), \lambda)(t) \\ \gamma^-(t) = \gamma^{-,\lambda}(t) &:= H(t) + v^u(V^u(\lambda), \lambda)(t) \end{aligned}$$

and have already verified that (γ^+, γ^-) satisfies all assumptions of definition 3. □

6.3.2 Existence of symmetric homoclinic solutions

Let us now define the bifurcation equation by

$$\kappa(\lambda) := \gamma^{+,\lambda}(0) - \gamma^{-,\lambda}(0),$$

whose zeros correspond to homoclinic orbits to the steady state zero. $\kappa(\lambda)$ can be considered as a map from $\mathbb{R} \cap B_\varepsilon(0)$ to $\tilde{K}$. Let us note that the solutions $v^{s/u}(V,\lambda)(\cdot)$ of the corresponding fixed point equations associated to $W^{s,+}(0)$ and $W^{u,-}(0)$ actually satisfy

$$\mathcal{R}v^s(V^s,\lambda)(t) = v^u(\mathcal{R}V^s,\lambda)(-t), \qquad \mathcal{R}v^u(V^u,\lambda)(-t) = v^s(\mathcal{R}V^u,\lambda)(t). \quad (39)$$

Indeed, $v^s(V^s,\lambda)(\cdot)$ admits the form $v^s(V^s,\lambda)(t)+H(t) = U(t)$, where $U(t)$ is a solution of (13) for $t \geqslant 0$. Hence $\mathcal{R}H(t) = H(-t)$ and $\mathcal{R}U(t) = U(-t,\mathcal{R}U(0))$, where $U(t,\mathcal{R}U(0))$ denotes the solution of (13) on $\mathbb{R}_-$ subject to the initial value $\mathcal{R}U(0)$ (which in particular exists!). Moreover, the following lemma holds.

Lemma 9
$\mathcal{R}\kappa(\lambda) = -\kappa(\lambda)$. *Hence,* $\kappa(\lambda) \in$ *Fix*$(-\mathcal{R})$.

Proof
Let us first note that on account of (39) and the representation (36) we conclude

$$\mathcal{R}y^s(V,\lambda) = y^u(\mathcal{R}V,\lambda)$$
$$\mathcal{R}k^s(V,\lambda) = k^u(\mathcal{R}V,\lambda).$$

Hence, also the implicitly defined maps $V^{s/u}(\lambda)$ satisfy $\mathcal{R}V^s(\lambda) = V^u(\lambda)$. By definition of $\kappa(\lambda)$ this proves the claim. □

On account of our chosen complement $\tilde{K}$ of $\tilde{\Xi}$, we can regard κ as a map with values in $\mathbb{R}$ by introducing a basis in the one-dimensional space $\tilde{K} \cap \mathrm{Fix}(-\mathcal{R})$. That is,

$$\kappa(\lambda) : \mathbb{R} \cap B_\varepsilon(0) \to \mathbb{R}.$$

The next assumption assures that when varying λ_1 of $\lambda = (\lambda_1,\lambda_2)$ the manifolds $W^{s,+}_\lambda(0)$ and $W^{u,-}_\lambda(0)$ split with non vanishing speed in the direction of the one-dimensional space $(\mathrm{Fix}(-\mathcal{R}) \cap \tilde{K})$.

Hypothesis 8
$\partial_{\lambda_1}\kappa'(\lambda_1,\lambda_2)\big|_{\lambda_1=0} \neq 0.$

We summarize our results in the following theorem.

Theorem 6
Assume that the hypotheses Hyp 1 *- Hyp* 8 *are valid. Then the existence of homoclinic orbits is a codimension-one-phenomenon. Hence, there exists a locally defined* C^2*-curve* $Hom = Hom(\lambda_2)$ *in the parameter space near* $\lambda = 0$ *with* $0 \in Hom$*, such that for exactly all parameter points on the curve* Hom *equation (2) possesses a (symmetric) homoclinic orbit* h^λ*. In particular,* h^λ *satisfies* $h^\lambda(t) = Rh^\lambda(-t)$*. Moreover, this homoclinic orbit converges to zero in forward and backward time with exponential rate.*

We can consider new parameters, such that $Hom = \{(0,\lambda_2) : |\lambda_2| \leqslant \varepsilon\}$. Indeed, let us define new parameters $(\tilde{\lambda},\lambda_2)$ via $(\lambda_1,\lambda_2) = (Hom(\lambda_2)+\tilde{\lambda},\lambda_2)$. We work with these parameters from now on but refrain from introducing new notation.

6.4 Homoclinic solutions to the center manifold

In this section we finally want to detect *all* homoclinic orbits to the center manifold. These solutions are induced by intersections of the center stable and center unstable manifold.
We now want to construct Lin-solutions (γ_c^+, γ_c^-) to the center manifold, which should have the following properties

a) $\gamma_c^{+/-}(\cdot)$ are solutions of (13) on $\mathbb{R}_+$, $\mathbb{R}_-$, respectively, whose orbits are close to the orbit of the homoclinic solution H.

b) $\gamma_c^+(0), \gamma_c^-(0) \in \tilde{\Sigma} + H(0)$

c) $\gamma_c^+(0) - \gamma_c^-(0) \in \tilde{K}$

d) $\gamma_c^+(0) \in W^{cs,+}(0)$ and $\gamma_c^-(0) \in W^{cu,-}(0)$.

Note that solutions starting near $H(0)$ in the center stable manifold $W_\lambda^{cs,+}(0)$, $\lambda \sim 0$, actually stay close to H for all $t \geqslant 0$. Indeed, all solutions on the center manifold are periodic solutions. Solutions in $W^{cs,+}(0)$ starting near $H(0)$ approach a solution on the center manifold, hence a periodic orbit, in forward time (we refer to [6] for a rigorous proof).
Let us now look for solutions

$$\gamma_c^+(t) = H(t) + v^{cs,+}(t)$$

in the center stable manifold $W^{cs,+}(0)$, which depend on λ and $V_0^{cs} \in \tilde{\mathcal{E}}_+^{cs}(0)$, where $\tilde{\mathcal{E}}_+^{cs}(0) \subset \tilde{X}$ denotes the closed subspace $\tilde{\mathcal{E}}_+^{cs}(0) := \tilde{\mathcal{E}}_+^{s}(0) + \tilde{Y}^c$. Then

$$\begin{aligned} v^{cs,+}(0,\cdot,\cdot) \ &: \ \left(\tilde{\mathcal{E}}_+^{cs}(0) \times \mathbb{R}\right) \cap B_\varepsilon(0) \to \tilde{\Sigma} \\ v^{cs,+}(0,\cdot,\cdot) \ &: \ (V^{cs}, \lambda) \mapsto V^{cs} + \int_\infty^0 \Phi_+^u(0,s)\mathcal{G}_{mod}(s, V_*^{cs}(s), \lambda)ds, \end{aligned}$$

where $V_*^{cs} = V_*^{cs}(\cdot, V^{cs}, \lambda) \in BC^{2\delta}([0,\infty), \tilde{X})$ denotes the unique fixed point of the integral equation corresponding to $W^{cs,+}(0)$. Similarly, setting $\tilde{\mathcal{E}}_-^{cu}(0) := \tilde{\mathcal{E}}_-^{u}(0) + \tilde{Y}^c$

$$\begin{aligned} v^{cu,-}(0,\cdot,\cdot) \ &: \ \left(\tilde{\mathcal{E}}_-^{cu}(0) \times \mathbb{R}\right) \cap B_\varepsilon(0) \to \tilde{\Sigma} \\ v^{cu,-}(0,\cdot,\cdot) \ &: \ (V^{cu}, \lambda) \mapsto V^{cu} + \int_{-\infty}^0 \Phi_+^s(0,s)\mathcal{G}_{mod}(s, V_*^{cu}(s), \lambda)ds, \end{aligned}$$

where $V_*^{cu} = V_*^{cu}(\cdot, V^{cu}, \lambda) \in BC^{2\delta}((-\infty,0], \tilde{X})$ denotes the unique fixed point of the integral equation corresponding to $W^{cu,-}(0)$. We can now write these values in the form

$$\begin{aligned} v^{cu,-}(0, V^{cu}, \lambda) &= V^u + V^c + z^s(V^{cu}, \lambda) + c^-(V^{cu}, \lambda) \\ v^{cs,+}(0, \tilde{V}^{cs}, \lambda) &= \tilde{V}^s + \tilde{V}^c + z^u(\tilde{V}^{cs}, \lambda) + c^+(\tilde{V}^{cs}, \lambda), \end{aligned}$$

where

$$\begin{aligned}(V^u, V^c, z^s, c^-) &\in \tilde{\mathcal{E}}^u_+(0) \oplus \tilde{Y}^c \oplus \tilde{\mathcal{E}}^s_+(0) \oplus C,\\ (\tilde{V}^s, \tilde{V}^c, z^u, c^+) &\in \tilde{\mathcal{E}}^s_+(0) \oplus \tilde{Y}^c \oplus \tilde{\mathcal{E}}^u_+(0) \oplus C.\end{aligned}$$

In order to guarantee that property b) in the definition of a Lin-solution is satisfied, we have to solve

$$z^u(\tilde{V}^s + \tilde{V}^c, \lambda) = V^u, \qquad z^s(V^u + V^c, \lambda) = \tilde{V}^s, \tag{40}$$

which can be done using the implicit function theorem. Hence, we conclude the existence of a C^1-function $(V^c, \tilde{V}^c, \lambda) \mapsto (V^s(V^c, \tilde{V}^c, \lambda), V^u(V^c, \tilde{V}^c, \lambda))$ and we define a bifurcation map via

$$\begin{aligned}\hat{\xi}^\infty &: \left(\tilde{Y}^c \times \tilde{Y}^c \times \mathbb{R}\right) \cap B_\varepsilon(0) \to \tilde{Y}^c \times C,\\ \hat{\xi}^\infty &: (V^c, \tilde{V}^c, \lambda) \mapsto (V^c - \tilde{V}^c) + c^-(V^u(V^c, \tilde{V}^c, \lambda), V^c, \lambda)\\ &\qquad -c^+(V^s(V^c, \tilde{V}^c, \lambda), V^c, \lambda).\end{aligned}$$

Since $c^- - c^+ \in C$, we have to choose $\tilde{V}^c = V^c$ in order to guarantee that at least the Y^c-part of $\hat{\xi}^\infty$ vanishes. Finally, we arrive at the reduced bifurcation map

$$\xi^\infty \ : \ (V^c, \lambda) \mapsto c^-(V^u(V^c, \lambda), V^c, \lambda) - c^+(V^s(V^c, \lambda), V^c, \lambda).$$

Note that ξ^∞ takes values in the one-dimensional subspace $C \subset \mathrm{Fix}(\mathcal{R})$. Hence, we may consider ξ^∞ as a map which takes values in $\mathbb{R}$ by choosing a basis in C. If we write $V^c = (V^c_+, V^c_-) \in Y^c$ with $V^c_+ \in \mathrm{Fix}(\mathcal{R})$ and $V^c_- \in \mathrm{Fix}(-\mathcal{R})$ we have the following lemma.

Lemma 10
$\xi^\infty(V^c_+, V^c_-, \lambda) = -\xi^\infty(V^c_+, -V^c_-, \lambda)$.

Proof
Note that on account of $\mathcal{R}z^u(\tilde{V}^s + V^c, \lambda) = z^s(\mathcal{R}\tilde{V}^s + \mathcal{R}V^c, \lambda)$ in equation (40) we also have that $\mathcal{R}V^u(V^c, \lambda) = V^s(\mathcal{R}V^c, \lambda)$. Hence,

$$\begin{aligned}&\mathcal{R}\left[c^-(V^u(V^c_+, V^c_-, \lambda), V^c_+, V^c_-, \lambda) - c^+(V^s(V^c_+, V^c_-, \lambda), V^c_+, V^c_-, \lambda)\right]\\ &= c^+(V^u(V^c_+, -V^c_-, \lambda), V^c_+, -V^c_-, \lambda) - c^-(V^s(V^c_+, -V^c_-, \lambda), V^c_+, -V^c_-, \lambda),\end{aligned}$$

which shows the claim on account of $c^+, c^- \in \mathrm{Fix}(\mathcal{R})$. □

Lemma 11
Under the above assumptions exactly the symmetric orbits correspond to the solutions $(V^c_+, V^c_-, \lambda) = (V^c_+, 0, \lambda)$ of $\xi^\infty(V^c_+, V^c_-, \lambda) = 0$ for which $V^c_- = 0$.

The proof of this lemma is straightforward, see also lemma 3.3.8 in [14]. Note that $D_1\xi^\infty(0,0,0) = D_2\xi^\infty(0,0,0) = 0$. On account of lemma 10 we can now write

$$\xi^\infty(V^c_+, V^c_-, \lambda) = V^c_- \cdot q(V^c_+, V^c_-, \lambda)$$

for a smooth function q for $(V^c_+, V^c_-, \lambda) \approx 0$. It is therefore reasonable to assume

Hypothesis 9
$D_{V_+^c} q(0,0,0) \neq 0$.

Hence, when this assumption is satisfied, ξ^∞ has the expansion

$$\xi^\infty(V_+^c, V_-^c, 0) = V_-^c \left(a \cdot V_+^c + b \cdot [V_-^c]^2 + h.o.t.\right)$$

for some $a \neq 0$ and $b \in \mathbb{R}$ for $V_-^c, V_+^c \approx 0$. Let us summarize our results in the following theorem.

Theorem 7
Under the above hypotheses Hyp 1-Hyp 9 we have the following scenario.

a) *For all sufficiently small parameters λ there exists a one-parameter family of symmetric homoclinic solutions $h^{sym,\kappa}$ of (2), parametrized by $\kappa \approx 0$. The map $\kappa \mapsto h^{sym,\kappa}$ is continuous as a map with values in $BC^0_{loc}(\mathbb{R}, \mathbb{R}^N)$. Moreover, $h^{sym,\kappa}$ approaches the equilibrium for some κ in forward and backward time if and only if $\lambda \in Hom$ (see the statement of theorem 6). Otherwise, $h^{sym,\lambda}$ approaches a unique nontrivial periodic orbit in forward and backward time.*

b) *For all sufficiently small parameters λ there exist two one-parameter families of unsymmetric homoclinic solutions $h^{asym,\kappa}, g^{asym,\kappa}$ of (2) with $h^{asym,\kappa}(t) = Rg^{asym,\kappa}(-t)$. Furthermore, for each fixed κ the solution $h^{asym,\kappa}$ approaches the equilibrium zero or a periodic orbit in forward and backward time.*

Remark
Let us relate this result to theorem 1. In fact, the only difference in the statement of theorem 7 is the fact that we have assumed hypothesis 5 which rules out case i) in theorem 1. Indeed, case i) occurs if the center stable and center unstable manifold intersect transversely. The above analysis can easily be adapted to this case. We omit the details.

Let us now discuss the role of the parameter λ_2 and focus on the existence of heteroclinic cycles between the steady state zero and a periodic solution. It suffices to look for heteroclinic solutions between the steady state and a periodic solution. Such a solution is induced by a zero $(0, V_-^c, \lambda)$ of $\xi^\infty = 0$. Indeed, $V_+^c = 0$ guarantees that the associated solutions lies on the strong stable manifold $W_+^s(0)$ of the steady state zero, see lemma 3.3.11 in [14]. One can now show that the assumption

$$\partial_{\lambda_2} q(0,0,(0,\lambda_2))\big|_{\lambda_2=0} \neq 0$$

implies the existence of a differentiable curve $HetCyc$ in the parameter plane such that for all parameters $\lambda \in HetCyc$ there exists a heteroclinic cycle between zero and a (possible trivial) periodic solution and we refer to Lemma 3.3.12 in [14] for more details.

7 Discussion

In this section we discuss various generalizations and the relevance of the above results for the Fermi-Pasta-Ulam and Klein-Gordon lattice.

7.1 Lattice-differential equations with Hamiltonian structure

Although we dealt with general reversible forward-backward delay equations (2) in this work, we now want to address the relevance of our results for lattice differential equations with an additional Hamiltonian structure, such as the Klein-Gordon lattice (3) or the Fermi-Pasta-Ulam lattice. This additional structure imposes restrictions on the resulting bifurcation scenario in the travelling wave equation (2). In the case of an ordinary differential equation, for example, an additional Hamiltonian structure typically prevents the transverse intersection of center stable and center unstable manifold along a homoclinic solution, see [14, 2]. Hence, case a) in (11) is ruled out and we expect case b) to occur. Let us now explain why we expect the same thing to happen here.

The Klein-Gordon lattice
Let us first consider the Klein-Gordon lattice

$$\ddot{u}_n = \gamma(u_{n+1} - 2u_n + u_{n-1}) + V'(u_n), \qquad n \in \mathbb{Z}, \tag{41}$$

$V'(0) = 0$ and $V''(0) \neq 0$, which induces a reversible forward-backward delay equation (2) via a travelling wave ansatz. Equation (41) is Hamiltonian with (formal) Hamiltonian given by

$$Ham = \sum_{n\in\mathbb{Z}} \frac{1}{2}v_n^2 + V(u_n) + \frac{1}{2}\gamma(u_{n+1} - u_n)^2.$$

As already observed before, the travelling wave equation reads

$$\begin{aligned} U'(\xi) &= W(\xi) \\ c^2W'(\xi) &= \gamma(U(\xi+1) - 2U(\xi) + U(\xi-1)) + V'(U(\xi)), \end{aligned} \tag{42}$$

where $c \neq 0$ denotes the travelling wave speed. This equation possesses a first integral I, given for $\phi, \psi \in C^0([-1,1],\mathbb{R})$ by

$$I(\phi(\cdot),\psi(\cdot)) = \psi(0) + (\gamma/c^2)\cdot\int_0^1 (\phi(\nu) - \phi(\nu-1))d\nu + (1/c^2)\int_0^1 V'(\phi(\nu))d\nu.$$

In fact, it is easy to verify that $\partial_\xi I(U(\xi+\cdot), W(\xi+\cdot)) = 0$ along any solution $(U(\xi), W(\xi))$ of (42). The function I now induces a first integral $\tilde{I} : \tilde{X} \to \mathbb{R}$ of the travelling wave equation in abstract form (17). We expect the existence of this first integral $\tilde{I}$ to prevent a transverse intersection of center stable and center unstable manifold along a homoclinic solution H. Let us give a formal argument why this is true and let us consider the dynamic on the center manifold $\mathcal{M}$ first. There, the existence of a family of periodic orbits implies that $\tilde{I}$ restricted to $\mathcal{M}$ possesses an extremum in 0, say a minimum, which corresponds to $\tilde{I} = 0$, say. Hence, $\tilde{I}$ takes non-negative values on $\mathcal{M}$. The surface corresponding to $\tilde{I} = 0$ intersects the Poincaré-section $\tilde{\Sigma}$. Since all orbits on the center stable and center unstable manifold $W^{cs,+}(0)$, $W^{cu,-}(0)$, respectively, starting sufficiently close to $H(0)$ approach a periodic orbit on the center manifold, $\tilde{I}$ also takes non-negative values on $W^{cs,+}(0) \cap \tilde{\Sigma}$ and $W^{cu,-}(0) \cap \tilde{\Sigma}$. If $H(0)$ corresponds to a regular point of $\tilde{I}$ we conclude that the submanifolds $W^{cs,+}(0) \cap \tilde{\Sigma}$ and $W^{cu,-}(0) \cap \tilde{\Sigma}$ are located "on the same side" of the manifold $\{u : \tilde{I}(u) = 0\} \cap \tilde{\Sigma}$. Hence,

$$T_{H(0)}W^{cs,+}(0), T_{H(0)}W^{cu,-}(0) \;\subset\; T_{H(0)}\{\tilde{I} = 0\}, \tag{43}$$

which prevents a transverse intersection of $W^{cs,+}(0)$ and $W^{cu,-}(0)$. Note, however, that we expect the vector space $T_{H(0)}\tilde{I} = 0$ to have codimension one in $\tilde{X}$ if $H(0)$ corresponds to a regular point of $\tilde{I}$. This would certainly be true if the value 0 corresponds to a regular value of the first integral I. Considering the specific preimage $U = 0$ shows that zero is in fact not a regular value. However, we would expect that the point $H(0)$ gives rise to a regular point and hence, a three-dimensional intersection of center stable and center unstable manifold (as in hypothesis 5) is indeed expected generically in this situation.

It has been shown in [11] that there exists a region in the two-dimensional (c, γ)-parameter plane, where the linearization at zero possesses exactly two simple eigenvalues $\pm i\omega$ on the imaginary axis. More precisely, if the positive parameters c, γ lie below the curve γc^2, then the linearization at zero possesses exactly two critical eigenvalues $\pm i\omega$, $\omega \neq 0$, see [11] for details. However, up to our knowledge the existence of homoclinic orbits of (41) in this parameter region is not known; at least not by analytical means.

Note that once a homoclinic solution exists, we expect that the resulting bifurcation scenario can be studied using the methods introduced in this work. In fact, even if we cannot determine the dimension of the space

$$T_{H(0)}W^{cs,+}_{\lambda=0}(0) \cap T_{H(0)}W^{u,-}_{\lambda=0}(0), \tag{44}$$

we still know a priori that it is finite dimensional. Indeed, it has been shown in [8] that the dimension of the space in (44) coincides with the dimension of the kernel of a certain Fredholm operator (which is finite by definition). The concepts used in this work therefore allow to relate the bifurcation scenario to the study of the zero set of a certain bifurcation function $\xi^\infty : \mathbb{R}^n \to \mathbb{R}^m$ (as far as only solutions in $W^{cs,+}(0) \cap W^{u,-}(0)$ are concerned that stay uniformly close to the primary homoclinic solution).

The Fermi-Pasta-Ulam lattice

Let us now consider the equation

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} + V'(u_{n+1} - u_n) + V'(u_{n-1} - u_n), \qquad n \in \mathbb{Z}. \tag{45}$$

Making a travelling wave ansatz we derive the equation

$$\begin{aligned} U'(\xi) &= W(\xi) \\ c^2 W'(\xi) &= U(\xi+1) - 2U(\xi) + U(\xi-1) \\ &\quad + V'(U(\xi+1) - U(\xi)) + V'(U(\xi-1) - U(\xi)), \end{aligned} \tag{46}$$

where c denotes the travelling wave speed. Again, there exists a first integral J given by

$$\begin{aligned} J(\phi(\cdot), \psi(\cdot)) &= \psi(0) + (1/c^2) \cdot \int_0^1 (\phi(\nu) - \phi(\nu-1)) d\nu \\ &+ (1/c^2) \int_0^1 V'(\phi(\nu) - \phi(\nu-1)) d\nu \end{aligned}$$

for $\phi, \psi \in C^0([-1,1], \mathbb{R})$. Hence, if there exists a homoclinic solution we expect that center stable and center unstable manifold don't intersect transversely at any point of the homoclinic orbit. As we have seen above, however, our argumentation depends strongly on the dynamic on the center manifold. What is promising about the Fermi-Pasta-Ulam lattice is the fact that the existence of solitary waves is known by analytical means, see [5].

In the case of the Fermi-Pasta-Ulam lattice zero is always a double, non semi-simple eigenvalue of the linearization at zero for all parameter values $c^2 > 0$. Moreover, as $c^2 \nearrow 1$ there is a pair of real eigenvalues $\pm\mu$ which approaches the imaginary axis and becomes purely imaginary as c^2 is increased through one, see lemma 1 in [12]. Therefore, the resulting bifurcation scenario does not fit exactly into the framework of this work. However, it should be possible to analyse the resulting bifurcation with the methods and techniques introduced in this work.

7.2 A double eigenvalue zero – The borderline case

We now consider a reversible forward-backward delay equation (2), where the critical eigenvalues $\pm i\omega$ of the linearization of at the steady state coalesce at zero and induce a double eigenvalue zero for $\lambda = 0$ (hence, hypothesis 3 is violated). More precisely, we assume that the characteristic equation possesses exactly one zero on the imaginary axis, namely zero, which is of second order. For the unfolding of the resulting bifurcation on the two-dimensional center manifold we need only one real parameter. Let us assume that it is controlled by the first component λ^1 of the two-dimensional parameter-vector $\lambda = (\lambda^1, \lambda^2)$. The second component λ^2 now accounts for a homoclinic solution h^{sym} to the steady state at the critical value $\lambda = 0$. We are now interested in the situation, where the critical eigenvalues at zero are purely imaginary for $\lambda^1 < 0$ and become real for $\lambda^1 > 0$. This kind of bifurcation in the case of ordinary differential equations has been studied by Champneys et al. [2]. The interesting fact in this case is the observation that the homoclinic orbit h^{sym} becomes structurally stable as λ^1 is increased through 0.

Let us recall the scenario in the case of an ordinary differential equation and make the additional assumption that center stable and center unstable manifold intersect transversely along the symmetric homoclinic orbit h^{sym} (we refer to [2] for more details).
The first step in analysing this bifurcation is to understand the reduced dynamic on the center manifold, which in normal form is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \lambda^1 x + x^2, \end{aligned}$$

where higher order terms have been truncated. In these coordinates, x denotes the variable which accounts for the $\mathrm{Fix}(\mathcal{R})$-part acting on the center eigenspace. The origin is a center for $\lambda^1 < 0$ and a saddle for $\lambda^1 > 0$. Hence, we expect the existence of a small symmetric homoclinic orbit for $\lambda^1 \neq 0$, which approaches the origin for $\lambda^1 > 0$ and a distinguished steady state for $\lambda^1 < 0$. In the latter case the origin is enclosed by the homoclinic orbit.
With the understanding of the reduced dynamics on the center manifold one can now show that generically the following scenario occurs:

Theorem [Champneys et al.]
Up to a change of parameters there exist fast decaying homoclinic solutions to the equilibrium exactly for those parameter values (λ^1, λ^2) *for which* $\lambda^2 = 0$. *For* $\lambda^1 < 0$ *no further homoclinic solutions to the trivial steady state exist.*
In the case $\lambda^1 = 0$ *there exists one homoclinic orbit to the equilibrium if* $\lambda^2 \leqslant 0$ *which is algebraically decaying for* $\lambda^2 < 0$.
For $\lambda^1 > 0$ *and* $\lambda^2 \leqslant 0$ *there exists one homoclinic orbit to the equilibrium. For* $\lambda^2 > 0$ *there exist two homoclinic orbits which coalesce in a saddle node bifurcation on some curve* $L := \{(\lambda^1, \lambda^2) : \lambda^2 \geqslant 0, \lambda^1 = a\lambda^2 + o(\lambda^2)\}$ *for some* $a > 0$.

If the ordinary differential equation additionally possesses an Hamiltonian structure, we have to abandon the assumption concerning a transverse intersection of center stable and unstable manifold (as explained in the first section of this chapter). However, as has been proved in [2], the theorem above remains true in this case as far as only *symmetric* homoclinic solutions are concerned (note that there may be more solutions which do not intersect $\mathrm{Fix}(R)$).
We expect that the above bifurcation scenario can be analysed in the more general case (2) completely analogous to the ODE-case by using the methods and results introduced in this work. As a technical point, let us stress the fact that the steady state is not stable with respect to the center dynamic in this case - a fact, which is relevant for the proof of the smoothness of the center stable and unstable manifold. However, it should be possible to overcome this technical problem, see the appendix II of [16].

7.3 Continuous delay

Our results are not limited to equations (2) with discrete delays. In fact, we can consider a general reversible forward-backward delay equation

$$\dot{U}(t) = \mathcal{F}(U(t+\cdot), \lambda), \tag{47}$$

$U(t+\cdot) \in C^0([-M, M], \mathbb{R}^N)$, instead. Equations of the more general form (47) also appear frequently as travelling wave equations of equations in elasticity [6, 9, 24, 25] or phase transitions [23]. The corresponding calculations in this case translate verbatim to this more general case.

8 Appendix: The weak* integral

In this section we want to clarify in which sense the integral

$$\int_0^t T(t,s)G(s)ds \tag{48}$$

is well defined, if $s \to G(s) = (g(s), 0)$ maps continuously into the space $\hat{X} = \mathbb{R}^N \times C^0([-M, M], \mathbb{C}^N)$ for some $M > 0$.

Assumption 1
Let $L(\cdot) \in BC^0(\mathbb{R}, L(C^0([-M, M], \mathbb{C}^N), \mathbb{C}^N))$ and let $L(t) \to L_\pm$ with respect to the operator norm as $t \to \pm\infty$, where $L_\pm \in L(C^0([-M, M], \mathbb{C}^N), \mathbb{C}^N)$. Consider

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} \xi(t) \\ \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} L(t)\phi(t, \cdot) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix}. \tag{49}$$

If the equations $\dot{y}(t) = L_\pm y_t$ are hyperbolic, (49) possesses an exponential dichotomy on $\mathbb{R}_+$ (respectively $\mathbb{R}_-$) with associated solution operators $\Phi^s_+(\tau, \sigma)$, $\Phi^u_+(\sigma, \tau)$ for $\tau \geq \sigma \geq 0$ (respectively $\Phi^s_-(\sigma, \tau)$, $\Phi^u_-(\tau, \sigma)$ and $\tau \leq \sigma \leq 0$). Otherwise equation (49) possesses a (center-) dichotomy on $\mathbb{R}_+$ with solution operators $\Phi^{cs}_+(t, s), \Phi^u_+(s, t)$ or $\Phi^s_+(t, s), \Phi^{cu}_+(s, t)$ for $t \geq s \geq 0$. We now consider the case that $T(t, s)$ is one of these solution operators on $\mathbb{R}_+$.

Let us now choose some element

$$(\eta, \psi) \in \tilde{Y} := \mathbb{C}^N \times L^1([-M, M], \mathbb{C}^N)$$

and note that

$$s \mapsto \langle T(t,s)G(s), (\eta, \psi) \rangle \in L^1([0, t], \mathbb{C}), \tag{50}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\tilde{Z} = \mathbb{C}^N \times L^\infty([-M, M], \mathbb{C}^N)$ and $\tilde{Y}$; that is

$$\langle (\xi, \phi), (\eta, \psi) \rangle = \xi \cdot \eta + \int_{-M}^{M} \phi(\theta)\psi(\theta)d\theta$$

for $(\xi, \phi) \in \tilde{Z}$ and $(\eta, \psi) \in \tilde{Y}$. Here, $\tilde{Z}$ can be identified with the dual space of $\tilde{Y}$. Hence, there exists a unique $Q \in \tilde{Z}$, such that

$$\langle Q, (\eta, \psi) \rangle = \int_0^t \langle T(t,s)G(s), (\eta, \psi) \rangle \, ds \tag{51}$$

for every $(\eta, \psi) \in \tilde{Y}$; see the appendix of [18].

Definition 4
We set $\int_0^t T(t,s)G(s)ds := Q$ *and call* Q *the weak** *integral.*

From now on we view the integral term in (48) as a weak* integral, which is an element of $\tilde{Y}^* = \tilde{Z}$ by definition. Note that if $s \mapsto G(s)$ is continuous and takes values in X, then the weak* integral coincides with the usual Riemann integral. Let us now prove that the integral is actually an element of $\tilde{X} = \{(\xi, \phi) \in \mathbb{C}^N \times C^0([-M, M], \mathbb{C}^N) : \phi(0) = \xi\}$. The next lemma has been proved in [7].

Lemma 12
For each fixed $t \geq 0$ *we have* $\int_0^t T(t,s)G(s)ds \in \tilde{X}$.

The weak integral actually depends continuously on t:

Lemma 13
The function $v : t \to \int_0^t T(t,s)G(s)ds$ *is continuous as a function from* $[0, \infty)$ *to* $\tilde{X}$ *and*

$$\|v(t)\|_{\tilde{X}} \leqslant \int_0^t Me^{\alpha(t-s)} ds \cdot \sup_{0 \leqslant s \leqslant t} \|G(s)\|_{\hat{X}},$$

if $T(t,s)$ *satisfies the estimate* $\|T(t,s)\|_{L(\tilde{Z},\tilde{Z})} \leqslant Me^{\alpha(t-s)}$ *for* $t \geq s \geq 0$ *and some* $\alpha \in \mathbb{R}$.

Proof
Note that the integral is well defined with values in $\tilde{X}$ by the previous lemma. Since the map $t \to \int_0^t T(t,s)G(s)ds$ is continuous when regarded with values in $\tilde{Y}^* = \tilde{Z}$ (see lemma 2.1, page 54 in [18]) and the norm of L^∞ restricted to C^0 coincides with the usual norm in C^0, the claim concerning continuity follows immediately by lemma 2.3 in [18]. □

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