STURM GLOBAL ATTRACTORS FOR S^1-EQUIVARIANT PARABOLIC EQUATIONS

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Abstract. We consider a semilinear parabolic equation of the form $u_t = u_{xx} + f(u, u_x)$ defined on the circle $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For a dissipative nonlinearity f this equation generates a dissipative semiflow in the appropriate function space, and the corresponding global attractor $\mathcal{A}_f$ is called a Sturm attractor. If $f = f(u, p)$ is even in p, then the semiflow possesses an embedded flow satisfying Neumann boundary conditions on the half-interval $(0, \pi)$. This is due to $O(2)$ equivariance of the semiflow and, more specifically, due to reflection at the axis through $x = 0, \pi \in S^1$. For general $f = f(u, p)$, where only $SO(2)$ equivariance prevails, we will nevertheless use the Sturm permutation σ introduced for the characterization of Neumann flows to obtain a purely combinatorial characterization of the Sturm attractors $\mathcal{A}_f$ on the circle. With this Sturm permutation σ we then enumerate and describe the heteroclinic connections of all Morse-Smale attractors $\mathcal{A}_f$ with m stationary solutions and q periodic orbits, up to $n := m + 2q \leq 9$.

Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday.

1. Introduction

Consider scalar semilinear parabolic equations of the form

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

defined on the interval $0 \leq x \leq 2\pi$ with periodic boundary conditions

$$u(t, 0) = u(t, 2\pi), \qquad u_x(t, 0) = u_x(t, 2\pi)\ , \tag{1.2}$$

or equivalently, defined on the circle $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Under suitable regularity and growth assumptions on the nonlinearity $f = f(x, u, u_x)$, such equations generate global semiflows on appropriate function spaces, that is, any solution $u(t, \cdot)$ of (1.1), (1.2) is defined for all time $t \geq 0$ (see, e.g., [Hen81, Paz83]). Here we assume

$$f \in C^2 \text{ is dissipative}\ . \tag{1.3}$$

This is guaranteed, for example, by sign and growth conditions of the form

$$f(x, v, 0) \cdot v < 0 \text{ for all large } |v|\ , \tag{1.4}$$

$$|f(x, v, p)| < C(|v|)(1 + |p|^\gamma)\ , \tag{1.5}$$

for all $(x, v, p) \in S^1 \times \mathbb{R}^2$, with some suitable constant $0 \leq \gamma < 2$, and a continuous function $C(|v|)$. In fact these specific conditions are sufficient (but not necessary) to ensure that (1.1), (1.2) generates a global dissipative semigroup

$$S_f^{\mathcal{P}}(t) : X^{\mathcal{P}} \to X^{\mathcal{P}}\ ,\ t \geq 0\ , \tag{1.6}$$

on a Sobolev space $X^{\mathcal{P}} = H^s(S^1),\ 3/2 < s < 2$, which embeds into $C^1(S^1)$. See, for example, [AF88, Mat88, MN97, FRW04, CR08] and the references included there. Here and onwards the superscript $\mathcal{P}$ stands for the periodic boundary conditions (1.2) on the interval $(0, 2\pi)$. We recall the definition of $X^{\mathcal{P}}$ in terms of the fractional power space for the Laplacian ∂_{xx} under periodic boundary conditions, that is $X^{\mathcal{P}} := \mathcal{D}((\mathrm{id} - \partial_{xx})^s_{\mathcal{P}})$. Fourier expansion representation of the elements $u \in X^{\mathcal{P}}$,

$$u = \sum_{k \geq 0} a_k \cos kx + b_k \sin kx \ , \tag{1.7}$$

provides the Sobolev norm $\| \cdot \|_{X^{\mathcal{P}}}$ as

$$\|u\|^2_{X^{\mathcal{P}}} = \sum_{k \geq 0} (1 + k^2)^s (a_k^2 + b_k^2) \ . \tag{1.8}$$

The dissipative property of the semigroup $S_f^{\mathcal{P}}(t), t \geq 0$, requires the existence of a fixed large ball in $X^{\mathcal{P}}$ in which any solution $u(t, \cdot)$ eventually stays for all time $t \geq t_0(u(0, \cdot))$. This property is ensured by conditions (1.4), (1.5). It entails the existence of a nonempty global attractor

$$\mathcal{A}_f^{\mathcal{P}} \subset X^{\mathcal{P}} \tag{1.9}$$

which is the maximal compact invariant subset of the state space $X^{\mathcal{P}}$ and consists of all globally defined bounded orbits of the semiflow, see [Hal88, BV92, HMO02]. In particular any solution $u(t) = u(t, \cdot)$ of (1.1), (1.2), has a nonempty ω-limit set in $\mathcal{A}_f^{\mathcal{P}}$. To emphasize the Sturm type nonlinear nodal properties of equation (1.1), we call the global attractor $\mathcal{A}_f^{\mathcal{P}}$ a *Sturm attractor*, in this setting.

In the case of separated boundary conditions the semiflow (1.6) is gradient and the ω-limit set of any solution is a unique equilibrium point, [Zel68, Mat78, HR92]. However, in our case of periodic boundary conditions the semiflow is not gradient-like in general. This provides for a more interesting geometric and dynamical structure of $\mathcal{A}_f^{\mathcal{P}}$ since, in addition to equilibria, the global attractor may also exhibit nonequilibrium periodic orbits, (cf. [AF88, Mat88, MN97]).

The absence of a variational structure for the semiflow complicates the study of the geometric and dynamical structure of $\mathcal{A}_f^{\mathcal{P}}$. To our rescue we have the *Sturm property*: for any given pair of solutions u_1, u_2 of (1.1), (1.2) and any fixed t, let $z^{\mathcal{P}} = z^{\mathcal{P}}(u_1(t, \cdot) - u_2(t, \cdot))$ denote the number of strict sign changes of the x-profiles $x \mapsto u_1(t, x) - u_2(t, x)$ of the difference $u_1 - u_2$. Then the *zero number*

$$t \mapsto z^{\mathcal{P}}(u_1(t, \cdot) - u_2(t, \cdot)) \ , \tag{1.10}$$

is a monotone nonincreasing function of time t and is finite for any $t > 0$, [Stu36, Mat82, Ang88]. Moreover $z^{\mathcal{P}}$ drops strictly at multiple zeros of the x-profile. It was Matano who first introduced the closely related concept of the lap number $z(u_x) + 1$, and thus initiated todays insight into the central importance of the zero number z for the global dynamics of PDEs like (1.1); [Mat82]. As a consequence, for example, the ω-limit set of any solution

$u(t) = u(t,\cdot)$ of (1.1), (1.2) either contains an equilibrium or is a periodic orbit, (cf. [FMP89]).

The set $\mathcal{E}_f^{\mathcal{P}}$ of equilibrium solutions of (1.1), (1.2) is the set of solutions of the spatially 2π-periodic ODE boundary value problem (here and where convenient we use $' = d/dx$)

$$v'' + f(x, v, v') = 0 \ , \ x \in S^1 \ , \tag{1.11}$$

corresponding to the stationary problem. To assure finiteness and nondegeneracy of the equilibria $v \in \mathcal{E}_f^{\mathcal{P}}$ we assume hyperbolicity. An equilibrium $v \in \mathcal{E}_f^{\mathcal{P}}$ is called *hyperbolic* if the linearized problem around v,

$$\lambda u = u_{xx} + \partial_p f(x, v(x), v_x(x))u_x + \partial_v f(x, v(x), v_x(x))u \ , \ x \in S^1 \ , \tag{1.12}$$

has no eigenvalue λ with zero real part, $Re(\lambda) = 0$. Then the unstable dimension of v, $i^{\mathcal{P}}(v)$, is given by the number of eigenvalues of (1.12) with strictly positive real part, counting algebraic multiplicities. In spite of the apparent absence of a variational context the unstable dimension $i^{\mathcal{P}}(v)$ is called the *Morse index* of v.

Similarly, to require nondegeneracy of a periodic orbit we assume hyperbolicity. Let $u(t) = u(t,\cdot)$ denote a nonstationary periodic solution of $S_f^{\mathcal{P}}(t)$ with minimal period $\tau > 0$ and initial value $u(0) = u_0 \in X^{\mathcal{P}}$. The *Floquet multipliers* of $u(t)$ are the eigenvalues of the evolution operator $P_\tau : X^{\mathcal{P}} \to X^{\mathcal{P}}$ defined by the linearization of (1.1), (1.2) around the periodic solution $u(t)$, at time $t = \tau$;

$$P_\tau = DS_f^{\mathcal{P}}(\tau)u_0 \ . \tag{1.13}$$

Then the periodic orbit $u = \{u(t) : 0 \leq t < \tau\}$ is *hyperbolic* if $\mu = 1$ is an algebraically simple eigenvalue of P_τ and is the unique Floquet multiplier of $u(t)$ on the complex unit circle. The (strong) unstable dimension $i^{\mathcal{P}}(u)$ of u is the number of characteristic multipliers of $u(t)$ with modulus $|\mu| > 1$, counting algebraic multiplicities. For details see, e.g., [Hen81, CR08]. As in the case of equilibria, $i^{\mathcal{P}}(u)$ is called the *Morse index* of the (hyperbolic) periodic orbit u. Note that the dimension of the unstable manifold is $i^{\mathcal{P}}(v)$, for a hyperbolic equilibrium v, but is $i^{\mathcal{P}}(u) + 1$, for a hyperbolic periodic orbit u.

Let

$$\mathrm{Sturm}^{\mathcal{P}}(x, u, u_x) \tag{1.14}$$

denote the *Sturm class* of dissipative C^2-nonlinearities $f = f(x, u, u_x)$, satisfying (1.3), for which all equilibria and periodic orbits are hyperbolic. The restricted Sturm classes

$$\mathrm{Sturm}^{\mathcal{P}}(u, u_x) \ , \ \mathrm{Sturm}^{\mathcal{P}}(x, u) \ , \ \ldots , \tag{1.15}$$

are also defined in the obvious way.

The characterization of Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(x, u, u_x)$ has been considered in the literature. See for example [CR08, JR10a, JR10b] and their references for some results on the geometric and dynamical properties of $\mathcal{A}_f^{\mathcal{P}}$. We remark that hyperbolicity of all equilibria and rotating waves is a generic property in the space $\mathcal{G}$ of functions $f = f(x, u, u_x)$ endowed with the Whitney C^2 topology, [JR10a]. This is equally true in the space $\mathcal{G}_0$ of functions $f = f(u, u_x)$. We also quote the following remarkable transversality property of the stable and unstable manifolds of hyperbolic periodic orbits:

Theorem 1: *(See* [CR08]*, Theorem 8.2.) The stable and unstable manifolds of two hyperbolic periodic orbits $u^\pm$ of (1.1), (1.2) always intersect transversely,*

$$W^u(u^-) \pitchfork W^s(u^+) \ . \tag{1.16}$$

In the case of equilibria for semilinear parabolic equations under separated boundary conditions such transversality results have been previously established in [Hen85, Ang86].

The transversality result of Theorem 1 still holds when one of the periodic orbits is replaced by a hyperbolic equilibrium. However, in general this result does not extend to the case of a pair of equilibria $v^\pm \in \mathcal{E}_f^{\mathcal{P}}$. Furthermore, while homoclinic behavior to periodic orbits is excluded, cf. [Nad98, CR08], homoclinic behavior to equilibria is known to occur, see [SF92]. Due to this main difficulty the characterization of Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(x, u, u_x)$ is still open.

The main goal of the present paper is a systematic study of heteroclinic orbits of the problem

$$u_t = u_{xx} + f(u, u_x) \ , \ x \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \ , \tag{1.17}$$

corresponding to (1.1), (1.2) with nonlinearity in the restricted Sturm class

$$f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x) \ , \tag{1.18}$$

where $f = f(u, u_x)$ is not allowed to depend on x, explicitly. In contrast with the general case, the geometric and dynamical structure of Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ is much more tractable. Our aim here is the presentation of a purely combinatorial characterization of the heteroclinic orbit structure in the Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ for f in the restricted class (1.18); see Theorem 3.

The important characteristic feature of our problem (1.17) is its S^1-equivariance with respect to shifts, due to the x-independent form of the nonlinearity $f = f(u, u_x)$. Indeed fix any shift by $\vartheta \in S^1$. Then $u(t, x + \vartheta)$ is a solution of (1.17) whenever $u(t, x)$ itself is. This S^1-equivariance is inherited as S^1-invariance by the Sturm global attractor $\mathcal{A}_f^{\mathcal{P}}$. Moreover, any periodic orbit of (1.17) is a *rotating wave*, that is, the corresponding solution has the form $u(t, x) = v(x - ct)$ rotating around S^1 with arbitrary constant speed $c \neq 0$, [AF88, Mat88, MN97]. Here v is any 2π-periodic solution of the ODE

$$v'' + f(v, v') + cv' = 0 \ , \ x \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \ , \tag{1.19}$$

again with $' = d/dx$. In the following we let $\mathcal{R}_f^{\mathcal{P}}$ denote the set of rotating waves of (1.17).

It follows from the Sturm property that, for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, any solution of (1.17) either approaches a single (hyperbolic) equilibrium solution or a (hyperbolic) rotating wave as $t \to +\infty$.

The same remains true in backwards time, for $t \to -\infty$, due to the backward uniqueness result for parabolic equations, see [Hen81]. In fact, the restriction of $S_f^{\mathcal{P}}(t)$ to the set $\mathcal{A}_f^{\mathcal{P}}$ defines a flow on $\mathcal{A}_f^{\mathcal{P}}$ and therefore it makes sense even for $t < 0$.

In addition, for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ homoclinic behavior to equilibria is also excluded, see [MN97]. Then, the global attractor $\mathcal{A}_f^{\mathcal{P}}$ decomposes as

$$\mathcal{A}_f^{\mathcal{P}} = \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}} \cup \mathcal{H}_f^{\mathcal{P}} \ , \tag{1.20}$$

where $\mathcal{H}_f^{\mathcal{P}}$ denotes the set of *heteroclinic orbits*, i.e. solutions $u(t, \cdot)$ which limit to two different elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ for $t \to \pm\infty$; see for example [FRW04]. In the following we write $v \rightsquigarrow w$, for $v, w \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$, if there exists a connecting heteroclinic orbit from v to w.

The set of equilibria $\mathcal{E}_f^{\mathcal{P}}$ for $f = f(u, u_x)$ is the set of spatially 2π-periodic solutions of the ODE

$$v'' + f(v, v') = 0 \ , \ x \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \ . \tag{1.21}$$

In general, this set is composed of two types of solutions: the set $\mathcal{Z}_f^{\mathcal{P}}$ of spatially homogeneous solutions, $v \equiv e$, corresponding to the zeros of $f(\cdot, 0)$,

$$f(e, 0) = 0 \ ; \tag{1.22}$$

and the set $\mathcal{F}_f^{\mathcal{P}}$ of spatially nonhomogeneous solutions of (1.21), $v = v(x)$ with $v' \not\equiv 0$. We use the letter $\mathcal{F}$ because we sometimes view nonhomogeneous equilibria as *frozen waves*, i.e. rotating waves with zero wave speed c; compare (1.21) with (1.19). However, spatially nonhomogeneous solutions are always nonhyperbolic in the S^1-equivariant case $f = f(u, u_x)$. In fact, in this case $u = v_x$ is an eigenfunction of the linearization at v

$$\lambda u = u_{xx} + f_p(v, v_x)u_x + f_v(v, v_x)u \ , \ x \in S^1 \ , \tag{1.23}$$

corresponding to the trivial eigenvalue $\lambda = 0$. We also remark that these solutions always occur in families of shifted copies around $x \in S^1$ due to the S^1-equivariance of (1.21). Therefore we caution the reader that for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, due to hyperbolicity, all equilibria $v \in \mathcal{E}_f^{\mathcal{P}}$ are spatially homogeneous. Hence, for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ we have that $\mathcal{F}_f^{\mathcal{P}} = \emptyset$ and

$$\mathcal{E}_f^{\mathcal{P}} = \mathcal{Z}_f^{\mathcal{P}} \ . \tag{1.24}$$

In fact a small perturbation $f(u, u_x) + \varepsilon u_x$ shows that $\mathcal{F}_f^{\mathcal{P}} = \emptyset$ is a generic property in the space $\mathcal{G}_0$.

The transversality result of Theorem 1 extends to all the hyperbolic elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$, equilibria and periodic orbits, as follows.

Theorem 2: *(See* [FRW04]*, Proposition 3.2.) The stable and unstable manifolds of two hyperbolic elements $v^\pm \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ of (1.17) always intersect transversely,*

$$W^u(v^-) \pitchfork W^s(v^+) \ . \tag{1.25}$$

This transversality and rigidity result is essential for the geometric description of the Sturm attractor $\mathcal{A}_f^{\mathcal{P}}$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ and is the main reason why it possesses the *Morse-Smale property*; see for example [HMO02] for details on this property.

Using transversality the authors have determined necessary and sufficient conditions for the existence of heteroclinic orbit connections between any pair of hyperbolic equilibria or rotating waves of the semiflow $S_f^{\mathcal{P}}(t)$ generated by (1.17). See [FRW04], and for partial results also [Miy04].

Due to the Morse-Smale rigidity property the heteroclinic connections in the global attractor $\mathcal{A}_f^{\mathcal{P}}$ are preserved when the nonlinearity f is changed by a smooth homotopy that preserves hyperbolicity of all equilibria and periodic orbits. Such a homotopy was used in [FRW04] to obtain a semiflow $S_f^{\mathcal{P}}(t)$ with an embedded semiflow satisfying Neumann boundary conditions on the half-interval $0 \leq x \leq \pi$. The heteroclinic connections were subsequently determined from adjacency relations which encode the configuration of the stationary states of this Neumann problem. Further on, in Section 4, we will review the appropriate adjacency relation for our problem (1.17), namely the notion of ($\mathcal{P}$)-adjacency; see [FRW04].

To distinguish the problem satisfying Neumann boundary conditions on the half-interval $0 \leq x \leq \pi$ from our case of periodic boundary conditions on the interval $0 \leq x \leq 2\pi$ we will use the superscript $\mathcal{N}$ whenever referring to the Neumann problem. We will also refer to this problem as the Neumann case ($\mathcal{N}$), while our problem (1.1), (1.2) will be called the periodic case ($\mathcal{P}$).

The embedded Neumann problem obtained by the homotopy process also possesses a Sturm attractor $\mathcal{A}_f^{\mathcal{N}}$. A characterization for this Neumann Sturm attractor $\mathcal{A}_f^{\mathcal{N}}$ in purely combinatorial terms is provided by a permutation defined on the set of its stationary states. Assuming hyperbolicity (but not necessarily homogeneity) of all equilibria $v_1, \ldots, v_n$ in $\mathcal{A}_f^{\mathcal{N}}$, the permutation $\sigma = \sigma_f^{\mathcal{N}} \in \mathcal{S}(n)$ is given by the ordering of the Neumann boundary values of the equilibria at $x = 0$ and $x = \pi$. To be specific, if the equilibria are labeled by the ordering at $x = 0$

$$v_1(0) < v_2(0) < \cdots < v_n(0) \ , \tag{1.26}$$

then $\sigma = \sigma_f^{\mathcal{N}}$ is defined by the labeling at $x = \pi$

$$v_{\sigma(1)}(\pi) < v_{\sigma(2)}(\pi) < \cdots < v_{\sigma(n)}(\pi) \ . \tag{1.27}$$

Such a permutation is called *Sturm permutation* and is an appropriate object for the combinatorial characterization of Sturm attractors. See [FR91, Fie94, FR96, Fie96, FR99, FR00] in the case of semilinear parabolic equations under separated boundary conditions and [Roc94] in the case of finite dimensional discretizations. We also refer to [Roc91, F&al02, Wol02a, Wol02b, HW05, FR08, FR09, FR10] for applications of this characterization, and to [FS02, Rau02] for surveys on the subject.

We will recall some of these results in Section 2. Here it suffices to mention that a characterization of Neumann Sturm attractors for some restricted classes of nonlinearities has recently been obtained in terms of particular Sturm permutations called integrable involutions; see [FRW11]. For f in restricted classes like $f = f(u)$ or $f = f(u, u_x)$ with even dependence on u_x the ODE

$$v'' + f(v, v') = 0 \tag{1.28}$$

has a first integral which we denote by $H = H(v, v')$; see [Rag10]. In this case there is an intimate relation between the Neumann solutions of the ODE (1.28) on the interval $0 \leq x \leq \pi$, alias the equilibria $v_j \in \mathcal{E}_f^{\mathcal{N}}$, and the 2π-periodic solutions of (1.28). This relation is behind all the results obtained for the periodic problem ($\mathcal{P}$) and is due to the cyclic reflection symmetric shape of the bounded level curves of H. In fact, extending any equilibrium solution $v_j(x)$ to $-\infty < x < \infty$ by reflection through the boundaries we obtain a periodic solution of (1.28) with possibly non-minimal period 2π. Conversely, from any 2π-periodic solution of (1.28) we obtain, after an appropriate x-shift, an equilibrium solution $v_j \in \mathcal{E}_f^{\mathcal{N}}$.

To prepare our main result we first define a *Sturm permutation* $\sigma = \sigma_f^{\mathcal{P}} \in \mathcal{S}(n)$ associated to our problem (1.17) for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$. Let us collect all the values e_j of the spatially homogeneous equilibria $v_j \equiv e_j \in \mathcal{E}_f^{\mathcal{P}}$ and all the maxima $\overline{w}_k$ and minima $\underline{w}_k$ of the rotating waves $w_k \in \mathcal{R}_f^{\mathcal{P}}$ in a vector $(\nu_1, \ldots, \nu_n)$ such that

$$\nu_1 < \nu_2 < \cdots < \nu_n \ . \tag{1.29}$$

Then, $\sigma_f^{\mathcal{P}} \in \mathcal{S}(n)$ is defined as the permutation of $\{\nu_1, \ldots, \nu_n\}$ corresponding to the product of all maximum/minimum 2-cycles $(\underline{w}_k\ \overline{w}_k)$ for the rotating waves w_k with odd $\ell_k = z^{\mathcal{P}}(w_{k,x})/2$.

$$\sigma_f^{\mathcal{P}} = \prod_{\ell_k \text{ odd}} (\underline{w}_k\ \overline{w}_k) \ . \tag{1.30}$$

Note that for a rotating wave $w_k \in \mathcal{R}_f^{\mathcal{P}}$ the number $z^{\mathcal{P}}(w_{k,x})$ is related to the *Matano lap number* introduced in [Mat82] in the setting of Neumann boundary conditions.

Our main result consists of the following theorem:

Theorem 3: *A Sturm permutation $\sigma \in \mathcal{S}(n)$ is a Sturm permutation $\sigma = \sigma_f^{\mathcal{P}}$ in the class $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ of x-periodic S^1-equivariant Sturm problems (1.17) if and only if σ is an integrable involution, as defined in Section 3.*

We postpone the proof of this theorem to Section 5. As a preparation we review in Section 2 the definition and characterization of Sturm permutations σ for the case of semilinear parabolic equations defined on the interval $0 \leq x \leq \pi$ and satisfying Neumann boundary conditions. We recall the Sturm class of nonlinearities $f \in \mathrm{Sturm}^{\mathcal{N}}(x, u, u_x)$ for the Neumann problem ($\mathcal{N}$). In Section 3 we review the characterization of Sturm permutations σ for the restricted (Hamiltonian) class of nonlinearities $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$. We recall the definition of integrable involutions, and also the extension of the characterization results to the restricted class of x-reversible nonlinearities in $\mathrm{Sturm}^{\mathcal{N}}(u, u_x)$, which are even in $p = u_x$. In Section 4 we review the adjacency relations for the equilibria $\mathcal{E}_f^{\mathcal{N}}$ of the Neumann flow and for the critical elements $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ of (1.17). In Theorem 7 we establish a convenient link between the two adjacency relations, and in Theorem 8 we recall the characterization of heteroclinic orbits for (1.17) in terms of adjacency.

Finally, in Section 6 we apply Theorems 3 and 8 to describe the heteroclinic orbit connections of a number of examples of Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ with $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$. In particular, we use the list of all Sturm permutations $\sigma \in \mathcal{S}(n)$ with $n \leq 9$ which are integrable involutions $\sigma = \sigma_f^{\mathcal{N}}$ in the Neumann problem $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ (cf. [FRW11]) to:

(a) enumerate all Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$, $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, with up to m equilibria and q rotating waves for $n = m + 2q \leq 9$; and
(b) determine their heteroclinic orbit connections.

2. Sturm permutations

In this section we recall some results for the semilinear parabolic equation

$$u_t = u_{xx} + f(x, u, u_x) \ , \ 0 \leq x \leq \pi \ , \tag{2.1}$$

subject to the Neumann boundary conditions

$$u_x(t, 0) = u_x(t, \pi) = 0 \ . \tag{2.2}$$

For a dissipative C^2-nonlinearity $f = f(x, u, u_x)$ this problem – the Neumann case ($\mathcal{N}$) – generates a global dissipative Neumann semiflow

$$S_f^{\mathcal{N}}(t) : X^{\mathcal{N}} \to X^{\mathcal{N}} \ , t \geq 0 \ , \tag{2.3}$$

on a Sobolev space $X^{\mathcal{N}} = H^s(0, \pi) \cap \{u : u_x = 0 \text{ at } x = 0, \pi\}$, $3/2 < s < 2$, which embeds in $C^1(0, \pi)$. Then, due to dissipativeness, the semiflow $S_f^{\mathcal{N}}(t)$ also has a Sturm attractor

$$\mathcal{A}_f^{\mathcal{N}} \subset X^{\mathcal{N}} \ . \tag{2.4}$$

Besides, $S_f^{\mathcal{N}}(t)$ possesses a gradient-like structure due to the existence of a Lyapunov function; see [Zel68, Mat88]. Therefore, the Neumann Sturm attractor $\mathcal{A}_f^{\mathcal{N}}$ decomposes as

$$\mathcal{A}_f^{\mathcal{N}} = \mathcal{E}_f^{\mathcal{N}} \cup \mathcal{H}^{\mathcal{N}} \ , \tag{2.5}$$

where $\mathcal{E}_f^{\mathcal{N}}$ denotes the set of equilibria of (2.1), (2.2) and $\mathcal{H}_f^{\mathcal{N}}$ denotes the set of heteroclinic orbits.

The set $\mathcal{E}_f^{\mathcal{N}}$ is the set of solutions of the ODE Neumann boundary value problem (again $' = d/dx$)

$$v'' + f(x, v, v') = 0 \ , \ 0 \leq x \leq \pi \ , \tag{2.6}$$

$$v'(0) = v'(\pi) = 0 \ . \tag{2.7}$$

Here, as in (1.12), an equilibrium $v \in \mathcal{E}_f^{\mathcal{N}}$ is called *hyperbolic* if $\lambda = 0$ is not an eigenvalue of the linearization at v, given by

$$\lambda u = u_{xx} + \partial_p f(x, v(x), v_x(x)) u_x + \partial_v f(x, v(x), v_x(x)) u \tag{2.8}$$

for $0 \leq x \leq \pi$ and satisfying the Neumann boundary conditions (2.2). The number $i^{\mathcal{N}}(v)$ of positive eigenvalues is the *Neumann Morse index* of v.

The Sturm class of nonlinearities

$$\mathrm{Sturm}^{\mathcal{N}}(x, u, u_x) \tag{2.9}$$

is defined as the set of dissipative $f = f(x, u, u_x)$ for which all equilibria of (2.1), (2.2) are hyperbolic. Likewise, the more restrictive classes

$$\mathrm{Sturm}^{\mathcal{N}}(u)\ ,\ \mathrm{Sturm}^{\mathcal{N}}(u, u_x)\ ,\ \ldots \tag{2.10}$$

are also defined. Unlike the equivariant S^1 case $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, hyperbolic equilibria in the x-independent Neumann case $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x)$ need not be spatially homogeneous.

As before, for $f \in \mathrm{Sturm}^{\mathcal{N}}(x, u, u_x)$ problem ($\mathcal{N}$) possesses only a finite number n of equilibria,

$$\mathcal{E}_f^{\mathcal{N}} = \{v_1, \ldots, v_n\}\ , \tag{2.11}$$

which turns out to be odd. The graphs of the equilibria $v_j \in \mathcal{E}_f^{\mathcal{N}}$,

$$\{(x, v_j(x), v_j'(x)) : 0 \le x \le \pi\}\ , \tag{2.12}$$

define a braid on n strands in $[0, \pi] \times \mathbb{R}^2$. Traversing the strands from $x = 0$ to $x = \pi$ the braid defines a permutation $\sigma = \sigma_f^{\mathcal{N}} \in \mathcal{S}(n)$. This is the Sturm permutation for problem ($\mathcal{N}$), introduced in [FR91], which is determined by the ordering of the Neumann boundary values of the equilibria at $x = 0$ and $x = \pi$. To be specific, we label the equilibria by their ordering at $x = 0$

$$v_1(0) < v_2(0) < \cdots < v_n(0)\ . \tag{2.13}$$

Then the *Sturm permutation* $\sigma = \sigma_f^{\mathcal{N}}$ is defined by the labeling at $x = \pi$

$$v_{\sigma(1)}(\pi) < v_{\sigma(2)}(\pi) < \cdots < v_{\sigma(n)}(\pi)\ . \tag{2.14}$$

This permutation is essential for the characterization of Sturm attractors. Its dynamical importance stems from the fact that Sturm attractors $\mathcal{A}_f^{\mathcal{N}}$ and $\mathcal{A}_g^{\mathcal{N}}$ are C^0-orbit equivalent if $\sigma_f^{\mathcal{N}} = \sigma_g^{\mathcal{N}}$, see [FR00]. In particular, as shown in [FR96], $\sigma_f^{\mathcal{N}}$ determines all heteroclinic orbit connections in $\mathcal{A}_f^{\mathcal{N}}$.

Sturm permutations are characterized, in purely combinatorial terms, as dissipative Morse meander permutations; see [FR99]. For a permutation $\sigma \in \mathcal{S}(n)$ to be *dissipative* requires $\sigma(1) = 1$ and $\sigma(n) = n$. The *Morse* property requires the integers (called *Morse numbers*)

$$i_j(\sigma) := \sum_{k=1}^{j-1} (-1)^{k+1} \operatorname{sign}(\sigma^{-1}(k+1) - \sigma^{-1}(k))\ ,\ 1 \le j \le n\ , \tag{2.15}$$

to be all nonnegative (empty sums denoting zero). In fact the Neumann Morse indices of the hyperbolic equilibria $v_j \in \mathcal{E}_f^{\mathcal{N}}$, i.e. the number of strictly positive eigenvalues λ of (2.8), satisfy:

$$i^{\mathcal{N}}(v_j) = i_j(\sigma)\ ,\ 1 \le j \le n\ . \tag{2.16}$$

To define the meander property consider a C^1 Jordan curve which intersects the horizontal axis transversely at exactly n points. Numbering the intersections by $1, 2, \ldots, n$ along the Jordan curve, let $\sigma(1), \sigma(2), \ldots, \sigma(n)$ denote the numbering along the horizontal axis. Any permutation $\sigma \in \mathcal{S}(n)$ which is obtained in this way is called *meander permutation*; see [Arn88].

Then we have the following characterization of Sturm permutations for the Neumann case ($\mathcal{N}$):

Theorem 4: *(See* [FR99]*, Theorem 1.2.) A permutation $\sigma \in \mathcal{S}(n)$ is a Sturm permutation $\sigma = \sigma_f^{\mathcal{N}}$ in the Sturm class $f \in \mathrm{Sturm}^{\mathcal{N}}(x, u, u_x)$ if and only if σ is a dissipative Morse meander permutation.*

3. Hamiltonian type

Here we consider again the Neumann case ($\mathcal{N}$) and recall our characterization [FRW11] of Sturm permutations in the more restrictive class of $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$. In this case the ODE (2.6) in the stationary problem for the equilibria $v \in \mathcal{E}_f^{\mathcal{N}}$ has the form

$$v'' + f(v) = 0 \ , \tag{3.1}$$

which corresponds to the integrable Hamiltonian planar system

$$v' = p \ , \ p' = -f(v) \ . \tag{3.2}$$

For this reason when $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ we say that the Sturm attractor $\mathcal{A}_f^{\mathcal{N}}$ is of *Hamiltonian type.*

The Hamiltonian function $H = H(v, p)$ for the pendulum equation (3.2) is

$$H(v,p) = \frac{1}{2}p^2 + F(v) \ , \tag{3.3}$$

where $\frac{1}{2}p^2$ is the kinetic energy and the potential F satisfies $dF(v)/dv = f(v)$. The level sets of H contain essential information regarding the shape of the equilibrium solutions in the phase plane (v, p). Via the nesting of equilibria v_j, the Hamiltonian imposes constraints on the Sturm permutation σ_f. In [FRW11] these constraints were used to characterize the Sturm permutations in the Hamiltonian class $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$.

Reversibility with respect to the reflection $x \mapsto -x$ is a second distinctive property of (3.1). This property immediately implies that the Sturm permutation $\sigma = \sigma_f^{\mathcal{N}}$ for $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ must necessarily be an *involution*, i.e. satisfies

$$\sigma = \sigma^{-1} \ . \tag{3.4}$$

Indeed, the involution $x \rightarrow \pi - x$ leaves (3.1) invariant and maps the set $\mathcal{E}_f^{\mathcal{N}}$ of equilibria to itself. The same transformation replaces σ by σ^{-1}, which proves (3.4). Because $\sigma = \sigma_f^{\mathcal{N}}$ is an involution it possesses a unique representation as a product of disjoint 2-cycles

$$\sigma = (\underline{c}_1 \ \overline{c}_1) \dots (\underline{c}_r \ \overline{c}_r) \ . \tag{3.5}$$

Sturm permutations $\sigma_f^{\mathcal{N}}$ in the class $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ are characterized as integrable involutions (cf. [FRW11]). The integrability property of an arbitrary involution $\sigma \in \mathcal{S}(n)$ is defined in terms of the nesting properties of its 2-cycles (3.5) as follows:

- For $\alpha \neq \beta$ the 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are *intersecting* if the corresponding open intervals in $\mathbb{R}$ possess a nonempty intersection, that is $(\underline{c}_\alpha, \overline{c}_\alpha) \cap (\underline{c}_\beta, \overline{c}_\beta) \neq \emptyset$;
- Intersecting 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are *nested* if one of these intervals strictly contains the other, i.e. if $(\underline{c}_\beta - \underline{c}_\alpha)(\overline{c}_\alpha - \overline{c}_\beta) > 0$;

- Nested 2-cycles ($\underline{c}_\alpha\ \overline{c}_\alpha$) and ($\underline{c}_\beta\ \overline{c}_\beta$) are *centered* if the mid-points of these intervals coincide, i.e. if $\underline{c}_\alpha + \overline{c}_\alpha = \underline{c}_\beta + \overline{c}_\beta$;
- In view of (2.16) a point j is called *σ-stable* if $i_j(\sigma) = 0$. Then the *σ-stable core* C_α of the 2-cycle ($\underline{c}_\alpha\ \overline{c}_\alpha$) is defined as the set
$$C_\alpha = \{j : i_j(\sigma) = 0,\ \underline{c}_\alpha < j < \overline{c}_\alpha\}\ , \tag{3.6}$$
and two nested 2-cycles ($\underline{c}_\alpha\ \overline{c}_\alpha$) and ($\underline{c}_\beta\ \overline{c}_\beta$) are *core-equivalent* if they share the same σ-stable core, $C_\alpha = C_\beta$.

Finally an arbitrary involution permutation $\sigma \in \mathcal{S}(n)$ is called *integrable* if the following three conditions all hold:

(I.1) intersecting 2-cycles are nested;
(I.2) core-equivalent 2-cycles are centered;
(I.3) non-nested 2-cycles are separated by at least one σ-stable point.

We recall the following characterization of Sturm permutations in the Hamiltonian class $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$:

Theorem 5: *(See* [FRW11]*, Theorem 1.) A Sturm permutation* $\sigma = \sigma_f^{\mathcal{N}} \in \mathcal{S}(n)$ *is in the Hamiltonian class* $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ *if and only if* $\sigma \in \mathcal{S}(n)$ *is an integrable involution.*

We remark that the integrability condition, with a complexity of the order $O(n^2)$ steps, is easy to verify for any given Sturm involution $\sigma \in \mathcal{S}(n)$.

For future reference we next recall the standard tool widely used to discuss periodic solutions of (3.1) – the period map; see for example [Ura64, Smo83, Sch90, Roc07]. Let $v = v(x,a)$ denote the solution of (3.1) with initial value $v(0,a) = a, v'(0,a) = 0$, and let $\mathcal{D} \subset \mathbb{R}$ denote the open set of real values $a \in \mathbb{R}$ such that $v = v(x,a)$ is a periodic nonconstant solution of (3.1). Then the *period map*
$$T_f^{\mathcal{N}} = T_f^{\mathcal{N}}(a) : \mathcal{D} \to \mathbb{R}_+ \tag{3.7}$$
is defined as the minimal period of the solution $v = v(x,a)$, that is
$$v(T_f^{\mathcal{N}}(a), a) = v(0,a) = a\ ,\ v'(T_f^{\mathcal{N}}(a), a) = v'(0,a) = 0\ , \tag{3.8}$$
and $T_f^{\mathcal{N}} > 0$ is minimal with these properties. Then $a \in \mathbb{R}$ corresponds to the left boundary value of a nonconstant equilibrium $v_j \in \mathcal{E}_f^{\mathcal{N}}$, i.e. $a = v_j(0)$, if and only if
$$T_f^{\mathcal{N}}(a) = 2\pi/\ell\ , \quad \text{for some} \quad \ell \in \mathbb{N}\ . \tag{3.9}$$
The period map was used in an essential way in the proof of Theorem 5.

The above results, established for the Hamiltonian case $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$, also extend to $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x)$ in the class of reversible nonlinearities $f = f(v,p)$, which are even in the second variable,
$$f(v,-p) = f(v,p)\ . \tag{3.10}$$
Indeed this ensures reversibility with respect to the reflection $x \mapsto -x$ of the ODE (2.6) corresponding to the stationary problem for the equilibria $v_j \in \mathcal{E}_f^{\mathcal{N}}$, which in this case has the form
$$v'' + f(v,v') = 0\ . \tag{3.11}$$

For this reason the class of nonlinearities f satisfying (3.10) is called *reversible* in [FRW11]. Here the class of reversible $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x)$ will be denoted symbolically by

$$\mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)\ . \tag{3.12}$$

The study of the Neumann problem ($\mathcal{N}$) in the case of reversible $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$ is analogous to the Hamiltonian case $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$. In fact, the discussion of the phase portrait for the planar system

$$v' = p\ ,\ \ p' = -f(v,p)\ , \tag{3.13}$$

corresponding to the ODE (3.11) in the reversible case, is analogous to the study of the planar system (3.2). In particular (3.11) is integrable as an ODE, and the set $\mathcal{C}$ of its periodic orbits is open and bounded in the plane (v, v') due to dissipativity sign condition (1.4) of f. Moreover, $\mathcal{C}$ is symmetric with respect to the v-axis and there is a period map $T_f^{\mathcal{N}}$ of class C^2,

$$T_f^{\mathcal{N}} : \mathcal{D}^{\mathcal{N}} \to \mathbb{R}_+ \tag{3.14}$$

with $\mathcal{D}^{\mathcal{N}} := \{a \in \mathbb{R} : (a, 0) \in \mathcal{C}\}$, which extends the period map (3.7) introduced for $f \in \mathrm{Sturm}^{\mathcal{N}}(u)$ to the case of reversible $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$. For the details see [FRW11].

These observations are condensed in the following

Theorem 5′: *A Sturm permutation $\sigma = \sigma_f^{\mathcal{N}} \in \mathcal{S}(n)$ is in the class of reversible $f \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$ if and only if σ is an integrable involution.*

4. Heteroclinic connections: S^1 versus Neumann

In this section we compare the characterization of heteroclinic orbits in Sturm attractors, both, under periodic and Neumann boundary conditions. We first recall the adjacency relations: for equilibria $v_j \in \mathcal{E}_f^{\mathcal{N}}$ in the Neumann problem ($\mathcal{N}$), and for equilibria and rotating waves $w_j \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ in the periodic problem ($\mathcal{P}$). We also recall a period map defined for problem ($\mathcal{P}$). Then we follow a homotopy which freezes all rotating waves and symmetrizes $f = f(v,p)$ while preserving the Morse-Smale structure of the Sturm attractor $\mathcal{A}_f^{\mathcal{P}}$. This homotopy establishes an embedding relation between problems ($\mathcal{P}$) and ($\mathcal{N}$), which we use as a bridge between the adjacency relations in both problems.

The adjacency relations, as developed by [Wol02b] for the Neumann problem ($\mathcal{N}$), involve the use of zero numbers. We recall that $z^{\mathcal{P}}$ in (1.10) is computed for $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In the following we let $z^{\mathcal{N}}$ denote the zero number computed on the restricted half-interval $x \in [0, \pi]$. Accordingly, the adjacency relation for the Neumann problem ($\mathcal{N}$) employs the zero number $z^{\mathcal{N}}$, while the adjacency relation for the periodic problem ($\mathcal{P}$) uses the zero number $z^{\mathcal{P}}$, see [FRW04].

Two different hyperbolic equilibria $v_1, v_2 \in \mathcal{E}_f^{\mathcal{N}}$ of the Neumann problem (2.1), (2.2) are called ($\mathcal{N}$)-*adjacent* if there does not exist another equilibrium $v \in \mathcal{E}_f^{\mathcal{N}}$ such that

$$z^{\mathcal{N}}(v_1 - v) = z^{\mathcal{N}}(v_2 - v) = z^{\mathcal{N}}(v_1 - v_2)\ ,\ \text{and} \tag{4.1}$$

$$v(0) \text{ is strictly between } v_1(0) \text{ and } v_2(0)\ . \tag{4.2}$$

Two hyperbolic Neumann equilibria $v_1, v_2 \in \mathcal{E}_f^{\mathcal{N}}$ possess a heteroclinic orbit if, and only if, they are ($\mathcal{N}$)-adjacent, see [Wol02b], Theorem 2.1.

In the case of periodic boundary conditions ($\mathcal{P}$) the notion of adjacency is entirely analogous. Two different hyperbolic equilibria or rotating waves of (1.17), $w_1, w_2 \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$, are called ($\mathcal{P}$)-*adjacent* if there does not exist another element $w \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ such that

$$z^{\mathcal{P}}(w_1 - w) = z^{\mathcal{P}}(w_2 - w) = z^{\mathcal{P}}(w_1 - w_2)\ , \text{ and} \tag{4.3}$$

$$\max_{x \in S^1} w(x) \text{ is strictly between } \max_{x \in S^1} w_1(x) \text{ and } \max_{x \in S^1} w_2(x)\ . \tag{4.4}$$

Here $k := z^{\mathcal{P}}(w_1 - w_2)$ must be even, due to the periodic boundary conditions. In [FRW04], Theorem 1.3, it is shown that two equilibria or rotating waves are connected by a heteroclinic orbit if, and only if, they are ($\mathcal{P}$)-adjacent.

The heteroclinic orbit connection results of [FRW04] are based on the relation between the solutions of the periodic problem ($\mathcal{P}$) on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and the solutions of a Neumann problem set up on the *full* interval $0 \leq x \leq 2\pi$. In the present paper, however, it is more advantageous to relate the solutions of our periodic problem ($\mathcal{P}$), (1.17), which is set up on the full interval $0 \leq x \leq 2\pi$, with the solutions of the Neumann problem ($\mathcal{N}$) set up on the *half-interval* $0 \leq x \leq \pi$. Therefore we need to adapt the heteroclinic orbit connection results of [FRW04] to the setting of the half-interval.

We first recall the "freezing" and "symmetrizing" homotopies that change the nonlinearity $f = f(u, u_x)$ preserving the heteroclinic structure by the Morse-Smale property. We start with problem ($\mathcal{P}$).

We define the *period map* $T_f^{\mathcal{P}}$ to be used with problem ($\mathcal{P}$) via the planar system

$$v' = p\ ,\ p' = -f(v,p) - cp \tag{4.5}$$

which corresponds to the rotating wave equation (1.19) with wave speed parameter $c \in \mathbb{R}$.

Let $\mathcal{C}^{\mathcal{P}} \subset \mathbb{R}^2$ denote the open set of initial conditions (v, p) for which there exists some $c \in \mathbb{R}$ such that (v, p) is a periodic point of (4.5). Note that dissipativity sign condition (1.4) of $f = f(v, p)$ implies that $\mathcal{C}^{\mathcal{P}}$ and the possible wave speeds c are bounded, [FRW04]. Inspired by Matano and Nakamura, [MN97], we define:

(i) the unique wave speed $c = c(v, p)$ such that (v, p) is a periodic point;
(ii) the minimal period

$$T_f^{\mathcal{P}} = T_f^{\mathcal{P}}(v, p) \tag{4.6}$$

of the periodic orbit through (v, p).

See also [FRW04], Lemma 4.2. Let $T_f^{\mathcal{P}}(a) = T_f^{\mathcal{P}}(a, 0)$ denote the restriction of $T_f^{\mathcal{P}}(v, p)$ to the open set of initial values (v, p) on the v-axis,

$$T_f^{\mathcal{P}} : \mathcal{D}^{\mathcal{P}} \to \mathbb{R}_+ \tag{4.7}$$

with $\mathcal{D}^{\mathcal{P}} := \{a \in \mathbb{R} : (a, 0) \in \mathcal{C}^{\mathcal{P}}\}$. Note that the function $c = c(v, p)$ is bounded. Moreover, by construction, $c(v, p)$ is in fact a first integral of

(4.5). Indeed all points on the periodic orbit through the point (v,p) share the same value $c = c(v,p)$.

The solution $w = w(x)$ of (4.5) with initial value $(v,p) = (a,0)$ for $a \in \mathcal{D}^{\mathcal{P}}$ and nonzero $c := c(a,0)$ corresponds to a rotating wave $w \in \mathcal{R}_f^{\mathcal{P}}$ of (1.17) if, and only if,

$$T_f^{\mathcal{P}}(a) = 2\pi/\ell, \quad \text{for some} \quad \ell \in \mathbb{N} \ . \tag{4.8}$$

Furthermore, this rotating wave is hyperbolic if and only if

$$\dot{T}_f^{\mathcal{P}}(a) := dT_f^{\mathcal{P}}(a)/da \neq 0 \ . \tag{4.9}$$

See [FRW04], Lemma 4.4. The integer number $\ell = \ell(w)$ introduced in (4.8) is called the *period lap number* of w, and is half of the Matano lap number,

$$\ell(w) = z^{\mathcal{P}}(w_x)/2 \ . \tag{4.10}$$

See, also [FRW11].

We recall that $\mathcal{C}^{\mathcal{P}}$ is bounded and that minimal periods are uniformly bounded from below [AR67, Yor69]. Hence hyperbolicity of all rotating waves implies finiteness of $\mathcal{R}_f^{\mathcal{P}}$. In the following, for $f \in \mathrm{Sturm}^{\mathcal{P}}(u,u_x)$, we denote by m the number of spatially homogeneous equilibria of (1.17) and by q the number of its rotating waves, i.e.

$$m := \#\mathcal{E}_f^{\mathcal{P}} \quad , \quad q := \#\mathcal{R}_f^{\mathcal{P}} \ . \tag{4.11}$$

Shifted spatial profile snapshots of the same rotating wave are considered identical, of course.

The "freezing" homotopy introduced in [FRW04] is a C^2-smooth homotopy $f^\tau,\ 0 \leq \tau \leq 1$, of the form

$$f^\tau(v,p) = f(v,p) + \tau c(v,p)p \ . \tag{4.12}$$

This homotopy, from $f^0 = f(v,p)$ to $h(v,p) := f^1$, "freezes" all rotating waves to zero wave speed as τ goes from 0 to 1. Most importantly the homotopy does not change the spatial profiles of the rotating waves and preserves the period map $T_f^{\mathcal{P}}$ in an open region $\mathcal{C}_0^{\mathcal{P}} \subset \mathcal{C}^{\mathcal{P}}$ which includes all interesting values of $T_f^{\mathcal{P}}$ in view of (4.8), say for example $T_f^{\mathcal{P}}(v,p) \leq 3\pi$. Hence, this homotopy produces a nonlinearity h for which the period map, in the region $\mathcal{C}_0^{\mathcal{P}}$, satisfies

$$T_h^{\mathcal{P}} = T_f^{\mathcal{P}} \ , \tag{4.13}$$

and all rotating waves become *frozen waves*. Therefore we obtain

$$\mathcal{R}_h^{\mathcal{P}} = \mathcal{F}_h^{\mathcal{P}} \ , \tag{4.14}$$

and the x-profiles $w = w(x)$ of the previously rotating waves become the x-profiles of frozen waves, alias spatially nonhomogeneous stationary solutions $w \in \mathcal{F}_h^{\mathcal{P}}$. Of course, here the frozen waves $w \in \mathcal{F}_h^{\mathcal{P}}$ are taken together with all their shifted snapshot copies $w(\cdot + \vartheta)$ for $\vartheta \in S^1$.

In addition, h satisfies $h(a,0) = f(a,0)$ for all $a \in \mathbb{R}$. Thus the zero sets $\mathcal{Z}^{\mathcal{P}}$ of $f(\cdot,0)$ and $h(\cdot,0)$ coincide,

$$\mathcal{Z}_h^{\mathcal{P}} = \mathcal{Z}_f^{\mathcal{P}} \tag{4.15}$$

and hyperbolicity of all spatially homogeneous equilibria $e \in \mathcal{Z}_h^{\mathcal{P}}$ is preserved. In fact, an easy computation shows that hyperbolicity of $e \in \mathcal{Z}_h^{\mathcal{P}}$ is equivalent to $h_v(e,0) \neq k^2, k \in \mathbb{Z}$ and this condition is ensured by $h_v(e,0) = f_v(e,0)$.

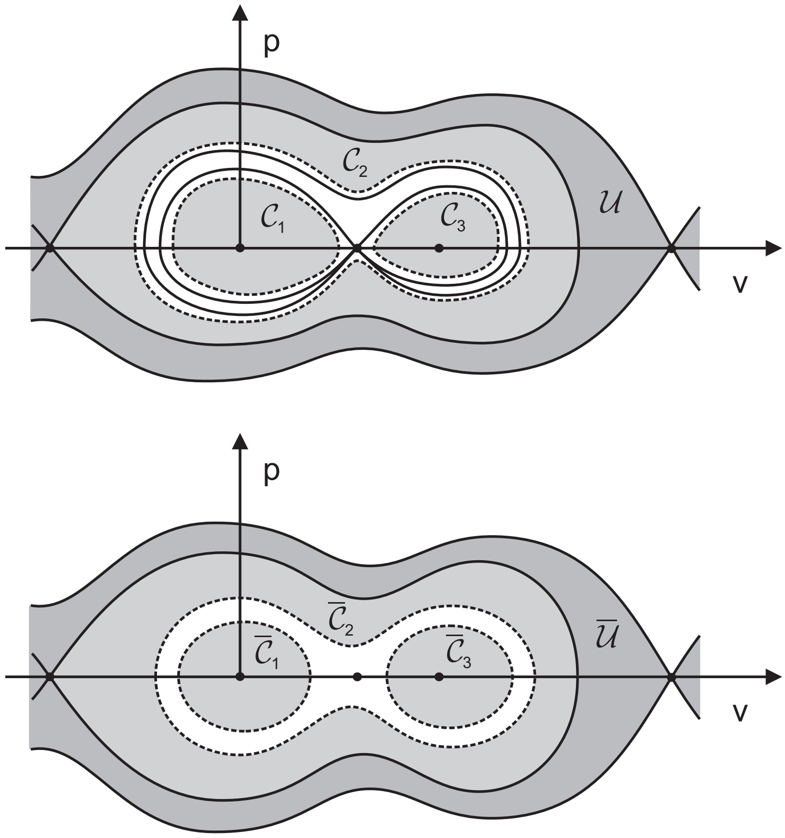

FIGURE 1. Top: A phase portrait corresponding to a frozen nonlinearity h. The region $\mathcal{C}_0^{\mathcal{P}} := \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is shaded in light gray. The region $\mathcal{U}$, in dark gray, corresponds to the set of orbits which intersect the v-axis exactly once. Bottom: A phase portrait corresponding to the symmetrized nonlinearity g. The region $Q^1(\mathcal{C}_0^{\mathcal{P}}) := \overline{\mathcal{C}}_1 \cup \overline{\mathcal{C}}_2 \cup \overline{\mathcal{C}}_3$ is shaded in light gray, and the region $Q^1(\mathcal{U}) = \overline{\mathcal{U}}$ is shaded in dark gray.

Therefore, the freezing homotopy does not affect the homogeneous equilibria. Moreover, it does not introduce any additional equilibria, frozen or rotating waves.

As remarked before, spatially nonhomogeneous equilibria $w \in \mathcal{F}_h^{\mathcal{P}}$ which arrise from hyperbolic rotating waves are not hyperbolic as equilibria. For this reason $h \notin \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, in general. However, in view of (4.13), (4.9), the equilibria $w \in \mathcal{F}_h^{\mathcal{P}}$ are *normally hyperbolic*, i.e. the trivial eigenvalue of the linearization at w is simple. This condition is sufficient to invoke the Morse-Smale property and claim the preservation of the heteroclinic connectivity along the homotopy. See [FRW04] for details.

The "symmetrizing" homotopy, also introduced in [FRW04], runs from the nonlinearity h to a reversible nonlinearity g, i.e. such that

$$g(v, -p) = g(v, p) \tag{4.16}$$

for all v, p. This homotopy symmetrizes the periodic orbits of the planar system (4.5) with respect to the v-axis and preserves the period map $T_h^{\mathcal{P}}$, again in the region $\mathcal{C}_0^{\mathcal{P}}$. The symmetrization homotopy is a family of diffeomorphisms $Q^\tau, 0 \le \tau \le 1$, of the phase plane (v,p) which takes the region $Q^0(\mathcal{C}_0^{\mathcal{P}}) = \mathcal{C}_0^{\mathcal{P}}$ to a region $Q^1(\mathcal{C}_0^{\mathcal{P}})$ which is symmetric with respect to the v-axis. This symmetric region is obtained from $\mathcal{C}_0^{\mathcal{P}}$ by the harmonic mean between orbit points (v,p) on the upper half-plane and their images $(v,-\tilde{p})$ from the lower half-plane. For details we refer to [FRW04]. An illustration is shown in Figure 1. We remark that this symmetrization extends to all the orbits which intersect the v-axis only once. Then the homotopy $g^\tau, 0 \le \tau \le 1$, with $g^0 = h$, is obtained from the phase portrait defined in $Q^\tau(\mathcal{C}_0^{\mathcal{P}})$. The final nonlinearity g is obtained by an arbitrary C^2-smooth extension of g^1 to the plane, preserving the reversibility symmetry (4.16). Then we have

$$T_g^{\mathcal{P}} \circ Q^1 = T_h^{\mathcal{P}} \ , \tag{4.17}$$

on $\mathcal{C}_0^{\mathcal{P}}$. Moreover, the symmetrization can be accomplished in such way that $g(a,0) = h(a,0)$ for all $a \in \mathbb{R}$, so that all the spatially homogeneous equilibria are preserved, i.e.

$$\mathcal{Z}_g^{\mathcal{P}} = \mathcal{Z}_h^{\mathcal{P}} \ , \tag{4.18}$$

together with their hyperbolicity. Again, dissipativity sign condition (1.4) is preserved and no new equilibria, frozen or rotating waves are introduced.

We emphasize that both the freezing and symmetrizing homotopies preserve the relative configuration in the phase plane (v,p) of the periodic orbits and fixed points corresponding to rotating waves and equilibria of problem $(\mathcal{P})$.

Since hyperbolicity fails at frozen waves, we have $g \notin \text{Sturm}^{\mathcal{P}}(u,u_x)$. However, normal hyperbolicity for $v \in \mathcal{F}_g^{\mathcal{P}}$ prevails. Indeed (4.17) restricted to the v-axis yields

$$T_g^{\mathcal{P}}(a) = T_h^{\mathcal{P}}(a) \ , \tag{4.19}$$

for all $a \in \mathcal{D}_0^{\mathcal{P}} := \{a \in \mathbb{R} : (a,0) \in \mathcal{C}_0^{\mathcal{P}}\}$. Hence preservation of heteroclinic connectivity ensues.

Let $v \in \mathcal{F}_g^{\mathcal{P}}$ denote a symmetrized frozen wave, i.e. a solution of

$$v'' + g(v,v') = 0 \ , \ x \in S^1 \ . \tag{4.20}$$

In fact, each frozen wave $v = v(x)$ is reflection symmetric in S^1, by reversibility of $g(v,p)$. Consider the extreme values

$$\overline{v} = \max_{x \in S^1} v(x) \quad , \quad \underline{v} = \min_{x \in S^1} v(x) \ . \tag{4.21}$$

Among the x-shifted copies of v in $\mathcal{F}_g^{\mathcal{P}}$ there are representatives $v_\pm(x) = v(x - s_\pm)$ which satisfy

$$v_\pm'(0) = 0 \quad , \quad \pm v_\pm''(0) < 0 \ . \tag{4.22}$$

That is, v_+, v_- correspond to the solutions which have their maximum and minimum value, respectively, occurring at $x = 0$:

$$v_+(0) = \overline{v} \quad , \quad v_-(0) = \underline{v} \ . \tag{4.23}$$

This is where we invoke the reversibility (4.16) of $g(v,p)$. By reflection symmetry of (4.20), periodicity $v_\pm(0) = v_\pm(2\pi)$ and $v_\pm'(0) = v_\pm'(2\pi) = 0$ imply

$$v_\pm(x) = v_\pm(2\pi - x) \ , \ v_\pm'(x) = -v_\pm'(2\pi - x) \ . \tag{4.24}$$

For $x = \pi$ we obtain

$$v'_\pm(\pi) = 0 \ . \tag{4.25}$$

Therefore $v_\pm(x)$ both satisfy Neumann boundary conditions on the half-interval $0 \le x \le \pi$ and, using the same notation for the restrictions of $v_\pm$ to the half-interval, we have $v_\pm \in \mathcal{F}_g^{\mathcal{N}} \subset \mathcal{E}_g^{\mathcal{N}}$.

Since we obtain exactly two distinct solutions $v_+, v_- \in \mathcal{F}_g^{\mathcal{N}}$ for each rotating wave $w \in \mathcal{R}_f^{\mathcal{P}}$, the Neumann problem possesses exactly $2q$ spatially nonhomogeneous solutions. In addition to these solutions, also the spatially homogeneous solutions $e \in \mathcal{Z}_g^{\mathcal{P}} = \mathcal{Z}_f^{\mathcal{P}}$ of (4.20) satisfy Neumann boundary conditions. Hence $v \equiv e \in \mathcal{Z}_g^{\mathcal{N}}$ and the Neumann problem possesses exactly m spatially homogeneous solutions. Therefore, by cutting the circle S^1 at $x = 0$ and $x = \pi$ we obtain a Neumann problem ($\mathcal{N}$) on the resulting half-interval with $n := m + 2q$ stationary solutions, $\mathcal{E}_g^{\mathcal{N}} = \mathcal{Z}_g^{\mathcal{N}} \cup \mathcal{F}_g^{\mathcal{N}}$.

Comparing the period maps for ($\mathcal{N}$) and ($\mathcal{P}$) we obviously have the identity $T_g^{\mathcal{N}} = T_g^{\mathcal{P}}$. Hence $g \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$. In addition, if $v \in \mathcal{F}_g^{\mathcal{N}}$ possesses the extreme values (4.21), then

$$T_g^{\mathcal{N}}(\overline{v}) = T_g^{\mathcal{N}}(\underline{v}) = 2\pi/\ell \ , \ \ell \in \mathbb{N} \ , \tag{4.26}$$

by (3.9). Moreover, the following alternative is satisfied:

$$\begin{aligned} v_+(\pi) &= v_+(0) = \overline{v} \ , \ v_-(\pi) = v_-(0) = \underline{v} \quad \text{if } \ell \text{ is even} \ , \\ v_+(\pi) &= v_-(0) = \underline{v} \ , \ v_-(\pi) = v_+(0) = \overline{v} \quad \text{if } \ell \text{ is odd} \ . \end{aligned} \tag{4.27}$$

The integer $\ell := 2\pi/T_g^{\mathcal{N}}(\overline{v})$ is the *period lap number* of the frozen wave v_+. Indeed $\ell = \ell(v_+) = \ell(v_-)$. Note that

$$\ell(v_\pm) = z^{\mathcal{P}}(v_{\pm,x})/2 = z^{\mathcal{N}}(v_{\pm,x}) + 1 \tag{4.28}$$

coincides with the Matano lap number on the Neumann half-interval $x \in [0, \pi]$.

We collect the properties of the composed freezing and symmetrizing homotopy in the following:

Lemma 6: *Let $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$. Then there exists a C^2 homotopy $f^\tau, 0 \le \tau \le 1$, from $f^0 = f(v,p)$ to a reversible $f^1 = g(v,p)$, which is even in p, such that:*

(i) *All spatially homogeneous solutions of the periodic problem* ($\mathcal{P}$) *are preserved,*

$$\mathcal{Z}_g^{\mathcal{P}} = \mathcal{Z}_f^{\mathcal{P}} = \mathcal{E}_f^{\mathcal{P}} \ , \tag{4.29}$$

and remain hyperbolic along the homotopy.

(ii) *All rotating waves $w_k \in \mathcal{R}_f^{\mathcal{P}}$ remain (normally) hyperbolic and become reflection symmetric frozen waves $v_k \in \mathcal{F}_g^{\mathcal{P}}$ which satisfy*

$$\begin{aligned} \max_{x \in S1} v_k(x) &= \max_{x \in S1} w_k(x) \ , \\ \min_{x \in S1} v_k(x) &= \min_{x \in S1} w_k(x) \ . \end{aligned} \tag{4.30}$$

Furthermore, the period map $T_f^{\mathcal{P}}$ is preserved along the homotopy, on a restricted domain $\mathcal{C}_0^{\mathcal{P}}$ which contains the periodic spatial profiles of *all rotating waves $w_k \in \mathcal{F}^{\mathcal{P}}_f$. Hence,*

$$T^{\mathcal{P}}_g(a) = T^{\mathcal{P}}_f(a) \quad \textit{for} \quad a \in \mathcal{D}^{\mathcal{P}}_0 \ , \tag{4.31}$$

where $\mathcal{D}^{\mathcal{P}}_0 = \{a \in \mathbb{R} : (a,0) \in \mathcal{C}^{\mathcal{P}}_0\}$. Therefore, all frozen waves $v_k \in \mathcal{F}^{\mathcal{P}}_g$ are normally hyperbolic. Moreover, no new frozen or rotating waves are introduced.

In general $g \notin \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ but, instead, $g \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$. In addition, the following relations between problems $(\mathcal{P})$ and $(\mathcal{N})$ are satisfied:

(iii) *$v \equiv e$ is a spatially homogeneous equilibrium $v \in \mathcal{Z}^{\mathcal{P}}_g$ if and only if also $v \equiv e \in \mathcal{Z}^{\mathcal{N}}_g$.*

(iv) *Each frozen wave $v \in \mathcal{F}^{\mathcal{P}}_g$ has two different Neumann representatives v_+, v_- defined by (4.23), (4.21). Denoting their restrictions to the half-interval $0 \le x \le \pi$ by v_+, v_- again, we have that $v \in \mathcal{F}^{\mathcal{P}}_g$ if and only if $v_+, v_- \in \mathcal{F}^{\mathcal{N}}_g$.*

(v) *The period maps $T^{\mathcal{N}}_g$ and $T^{\mathcal{P}}_g$ are identical,*

$$T^{\mathcal{N}}_g = T^{\mathcal{P}}_g \ . \tag{4.32}$$

(vi) *The boundary values of $v_+, v_- \in \mathcal{F}^{\mathcal{N}}_g$ on the half-interval $0 \le x \le \pi$ satisfy (4.27), (4.26).*

Let us relate the Neumann semiflow $S^{\mathcal{N}}_g(t)$ on $X^{\mathcal{N}} = H^s(0,\pi) \cap \{u : u_x = 0 \text{ at } x = 0, \pi\}$ with the S^1 semiflow $S^{\mathcal{P}}_g(t)$ on $X^{\mathcal{P}} = H^s(S^1) = H^s(\mathbb{R}/2\pi\mathbb{Z})$. Recall that $g = g(v,p)$ is even in p, i.e. reversible. We first observe the isometric equivalence

$$X^{\mathcal{N}} \cong X^{\mathcal{P}} \cap \{u : \text{ all } b_k = 0\} \tag{4.33}$$

in terms of the Fourier expansion (1.7) of u in Section 1. Here we have represented elements $u \in X^{\mathcal{N}}$ as

$$u = \sum_{k \ge 0} a_k \cos kx \ , \tag{4.34}$$

due to their L^2 eigenfunction expansion for the Laplacian ∂_{xx}, and we can define $X^{\mathcal{N}}$ by the Fourier coefficients a_k via the bounded norm

$$\|u\|^2_{X^{\mathcal{N}}} := \sum_{k \ge 0} (1+k^2)^s a_k^2 \ . \tag{4.35}$$

Thanks to Fourier, this follows directly from the definition of the fractional power space $X^{\mathcal{N}} := \mathcal{D}((\mathrm{id} - \partial^2_{xx})^s)$ under Neumann boundary conditions. Alternatively, and circumventing the above slight abuse of notation, we may define the equivalence (4.33) by explicit reflection through the boundary as the lifting

$$\begin{aligned} E : X^{\mathcal{N}} &\to X^{\mathcal{P}} \cap \{u : \text{ all } b_k = 0\} \\ u &\mapsto \tilde{u} \end{aligned} \tag{4.36}$$

where $\tilde{u}(x) := u(x)$, for $0 \le x \le \pi$, and $\tilde{u}(x) := u(2\pi - x)$, for $\pi \le x \le 2\pi$. In particular the lift E commutes with the respective semiflows

$$S^{\mathcal{P}}_g(t) E u = E S^{\mathcal{N}}_g(t) u \ , \tag{4.37}$$

for all $u \in X^{\mathcal{N}}$. Indeed this follows from uniqueness of both semiflows and invariance of the reflection invariant subspace $X^{\mathcal{P}} \cap \{u : \text{ all } b_k = 0\}$ under the $O(2)$-equivariant semiflow $S_g^{\mathcal{P}}(t)$. The latter fact hinges on x-reversibility of g, of course.

Remark that, by Lemma 6, (iii) and (iv), each equilibrium or rotating wave $w \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ of $(\mathcal{P})$ is represented by an equilibrium $v_+ \in \mathcal{E}_g^{\mathcal{N}}$ of $(\mathcal{N})$ on the half-interval $x \in [0, \pi]$ with maximum value at $x = 0$. Comparing the adjacency definitions (4.2) in $(\mathcal{N})$ and (4.4) in $(\mathcal{P})$ we obtain the following correspondence:

Theorem 7: *Let $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ and $g \in \mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$ denote two homotopy related nonlinearities as in Lemma 6. Let $w^1, w^2 \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ denote two different equilibria or rotating waves of problem $(\mathcal{P})$ given by (1.17), and let $v_+^1, v_+^2 \in \mathcal{E}_g^{\mathcal{N}}$ denote the corresponding representative equilibria of problem $(\mathcal{N})$, i.e. the frozen Neumann solutions. Then w^1, w^2 are $(\mathcal{P})$-adjacent if and only if v_+^1, v_+^2 are $(\mathcal{N})$-adjacent. Moreover*

$$z^{\mathcal{N}}(v_+^2 - v_+^1) = z^{\mathcal{P}}(w^2 - w^1)/2 \ . \tag{4.38}$$

Proof: As in the previous Lemma, we let $v_+^j \in \mathcal{E}_g^{\mathcal{P}}$ denote the frozen and symmetrized representatives of the elements $w^j \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$, using the same notation as for their restrictions to the half-interval, $v_+^j \in \mathcal{E}_g^{\mathcal{N}}$. We recall that the composed freezing and symmetrizing homotopy preserves the relative configuration in the phase plane (v, p) of the periodic orbits and fixed points corresponding to rotating waves and equilibria of problem $(\mathcal{P})$. In particular, (4.30) implies that the periodic orbits of v_+^1 and v_+^2 have the same relative position in the phase plane (v, p) as the periodic orbits of w^1 and w^2. If these orbits are not nested then

$$z^{\mathcal{P}}(v_+^2 - v_+^1) = z^{\mathcal{P}}(w^2 - w^1) = 0 \ . \tag{4.39}$$

If, without loss, the orbit of w^1 is in the interior of the orbit of w^2, then the phase plane argument of [MN97], Lemma 4.8, implies that

$$z^{\mathcal{P}}(w^2 - w^1) = z^{\mathcal{P}}(w_x^2) \ , \tag{4.40}$$

and also

$$z^{\mathcal{P}}(v_+^2 - v_+^1) = z^{\mathcal{P}}(v_{+,x}^2) \ . \tag{4.41}$$

Then (4.31) shows that

$$z^{\mathcal{P}}(w_x^2) = 4\pi / T_f^{\mathcal{P}}(\overline{a}) = 4\pi / T_g^{\mathcal{P}}(\overline{a}) = z^{\mathcal{P}}(v_{+,x}^2) \ , \tag{4.42}$$

where $\overline{a} := \max_{x \in S^1} w^2(x) = \max_{x \in S^1} v_+^2(x)$. Hence

$$z^{\mathcal{P}}(v_+^2 - v_+^1) = z^{\mathcal{P}}(w^2 - w^1) \ . \tag{4.43}$$

Since the periodic problem is set up on the full interval $x \in [0, 2\pi]$ while the Neumann problem is set up on the half-interval $x \in [0, \pi]$, we indeed have (4.38). In particular $k := z^{\mathcal{P}}(w^1 - w^2)$ is even.

If w^1, w^2 are not $(\mathcal{P})$-adjacent, then there is an equilibrium or rotating wave $w \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ of (1.17) with

$$z^{\mathcal{P}}(w^1 - w) = z^{\mathcal{P}}(w^2 - w) = k \tag{4.44}$$

and satisfying (4.4). The corresponding representatives v_+^1, v_+^2 and v_+ in $\mathcal{E}_g^{\mathcal{N}}$, which are equilibria of the Neumann problem ($\mathcal{N}$) with maximum values occurring at $x = 0$, satisfy

$$v_+(0) \text{ is strictly between } v_+^1(0) \text{ and } v_+^2(0)\ . \tag{4.45}$$

By the phase plane argument ([MN97], Lemma 4.8) we have

$$z^{\mathcal{N}}(v_+^1 - v_+) = z^{\mathcal{N}}(v_+^2 - v_+) = k/2\ . \tag{4.46}$$

Hence, in view of (4.38), v_+^1, v_+^2 are not ($\mathcal{N}$)-adjacent, cf. (4.1), (4.2).

Conversely, suppose v_+^1, v_+^2 are not ($\mathcal{N}$)-adjacent. Then there exists an equilibrium solution $v \in \mathcal{E}_g^{\mathcal{N}}$ of the Neumann problem satisfying

$$z^{\mathcal{N}}(v_+^1 - v) = z^{\mathcal{N}}(v_+^2 - v) = z^{\mathcal{N}}(v_+^1 - v_+^2) \tag{4.47}$$

and $v(0)$ strictly between $v_+^1(0)$ and $v_+^2(0)$. Without loss of generality we assume

$$v_+^1(0) < v(0) < v_+^2(0)\ . \tag{4.48}$$

Let $\tilde{v}_+^1, \tilde{v}_+^2$ and $\tilde{v}$ denote the reflection extensions through the boundary of v_+^1, v_+^2 and v, respectively. These are frozen waves or spatially homogeneous equilibria of the frozen symmetrized periodic problem ($\mathcal{P}$),

$$\tilde{v}_+^1, \tilde{v}_+^2, \tilde{v} \in \mathcal{E}_g^{\mathcal{P}} = \mathcal{Z}_g^{\mathcal{P}} \cup \mathcal{F}_g^{\mathcal{P}}\ . \tag{4.49}$$

Then (4.47) implies

$$z^{\mathcal{P}}(\tilde{v}_+^1 - \tilde{v}) = z^{\mathcal{P}}(\tilde{v}_+^2 - \tilde{v}) = z^{\mathcal{P}}(\tilde{v}_+^1 - \tilde{v}_+^2) = k \tag{4.50}$$

with $k := 2z^{\mathcal{N}}(v_+^1 - v_+^2)$. Moreover, if $\tilde{v}$ is spatially nonhomogenous, then $\tilde{v}(0)$ is either the maximum or the minimum value of $\tilde{v}$. Respectively, $\tilde{v} \in \mathcal{F}_g^{\mathcal{P}}$ is the representative v_+, or v_-, of a frozen wave corresponding to a rotating wave $w \in \mathcal{R}_f^{\mathcal{P}}$. Therefore, by the invariance of $z^{\mathcal{P}}(\tilde{v}_+^j(\cdot + \vartheta) - \tilde{v}(\cdot))$ with respect to phase shifts $\vartheta \in S^1$ (see [AF88, MN97, FRW04] again) we find that

$$z^{\mathcal{P}}(w^1 - w) = z^{\mathcal{P}}(w^2 - w) = z^{\mathcal{P}}(w^1 - w^2)\ . \tag{4.51}$$

The same holds true if $\tilde{v} \equiv e \in \mathcal{Z}_g^{\mathcal{P}}$ is spatially homogeneous, in which case $w \equiv e \in \mathcal{E}_f^{\mathcal{P}}$ is the corresponding spatially homogeneous equilibrium. Furthermore, we also have

$$\max_{x\in S^1} w^1(x) = v_+^1(0) \quad , \quad \max_{x\in S^1} w^2(x) = v_+^2(0)\ . \tag{4.52}$$

If $k > 0$ we conclude from $\tilde{v}(0) = v(0) > v_+^1(0)$ and $z^{\mathcal{P}}(v_+^1 - \tilde{v}) = k$ that $\tilde{v} = v_+$. Hence

$$\max_{x\in S^1} w(x) = \max_{x\in S^1} \tilde{v}(x) = v(0)\ . \tag{4.53}$$

Consequently, we have

$$\max_{x\in S^1} w^1(x) < \max_{x\in S^1} w(x) < \max_{x\in S^1} w^2(x)\ . \tag{4.54}$$

If $k = 0$ this holds true trivially. Hence w blocks adjacency of w^1 and w^2 which, therefore, cannot be ($\mathcal{P}$)-adjacent. This concludes the proof. □

Adjacency in $(\mathcal{N})$ determines the heteroclinic connectivity of the elements in $\mathcal{E}_g^{\mathcal{N}}$. The same occurs in $(\mathcal{P})$ for the elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$. In fact, we recall the following relation between adjacency and connectivity in $(\mathcal{P})$:

Theorem 8: *(See* [FRW04], *Theorems 1.3 and 1.4.) Again let* $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ *and consider any two different (hyperbolic) elements* $w^{\pm} \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$. *Then there is a heteroclinic orbit connection between* w^+ *and* w^- *if, and only if,* w^+ *and* w^- *are* $(\mathcal{P})$*-adjacent. Moreover, the connecting orbits run in the direction of decreasing Morse indices, that is,* $w^+ \leadsto w^-$ *only if*

$$i^{\mathcal{P}}(w^+) > i^{\mathcal{P}}(w^-) \ . \tag{4.55}$$

The two previous theorems together show that, for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$, two different (hyperbolic) equilibria or rotating waves $w^1, w^2 \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ of (1.17) on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ are connected by a heteroclinic orbit if, and only if, the corresponding frozen Neumann solutions $v_+^1, v_+^2 \in \mathcal{E}_g^{\mathcal{N}}$ are $(\mathcal{N})$-adjacent on the half-interval $[0, \pi]$.

For completeness we next collect various aspects of this relation between the equilibria and rotating waves of our periodic problem $(\mathcal{P})$ and the corresponding frozen equilibria of the Neumann problem $(\mathcal{N})$.

Theorem 9:

(i) Let $f(e, 0) = 0$ *define a homogeneous equilibrium of* $(\mathcal{P})$ *or* $(\mathcal{N})$. *Then* e *is an equilibrium both for* $(\mathcal{P})$ *and* $(\mathcal{N})$. *Moreover,* e *is hyperbolic if and only if* $f_v(e, 0) \neq k^2, k \in \mathbb{Z}$. *The respective Morse indices* $i^{\mathcal{P}}(e)$ *for* $e \in \mathcal{E}_f^{\mathcal{P}}$ *and* $i^{\mathcal{N}}(e)$ *for* $e \in \mathcal{E}_f^{\mathcal{N}}$ *satisfy:*

$$\begin{aligned} &i^{\mathcal{P}}(e) = 2i^{\mathcal{N}}(e) - 1 = 1 + 2[\sqrt{f_v(e,0)}] > 0 \qquad \text{if } f_v(e,0) > 0 \ , \\ &i^{\mathcal{P}}(e) = i^{\mathcal{N}}(e) = 0 \qquad \text{if } f_v(e,0) < 0 \ , \end{aligned} \tag{4.56}$$

where the floor function $[\cdot]$ *denotes the integer part.*

(ii) Let $w \in \mathcal{R}_f^{\mathcal{P}}$ *denote a rotating wave of the periodic problem* $(\mathcal{P})$ *with nonlinearity* $f = f(v, p)$. *Also for problem* $(\mathcal{P})$, *but with a reversible nonlinearity* $g = g(v, p)$ *obtained from* f *by the freezing and symmetrizing homotopy of Lemma 6, let* $v \in \mathcal{F}_g^{\mathcal{P}}$ *represent a nonhomogeneous frozen wave equilibrium corresponding to* w. *In addition, let* $v_{\pm} \in \mathcal{F}_g^{\mathcal{P}}$ *denote* x*-shifted copies of* v *such that*

$$\begin{aligned} v_+(0) &= \max_{x \in S^1} v(x) = \max_{x \in S^1} w(x) \ , \\ v_-(0) &= \min_{x \in S^1} v(x) = \min_{x \in S^1} w(x) \ . \end{aligned} \tag{4.57}$$

Then the rotating wave w *is (normally) hyperbolic for the problem* $(\mathcal{P})$ *with nonlinearity* f *if, and only if, the frozen representation* $v_+ \in \mathcal{E}_g^{\mathcal{N}}$ *in the half-interval* $x \in [0, \pi]$ *is a hyperbolic equilibrium of the Neumann problem* $(\mathcal{N})$ *with nonlinearity* g. *This statement also holds with* v_+ *replaced by* v_-.

Let $\ell(w)$ *denote the period lap number of the rotating wave* $w \in \mathcal{R}_f^{\mathcal{P}}$; *see (4.10). Then, the period lap number of the frozen waves* $v, v_{\pm} \in \mathcal{F}_g^{\mathcal{P}}$ *satisfy*

$$\ell(v_+) = \ell(v_-) = \ell(v) = \ell(w) \ . \tag{4.58}$$

Moreover, the Morse indices of $w \in \mathcal{R}_f^{\mathcal{P}}$ *and* $v_+ \in \mathcal{E}_g^{\mathcal{N}}$*, for problems* $(\mathcal{P})$ *and* $(\mathcal{N})$ *respectively, satisfy*

$$i^{\mathcal{N}}(v_+) \quad \in \quad \{\ell(v_+), \ell(v_+)+1\} \ , \quad \textit{and} \tag{4.59}$$

$$i^{\mathcal{P}}(w) \quad = \quad \begin{cases} 2i^{\mathcal{N}}(v_+) - 1 & \textit{if } \ i^{\mathcal{N}}(v_+) = \ell(v_+) \ , \\ \\ 2i^{\mathcal{N}}(v_+) - 2 & \textit{if } \ i^{\mathcal{N}}(v_+) = \ell(v_+) + 1 \ . \end{cases} \tag{4.60}$$

In addition $i^{\mathcal{N}}(v_+) = i^{\mathcal{N}}(v_-)$ *and (4.59), (4.60) also hold with* v_+ *replaced by* v_-*.*

Proof of Theorem 9(i): The proof of (i) is an elementary computation involving sines and cosines in the case of $(\mathcal{P})$ and only cosines in the case of $(\mathcal{N})$. As a remark we point out that (4.56) implies that the Morse index $i^{\mathcal{P}}(e)$ of an unstable homogeneous hyperbolic equilibrium $e \in \mathcal{E}_f^{\mathcal{P}}$ of $(\mathcal{P})$ is always odd. $\square$

As an outline to the proof of (ii) we recall the relations (4.31), (4.32) between period maps of $(\mathcal{P})$ and $(\mathcal{N})$ in Lemma 6, i.e.

$$T_g^{\mathcal{N}}(a) = T_g^{\mathcal{P}}(a) = T_f^{\mathcal{P}}(a) \ \text{ for } \ a \in \mathcal{D}_0^{\mathcal{P}} \ . \tag{4.61}$$

We use these identities to show that hyperbolicity of the equilibrium $v_+ \in \mathcal{E}_g^{\mathcal{N}}$ of $(\mathcal{N})$ is equivalent to hyperbolicity of $w \in \mathcal{R}_f^{\mathcal{P}}$ in case w is a rotating wave of $(\mathcal{P})$, or its normal hyperbolicity in case w is a frozen wave.

The Neumann case $(\mathcal{N})$ is considered in [FRW11], Lemma 2, and refers to similar results appearing in the literature, in particular [Smo83, BF88, BF89, FRW04]. We restate this result in the following Proposition.

Proposition 10: *Let* $v_+, v_- \in \mathcal{E}_g^{\mathcal{N}}$ *denote equilibrium solutions of the Neumann problem* $(\mathcal{N})$ *in the half-interval* $x \in [0, \pi]$ *corresponding to* x*-shifted copies of a frozen wave* $v \in \mathcal{F}_g^{\mathcal{P}}$ *with* $v_+(0) = \max_{x \in S^1} v(x)$*,* $v_-(0) = \min_{x \in S^1} v(x)$*. Then, hyperbolicity of* $v_+ \in \mathcal{E}_g^{\mathcal{N}}$ *(and also of* $v_- \in \mathcal{E}_g^{\mathcal{N}}$*) occurs if and only if*

$$\dot{T}_g^{\mathcal{N}}(a) \neq 0 \tag{4.62}$$

for $a := v_+(0)$*. Also* $i^{\mathcal{N}}(v_+) = i^{\mathcal{N}}(v_-)$ *in the hyperbolic case. Moreover,*

$$i^{\mathcal{N}}(v_+) = \begin{cases} \ell(v_+) & \textit{if } \ \dot{T}_g^{\mathcal{N}}(a) > 0 \ , \\ \\ \ell(v_+) + 1 & \textit{if } \ \dot{T}_g^{\mathcal{N}}(a) < 0 \ , \end{cases} \tag{4.63}$$

where the period lap number $\ell(v_+)$ *of* v_+ *satisfies* $\ell(v_+) = 2\pi / T_g^{\mathcal{N}}(a)$*.*

For the benefit of the reader we include the proof of this well known result.

Proof: We start with a reversible C^2-smooth nonlinearity $g = g(v, p)$, even in p, and let $v = v(\cdot, \alpha)$ denote the solution of the ODE

$$v'' + g(v, v') = 0 \ , \tag{4.64}$$

with initial conditions

$$v(0, \alpha) = \alpha \quad , \quad v'(0, \alpha) = 0 \ . \tag{4.65}$$

By definition (3.14) of the period map $T_g^{\mathcal{N}} : \mathcal{D}^{\mathcal{N}} \to \mathbb{R}_+$ we also have that

$$v(kT_g^{\mathcal{N}}(\alpha), \alpha) = \alpha \quad , \quad v'(kT_g^{\mathcal{N}}(\alpha), \alpha) = 0 \tag{4.66}$$

for $\alpha \in \mathcal{D}^{\mathcal{N}}$ and any integer $k \in \mathbb{N}$. Moreover, in view of the reversibility of g and by the arguments (4.24)-(4.25) we obtain

$$v'(kT_g^{\mathcal{N}}(\alpha)/2, \alpha) = 0 \ . \tag{4.67}$$

Differentiating this equation with respect to α and defining

$$\overline{v}(\cdot, \alpha) := \partial v(\cdot, \alpha)/\partial \alpha \tag{4.68}$$

we have

$$\overline{v}'(kT_g^{\mathcal{N}}(\alpha)/2, \alpha) + v''(kT_g^{\mathcal{N}}(\alpha)/2, \alpha)\, k\, \dot{T}_g^{\mathcal{N}}(\alpha)/2 = 0 \ . \tag{4.69}$$

For $a \in \mathcal{D}^{\mathcal{N}}$ such that $T_g^{\mathcal{N}}(a) = 2\pi/k$ and $v''(0, a) < 0$ we have that $a = v(0, a)$ is the maximum value of the 2π-periodic solution v of (4.64). In this case, the restriction of $v(\cdot, a)$ to the interval $x \in [0, \pi]$ is $v_+ \in \mathcal{E}_g^{\mathcal{N}}$. In addition,

$$\operatorname{sign} v''(\pi, a) = (-1)^{\ell} \operatorname{sign} v''(0, a) = (-1)^{\ell+1} \ , \tag{4.70}$$

where $\ell := \ell(v_+)$ is the period lap number of the equilibrium v_+; see (4.28). Together with (4.69) this shows that $\dot{T}_g^{\mathcal{N}}(a) \neq 0$ implies

$$\operatorname{sign} \overline{v}'(\pi, a) = (-1)^{\ell} \operatorname{sign} \dot{T}_g^{\mathcal{N}}(a) \ . \tag{4.71}$$

To prove the specific claims (4.63) we establish a relation between $\overline{v}'(\pi, a)$ and the Morse index $i^{\mathcal{N}}(v_+)$. Towards this objective we first remark that $\overline{v}$ solves the linearization

$$\mathcal{L}\overline{v} := \overline{v}'' + g_p(v(x), v'(x))\overline{v}' + g_v(v(x), v'(x))\overline{v} = 0 \ , \tag{4.72}$$

with initial conditions

$$\overline{v}(0, \alpha) = 1 \quad , \quad \overline{v}'(0, \alpha) = 0 \ ; \tag{4.73}$$

see (4.65). But v_x also solves $\mathcal{L}v_x = 0$ and is linearly independent from $\overline{v}$, by (4.65), (4.73). Hence the classical Sturm comparison theorem implies that the zeros of $\overline{v}$ and v_x alternate. See, for example, [CL55] or [Har64]. Therefore, we obtain

$$z^{\mathcal{N}}(\overline{v}(\cdot, a)) = z^{\mathcal{N}}(v_x) + 1 = \ell(v_+) \ . \tag{4.74}$$

More generally, let ψ_μ denote the solution of the eigenvalue problem

$$\mathcal{L}\psi_\mu = \mu\psi_\mu \tag{4.75}$$

with initial conditions

$$\psi_\mu(0) = 1 \quad , \quad \psi_\mu'(0) = 0 \ . \tag{4.76}$$

Then we have

$$\overline{v}(\cdot, a) = \psi_0 \ . \tag{4.77}$$

Note that (4.69) with $\dot{T}_g^{\mathcal{N}}(a) = 0$ yields $\overline{v}'(\pi, a) = 0$. In this case ψ_0 restricted to $x \in [0, \pi]$ is an eigenfunction of the linearization of the Neumann problem ($\mathcal{N}$) around the equilibrium v_+ corresponding to the eigenvalue $\mu = 0 = \lambda_\ell$. In particular, this yields the hyperbolicity claim (4.62) for the Neumann equilibrium v_+ on the half-interval $x \in [0, \pi]$.

Again by Sturm comparison, the nonzero vector $\Psi_\mu := (\psi_\mu, \psi'_\mu)$ rotates more slowly than $\overline{V} := (\overline{v}, \overline{v}')$ for positive eigenvalue parameters μ, whereas it rotates faster for negative μ. Therefore (4.73) and (4.74) imply that the eigenvalues λ_k of the Sturm-Liouville Neumann linearization $\mathcal{L}$ satisfy

$$\begin{array}{lcl} \lambda_\ell > 0 > \lambda_{\ell+1} & \Leftrightarrow & \operatorname{sign} \overline{v}'(\pi, a) = (-1)^{\ell+1} , \\ \lambda_\ell < 0 < \lambda_{\ell-1} & \Leftrightarrow & \operatorname{sign} \overline{v}'(\pi, a) = (-1)^{\ell} . \end{array} \tag{4.78}$$

Combined with (4.71), this is equivalent to

$$\begin{array}{lcl} i^{\mathcal{N}}(v_+) = \ell(v_+) + 1 & \Leftrightarrow & \dot{T}^{\mathcal{N}}_g(a) < 0 , \\ i^{\mathcal{N}}(v_+) = \ell(v_+) & \Leftrightarrow & \dot{T}^{\mathcal{N}}_g(a) > 0 , \end{array} \tag{4.79}$$

which proves claim (4.63). The claims on v_- follow from (4.26) and the same Sturm comparison argument. $\square$

Proof of Theorem 9(ii): To prove (ii) we use (4.61) to establish the relation between hyperbolicity of the Neumann equilibria $v_\pm \in \mathcal{E}^{\mathcal{N}}_g$ of ($\mathcal{N}$) and (normal) hyperbolicity of the rotating or frozen wave $w \in \mathcal{R}^{\mathcal{P}}_f \cup \mathcal{F}^{\mathcal{P}}_f$ of ($\mathcal{P}$). Hyperbolicity in the Neumann case ($\mathcal{N}$) is the subject of Proposition 10. In the case of problem ($\mathcal{P}$), (normal) hyperbolicity of $w \in \mathcal{R}^{\mathcal{P}}_f$ occurs if, and only if, $\dot{T}^{\mathcal{P}}_f(\overline{w}) \neq 0$ for $\overline{w} := \max_{x \in S^1} w(x)$, see also (4.9). For the proof of this result see [FRW04], Lemmas 4.3 and 4.4. Moreover, by [FRW04], Lemma 5.3, the following alternative holds

$$i^{\mathcal{P}}(w) = \begin{cases} 2\ell(w) - 1 & \text{if } \dot{T}^{\mathcal{P}}_f(\overline{w}) > 0 , \\ & \\ 2\ell(w) & \text{if } \dot{T}^{\mathcal{P}}_f(\overline{w}) < 0 , \end{cases} \tag{4.80}$$

where $\ell(w) = z^{\mathcal{P}}(w_x)/2$ is the period lap number of w, (4.10). By (4.8) we have $\ell(w) = 2\pi / T^{\mathcal{P}}_f(\overline{w})$. Then, equality of the period lap numbers (4.58) follows from (4.61). Finally, a comparison between (4.80) and (4.63) via

$$\ell(v_+) = 2\pi / T^{\mathcal{N}}_g(\overline{w}) = 2\pi / T^{\mathcal{P}}_f(\overline{w}) = \ell(w) \tag{4.81}$$

yields (4.60) and completes the proof. $\square$

5. Proof of the main result

This section is dedicated to the proof of Theorem 3, which is the main result already stated in Section 1. In preparation for this proof, we recall the definition of the Sturm permutation $\sigma^{\mathcal{P}}_f$ in the spatially periodic setting of $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$. Let $(\nu_1, \ldots, \nu_n)$ denote the vector whose entries, ordered by $\nu_1 < \nu_2 < \cdots < \nu_n$, correspond to the collected values of the spatially homogeneous equilibria $e_j \in \mathcal{E}^{\mathcal{P}}_f, j = 1, \ldots, m$, and the maxima $\overline{w}_k$ and minima $\underline{w}_k$ of the rotating waves $w_k \in \mathcal{R}^{\mathcal{P}}_f, k = 1, \ldots, q$. Of course $n = m + 2q$. Then, the cycle decomposition of the permutation $\sigma^{\mathcal{P}}_f \in \mathcal{S}(n)$ of $\{\nu_1, \ldots, \nu_n\}$ consists of all maximum/minimum 2-cycles $(\underline{w}_k\ \overline{w}_k)$ for those rotating waves w_k with odd period lap number $\ell_k := \ell(w_k)$, see (4.10):

$$\sigma^{\mathcal{P}}_f = \prod_{\ell_k \text{ odd}} (\underline{w}_k\ \overline{w}_k) . \tag{5.1}$$

Proof of Theorem 3: Let $g = g(v,p)$ denote the reversible nonlinearity obtained from $f = f(v,p)$ by the freezing and symmetrizing homotopy of Lemma 6. Our objective is to identify the Sturm permutation $\sigma_f^{\mathcal{P}}$ defined for $f \in \mathrm{Sturm}^{\mathcal{P}}(u,u_x)$ as the Sturm permutation $\sigma_g^{\mathcal{N}}$ derived from the frozen and symmetrized $g \in \mathrm{Sturm}^{\mathcal{N}}(u,u_x^2)$,

$$\sigma_f^{\mathcal{P}} = \sigma_g^{\mathcal{N}}. \tag{5.2}$$

Indeed suppose (5.2) has been proved. Theorem 5′ applied to $\sigma_g^{\mathcal{N}}$ then shows that the Sturm permutation $\sigma_f^{\mathcal{P}}$ is necessarily an integrable involution. Moreover, this condition is also sufficient. In fact, by Theorem 5 all Sturm integrable involutions are realized by nonlinearities $g \in \mathrm{Sturm}^{\mathcal{N}}(u)$. Therefore, all Sturm integrable involutions are realized by nonlinearities $f \in \mathrm{Sturm}^{\mathcal{P}}(u,u_x)$ of the form $f(u,p) = g(u) + cp$ with $g \in \mathrm{Sturm}^{\mathcal{N}}(u)$ and $c \neq 0$. This proves Theorem 3, up to the $\mathcal{P} - \mathcal{N}$ claim (5.2) above.

To prove (5.2) we scrutinize the freezing and symmetrizing homotopy of Lemma 6. The homogeneous equilibria $e_j, j = 1,\dots,m$, in $\mathcal{E}_g^{\mathcal{P}}$ are also zeros of $g(\cdot,0)$, hence equilibria of the Neumann problem ($\mathcal{N}$) on the half-interval $x \in [0,\pi]$;

$$e_j \in \mathcal{E}_g^{\mathcal{N}}\ ,\ j = 1,\dots,m\ . \tag{5.3}$$

Let $v_k \in \mathcal{F}_g^{\mathcal{P}}, k = 1,\dots,q$, denote the reflection symmetric frozen waves corresponding to the rotating waves $w_k \in \mathcal{R}_f^{\mathcal{P}}, k = 1,\dots,q$; see Lemma 6, (ii). Moreover, let $v_{k,\pm} \in \mathcal{F}_g^{\mathcal{N}}, k = 1,\dots,q$, denote the representatives of the frozen waves v_k which satisfy Neumann boundary conditions on the half-interval $x \in [0,\pi]$; see Lemma 6, (iv). By (4.23), (4.21) and (4.30) we have

$$\begin{aligned} v_{k,+}(0) &= \overline{w}_k = \max_{x\in S^1} w_k(x)\ ,\\ v_{k,-}(0) &= \underline{w}_k = \min_{x\in S^1} w_k(x)\ . \end{aligned} \tag{5.4}$$

To compute the Sturm permutation $\sigma_g^{\mathcal{N}}$, let $\{\hat{v}_1,\dots,\hat{v}_n\}$, denote the set of equilibria of the Neumann problem, $\hat{v}_r \in \mathcal{E}_g^{\mathcal{N}},\ r = 1,\dots,n$, ordered by their initial values

$$\hat{v}_1(0) < \hat{v}_2(0) < \cdots < \hat{v}_n(0)\ . \tag{5.5}$$

Then, we have $\hat{v}_r = \nu_r$ for $1 \le r \le n$ and one of the following three possibilities occur:

- $\hat{v}_r$ is a spatially homogeneous equilibrium $e_j \in \mathcal{E}_g^{\mathcal{N}}$, in which case
$$\hat{v}_r(\pi) = \hat{v}_r(0)\ ; \tag{5.6}$$
- $\hat{v}_r$ is a frozen wave representative $v_{k,\pm} \in \mathcal{E}_g^{\mathcal{N}}$ such that ℓ_k is even, in which case again we have (5.6);
- $\hat{v}_r$ is a frozen wave representative $v_{k,\pm} \in \mathcal{E}_g^{\mathcal{N}}$ such that ℓ_k is odd, in which case we have the alternative
$$\begin{aligned} &\hat{v}_r(0) = \overline{w}_k \ \text{ and } \ \hat{v}_r(\pi) = \underline{w}_k \quad \text{ if } \ \hat{v}_r = v_{k,+}\ ,\\ &\hat{v}_r(0) = \underline{w}_k \ \text{ and } \ \hat{v}_r(\pi) = \overline{w}_k \quad \text{ if } \ \hat{v}_r = v_{k,-}\ . \end{aligned} \tag{5.7}$$

Since $\sigma = \sigma_g^{\mathcal{N}}$ is defined by (2.14),

$$\hat{v}_{\sigma(1)}(\pi) < \hat{v}_{\sigma(2)}(\pi) < \cdots < \hat{v}_{\sigma(n)}(\pi)\ , \tag{5.8}$$

we conclude from (5.6), (5.7) that $\sigma_f^{\mathcal{P}} = \sigma_g^{\mathcal{N}}$. This identifies $\sigma_f^{\mathcal{P}}$ as $\sigma_g^{\mathcal{N}}$ and completes the proof of Theorem 3. □

6. Sturm Attractors in the class $\mathrm{Sturm}^{\mathcal{P}}(u, u_x)$

Sturm permutations $\sigma_f^{\mathcal{P}}$ encode essential information on the Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$. In particular, $\sigma_f^{\mathcal{P}}$ in the class of $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ collects ODE information regarding the equilibria and rotating waves of $\mathcal{A}_f^{\mathcal{P}}$ such as Morse indices, zero numbers, adjacency relations and period lap numbers.

In the following we represent the PDE Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ in the spatially periodic class of S^1-equivariant $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ by *connection graphs* $\mathcal{G}_f^{\mathcal{P}}$. The graphs $\mathcal{G}_f^{\mathcal{P}}$ are directed acyclic with $m + q$ vertices of two types: m vertices corresponding to the homogeneous equilibria $v_1, \ldots, v_m \in \mathcal{E}_f^{\mathcal{P}}$, represented by black dots; and q vertices corresponding to the rotating waves $w_1, \ldots, w_q \in \mathcal{R}_f^{\mathcal{P}}$, represented by circles with Matano's period lap numbers $\ell(w_k) = z^{\mathcal{P}}(w_{k,x})/2$ attached. The directed edges of $\mathcal{G}_f^{\mathcal{P}}$ correspond to heteroclinic orbit connections between adjacent elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$. By transversality, heteroclinic connectivity in $\mathcal{A}_f^{\mathcal{P}}$ is transitive; see for example [HMO02]. Moreover, Theorem 8 implies that heteroclinic orbits can only run from higher to strictly lower Morse indices. Hence the connection graph $\mathcal{G}_f^{\mathcal{P}}$ comes with a natural flow defined edge orientation. In addition, we invoke transitivity to represent only heteroclinic connections $v^+ \leadsto v^-$ between elements $v^{\pm} \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ which are not transitively connected via additional elements $w \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$, eg. $v^+ \leadsto w \leadsto v^-$. We call such heteroclinic connections *minimal* since they are minimal with respect to the transitivity of $\leadsto$. Therefore, an edge between two vertices $v^+, v^- \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ represents a heteroclinic orbit connection $v^+ \leadsto v^-$ which is minimal. By transitivity, all heteroclinic connections in $\mathcal{A}_f^{\mathcal{P}}$ can be inferred from the minimal heteroclinic connections.

The (minimal) *flow-defined order* of a Morse decomposition is an equivalent notion; see for example [Mis95]. Indeed the homogeneous equilibria $\mathcal{E}_f^{\mathcal{P}}$ and the rotating wave circles $\mathcal{R}_f^{\mathcal{P}}$ together define a Morse decomposition of the global attractor $\mathcal{A}_f^{\mathcal{P}}$, due to the absence of homoclinic orbits and of heteroclinic cycles among them. The flow-defined order is the minimal transitive order relation compatible with the directed heteroclinic orbits $u(t),\ t \in \mathbb{R}$, among the distinct elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$. In other words, the flow-defined order prefers to order as few pairs as possible: only if there exists a directed heteroclinic path between two distinct elements of $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ this pair is ordered accordingly. Therefore the flow-defined order is indeed equivalent to the above orientation of the acyclic connection graph $\mathcal{G}_f^{\mathcal{P}}$ of the global attractor $\mathcal{A}_f^{\mathcal{P}}$.

In the Neumann class $\mathrm{Sturm}^{\mathcal{N}}(x, u, u_x)$, Sturm attractors $\mathcal{A}_{f_1}^{\mathcal{N}}, \mathcal{A}_{f_2}^{\mathcal{N}}$ with the same Sturm permutation $\sigma_{f_1}^{\mathcal{N}} = \sigma_{f_2}^{\mathcal{N}}$ are flow equivalent, cf. [FR00]. We suspect that this also holds in the spatially periodic class $\mathrm{Sturm}^{\mathcal{P}}(u, u_x)$.

Alas, a proof is elusive even though transversality reduces flow equivalence to the construction of homotopies. By Lemma 6 we have reduced $\mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ to the reversible/integrable class $\mathrm{Sturm}^{\mathcal{N}}(u, u_x^2)$, but we are lacking a homotopy in this class, as in the original equivariant class. Nevertheless, Theorem 8 shows that $\sigma_{f_1}^{\mathcal{P}} = \sigma_{f_2}^{\mathcal{P}}$ implies *connection equivalence* of $\mathcal{A}_{f_1}^{\mathcal{P}}, \mathcal{A}_{f_2}^{\mathcal{P}}$, in the sense that

$$\mathcal{G}_{f_1}^{\mathcal{P}} = \mathcal{G}_{f_2}^{\mathcal{P}} \ . \tag{6.1}$$

Therefore, a list of all integrable Sturm involutions $\sigma_f^{\mathcal{P}}$ provides a first step towards a classification of the Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$.

We represent permutations $\sigma \in \mathcal{S}(n)$ of $\{1, 2, \ldots, n\}$ by cycle notation. Let $\iota(n)$ denote the number of integrable Sturm involutions $\sigma \in \mathcal{S}(n)$. As the number of meander permutations in $\mathcal{S}(n)$ increases exponentially with n (see [LZ92]), the number $\iota(n)$ might also get very large with increasing n. Table 1 shows the first six numbers in the sequence. Table 2 lists, up to trivial equivalences induced by $x \mapsto -x$, $u \mapsto -u$, all Sturm permutations $\sigma_{n,\kappa} \in \mathcal{S}(n), 1 \le \kappa \le \iota(n), n \le 7$, which are integrable involutions. See [FRW11] for details on these lists.

n	1	3	5	7	9	11
$\iota(n)$	1	1	2	4	10	23

TABLE 1. Number $\iota(n)$ of Sturm permutations $\sigma \in \mathcal{S}(n)$ which are integrable involutions.

$$\begin{aligned}
&n = 1 : \\
\sigma_{1,1} &= \{1\} = \mathrm{id} \ ; \\[1em]
&n = 3 : \\
\sigma_{3,1} &= \{1, 2, 3\} = \mathrm{id} \ ; \\[1em]
&n = 5 : \\
\sigma_{5,1} &= \{1, 2, 3, 4, 5\} = \mathrm{id} \ ; \\
\sigma_{5,2} &= \{1, 4, 3, 2, 5\} = (2\ 4) \ ; \\[1em]
&n = 7 : \\
\sigma_{7,1} &= \{1, 2, 3, 4, 5, 6, 7\} = \mathrm{id} \ ; \\
\sigma_{7,2} &= \{1, 2, 3, 6, 5, 4, 7\} = (4\ 6) \ ; \\
\sigma_{7,3} &= \{1, 6, 3, 4, 5, 2, 7\} = (2\ 6) \ ; \\
\sigma_{7,4} &= \{1, 6, 5, 4, 3, 2, 7\} = (2\ 6)(3\ 5) \ ;
\end{aligned}$$

TABLE 2. List of all integrable Sturm involutions $\sigma_{n,\kappa} \in \mathcal{S}(n), 1 \le \kappa \le \iota(n), n \le 7$, up to trivial equivalence.

By Theorem 3, Tables 1 and 2 also provide complete lists of all Sturm permutations $\sigma_f^{\mathcal{P}}$ in the spatially periodic class $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$.

All necessary information to construct the connection graphs $\mathcal{G}_f^{\mathcal{P}}$ of the associated Sturm attractors $\mathcal{A}_f^{\mathcal{P}}$ in that periodic class is contained in the Sturm permutation $\sigma_f^{\mathcal{P}}$. This includes not only ODE information like the Morse indices and the period lap numbers. It also contains PDE information like the heteroclinic adjacency relations of all equilibria and rotating waves in $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$. With the objective of constructing the connection graphs $\mathcal{G}_f^{\mathcal{P}}$ from the Sturm integrable involution $\sigma = \sigma_f^{\mathcal{P}}$ we proceed in five steps to sequentially determine all:

1. Stable homogeneous equilibria;
2. Unstable homogeneous equilibria;
3. Rotating waves;
4. Heteroclinic orbit connections;
5. Period lap numbers and Morse indices.

Step 1: Stable homogeneous equilibria.

Given an integrable Sturm involution $\sigma = \sigma_f^{\mathcal{P}}$, we first compute the Morse numbers $i_k(\sigma)$ according to (2.15). Then we determine the σ-stable points, i.e. the points k such that $i_k(\sigma) = 0$; see Section 3. By [FRW11], Lemma 7, all σ-stable points k are fixed points of σ, that is $\sigma(k) = k$. This immediately identifies the stable equilibria of $\mathcal{A}_f^{\mathcal{P}}$, alias the ODE saddles of (4.20). Indeed, by Theorem 9, (4.56), and (2.16) the stable equilibria correspond to the solutions $v_k \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ for which

$$i^{\mathcal{P}}(v_k) = i^{\mathcal{N}}(v_k) = i_k(\sigma) = 0 \ . \tag{6.2}$$

We recall that only equilibria of $\mathcal{A}_f^{\mathcal{P}}$ can be stable, [AF88]. Moreover, all equilibria are spatially homogeneous, $\mathcal{E}_f^{\mathcal{P}} = \mathcal{Z}_f^{\mathcal{P}}$.

Step 2: Unstable homogeneous equilibria.

Next, we determine the remaining (unstable homogeneous) equilibria. We remark that between each pair of consecutive stable equilibria there is exactly one PDE unstable equilibrium, alias an ODE center of (4.20). Let $j < k$ denote the ordering of two successive stable equilibria, i.e.

$$i_j(\sigma) = i_k(\sigma) = 0 \ \text{ and } \ i_c(\sigma) > 0 \text{ for all } j < c < k \ . \tag{6.3}$$

Of course j and k are fixed by the permutation σ and, by [FRW11], Lemma 1, they are both odd. By the meander property of the permutation σ we have a Jordan curve which intersects the horizontal axis transversely at exactly n points; see Section 2. In our stylized version this Jordan curve always intersects verticaly the horizontal axis and is composed of half-circles. This curve is called a *meander* (see [Arn88]) and the intersection points are numbered by $1, 2, \ldots, n$ along the meander, and by $\sigma(1), \sigma(2), \ldots, \sigma(n)$ along the horizontal axis. The *meander section* $\mu(j,k)$ corresponding to the interval $j, j+1, \ldots, k$ is also a Jordan curve which yields the section $\sigma(j), \sigma(j+1), \ldots, \sigma(k)$ of the permutation σ.

The Morse property $i_c(\sigma) \geq 0$, for $j < c < k$, implies

$$\sigma(j+1) > j \ . \tag{6.4}$$

In fact, since $\sigma = \sigma^{-1}$ is an involution, from (2.15) we obtain

$$i_{j+1}(\sigma) = i_j(\sigma) + (-1)^{j+1}\,\mathrm{sign}(\sigma(j+1) - \sigma(j)) \ . \tag{6.5}$$

Hence $i_j(\sigma) = 0$, $\sigma(j) = j$ odd and $i_{j+1}(\sigma) > 0$ yields (6.4). Therefore $\sigma(j+1)$ is located to the right of j on the horizontal axis. Similarly we have

$$\sigma(k-1) < k \tag{6.6}$$

and conclude that $\sigma(k-1)$ is located to the left of k on the horizontal axis.

To determine the (unique) unstable homogeneous equilibrium located between the (stable) equilibria v_j and v_k we consider the alternative $k = j+2$ or $k > j+2$ (alias $k \geq j+4$).

If $k = j+2$ by (6.4) and (6.6) we have that $\sigma(j+1) = j+1$ is fixed as well by the permutation. In this case, $v_{j+1} = v_{k-1}$ is the unique (homogeneous) equilibrium between v_j and v_k and is unstable with

$$i^{\mathcal{P}}(v_{j+1}) = i^{\mathcal{N}}(v_{j+1}) = i_{j+1}(\sigma) = 1 \ , \tag{6.7}$$

by (4.56). See Figure 2 (i) for an illustration of the meander section in this case.

If $k > j+2$, we first remark that $j+1$ and $k-1$ cannot both be fixed by σ. To show this we argue by contradiction.

Assume that $j+1$ and $k-1$ are both fixed. Then

$$i_{k-1}(\sigma) = i_{j+1}(\sigma) = 1 \ , \tag{6.8}$$

by (2.15); see illustration in Figure 2 (ii).

Let $I(j+1, k-1)$ denote the section of the horizontal axis between $j+1$ and $k-1$. If the meander section $\mu(j+1, k-1)$ does not intersect $I(j+1, k-1)$ then the union of $\mu(j+1, k-1)$ with $I(j+1, k-1)$ is a closed Jordan curve

$$J(j+1, k-1) = \mu(j+1, k-1) \cup I(j+1, k-1) \ . \tag{6.9}$$

In this case the rotation $\delta(j+1, k-1)$ of the tangent to the meander section $\mu(j+1, k-1)$ between the extreme points $j+1$ and $k-1$ is equal to the total rotation around the (piecewise smooth) Jordan curve $J(j+1, k-1)$, which is $\pm 2\pi$; see [doC76]. From the relation between the tangent rotation and the Morse numbers, see for example [Roc91], we have

$$\delta(j+1, k-1)/\pi = i_{k-1}(\sigma) - i_{j+1}(\sigma) \ . \tag{6.10}$$

Then (6.8) implies $\delta(j+1, k-1)/\pi = 0$ which contradicts the total rotation of $\pm 2\pi$.

Hence the meander section $\mu(j+1, k-1)$ must intersect $I(j+1, k-1)$ at some point c_0, $j+1 < c_0 < k-1$. Assume that c_0 is the first such point along the meander $\mu(j+1, k-1)$, see again Figure 2 (ii). Then the union of the meander section $\mu(j+1, c_0)$ with $I(j+1, c_0)$ is also a closed Jordan curve,

$$J(j+1, c_0) = \mu(j+1, c_0) \cup I(j+1, c_0) \ , \tag{6.11}$$

with the orientation provided by the meander section $\mu(j+1, c_0)$. Moreover, as in (6.10) the rotation $\delta(j+1, c_0)$ of the tangent to the meander section $\mu(j+1, c_0)$ between the extreme points $j+1$ and c_0 is

$$\delta(j+1, c_0)/\pi = i_{c_0}(\sigma) - i_{j+1}(\sigma) \ . \tag{6.12}$$

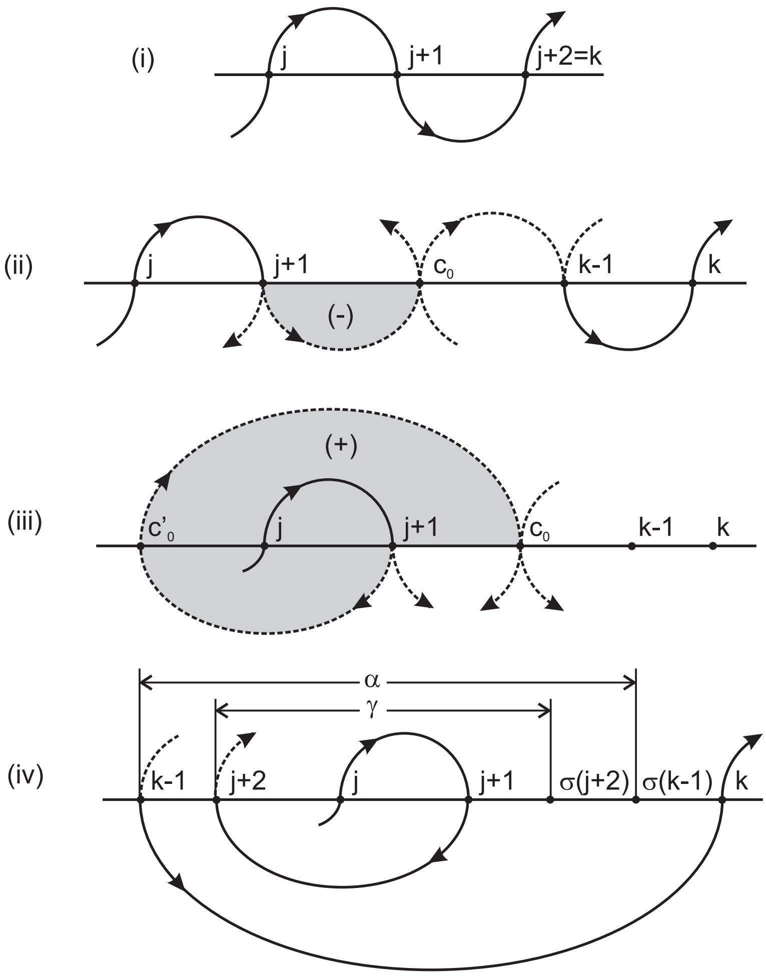

FIGURE 2. Illustration of meander sections $\mu(j,k)$ in the following cases: (i) $k = j+2$; (ii) $k > j+2$ and $J(j+1,c_0)$ (with shaded interior) oriented counter-clockwise $(-)$; (iii) $k > j+2$ and $J(j+1,c_0)$ oriented clockwise $(+)$; (iv) 2-cycles α and γ for $k > j+2$ and $j+1$ fixed by σ.

This implies that the orientation of the Jordan curve $J(j+1,c_0)$ is not counter-clockwise. Indeed, the total rotation of the tangent around the Jordan curve yields $\delta(j+1,c_0) = -2\pi$. Then $i_{j+1} = 1$ implies $i_{c_0}(\sigma) = 0$ which corresponds to a σ-stable point c_0 between j and k contradicting the assumption $i_{c_0}(\sigma) > 0$ of (6.3).

We show next that the orientation of $J(j+1,c_0)$ cannot be clockwise either. In fact, in that case there must exist a point c_0' on the meander section $\mu(j+1,c_0)$, i.e. $j+1 < c_0' < c_0$, which is located on the horizontal axis to the left of $j+1$ (and j). See the illustration in Figure 2 (iii). This implies that c_0' is in a 2-cycle α_0 of σ with

$$\sigma(c_0') < j < j+1 < c_0' < k-1 < k\ . \tag{6.13}$$

Arguing symmetrically for the σ-fixed point $k-1$ we have a last point c_1 along the meander $\mu(j+1,k-1)$ at which $\mu(j+1,k-1)$ intersects $I(j+1,k-1)$

and we obtain a point c_1', with $c_1 < c_1' < k-1$, which is in a 2-cycle α_1 of σ with

$$j < j+1 < c_1' < k-1 < k < \sigma(c_1') \ . \tag{6.14}$$

By (6.13) and (6.14) the 2-cycles α_0 and α_1 are non-nested. Therefore they must be separated by a σ-stable point c between j and k, by the integrability condition (I.3) of σ. See Section 3. This contradicts the assumption $i_c(\sigma) > 0$ of (6.3) and shows that $j+1$ and $k-1$ cannot both be fixed by σ.

Let $j+1$ be fixed by σ. Then the point $k-1$ belongs to a 2-cycle $\alpha = (c_\alpha\ k-1)$ of σ. Since k is σ-stable, c_α satisfies $c_\alpha = \sigma(k-1) < k-1$. See illustration in Figure 2 (iv). Remark that the point $j+2$ belongs to a 2-cycle γ of σ with $\sigma(j+2) < j+2$. Therefore, to avoid a forbidden σ-stable point, the integrability condition (I.3) of σ implies that α and γ are nested, hence $\sigma(k-1) < j < j+1$. Moreover, the same argument implies that there are no 2-cycles comprised between j and k. Then v_{j+1} is the unstable equilibrium and has Morse index

$$i^{\mathcal{P}}(v_{j+1}) = 2i^{\mathcal{N}}(v_{j+1}) - 1 = 2i_{j+1}(\sigma) - 1 > 0 \tag{6.15}$$

by (4.56).

Arguing symmetrically if $k-1$ is fixed by σ, we obtain the unstable equilibrium v_{k-1} with Morse index

$$i^{\mathcal{P}}(v_{k-1}) = 2i^{\mathcal{N}}(v_{k-1}) - 1 = 2i_{k-1}(\sigma) - 1 > 0 \ . \tag{6.16}$$

Finally, if $j+1$ also belongs to a 2-cycle β, the above argument symmetrically implies that $\beta = (j+1\ c_\beta)$ with $j+1 < c_\beta = \sigma(j+1)$. Hence, the points $j+1$ and $k-1$ belong to the 2-cycles,

$$\alpha = (c_\alpha\ k-1) \quad \text{and} \quad \beta = (j+1\ c_\beta) \ , \tag{6.17}$$

and the integrability condition (I.3) of σ implies

$$c_\alpha = \sigma(k-1) < c_\beta = \sigma(j+1) \ , \tag{6.18}$$

again to avoid a forbidden σ-stable point. Therefore, either α and β are identical, or they are nested by the integrability condition (I.1) of σ. In the first case $\alpha = \beta$ implies $j+1 = \sigma(k-1)$ and $k-1 = \sigma(j+1)$. See the illustration in Figure 3 (a). If $\alpha \neq \beta$ the following alternative applies: either α is the inner 2-cycle, i.e. $\alpha \subset \beta$ in the obvious notation, in which case

$$c_\beta < j+1 < c_\alpha < k-1 \ ; \tag{6.19}$$

or β is the inner 2-cycle, i.e. $\alpha \supset \beta$, in which case

$$j+1 < c_\beta < k-1 < c_\alpha \ . \tag{6.20}$$

See the illustration in Figure 3 (b) and (c), respectively.

Note that all 2-cycles comprised between j and k are centered. In fact, they are core-equivalent by the absence of σ-stable points. Hence they are centered by the integrability condition (I.2) of σ. Therefore, the middle point of the innermost 2-cycle is a fixed point of σ and corresponds to the unstable homogeneous equilibrium.

If $\alpha \subset \beta$ then $c := (j+1+c_\alpha)/2$ is the middle point of α. Remark that $c_\alpha = \sigma(j+1)$ and $j+1$ have the same even/odd parity by [FRW11], Lemma 6. Then v_c is the unstable homogeneous equilibrium with Morse index

$$i^{\mathcal{P}}(v_c) = 2i^{\mathcal{N}}(v_c) - 1 = 2i_c(\sigma) - 1 > 0 \ , \tag{6.21}$$

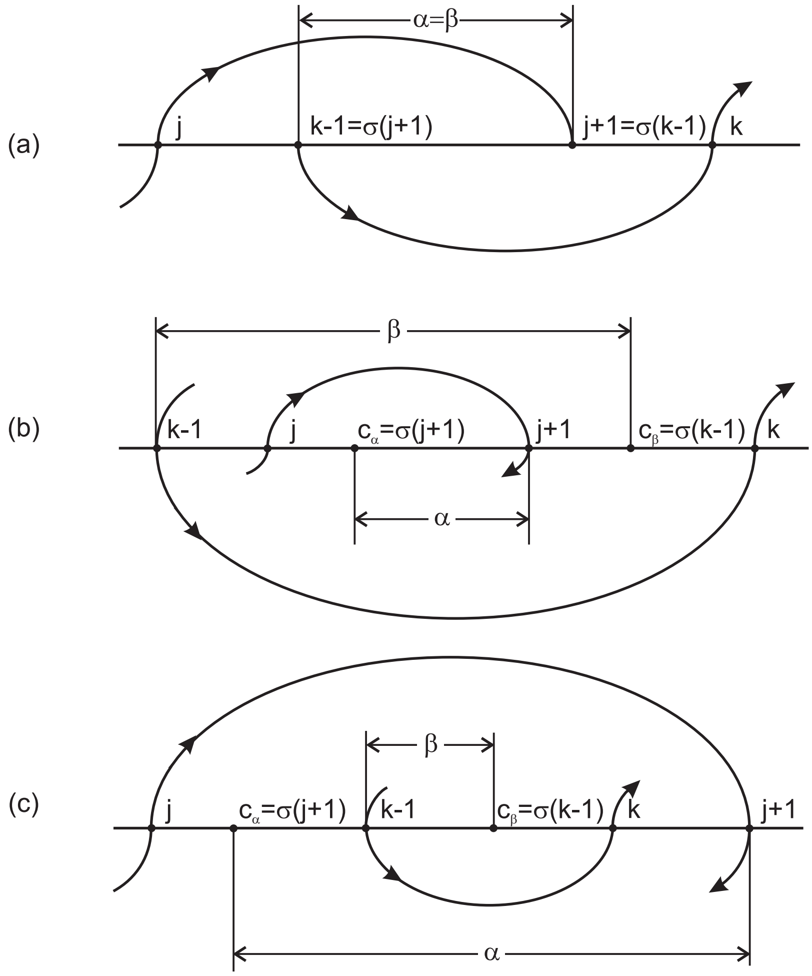

FIGURE 3. Relative position of the 2-cycles $\alpha = (c_\alpha\ j+1)$ and $\beta = (k-1\ c_\beta)$ of σ: (a) $\alpha = \beta$; (b) $\alpha \subset \beta$; (c) $\alpha \supset \beta$.

again by (4.56). The same result applies if $\alpha = \beta$ with $c := (j+k)/2$. If $\alpha \supset \beta$ then the middle point of β is $c := (c_\beta + k - 1)/2$ and the same result ensues.

This completes the localization of all the unstable (spatially homogeneous) equilibria and the computation of their Morse indices. Summarizing, if either $j+1$ or $k-1$ is a fixed point of σ then the unstable homogeneous equilibrium between v_j and v_k is, respectively, v_{j+1} or v_{k-1} with Morse index given by (6.15) or (6.16). If both $j+1$ and $k-1$ belong to 2-cycles of σ then the unstable homogeneous equilibrium between v_j and v_k is v_c, where c corresponds to the middle point of the innermost 2-cycle, and its Morse index is given by (6.21).

Step 3: Rotating waves.

We proceed by determining the rotating waves of $\mathcal{A}_f^{\mathcal{P}}$, i.e. the 2π-periodic orbits of (4.20). After exhausting, in the previous steps, the set of fixed points of σ corresponding to the (spatially homogeneous) equilibria we are left with an even number of points corresponding to the rotating waves. Recall that each rotating wave $w_k \in \mathcal{R}_f^{\mathcal{P}}$ corresponds to a pair of points representing its maximum and minimum values, $\overline{w}_k$ and $\underline{w}_k$ respectively. Therefore our next objective is to match the pair of points corresponding to each rotating wave.

By construction, each 2-cycle of σ corresponds to a rotating wave $w_k \in \mathcal{R}_f^{\mathcal{P}}$ with odd period lap number. In this case the 2-cycles of σ immediately identify the matching pair of points corresponding to the same rotating wave. All the remaining points are fixed by σ and correspond to the rotating waves $w_k \in \mathcal{R}_f^{\mathcal{P}}$ with even period lap number. The relative position of the 2-cycles of σ and of the σ-fixed points corresponding to the (homogeneous) equilibria determine the matching pairs among these remaining fixed points of σ. In fact, the nesting in the ODE phase plane (v, p) of the periodic orbits corresponding to rotating waves induces a *regular bracket structure* on the matching pairs of points representing the rotating waves. This is the structure of the parentheses ordinarily used in the arithmetic expressions to indicate the hierarchical order of computations, [Lan03]. The matching of left and right brackets uniquely determines the pair of points representing the same rotating wave. See [FRW11] for more details.

Step 4: Heteroclinic orbit connections.

With all equilibria and rotating waves $w_k \in \mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ determined from $\sigma = \sigma_f^{\mathcal{P}}$ we turn to the heteroclinic orbit connections between them. These are determined from the ($\mathcal{P}$)-adjacency relations as asserted by Theorem 8. Let w^1, w^2 denote two different equilibria or rotating waves in $\mathcal{E}_f^{\mathcal{P}} \cup \mathcal{R}_f^{\mathcal{P}}$ and let $v_+^1, v_+^2 \in \mathcal{E}_g^{\mathcal{N}}$ denote the corresponding representative Neumann equilibria of the frozen symmetrized problem ($\mathcal{N}$) on the half-interval $x \in [0, \pi]$. Of course $v_+^j = w^j \equiv e$ in case of an equilibrium $w^j \in \mathcal{E}_f^{\mathcal{P}}$, which we recall is spatially homogeneous. We now use that a heteroclinic orbit (of ($\mathcal{P}$) with nonlinearity f) from w^1 to w^2 exists if and only if a heteroclinic orbit (of ($\mathcal{N}$) with nonlinearity g) exists from the frozen and symmetrized Neumann representative v_+^1 to v_+^2; see Theorem 7.

Let $v_1, \ldots, v_n \in \mathcal{E}_g^{\mathcal{N}}$ denote the Neumann equilibria of the problem ($\mathcal{N}$). To establish heteroclinic ($\mathcal{N}$)-adjacency of equilibria $v_j \in \mathcal{E}_g^{\mathcal{N}}$ we consider the differences $v_j - v_k$ for all equilibria $v_j \neq v_k$ and compute the zero numbers $z^{\mathcal{N}}(v_j - v_k)$. We recall that the Sturm permutation $\sigma_g^{\mathcal{N}} = \sigma_f^{\mathcal{P}}$ determines these zero numbers; see [FR96], Proposition 2.1. Specifically

$$z^{\mathcal{N}}(v_j - v_k) = z_{jk}(\sigma) \ , \tag{6.22}$$

where

$$\begin{aligned} z_{jk}(\sigma) := i_j(\sigma) + \frac{1}{2}[(-1)^k \operatorname{sign}(\sigma(k) - \sigma(j)) - 1] \\ + \sum_{c=j+1}^{k-1} (-1)^c \operatorname{sign}(\sigma(c) - \sigma(j)) \ , \end{aligned} \tag{6.23}$$

for $j < k$, and $z_{kj}(\sigma) = z_{jk}(\sigma)$. In particular for two successive equilibria, e.g. $k = j + 1$, we have

$$z^{\mathcal{N}}(v_j - v_{j+1}) = \min\{i_j(\sigma), i_{j+1}(\sigma)\} \; . \tag{6.24}$$

For details we refer to [FR96].

Step 5: Period lap numbers and Morse indices.

In this final step we determine the Matano period lap numbers $\ell(w) = z^{\mathcal{P}}(w_x)/2$ and the Morse indices $i^{\mathcal{P}}(w)$ of all rotating waves w of $\mathcal{A}_f^{\mathcal{P}}$. By (4.58) a rotating wave $w \in \mathcal{R}_f^{\mathcal{P}}$ and the corresponding symmetrized frozen wave $v_+ \in \mathcal{F}_g^{\mathcal{P}}$ have the same period lap number, $\ell(w) = \ell(v_+)$. Let $v^0 \in \mathcal{Z}_g^{\mathcal{P}} \cup \mathcal{F}_g^{\mathcal{P}}$ denote a homogeneous equilibrium or symmetrized frozen wave of the spatially periodic problem ($\mathcal{P}$) with $v^0 \neq v_+$. Choose v^0 such that on the phase plane (v, p) the periodic orbit of v_+ encircles the orbit of v^0. Then, by [MN97], Lemma 4.8, we have

$$z^{\mathcal{P}}(v_+ - v^0) = z^{\mathcal{P}}(v_{+,x}) \; . \tag{6.25}$$

By (4.28) this implies

$$\ell(v_+) = z^{\mathcal{P}}(v_+ - v^0)/2 \; . \tag{6.26}$$

Therefore Theorem 7, (4.38) implies

$$\ell(w) = \ell(v_+) = z^{\mathcal{N}}(v_+ - v_+^0) \; . \tag{6.27}$$

This determines the Matano period lap number of the rotating wave $w \in \mathcal{R}_f^{\mathcal{P}}$. Note that the zero number $z^{\mathcal{N}}(v_+ - v_+^0)$ was already computed in the previous step.

Finally, the Morse indices of all rotating waves $w_k \in \mathcal{R}_f^{\mathcal{P}}$ are determined from σ using Theorem 9. Let $v_j \in \mathcal{E}_g^{\mathcal{N}}$ denote the equilibrium of the frozen symmetrized problem ($\mathcal{N}$) corresponding to the restriction of $v_{k,+} \in \mathcal{E}_g^{\mathcal{P}}$ to the half-interval, $v_j = v_{k,+}$ in $0 \leq x \leq \pi$. Then $i^{\mathcal{N}}(v_j) = i_j(\sigma)$ and (4.60) implies

$$i^{\mathcal{P}}(w) = \begin{cases} 2i_j(\sigma) - 1 & \text{if } \; i_j(\sigma) = \ell(w) \; , \\ \\ 2i_j(\sigma) - 2 & \text{if } \; i_j(\sigma) = \ell(w) + 1 \; . \end{cases} \tag{6.28}$$

This completes the recipe for the construction of the connection graph $\mathcal{G}_f^{\mathcal{P}}$ from the Sturm permutation $\sigma = \sigma_f^{\mathcal{P}}$. Let us denote by $\mathcal{A}_{n,\kappa}^{\mathcal{P}}$ and $\mathcal{G}_{n,\kappa}^{\mathcal{P}}$ the Sturm attractor and connection graph of (1.17) corresponding to $\sigma_{n,\kappa} \in \mathcal{S}(n)$ with $n = m + 2q$.

The Sturm attractors $\mathcal{A}_{1,1}^{\mathcal{P}}$ and $\mathcal{A}_{3,1}^{\mathcal{P}}$ are particularly simple. They correspond to $(m, q) = (1, 0)$ and $(m, q) = (3, 0)$, respectively, and have dimensions $\dim \mathcal{A}_{1,1}^{\mathcal{P}} = 0$ and $\dim \mathcal{A}_{3,1}^{\mathcal{P}} = 1$.

For $n = 5$, the two cases $\mathcal{A}_{5,1}^{\mathcal{P}}$ and $\mathcal{A}_{5,2}^{\mathcal{P}}$ correspond to $(m, q) = (5, 0)$ and $(m, q) = (3, 1)$, respectively. Their dimensions are $\dim \mathcal{A}_{5,1}^{\mathcal{P}} = 1$ and $\dim \mathcal{A}_{5,2}^{\mathcal{P}} = 3$. Figure 4 shows stylized meanders corresponding to the Sturm permutations $\sigma_{5,1}$ and $\sigma_{5,2}$, and the nesting configuration in the ODE phase plane (v, p) of the PDE homogeneous equilibria and periodic orbits of (4.20) corresponding to the equilibria and rotating waves of (1.17).

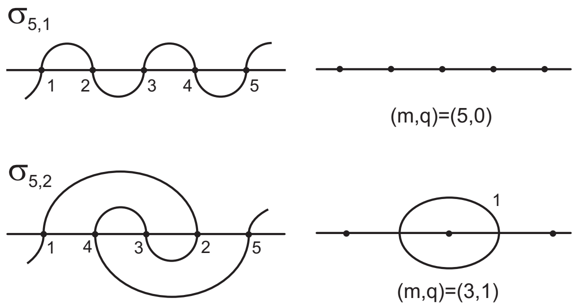

FIGURE 4. Meanders and phase plane configurations for $\sigma_{5,1}$ and $\sigma_{5,2}$. Left: Stylized meanders. Right: Phase plane configurations (m and q denote the number of equilibria and 2π-periodic orbits, respectively). The number attached to the periodic orbit denotes its Matano period lap number.

The connection graphs for the Sturm attractors with $n \leq 5$ are shown in Figure 5.

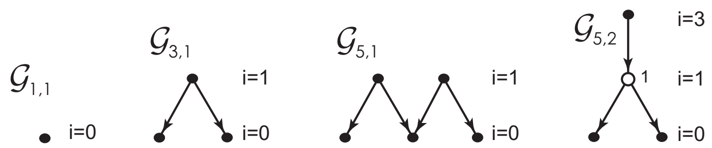

FIGURE 5. Connection graphs $\mathcal{G}^{\mathcal{P}}_{n,\kappa}$ for Sturm attractors $\mathcal{A}^{\mathcal{P}}_{n,\kappa}$ with $1 \leq \kappa \leq \iota(n)$ and $n \leq 5$.

In the following we denote by $\mathcal{A}^{\mathcal{P}}_1 \sqcup \mathcal{A}^{\mathcal{P}}_2$ the connected sum of $\mathcal{A}^{\mathcal{P}}_1$ and $\mathcal{A}^{\mathcal{P}}_2$ obtained by gluing the two attractors via the identification of the maximal equilibrium of $\mathcal{A}^{\mathcal{P}}_1$ with the minimal equilibrium of $\mathcal{A}^{\mathcal{P}}_2$. Indeed dissipativeness of $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ imply that such highest and lowest elements of $\mathcal{E}^{\mathcal{P}}_f \cup \mathcal{R}^{\mathcal{P}}_f$ exist in $\mathcal{A}^{\mathcal{P}}_f$, due to monotonicity of the semiflow. In fact these extreme objects are stable and hence coincide with spatially homogeneous equilibria.

Likewise we define $\mathcal{G}^{\mathcal{P}}_1 \sqcup \mathcal{G}^{\mathcal{P}}_2$ as the connection graph associated with $\mathcal{A}^{\mathcal{P}}_1 \sqcup \mathcal{A}^{\mathcal{P}}_2$. Then, we can easily see that

$$\mathcal{G}^{\mathcal{P}}_{5,1} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \,. \tag{6.29}$$

In fact, this is already visible in the corresponding permutations. At the meander level this connected sum corresponds to a concatenation of the respective meanders by their extreme points.

The connection graph $\mathcal{G}^{\mathcal{P}}_{5,2}$ corresponds to a Sturm attractor $\mathcal{A}^{\mathcal{P}}_{5,2}$ with three equilibria and one rotating wave. From our list of integrable Sturm involutions it is the simplest Sturm attractor $\mathcal{A}^{\mathcal{P}}_{n,\kappa}$ with exactly one rotating wave.

The rotating wave w of $\mathcal{A}^{\mathcal{P}}_{5,2}$, with period lap number $\ell(w) = 1$ and Morse index $i^{\mathcal{P}}(w) = 1$, possesses an unstable manifold of dimension $\dim W^u(w) =$

$i^{\mathcal{P}}(w)+1=2$. Moreover, the middle unstable equilibrium e_2, with Morse index $i^{\mathcal{P}}(e_2)=3$, possesses heteroclinic orbit connections to the rotating wave w, $e_2 \leadsto w$. Since $\operatorname{codim} W^s(w) = i^{\mathcal{P}}(w) = 1$, by transversality we have

$$\dim(W^u(e_2)\cap W^s(w)) = \dim W^u(e_2) - \operatorname{codim} W^s(w) = 2\ . \qquad (6.30)$$

Also $w \leadsto e_1$, $w \leadsto e_3$ and the respective connecting sets are two dimensional manifolds,

$$\dim(W^u(w)\cap W^s(e_1)) = \dim(W^u(w)\cap W^s(e_3)) = \dim W^u(w) = 2\ . \quad (6.31)$$

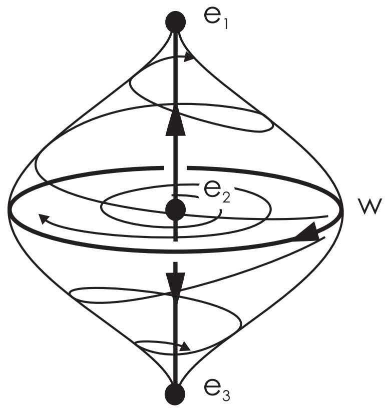

FIGURE 6. A geometric representation of the Sturm "spindle" attractor $\mathcal{A}^{\mathcal{P}}_{5,2}$, with $m=3$ and $q=1$. See also [KW01].

A geometric model studied by [KW01] in the technically somewhat different setting of positive feedback delay equations is the *spindle attractor*. See Figure 6. The spindle attractor $\overline{W}$ is the closure of the unstable manifold of a central equilibrium $\overline{e}_0$ and is split by an invariant disk $\overline{S}$ into the basins of attraction toward the tips $\overline{e}_+, \overline{e}_-$. The invariant disk $\overline{S}$ corresponds to heteroclinic orbit connections between $\overline{e}_0$ and a unique periodic orbit $\overline{w}$. Clearly the connection graph of the spindle attractor coincides with our connection graph $\mathcal{G}^{\mathcal{P}}_{5,2}$. We suspect, but did not prove, that in fact the global PDE attractor $\mathcal{A}^{\mathcal{P}}_{5,2}$ is C^0 orbit equivalent to the spindle attractor.

For $n=7$ we have four connection graphs. The corresponding stylized meanders and ODE phase plane nesting configurations are presented in Figure 7. All the connection graphs $\mathcal{G}^{\mathcal{P}}_{n,\kappa}$ for $n=7$ are shown in Figure 8.

We can immediately see that the first two cases correspond to the concatenation of previous ones. In fact, we have

$$\mathcal{G}^{\mathcal{P}}_{7,1} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1}\ , \qquad (6.32)$$

$$\mathcal{G}^{\mathcal{P}}_{7,2} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{5,2}\ . \qquad (6.33)$$

Each of the other two cases, $\mathcal{A}^{\mathcal{P}}_{7,3}$ and $\mathcal{A}^{\mathcal{P}}_{7,4}$, possesses two rotating waves: $q=2$ and $m=3$. In the ODE phase plane (v,p) the periodic orbits associated to these rotating waves are nested, see Figure 7. Therefore we denote by w_1 and w_2 the rotating waves associated to the innermost and outermost periodic orbits, respectively.

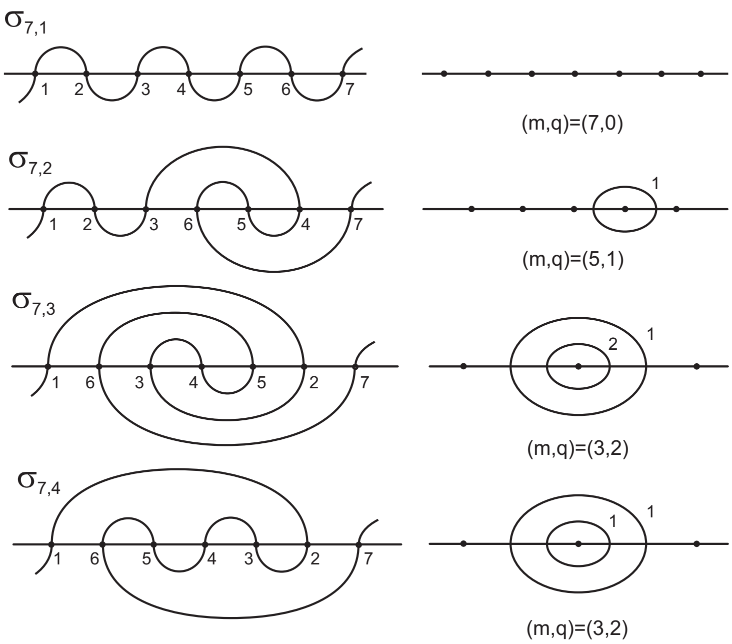

FIGURE 7. Meanders and phase plane configurations for all integrable Sturm involutions $\sigma_{n,\kappa}$ with $n = 7$. Left: Stylized meanders. Right: Phase plane configurations. The periodic orbits have the Matano period lap number attached.

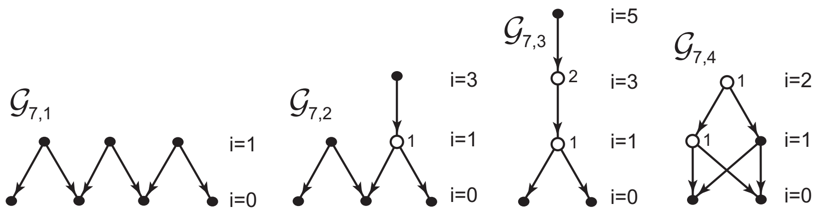

FIGURE 8. Connection graphs $\mathcal{G}^{\mathcal{P}}_{n,\kappa}$ for the Sturm attractors $\mathcal{A}^{\mathcal{P}}_{n,\kappa}$ with $1 \leq k \leq \kappa(n)$ and $n = 7$.

The attractor $\mathcal{A}^{\mathcal{P}}_{7,3}$ has dimension $\dim \mathcal{A}^{\mathcal{P}}_{7,3} = 5$. In fact the unstable middle equilibrium, e_2, has Morse index $i^{\mathcal{P}}(e_2) = 5$. The (innermost) rotating wave w_1 has period lap number $\ell(w_1) = 2$ and Morse index $i^{\mathcal{P}}(w_1) = 3$, while the (outermost) rotating wave w_2 has period lap number $\ell(w_2) = 1$ and Morse index $i^{\mathcal{P}}(w_2) = 1$. We have $e_2 \rightsquigarrow w_1$, $w_2 \rightsquigarrow e_1$, $w_2 \rightsquigarrow e_3$ and, due to transversality, the connecting sets are two dimensional manifolds,

$$\begin{aligned} \dim(W^u(e_2) \cap W^s(w_1)) &= \dim W^u(e_2) - \operatorname{codim} W^s(w_1) \\ &= i^{\mathcal{P}}(e_2) - i^{\mathcal{P}}(w_1) = 2 \ , \end{aligned} \tag{6.34}$$

$$\begin{aligned} \dim(W^u(w_2) \cap W^s(e_1)) = \dim(W^u(w_2) \cap W^s(e_3)) &= \dim W^u(w_2) \\ &= i^{\mathcal{P}}(w_2) + 1 = 2 \ . \end{aligned} \tag{6.35}$$

We also have $w_1 \rightsquigarrow w_2$ and, since the two rotating waves have different periods, it should be interesting to study the geometry of the respective

$$\begin{aligned}
\sigma_{9,1} &= \{1,2,3,4,5,6,7,8,9\} = \mathrm{id}\ ;\\
\sigma_{9,2} &= \{1,2,3,4,5,8,7,6,9\} = (6\ 8)\ ;\\
\sigma_{9,3} &= \{1,2,3,6,5,4,7,8,9\} = (4\ 6)\ ;\\
\sigma_{9,4} &= \{1,2,3,8,5,6,7,4,9\} = (4\ 8)\ ;\\
\sigma_{9,5} &= \{1,2,3,8,7,6,5,4,9\} = (4\ 8)(5\ 7)\ ;\\
\sigma_{9,6} &= \{1,4,3,2,5,8,7,6,9\} = (2\ 4)(6\ 8)\ ;\\
\sigma_{9,7} &= \{1,8,3,4,5,6,7,2,9\} = (2\ 8)\ ;\\
\sigma_{9,8} &= \{1,8,3,6,5,4,7,2,9\} = (2\ 8)(4\ 6)\ ;\\
\sigma_{9,9} &= \{1,8,7,4,5,6,3,2,9\} = (2\ 8)(3\ 7)\ ;\\
\sigma_{9,10} &= \{1,8,7,6,5,4,3,2,9\} = (2\ 8)(3\ 7)(4\ 6)\ ;
\end{aligned}$$

TABLE 3. List of integrable involutive Sturm permutations $\sigma \in S(9)$.

connecting set. This set is a three dimensional manifold. In fact, since $\dim W^u(w_1) = i^{\mathcal{P}}(w_1)+1 = 4$ and $\operatorname{codim} W^s(w_2) = i^{\mathcal{P}}(w_2) = 1$, by transversality we obtain

$$\dim(W^u(w_1) \cap W^s(w_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(w_2) = 3\ . \tag{6.36}$$

We recall that all the possible asymptotic phases are realized by heteroclinic orbits, (see [FRW04] for details).

In spite of exhibiting also two rotating waves, the Sturm attractor $\mathcal{A}^{\mathcal{P}}_{7,4}$ is quite different from the previous one. Both rotating waves w_1 and w_2 have the same period lap number, $\ell(w_1) = \ell(w_2) = 1$, and the corresponding Morse indices differ by one, $i^{\mathcal{P}}(w_1) = 2$ and $i^{\mathcal{P}}(w_2) = 1$. In this case we have $\dim W^u(w_1) = i^{\mathcal{P}}(w_1) + 1 = 3$, $\dim W^u(w_2) = i^{\mathcal{P}}(w_2) + 1 = 2$ and the unstable middle equilibrium e_2 has Morse index $i^{\mathcal{P}}(e_2) = 1$. Therefore, the Sturm attractor has dimension $\dim \mathcal{A}^{\mathcal{P}}_{7,4} = 3$. Moreover, $w_1 \leadsto w_2$, $w_1 \leadsto e_2$ and the respective connecting sets are two dimensional manifolds. Indeed, since $\operatorname{codim} W^s(w_2) = i^{\mathcal{P}}(w_2) = 1$ and $\operatorname{codim} W^s(e_2) = i^{\mathcal{P}}(e_2) = 1$, tranversality implies

$$\dim(W^u(w_1) \cap W^s(w_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(w_2) = 2\ , \tag{6.37}$$

$$\dim(W^u(w_1) \cap W^s(e_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(e_2) = 2\ . \tag{6.38}$$

Also $w_2 \leadsto e_1$, $w_2 \leadsto e_3$ with connecting sets which are two dimensional manifolds, and $e_2 \leadsto e_1$, $e_2 \leadsto e_3$ with one dimensional manifold connections.

For $n = 9$ the list of Sturm attractors $\mathcal{A}^{\mathcal{P}}_{n,\kappa}$ and connection graphs $\mathcal{G}^{\mathcal{P}}_{n,\kappa}$ becomes even more variegated, while still manageable. Up to trivial equivalence, there are exactly 10 Sturm permutations which are integrable involutions. These are listed in Table 3, see [FRW11] for details.

For presentation purposes we organize these Sturm permutations into four sets P_q according to the number q of rotating waves of the corresponding Sturm attractors:

$$\begin{aligned}
P_0 &= \{\sigma_{9,1}\}\ ,\\
P_1 &= \{\sigma_{9,2}, \sigma_{9,3}\}\ ,\\
P_2 &= \{\sigma_{9,4}, \sigma_{9,5}, \sigma_{9,6}, \sigma_{9,9}\}\ ,\\
P_3 &= \{\sigma_{9,7}, \sigma_{9,8}, \sigma_{9,10}\}\ .
\end{aligned} \tag{6.39}$$

The first set P_0 corresponds to Sturm attractors without rotating waves. There is, of course, one attractor $\mathcal{A}^{\mathcal{P}}_{9,1}$ with nine equilibria and no rotating waves, corresponding to $\sigma^{\mathcal{P}}_f = \sigma_{9,1} = \mathrm{id}$. It has $\dim \mathcal{A}^{\mathcal{P}}_{9,1} = 1$, and one easily verifies that

$$\mathcal{G}^{\mathcal{P}}_{9,1} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \ . \tag{6.40}$$

See Figure 9.

The second set P_1 consists of two Sturm permutations. The corresponding attractors $\mathcal{A}^{\mathcal{P}}_{9,2}$ and $\mathcal{A}^{\mathcal{P}}_{9,3}$ possess one rotating wave and seven homogeneous equilibria each. They both satisfy $\dim \mathcal{A}^{\mathcal{P}}_{9,2} = \dim \mathcal{A}^{\mathcal{P}}_{9,3} = 3$, and their connection graphs can be written as

$$\mathcal{G}^{\mathcal{P}}_{9,2} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{5,2} \ , \quad \mathcal{G}^{\mathcal{P}}_{9,3} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{5,2} \sqcup \mathcal{G}^{\mathcal{P}}_{3,1} \ . \tag{6.41}$$

The meanders and phase plane configurations obtained from the Sturm permutations in P_0 and P_1 are shown in Figure 9. The connection graphs of the corresponding Sturm attractors are presented in Figure 10.

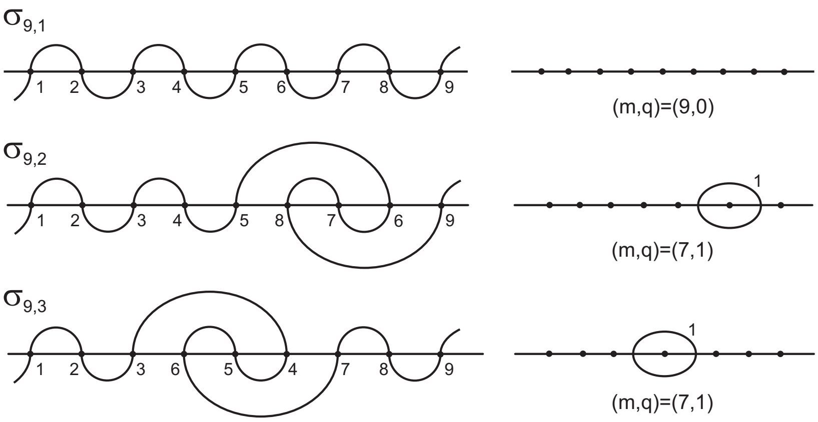

FIGURE 9. Meanders and phase plane configurations for the Sturm permutations in P_0 and P_1. Left: Stylized meanders. Right: Phase plane configurations.

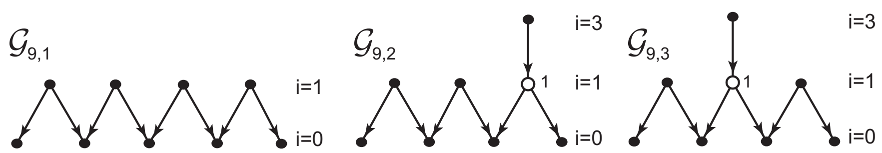

FIGURE 10. Connection graphs $\mathcal{G}^{\mathcal{P}}_{9,\kappa}$ for the Sturm permutations in P_0 and P_1.

The third set of Sturm permutations P_2 consists of four permutations corresponding to Sturm attractors with exactly two rotating waves and five equilibria. The connection graphs of the first three Sturm attractors are concatenations of previous ones. We have

$$\mathcal{G}^{\mathcal{P}}_{9,4} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{7,3} \ , \quad \mathcal{G}^{\mathcal{P}}_{9,5} = \mathcal{G}^{\mathcal{P}}_{3,1} \sqcup \mathcal{G}^{\mathcal{P}}_{7,4} \ , \quad \mathcal{G}^{\mathcal{P}}_{9,6} = \mathcal{G}^{\mathcal{P}}_{5,2} \sqcup \mathcal{G}^{\mathcal{P}}_{5,2} \ , \tag{6.42}$$

and $\dim \mathcal{A}^{\mathcal{P}}_{9,4} = 5$, $\dim \mathcal{A}^{\mathcal{P}}_{9,5} = \dim \mathcal{A}^{\mathcal{P}}_{9,6} = 3$.

The connection graph $\mathcal{G}^{\mathcal{P}}_{9,9}$ of the fourth attractor $\mathcal{A}^{\mathcal{P}}_{9,9}$ does not decompose into previous ones. It has a special geometrical interest because in the phase plane the periodic orbits corresponding to the rotating waves are nested and both encircle the three middle equilibria. The meanders and phase plane configurations associated to the set P_2 of Sturm permutations are shown in Figure 11.

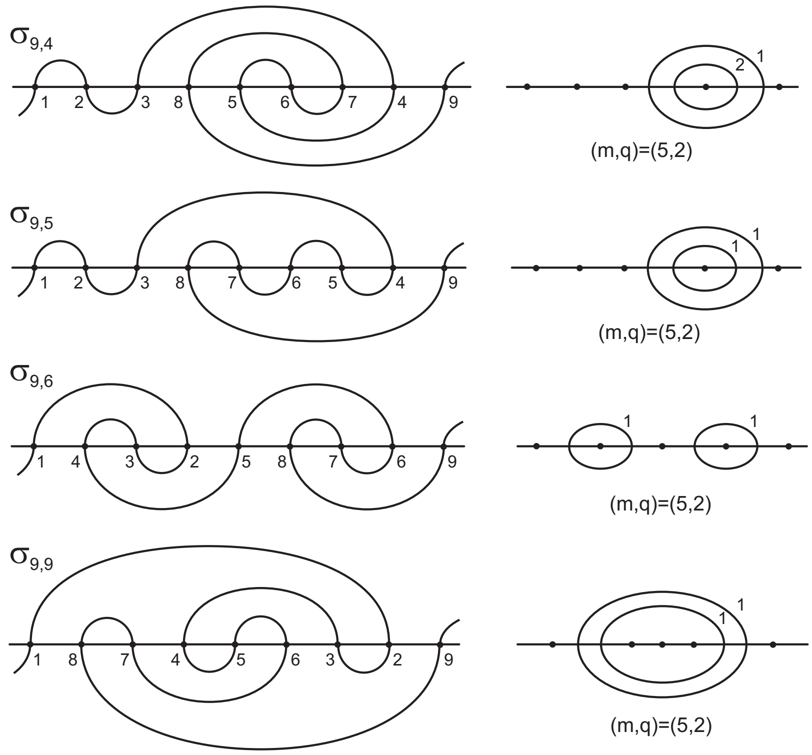

FIGURE 11. Meanders and phase plane configurations for the set P_2 of Sturm permutations. Left: Stylized meanders. Right: Phase plane configurations with attached Matano period lap numbers.

In several aspects the Sturm attractor $\mathcal{A}^{\mathcal{P}}_{9,9}$ is similar to $\mathcal{A}^{\mathcal{P}}_{7,4}$. We denote again by w_1 and w_2 the rotating waves corresponding to the innermost and the outermost periodic orbits, respectively. Then both have the same period lap number $\ell(w_1) = \ell(w_2) = 1$ and their Morse indices differ by one, $i^{\mathcal{P}}(w_1) = 2$ and $i^{\mathcal{P}}(w_2) = 1$. Again we have $\dim W^u(w_1) = i^{\mathcal{P}}(w_1) + 1 = 3$ and the attractor has dimension $\dim \mathcal{A}^{\mathcal{P}}_{9,9} = 3$. The middle equilibrium e_3 is stable, $i^{\mathcal{P}}(e_3) = 0$, and the two unstable equilibria have Morse indices $i^{\mathcal{P}}(e_2) = i^{\mathcal{P}}(e_4) = 1$. With respect to heteroclinic orbit connections in $\mathcal{A}^{\mathcal{P}}_{9,9}$ we have $w_1 \leadsto w_2$, $w_1 \leadsto e_2$ and $w_1 \leadsto e_4$ for which the corresponding connecting sets are two dimensional manifolds,

$$\dim(W^u(w_1) \cap W^s(w_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(w_2) = 2 \ , \tag{6.43}$$

$$\dim(W^u(w_1) \cap W^s(e_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(e_2) = 2 \ , \tag{6.44}$$

$$\dim(W^u(w_1) \cap W^s(e_4)) = \dim W^u(w_1) - \operatorname{codim} W^s(e_4) = 2 \ . \tag{6.45}$$

Moreover, we also have $w_2 \rightsquigarrow e_1$, $w_2 \rightsquigarrow e_5$ with two dimensional connecting manifolds, and $e_2 \rightsquigarrow e_1$, $e_2 \rightsquigarrow e_3$, $e_4 \rightsquigarrow e_1$, $e_4 \rightsquigarrow e_5$ with one dimensional heteroclinic connections.

All the connection graphs $\mathcal{G}^{\mathcal{P}}_{9,\kappa}$ associated to the Sturm permutations in P_2 are presented in Figure 12.

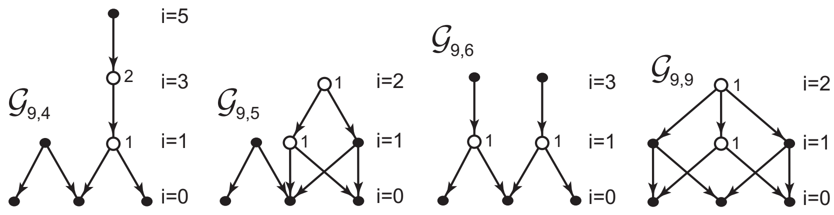

FIGURE 12. Connection graphs $\mathcal{G}^{\mathcal{P}}_{9,\kappa}$ for the set P_2 of Sturm permutations.

Finally, the set of Sturm permutations P_3 corresponds to three Sturm attractors $\mathcal{A}^{\mathcal{P}}_{9,7}$, $\mathcal{A}^{\mathcal{P}}_{9,8}$, and $\mathcal{A}^{\mathcal{P}}_{9,10}$, each with exactly three rotating waves and three equilibria. In all three cases the ODE periodic orbits which generate the rotating waves are nested and encircle the middle equilibrium e_2. Otherwise the attractors are completely different. The meanders and phase plane configurations associated to this set P_3 of Sturm permutations are depicted in Figure 13.

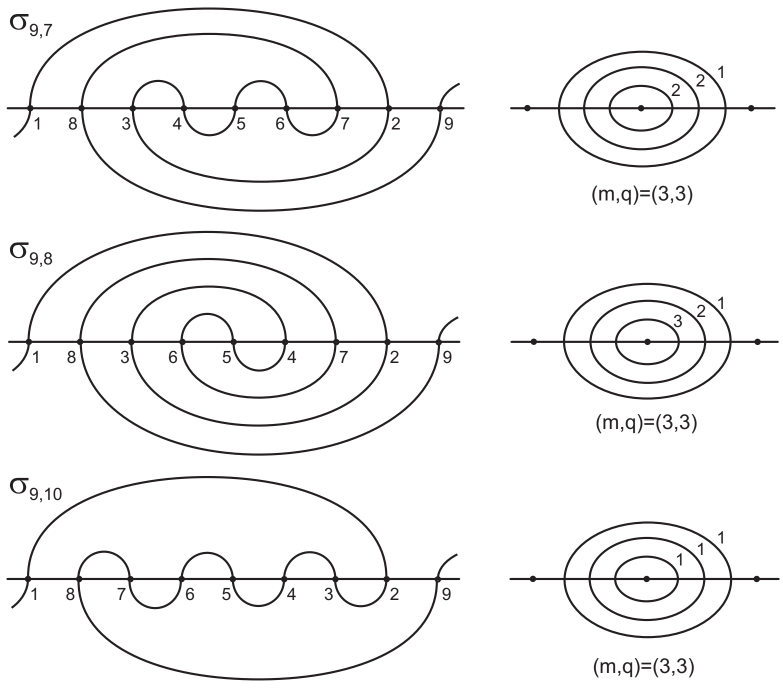

FIGURE 13. Meanders and phase plane configurations for the set P_3 of Sturm permutations. Left: Stylized meanders. Right: Phase plane configurations with attached Matano period lap numbers.

Let w_1, w_2 and w_3 denote the rotating waves corresponding to the innermost, the middle and the outermost periodic orbits, respectively. For the Sturm attractor $\mathcal{A}^{\mathcal{P}}_{9,7}$, the period lap numbers of the rotating waves are $\ell(w_1) = \ell(w_2) = 2$ and $\ell(w_3) = 1$, while the Morse indices are $i^{\mathcal{P}}(w_1) = 4$, $i^{\mathcal{P}}(w_2) = 3$ and $i^{\mathcal{P}}(w_3) = 1$. Moreover, the middle equilibrium e_2 has Morse index $i^{\mathcal{P}}(e_2) = 3$. Therefore $\dim W^u(w_1) = i^{\mathcal{P}}(w_1) + 1 = 5$ and the attractor has dimension $\dim \mathcal{A}^{\mathcal{P}}_{9,7} = 5$. In $\mathcal{A}^{\mathcal{P}}_{9,7}$ we have the heteroclinic orbit connections $w_1 \leadsto w_2$, $w_1 \leadsto e_2$ and $e_2 \leadsto w_3$ for which the connecting sets are two dimensional manifolds,

$$\dim(W^u(w_1) \cap W^s(w_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(w_2) = 2\ , \tag{6.46}$$

$$\dim(W^u(w_1) \cap W^s(e_2)) = \dim W^u(w_1) - \operatorname{codim} W^s(e_2) = 2\ , \tag{6.47}$$

$$\dim(W^u(e_2) \cap W^s(w_3)) = i^{\mathcal{P}}(e_2) - i^{\mathcal{P}}(w_3) = 2\ . \tag{6.48}$$

In addition $w_2 \leadsto w_3$ and the connecting set is a three dimensional manifold,

$$\dim(W^u(w_2) \cap W^s(w_3)) = i^{\mathcal{P}}(w_2) + 1 - i^{\mathcal{P}}(w_3) = 3\ , \tag{6.49}$$

Note that by transitivity also $w_1 \leadsto w_3$, and the connecting set is a four dimensional manifold,

$$\dim(W^u(w_1) \cap W^s(w_3)) = i^{\mathcal{P}}(w_1) + 1 - i^{\mathcal{P}}(w_3) = 4\ . \tag{6.50}$$

The heteroclinic orbit connections $w_3 \leadsto e_1$, $w_3 \leadsto e_3$, which complete the edges of the connection graph $\mathcal{G}^{\mathcal{P}}_{9,7}$, are two dimensional manifolds,

$$\begin{aligned} \dim(W^u(w_3) \cap W^s(e_1)) = \dim(W^u(w_3) \cap W^s(e_3)) &= \dim W^u(w_3) \\ &= i^{\mathcal{P}}(w_3) + 1 = 2\ . \end{aligned} \tag{6.51}$$

In the case of the Sturm attractor $\mathcal{A}^{\mathcal{P}}_{9,8}$, the inner rotating wave has period lap number $\ell(w_1) = 3$ and Morse index $i^{\mathcal{P}}(w_1) = 5$, the middle one has period lap number $\ell(w_2) = 2$ and Morse index $i^{\mathcal{P}}(w_2) = 3$, and the outer one has period lap number $\ell(w_3) = 1$ and Morse index $i^{\mathcal{P}}(w_3) = 1$. Moreover, the unstable middle equilibrium has index $i^{\mathcal{P}}(e_2) = 7$ and the Sturm attractor has dimension $\dim \mathcal{A}^{\mathcal{P}}_{9,8} = 7$. The heteroclinic orbit connections corresponding to $w_1 \leadsto w_2$, $w_2 \leadsto w_3$ are three dimensional manifolds,

$$\begin{aligned} \dim(W^u(w_1) \cap W^s(w_2)) &= \dim W^u(w_1) - \operatorname{codim} W^s(w_2) \\ &= i^{\mathcal{P}}(w_1) + 1 - i^{\mathcal{P}}(w_2) = 3\ , \end{aligned} \tag{6.52}$$

$$\begin{aligned} \dim(W^u(w_2) \cap W^s(w_3)) &= \dim W^u(w_2) - \operatorname{codim} W^s(w_3) \\ &= i^{\mathcal{P}}(w_2) + 1 - i^{\mathcal{P}}(w_3) = 3\ , \end{aligned} \tag{6.53}$$

The remaining edges of the connection graph $\mathcal{G}^{\mathcal{P}}_{9,8}$ correspond to $e_2 \leadsto w_1$, $w_3 \leadsto e_1$, $w_3 \leadsto e_3$ and are two dimensional manifolds,

$$\begin{aligned} \dim(W^u(e_2) \cap W^s(w_1)) &= \dim W^u(e_2) - \operatorname{codim} W^s(w_1) \\ &= i^{\mathcal{P}}(e_2) - i^{\mathcal{P}}(w_1) = 2\ , \end{aligned} \tag{6.54}$$

$$\begin{aligned} \dim(W^u(w_3) \cap W^s(e_1)) = \dim(W^u(w_3) \cap W^s(e_3)) &= \dim W^u(w_3) \\ &= i^{\mathcal{P}}(w_3) + 1 = 2\ . \end{aligned} \tag{6.55}$$

Finally, in the case of the Sturm attractor $\mathcal{A}^{\mathcal{P}}_{9,10}$ all the rotating waves have the same period lap number, $\ell(w_1) = \ell(w_2) = \ell(w_3) = 1$. Yet, while w_1, w_3 have Morse indices $i^{\mathcal{P}}(w_1) = i^{\mathcal{P}}(w_3) = 1$, the rotating wave w_2 has Morse index $i^{\mathcal{P}}(w_2) = 2$. Moreover, the unstable middle equilibrium e_2 has

Morse index $i^{\mathcal{P}}(e_2) = 3$ and the attractor has dimension $\dim \mathcal{A}^{\mathcal{P}}_{9,10} = 3$. The connecting sets corresponding to the edges of the connection graph $\mathcal{G}^{\mathcal{P}}_{9,10}$ are all two dimensional manifolds. In fact, we have the connections $w_2 \leadsto w_1$ and $w_2 \leadsto w_3$ for which

$$\dim(W^u(w_2) \cap W^s(w_1)) = i^{\mathcal{P}}(w_2) + 1 - i^{\mathcal{P}}(w_1) = 2 \,, \tag{6.56}$$

$$\dim(W^u(w_2) \cap W^s(w_3)) = i^{\mathcal{P}}(w_2) + 1 - i^{\mathcal{P}}(w_3) = 2 \,. \tag{6.57}$$

Also the middle equilibrium e_2 satisfies $e_2 \leadsto w_1$ and

$$\dim(W^u(e_2) \cap W^s(w_1)) = i^{\mathcal{P}}(e_2) - i^{\mathcal{P}}(w_1) = 2 \,. \tag{6.58}$$

Moreover, the edges of the connection graph $\mathcal{G}^{\mathcal{P}}_{9,10}$ are completed by $w_1 \leadsto e_1$, $w_1 \leadsto e_3$, $w_3 \leadsto e_1$ and $w_3 \leadsto e_3$ for which

$$\dim(W^u(w_1) \cap W^s(e_1)) = \dim(W^u(w_1) \cap W^s(e_3)) = i^{\mathcal{P}}(w_1) + 1 = 2 \,, \tag{6.59}$$

$$\dim(W^u(w_3) \cap W^s(e_1)) = \dim(W^u(w_3) \cap W^s(e_3)) = i^{\mathcal{P}}(w_3) + 1 = 2 \,. \tag{6.60}$$

All the connection graphs $\mathcal{G}^{\mathcal{P}}_{9,\kappa}$ associated to the Sturm permutations in P_3 are shown in Figure 14.

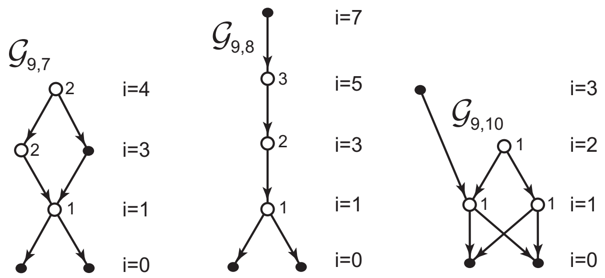

FIGURE 14. Connection graphs $\mathcal{G}^{\mathcal{P}}_{9,\kappa}$ for the set P_3 of Sturm permutations.

This completes the description, up to trivial equivalence, of all the Sturm connection graphs $\mathcal{G}^{\mathcal{P}}_f$ for $f \in \mathrm{Sturm}^{\mathcal{P}}(u, u_x)$ with m equilibria, q rotating waves, and $n = m + 2q \leq 9$.

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