Abstract

This thesis presents a dynamical systems approach for handling uniqueness problems in nonautonomous planar systems. The method is shown while being applied to a problem of uniqueness of a ground state of the continuum limit of a strictly anharmonic multi-dimensional lattice. Although this problem, and even a more general version of it, was already solved by Pucci and Serrin, we believe that our method sheds a new light on it.

Our solution to the uniqueness problem has two parts. In the first one we deal with the region $u > 0$ and give a uniqueness result for it. The second part presents a local result, i.e. that is independent of the initial condition, when analyzing what happens near the origin. In addition we discuss Pucci and Serrin's solution to the problem, and present it while using some concepts and results from the dynamical systems approach.

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Dynamical Systems Approach to Uniqueness Problems

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1 Introduction

This thesis presents a dynamical systems approach for handling uniqueness problems in‘ nonautonomous planar systems. The method is shown while being applied to a problem of uniqueness of a ground state of the continuum limit of a strictly anharmonic multi-dimensional lattice. Although this problem, and even a more general version of it, was already solved by Pucci and Serrin [10], we believe that our method sheds a new light on it.

The paper begins with an introduction to dynamical systems and phase plane analysis in the second section. The third section reviews Rosenau and Schochet's [9] discussion of lattice and continuum dynamics and their dynamical systems approach to the existence problem for a multi-dimensional continuum limit, including most details of the proof. Sections four and five phrase the uniqueness problem and present a dynamical systems approach to its solution. Section four deals with the region $u > 0$, while section five completes the proof by presenting a local result analyzing what happens near the origin. Finally, the last section focuses on Pucci and Serrin's [10] solution to the problem, and presents it while using some concepts and results from the dynamical systems approach.

2 Dynamical Systems and Phase-Plane Analysis

This section presents important definitions and results from dynamical systems theory, and in specific those related to phase space and phase plane analysis. The proofs are mostly classic, well known results of the field and were therefore omitted. The reader can find those proofs in most basic dynamical systems books, e.g. [6].

Consider an ODE system of dimension d

$$\vec{u}' = \vec{f}(\vec{u}, t), \tag{2.1}$$

where $\vec{f}$ will usually be assumed to be at least C^1. Several of the difficulties arising in the specific system studied here is that the corresponding $\vec{f}$ is not C^1 everywhere. The **phase space** $\mathbb{R}^d$ is the space of all possible values $\vec{u}$ of the system. The behavior of solutions in the phase space is dictated by the function $\vec{f}(\vec{u}, t)$. As a solution develops through time, it forms a curve in the phase space called a **trajectory**. Phase space analysis focuses on the study of those trajectories of different solutions, as their behavior determines many qualitative properties of the system. An important concept in phase-space analysis is that of **equilibrium points** or **fixed points**, i.e., the points $(\vec{u}_*, t_*)$ for which $\vec{f}(\vec{u}_*, t_*) = 0$.

In this work we mostly restrict our attention to the case $d = 2$, for which the phase space is called the **phase plane**. It is then convenient to let u and v denote the components of that plane, which allows the system to be written in the form

$$\begin{aligned} u' &= f(u, v, t) \\ v' &= g(u, v, t). \end{aligned} \tag{2.2}$$

As will be seen later in the section, several important results are valid only for this case.

It is convenient to examine the autonomous case

$$\begin{aligned} u' &= f(u, v) \\ v' &= g(u, v). \end{aligned} \tag{2.3}$$

on its own. One reason is that if $(u(t), v(t))$ is a solution of such a system, then so is $(u(t - t_0), v(t - t_0))$ for any t_0. Although the solutions for different values of t_0 are different, their trajectories are identical. Even more importantly, we will see later in the section that in the planar autonomous case trajectories cannot cross each other, a property which is very useful in the study of qualitative properties of solutions. When the system is autonomous then $(f(u, v), g(u, v))$ defines a **vector field** in the phase plane, i.e. assigns a vector $(f(u, v), g(u, v))$ to each point in the plane.

2.1 Fences, Funnels and Antifunnels

The results in this subsection are phrased for planar systems only since, as we mentioned in the previous subsection, we limit ourself to the case $d = 2$. A **fence** is a curve such that a solution cannot pass from one side of the curve to the other, possibly in one direction only. Formally:

Definition 2.1. *Let I be a t-interval, i.e., some interval for the variable t. A C^1 curve γ is a **fence** for system (2.3) if $(f, g) \cdot \nu \geq 0$ everywhere along the curve for every $t \in I$, where ν is a continuous normal to the curve. The **bounded** side is the side towards which the normal points.*

Remark: If the normal to a curve is always perpendicular to the vector field, then both sides are considered as bounded sides. In the autonomous case this happens precisely when the curve is a solution curve. Hence the following theorem shows in particular that two trajectories of an autonomous system cannot cross.

Theorem 2.2. ***(Fence Theorem)*** *A trajectory on the bounded side of a fence cannot cross the fence.*

One case in which fences are valuable is of a region that is bounded by fences on two sides. Funnels and antifunnels are both example of such regions.

Definition 2.3. *Let some region F be confined by fences on two sides. If F is the bounded side for both fences, then it is a **funnel**.*

Definition 2.4. *Let some region F be confined by fences with strict inequalities on two sides. If for both fences F is the not the bounded side, then F is an **antifunnel***

Three theorems come in handy when working with funnels and antifunnels. As was mentioned in the beginning of the section, their proof can be found in [6].

Theorem 2.5. ***(Funnel Theorem)*** *Let I be a u-interval in which the functions $\alpha(u)$ and $\beta(u)$ determine a funnel for all $u \in I$. Let $s = (u, v)$ be solutions to (2.3). If for some u_*, s is in the funnel, then s is in the funnel for all $u > u_*$ in I for which s is defined.*

Theorem 2.6. ***(Antifunnel Theorem: Existence)*** *Let I be some u-interval, and let the functions $\alpha(u)$ and $\beta(u)$ determine an antifunnel for all $u \in I$. Then there exists a solution to (2.3) that lies in the antifunnel for all $u \in I$.*

In certain conditions we can guarantee the uniqueness of such a solution

Theorem 2.7. ***(Antifunnel Theorem: Uniqueness)*** *Let $I = [u_0, u_1]$ be some u-interval, and let the functions $\alpha(u)$ and $\beta(u)$ determine an antifunnel for all $u \in I$ in the phase plane of (2.3). If the antifunnel is narrowing, i.e.,*

$$\lim_{u \to u_1} |\alpha(u) - \beta(u)| = 0, \tag{2.4}$$

and if f is nonzero, and $\frac{\partial(g/f)}{\partial v} \geq 0$ in the antifunnel, then there exists one and only one solution to (2.3) in $\mathbb{R}^2$, that stays in the antifunnel.

2.2 Linear Systems and Linearization

The results in the previous subsection were phrased for the autonomous equation, but are nevertheless valid for the nonautonomous case as well. As was mentioned before, certain results are only valid for autonomous equation

$$u' = f(u) \tag{2.5}$$

or for its corresponding two dimensional form

$$\begin{aligned} u' &= f(u, v) \\ v' &= g(u, v). \end{aligned} \tag{2.6}$$

Examine the system:

$$u' = Au, \tag{2.7}$$

where A is a $2x2$ real matrix, and the only fixed point is zero. Let λ_1, λ_2 be the two eigenvalues of A. The behavior of solutions around the fixed point is determined by the signs of the λs and by their types: real or complex. There are three cases of possible eigenvalues:
1. real and distinct eigenvalues.
2. complex eigenvalues (necessarily distinct since they are complex conjugate)
3. a double eigenvalue (real)

We won't analyze all the available cases, but will be satisfied with those of distinct and nonzero eigenvalues. In the case of two real distinct eigenvalues, the corresponding eigenvectors v_1 and v_2 form a basis of $\mathbb{R}^2$. The fixed points are classified in the following way:
1. **Source**: $0 < \lambda_1 < \lambda_2$. In that case all trajectories except the trivial one tend to ∞ as $t \to \infty$, and to 0 as $t \to -\infty$.
2. **Saddle**: $\lambda_1 < 0 < \lambda_2$. In that case there are two trajectories tending to zero as $t \to \infty$, and two trajectories tending to zero as $t \to -\infty$. All the other nontrivial solutions are superpositions of those motions, and tend to $\pm\infty$ as t goes to $\pm\infty$ (not necessarily correspondingly).
3. **Sink**: $\lambda_1 < \lambda_2 < 0$. In that case all trajectories tend to zero as $t \to \infty$, and to $\pm\infty$ as $t \to -\infty$.

In the complex eigenvalues case, let the two eigenvalues be $\alpha \pm i\beta$, where $\beta > 0$. The fixed points are classified in the following way:

1. **Spiral sink**: $\alpha < 0$. In that case the trajectories spiral into $(0, 0)$.
2. **Center**: $\alpha = 0$. In that case the trajectories turn periodically on ellipses, each of which is centered at $(0, 0)$.
3. **Spiral source**: $\alpha > 0$. In that case the trajectories spiral out to ∞.
A graphical representation of the different types of points can be seen in Figure 2.1.

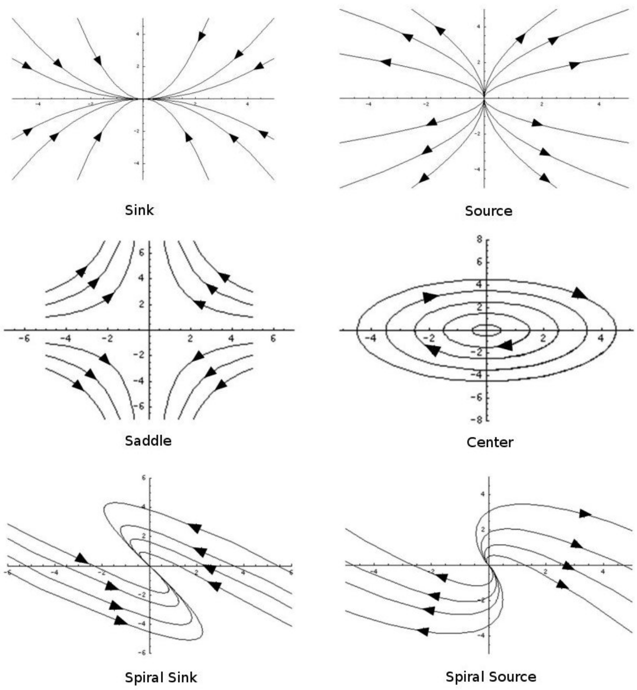

Figure 2.1: Fixed Point Classification

Now consider a nonlinear system having a fixed point u_*. The change of variable $u \to u - u_*$ moves the fixed point to the origin. The system may then be written, using a Taylor series expansion, in the form:

$$\frac{d}{dt}u = Au + g(u), \tag{2.8}$$

where $g(u) = O(|u^2|)$ as $u \to 0$. In order to make the nonlinear part $g(x)$ small compared to the linear part Ax near the origin, We demand that A is a nonsingular matrix, so that $|Au| > c|u|$. It turns out that in most cases, solutions that are close to the origin behave similarly to solutions of the linear part of (2.8). In specific, this is true as long as the real part of the eigenvalues is nonzero. Before phrasing this as a theorem, we have to define the terms 'sink', 'source' and 'saddle' for nonlinear case (2.8):

Let u_0 be a zero of an autonomous differential equation $u' = f(u)$ in $\mathbb{R}^2$.
1. u_0 is a **sink** if there is a neighborhood U of u_0 such that any solution $u(t)$ with $u(t_0) \in U$, remains in U for $t \geq t_0$, and $\lim_{t\to\infty} u(t) = u_0$.
2. u_0 is a **source** if there is a neighborhood U of u_0 such that any solution

$u(t)$ with $u(t_0) \in U$, was in U for $t \leq t_0$, and $\lim_{t \to -\infty} u(t) = u_0$.
3. u_0 is a **saddle** if there are precisely two trajectories that tend to u_0 as $t \to \infty$, and precisely two trajectories that tend to u_0 as $t \to -\infty$.

Theorem 2.8. *If the eigenvalues of A in (2.8) both have positive real part, both have negative real part, or have opposite signs, then zero is a source, sink or saddle for equation (2.8) correspondingly.*

2.3 Poincare-Bendixon Theorem

No introduction to phase-plane analysis will be complete without the famous Poincare-Bendixon theorem. Before introducing that theorem, it is necessary to define the concept of 'limit set'.

Definition 2.9. ***(Limit set)*** *The ω-limit set of a point $\vec{u}_0$ is:*

$$L_\omega(\vec{u}_0) := \{\lim \vec{u}(t_j, \vec{u}_0) \mid t_j \to \infty\}. \tag{2.9}$$

The corresponding definition for the α-limit set:

$$L_\alpha(\vec{u}_0) := \{\lim \vec{u}(t_j, \vec{u}_0) \mid t_j \to -\infty\}. \tag{2.10}$$

The names α and ω come from the fact that those are the first and last letters in the Greek alphabet.

For a trajectory T, $L_\omega(T) := L_\omega(\vec{u}_0)$ for $\vec{u}_0 \in T$. In the same manner: $L_\alpha(T) := L_\alpha(\vec{u}_0)$ for $\vec{u}_0 \in T$. Note that both definitions are independent of the choice of $\vec{u}_0$.

With those definitions in hand, we can now state the two versions of the Poincare-Bendixon Theorem.

Theorem 2.10. ***(Poincare Bendixon)*** *Let $\vec{u}(t)$ be a solution to the differential equation $\vec{u}' = \vec{f}(\vec{u})$ in $\mathbb{R}^2$, defined and bounded for $t \geq t_0$. If $L_\omega(\vec{u})$ contains no fixed point of $\vec{f}$ then it is a periodic trajectory.*

Theorem 2.11. ***(Strong Poincare Bendixon)*** *Let $\vec{u}(t)$ be a solution to the differential equation $\vec{u}' = \vec{f}(\vec{u})$ in $\mathbb{R}^2$, defined and bounded for $t \geq t_0$. Let $\vec{u}_0$ be a point of the solution in the phase plane. If $L_\omega(\vec{u}_0)$ is not a periodic trajectory, then $\vec{f}$ vanishes on $L_\omega(\vec{u}_*)$ and $L_\alpha(\vec{u}_*)$ for every $\vec{u}_* \in L_\omega(\vec{u}_0)$.*

3 Strictly Anharmonic FPU Lattices

This section introduces work of Rosenau and Schochet concerning solutions of strictly anharmonic FPU lattices. Every reference to Rosenau and Schochet here refers to [11] unless written otherwise.

In the summer of 1953 E. Fermi, J. Pasta, and S.M. Ulam conducted a series of computer simulations of physical systems containing a non-linear term. Their motivation was based on two main beliefs. The first claimed that future fundamental theories in physics may involve non-linear operators and equations. The latter said that a computing machine would be a great tool in understand asymptotic long-time behavior of solutions. As a first example was chosen a vibrating string, in which in addition to the usual linear term, non-linear ones were also included. The conjecture was that for very long times the influence of the initial state would fade, and the system would become more or less random. Surprisingly enough, what happened is that instead of having energy flowing consistently from the initial state, it was found out that the solution remains more or less close to it. At some large time the system even got back to almost its initial state. The Fermi-Pasta-Ulam (FPU) problem is the paradox between the expected solution and the one we actually got.

The behavior of that last described solution resembles the behavior of a **breather**, a term that describes oscillatory solutions which are periodic in time and localized in space. This term first came up from a particular solution of the sine-Gordon Equation (SGE). The name was given since the oscillation, combined with localization and periodicity, gives the solution an illusion of 'breathing'. When talking about the discrete case, in which a system can be viewed as a lattice, a breather is called a **discrete breather** but is also periodic in time and localized in space. There are an very large number of results about discrete breathers, a review of those can be found in FW [4].

Rosenau and Schochet considered lattices that are strictly anharmonic, i.e. have no linear interaction term, and have a suitable quartic site potential appended. When the lattice is dense, the distance between particles becomes very small, and they can be modeled using the continuum limit. In this limit the system is described by a PDE. Rosenau and Schochet focused on the discovery of discrete breather and breather solutions, both for the discrete lattice, and for its continuum limit.

The order of the section is as follows: first the problem is introduced in one dimension (1D). In this dimension we present existence results for almost-compact breathers (breather that decays to zero faster than exponentially) in a discrete system, and strictly compact breathers in the continuum limit. Next, the problem is defined for hexagonal lattices in dimension two (2D), which supports a compact radial breather in its continuum limit. Full technical details for the results of that case will be supplied, as they are relevant for us later in the paper.

3.1 One dimensional lattices

The original system for the 1D case is presented using physical variables. However, those are not of interest to us, so we begin with the form that the discrete oscillator chain gets after normalization:

$$\ddot{u_n} + u_n = \frac{\left(\frac{u_{n+1}-u_n}{h}\right)^3 - \left(\frac{u_n - u_{n-1}}{h}\right)^3}{h} + u_n^3, \tag{3.1}$$

where h measures how far the system is from being described by the normalized continuum limit equation. This equation is obtained by replacing $u_n(t)$ with $u(t, nh)$, and taking the formal limit as $h \to 0$:

$$u_{tt} + u = \partial[(u_x)^3] + u^3. \tag{3.2}$$

Breathers for equations such as (3.1) and (3.2) are solutions that are periodic in time and localized in space [7]. Using the space-time separability of both the equations, Rosenau and Schochet calculated the solution of the continuum case (3.2) explicitly, and showed that such a breather solution exists for that case.

For the discrete system (3.1) the solution does not seem to have an explicit formula. However, using a one-parameter shooting method it was possible to prove that this equation has two breather solutions. Both of them decrease monotonically to zero to the right of their center of symmetry, and converge everywhere as h tends to zero to the continuum compact wave. Discrete breather spatial variable cannot be compactly supported. However, the discrete profile decays to zero at a double exponential rate, which is, as we mentioned before, an almost-compact breather solution.

3.2 Two dimensional lattices

Rosenau and Schochet showed that in the 2D case, a rectangular lattice induced a non-isotropic limit equation of motion unless the potentials are quadratic, and a limit equation that contains linear terms when the potentials are quadratic. Since our interest is in purely nonlinear equation of motions, a hexagonal lattice, i.e., an equilateral triangular lattice for which the nearest neighbors of each grid point form a regular hexagon, was used. The underlying discrete Hamiltonian is:

$$H = \Sigma \left\{ \frac{1}{2}(\dot{u}_{m,n})^2 + \Phi(u_{m,n}) + \Sigma P \left(\frac{u_{j,k} - u_{n,n}}{l} \right) \right\}. \tag{3.3}$$

We will be interested in the investigation of the continuum limit equation of a hexagonal lattice that has purely quartic interactions:

$$u_{tt} + u = \nabla \cdot [(\nabla u)^2 \nabla u] + u^3. \tag{3.4}$$

The rest of this section shows that a compact breather solution exists for this case as well.

3.3 Separation of variables

The first step towards proving the existence of a compact breather solution to the continuum hexagonal equation (3.4), is to separate variables. I.e. represent its solution as:

$$u(t, x) = \phi(t)U(x). \tag{3.5}$$

Substituting this into (3.4) yields the ODE for the temporal variable:

$$\ddot{\phi} + \phi = \phi^3. \tag{3.6}$$

For the spatial part we obtain

$$\nabla \cdot [|\nabla u|^2 \nabla u] + u^3 - u = 0. \tag{3.7}$$

The solutions of (3.6), which can be expressed in terms of elliptic integrals, are periodic if $|\phi(0)| < 1$, constant if $|\phi(0)| = 1$, and blow up in finite time if $|\phi(0)| > 1$, assuming that the other initial condition is $\dot{\phi} = 0$. Hence, every spatially localized solution u of (3.7) yield breather solution, stationary solution and blowing-up solution of the continuum hexagonal equation (3.4).

For the spatial equation (3.7), radial solutions satisfy the ODE

$$[u'(r)^3]' + \frac{d-1}{r} u'(r)^3 + u(r)^3 - u(r) = 0, \tag{3.8}$$

where d is the spatial dimension. Our focus in this thesis is on positive solutions of that equation. Specifically, we look for solutions satisfying the conditions:
1. $u'(0) = 0$, and $u(0) > 0$.
2. $u(r_*) = 0 = u'(r_*)$ for some finite positive $r_* > 0$.
3. $u(r) > 0 > u'(r)$ for $0 < r < r_*$.
4. $(\frac{du}{dr})^3$ is differentiable at $r = 0$ and $r = r_*$.

3.4 Existence of multidimensional continuum breathers

This section focuses on the existence result of radial solutions for the spatial equation for the continuum 2D lattice equation (3.8). The result presented here is based on Appendix C in Rosenau and Schochet [11]. It should be noted that a more general existence result for that problem was proved by FLS [5]. However, just as this thesis uses the dynamical systems approach to the uniqueness problem, Rosenau and Schochet use that approach to the existence problem. Most of their proofs are repeated here since the technical details will be used later on. Some proofs, however, were omitted, and can be found in the original article.

3.4.1 A transformed system.

Any solution of (3.8) satisfies (3.7) provided that $u'(0) = 0$ so that the corresponding solution of (3.7) will be smooth at the origin. Equation (3.8) is transfered to a first-order system upon defining $v := u'(r)^3$:

$$\begin{aligned} u' &= v^{\frac{1}{3}}, \\ v' &= -\tfrac{d-1}{r}v + u - u^3. \end{aligned} \tag{3.9}$$

The presence of the term $\frac{v}{r}$ in the second equation of (3.9) implies that v must vanish at r=0 in order for solutions to be smooth. This, in turn, means that term $v^{\frac{1}{3}}$ appearing in the first equation is not smooth at $r = 0$. However, we desire a system whose only singularity at $r = 0$ is a simple pole like that in the term $\frac{v}{r}$. This will be achieved with rescaling the v variable as

$$v = -rW, \tag{3.10}$$

where the minus sign is used to assure W's positivity. The spatial variable is rescaled to

$$R := \frac{3}{4} r^{\frac{4}{3}}. \tag{3.11}$$

The constant factor in the last equation has been included for convenience as it simplifies the numerical factors appearing in the resulting equations.
Remark: The name W (big 'w') was chosen to stay consistent with Rosenau and Schochet's paper. There they used the negative variable w, which is actually $w = -W$. For our work it is more comfortable to work with a positive variable, and therefore the theory is presented using W.

Under these changes of variables, system (3.9) transforms to

$$\begin{aligned} u' &= -W^{\frac{1}{3}}, \\ W' &= \tfrac{3}{4R}[u^3 - u - dW], \end{aligned} \tag{3.12}$$

where the derivatives are with respect to R. System (3.12) implies a condition for smoothness, which is obtained from the equation for W:

$$W(0) = \frac{[u(0)^3 - u(0)]}{d}. \tag{3.13}$$

$W(0)$ is different than zero as long as $u(0) \notin -1, 0, 1$. Therefore as long as this happens the term $W^{\frac{1}{3}}$ in the first equation will be smooth at $R = 0$.

Local existence near R=0. The equation for W in (3.12) is linear. Therefore, it can be solved to yield

$$W(R) = \frac{c}{R^{4d/3}} + \frac{4}{3R^{4d/3}} \int_0^R s^{\frac{4d}{3}-1}[u(S)^3 - u(S)]ds. \tag{3.14}$$

We look for a solution that remains bounded at $R = 0$. Therefore, the constant of integration c in (3.14) must vanish. The equation reduces to

$$W(R) = \frac{4}{3R^{4d/3}} \int_0^R s^{\frac{4d}{3}-1}[u(S)^3 - u(S)]ds. \tag{3.15}$$

This can be plugged into the equation for u' in (3.12). Integrating yields

$$u = u_0 - \left(\frac{4}{3}\right)^{\frac{1}{3}} \int_0^R \frac{1}{\rho^{4d/9}} \left[\int_0^\rho S^{\frac{4d}{3}-1}[u(S)^3 - u(S)]dS\right]^{1/3} d\rho. \tag{3.16}$$

Using the contraction-mapping fixed-point theorem in standard fashion, the integral equations (3.15) and (3.16) can be used to prove the local existence of solutions to system (3.12) near $R = 0$.

Theorem 3.1. *Take some $u_0 > 1$. Then, there exists a solution to (3.12) with initial conditions $u(0) = u_0$ and (3.13) on some interval $[0, \delta]$.*

The details of the proof for that theorem can be found in [11]. In addition the solutions depend continuously on their initial condition, since by Proposition 1.2 in [13] the contraction-mapping theorem yields continuous dependence on parameter, and an initial condition can be considered as an implicit parameter.

3.5 Existence theorem for multidimensional breathers

In the last subsection we show existence of a breather profile.

Theorem 3.2. *Let $d > 1$. There exists a solution to the ODE (3.8) satisfying:*
1. $u'(0) = 0$, and $u(0) > 0$.
2. $u(r_*) = 0 = u'(r_*)$ for some finite positive $r_* > 0$.
3. $u(r) > 0 > u'(r)$ for $0 < r < r_*$.
4. $(\frac{du}{dr})^3$ is differentiable at $r = 0$ and $r = r_*$.

Remark: To see that we indeed achieve a compact breather, one need to define the function $U(x)$ to equal $u(|x|)$ for $|x| < r_*$, and to vanish for $|x| > r_*$. Then, the properties of u stated in the theorem ensure that U is a compact breather solution to (3.7).

Equation (3.8) corresponds to (3.9), which in turn corresponds to (3.12). Therefore, it is sufficient to prove the following theorem:

Theorem 3.3. *For some positive $R_* > 0$ there exists a solution to system (3.12) on $(0, R_*]$ that is continuous on $[0, R_*]$ and satisfies:*
1. $u(0) > 0$.
2. $u(R), W(R) > 0$ *on* $(0, R_*)$.
3. $u(R_*) = 0 = W(R_*)$

Remark: The differentiability of $(\frac{du}{dr})^3$ with respect to r at $r = 0$ can be seen from (3.9) and (3.10) plus the continuity of $W(R)$ at $R = 0$. Its differentiability at $r = r_*$ can be seen from (3.9) and (3.10) plus the fact that (3.12) is satisfied at $R = R_*$.

Proof. Denote by C the portion of the curve $\frac{(u^3-u)}{d}$ lying in the region $u > 1$. By Theorem 3.1, for every point P on the curve C, there exists a solution to (3.12) starting at P at $R = 0$, for R in some interval $[0, \delta]$. Define Ω to be the region bounded by the positive W-axis, the line segment $[0, 1]$ on the u-axis, and the curve C. Since the system (3.9) is smooth in Ω, the continuation theorem for ODEs shows that solutions continue to exist as long as they remain in Ω. Furthermore, although the system (3.9) is non-autonomous, the right side of that system still defines a vector field, albeit a time-dependent one. For all positive "times" R, that vector field shows that both u' and W' are negative as long as the solution remains in the region Ω. Note that the vector field points into Ω along C for $R > 0$. That and the direction of u' and W', shows that the solutions exists and remains in Ω until it reaches either the positive W axis, the origin, or the line segment $(0, 1]$ on the u-axis. There only remains to show that for some point P on C, the solution starting at P at "time" $R = 0$, reaches the origin, and in a finite time.

Subtracting $\frac{3}{4}RW'$ times u' from $\frac{3}{4}Ru'$ times W' gives zero: $\frac{3}{4}RW'u' - \frac{3}{4}Ru'W' = 0$. Plugging this expression into the equations for u', W' from (3.12), and integrating yields

$$\begin{aligned} &u'u^3 - u'u + dW^{4/3} + \tfrac{4}{3}RW'W^{1/3} = 0 \Rightarrow \\ &\mathscr{E} := RW(R)^{4/3} + \tfrac{1}{4}u(R)^4 - \tfrac{1}{2}u(R)^2 = \\ &E - (d-1)\textstyle\int_0^R W(S)^{4/3}dS, \end{aligned} \tag{3.17}$$

where E is a constant of integration.

The function $\mathscr{E}$ takes negative values on the line segment $(0,1]$ of the u-axis, vanishes at the origin, and takes positive values on the positive W axis for any nonnegative R. By (3.17), $\mathscr{E}$ is nonpositive initially for $u_0 < \sqrt{2}$ and decreases with time. But solutions that begin at $R = 0$ on C with $u_0 < \sqrt{2}$, have negative values of $\mathscr{E}$ for positive R. Therefore they cannot approach the origin (nor the positive W axis).

The next functional aids us in proving that solutions starting on C with large values of u_0 do reach the positive W axis:

$$\mathscr{F} := \mathscr{E} - (d-1)uW. \tag{3.18}$$

Formula (3.17) shows that $\mathscr{E}' = -(d-1)W^{4/3}$. Combining this with the first equation of (3.12), shows that

$$\mathscr{F}' = -(d-1)uW', \tag{3.19}$$

which is positive since u is positive and W' is negative in the region Ω. The term added to $\mathscr{E}$ to obtain $\mathscr{F}$, namely $(d-1)uW$, vanishes on the axes. Therefore, $\mathscr{F}$ is also negative on the line segment $(0,1]$ of the u-axis, zero at the origin, and positive on the positive W axis like $\mathscr{E}$. $\mathscr{F}$ is increasing, and therefore if a solution ever has a positive value of $\mathscr{F}$, then it must reach the positive W axis.

Calculating $\mathscr{F}$ on the initial curve C for $R = 0$ shows that it is negative there. The next argument shows that for u_0 large enough, the value of $\mathscr{F}$ on the corresponding solution eventually becomes positive. First, for R sufficiently large the functional $\mathscr{F}$ is positive on a line $W = c > 0$. For example, for $R = 1$:

$$\mathscr{F}\Big|_{W=8} = 16 + \frac{u^4}{4} - \frac{u^2}{2} - 8u = \left(\frac{8}{3} - \frac{3}{2}u\right)^2 + \left(\frac{11}{4} - \frac{1}{2}u^2\right)^2 + \frac{191}{144}$$

Since $\mathscr{F}$ is increasing in R, this shows that $\mathscr{F}$ is positive on the line $W = 8$ for all $R \geq 1$. Any solution for which $W > 8$ at time one must either reach the W axis above $W = 8$ or else, reach the line $W = 8$. Therefore, any such solution must eventually have a positive value of $\mathscr{F}$, and so reach the positive W axis.

Next we'll show that for u_0 sufficiently large, the solution must either reach the positive W axis, or remain above $W = 8$ through time $R = 1$. W is decreasing and therefore

$$|u'| \leq |W(0)|^{1/3} < \left(\frac{u_0^3}{d}\right)^{1/3} = \frac{u_0}{d^{1/3}}$$

Hence, through time $R = 1$, $u \geq u_0(1 - \frac{1}{d^{1/3}})$, and for u_0 large enough, $u > 1/\sqrt{3}$ at that time. Combining this with the fact that $u^3 - u$ increases for $u > 1/\sqrt{3}$, and plugging those estimates in (3.15) shows that through $R = 1$

$$\begin{aligned} W(R) &= \tfrac{4}{3R^{4d/3}} \int_0^R s^{\frac{4d}{3}-1}[u(S)^3 - u(S)]ds \\ &\geq \tfrac{4}{3R^{4d/3}} \int_0^R s^{\frac{4d}{3}-1}\left\{\left[u_0\left(1 - \tfrac{1}{d^{1/3}}\right)\right]^3 - u_0\left(1 - \tfrac{1}{d^{1/3}}\right)\right\}ds \\ &\geq \tfrac{1}{d}\left\{\left[u_0\left(1 - \tfrac{1}{d^{1/3}}\right)\right]^3 - u_0\left(1 - \tfrac{1}{d^{1/3}}\right)\right\}, \end{aligned} \tag{3.20}$$

which is bigger than 8 for u_0 sufficiently large.

We showed that some solutions leave Ω along the positive u axis, and other leave along the positive W axis. This, and the continuity of solutions on their initial data as was noted after theorem 3.1, allows us to show that there exists a solution that reaches the origin. This is true even though the system (3.12) is nonautonomous, so that the projection onto the $u - w$ plane of trajectories may cross one another.

Consider the infimum u_* of all $k > 1$, such that for all $u_0 > k$, the solution reaches the positive W axis. We have shown above that the set over which the infimum is taken is nonempty and is bounded from below by a number greater than one. Hence, u_* exists and is greater than one. Then there exist $u_0^{(j)} \to u_*$, such that the solutions with initial value u_0 reach the positive

W axis, and so have $\mathscr{E}$ positive on their entire trajectories. The solutions depend continuously on u_0 as was noted after theorem 3.1.

Hence the solution starting at u_* must also have $\mathscr{E} \geq 0$. If that solution had $\mathscr{E}$ strictly positive, then it would reach the positive W axis. In that case, the same continuity argument would show that solutions with slightly smaller u_0 would also reach the positive W axis, which would contradict the fact the u_* is the infimum. Hence $\mathscr{E}$ must tend to zero, but remain nonnegative on the solution starting at u_*. This shows that it must tend to the origin.

Finally, we'll show that a solution that reaches the origin must do it in a finite time. Since $\mathscr{E}$ is nonnegative along a solution tending to the origin, for $u < 1$ we have that

$$RW^{4/3} \geq \frac{1}{2}u^2[1 - \frac{1}{2}u^2] \geq \frac{1}{4}u^2. \tag{3.21}$$

Solving that equation for $W^{1/3}$ yield

$$W^{1/3} \geq c\frac{u^{\frac{1}{2}}}{R^{\frac{1}{4}}}. \tag{3.22}$$

Plugging (3.22) into the first equation of (3.12) and integrating from some time R_1 at which $u = u_1 \leq 1$ to R yield

$$2u^{\frac{1}{2}} - 2u_1^{\frac{1}{2}} \leq k(R_1^{\frac{3}{4}} - R^{\frac{3}{4}}), \tag{3.23}$$

which shows that u reaches zero at a finite value of R. □

3.6 Existing methods for uniqueness problem

Before proving the uniqueness of a solution to (3.7), we discuss in this section some former work and existing methods for uniqueness proofs of equations similar to ours. The main equation in this work, equation (3.7) is a special case of

$$\Delta_m u + f(u) = 0, \tag{3.24}$$

where $\Delta_m u$ denotes the degenerate m-Laplace operator $div(|Du|^{m-2}Du)$ with $m > 1$, and $f(u)$ is a continuously differentiable function defined for $0 < u < \infty$. This equation is, in turn, a special case of the more general equation

$$div(A(|Du|)Du) + f(u) = 0, \tag{3.25}$$

where $A \in C^1(0,\infty)$. A uniqueness results to equation (3.25) was achieved, under certain conditions on f and A, by Pucci and Serrin [10]. The main tool that they used is a cleverly-chosen function P, a version of which is also used in Lemma 4.8 of our proof. Section 6 reviews Pucci and Serrin's solution, while also pointing out where the dynamical systems approach could have been used there.

Pucci and Serrin uniqueness result applies in particular to the important canonical nonlinearity

$$f(u) = -u^p + u^q, \tag{3.26}$$

for $0 < p \le m^* - 1$, $p < q < m^*$, and m_* is

$$m^* = \begin{cases} \frac{(m-1)n+m}{n-m} \quad if \quad m < n; \\ \infty \quad if \quad m \ge n. \end{cases}$$

The question of uniqueness in the entire range $-1 < p < q < m^*$ is left open. Serrin and Tang [12] managed to proof a uniqueness result for the entire range for the less general equation (3.24), by using the same function P, which is simpler on that equation, with the aid of a separation theorem.

Berestycki and Lions [1] proved a very general existence theorem for the semi-linear case ($m = 2$ of (3.24))

$$\Delta u + f(u) = 0, n > 2, \tag{3.27}$$

where the nonlinearity (3.26) was treated as a canonical example. They asked, as a major open problem, which nonlinearities f admit uniqueness of ground states for the semilinear elliptic equation (3.27) with $n > 2$. The first result in this problem was achieved by Coffman [2] for the case $p = 3, q = 1$ and $n = 3$. Coffman proof used a series of special identities, which sadly, as he himself remarked, cannot be extended to other choices of n or other power of u in f. Later, Mcleod and Serrin [9] improved Coffman's method to include more general functions f. Based on that previous work, Kwong [8] proved uniqueness for (3.26) with $p = 1$ and $1 < q < (n+2)/(n-2)$. His proof was based on a version of Sturmian comparison theorem, which, sadly, is not valid for other cases than $m = 2$ (and especially our case, $m = 4$).

4 Uniqueness Result Part I: No Intersection in $u > 0$ For $2 \leq d \leq 4$

It is now time to present the main theorem of this thesis, which will be proved in this section and following one. Keep in mind that the main goal of this thesis is not only to prove that theorem, but also to do it with the dynamical systems approach while comparing that to an older existing approach (see Section 6).

Main Theorem 4.1. *Assume $2 \leq d \leq 4$. There exists a unique solution to the ODE (3.8) satisfying:*
1. $u'(0) = 0$, and $u(0) > 0$.
2. $u(r_*) = 0 = u'(r_*)$ for some finite positive $r_* > 0$.
3. $u(r) > 0 > u'(r)$ for $0 < r < r_*$.
4. $(\frac{du}{dr})^3$ is differentiable at $r = 0$ and $r = r_*$.

Like in the existence result, equation (3.8) corresponds to (3.9), which in turn corresponds to (3.12). Therefore, it is sufficient here as well to prove the following theorem:

Theorem 4.2. *For some positive $R_* > 0$ there exists a unique solution to system (3.12) on $(0, R_*]$ that is continuous on $[0, R_*]$ and satisfies:*
1. $u(0) > 0$.
2. $u(R), W(R) > 0$ on $(0, R_*)$.
3. $u(R_*) = 0 = W(R_*)$

The existence result for this problem was already proved in Theorem 3.2, therefore only the uniqueness result is left. The demand for $W(R)$ to be positive follows from the fact that $\mathscr{E}$ from equation (3.17) is a decreasing function of R, that is negative on the interval $(0, 1]$ of the u axis (where $W = 0$) and vanishes at the origin.

Assume that two solutions s_1 and s_2 reach the origin. There are three ways for that to happen:
1. The solutions reach the origin at different times.
2. The solutions reach the origin at the same time but are not identical in

any neighborhood of the origin.
3. At some time R, the solutions consolidated into one curve, which later reaches the origin. That possibility is eliminated by the uniqueness theorem, that holds for every $R > 0$ except at the origin, since the vector field is smooth there.

This section proves two related results. The first one is that trajectories (u, W) of the non-autonomous system (3.12) that begin on the initial curve C defined above can't cross each other in the region Ω. Throughout this section solutions are assumed to begin on the curve C at time zero, so this result will be phrased in the form "$u - W$ trajectories cannot cross". Of course, solutions that do not start at the origin can cross; for example, take two solutions starting at the same point (u, W) at different times. The fact that the solutions under consideration start on the initial curve C enters into the argument most prominently in formula (4.17).

The second result claims that if two solutions reach the origin, then they must do so at the same time. In other words, the first case is impossible. The proof of the theorem will be completed in the next section by showing that the second case is also impossible.

We work with the system (3.12). Unless specified otherwise, solutions start on the initial curve $W_0 = \frac{u_0^3 - u_0}{d}$ with $u_0 > 1$. It was shown in previous sections that on that curve, the solutions are well defined for a small neighborhood of $R \in [0, \delta]$ for some $\delta > 0$. We work, as in sections 3.5, in the region Ω that is confined between the initial curve ,$u \geq 0$ and $W \geq 0$. In that region $u' \leq 0$ and $W' \leq 0$. We handle only cases in which $d \geq 2$.

One notation that will be used a lot is of $R(u)$ which is the inverse function of $u(R)$. The function exists as long as u is inside of Ω (e.g. for $u \leq u_0$), since u' is always negative there. Hence we may also consider W as a function of u given by $W(u) = W(R(u))$. In fact, since W is also strictly monotone we can equally consider u as a function of W. When considering two solutions $u_1(R)$ and $u_2(R)$ we will let $R_1(u)$ and $R_2(u)$ denote the corresponding inverses.

Definition 4.1. *Let s_1, s_2 be two solutions such that $u_1(0) < u_2(0)$.*
1. *s_2 is **above** s_1 at u for the $u - W$ system (3.12) if $W_2(u) > W_1(u)$ or if $W_1(u)$ doesn't exist. The latter case means that $u > u_1(0)$ and the definition*

*follows since s_2 is above s_1 at $u_1(0)$. Under those conditions s_1 is **below** s_2 at u for the $u-W$ system.*
2. *s_1 at u for the $u-z$ system (4.3) if $z_2(u) > z_1(u)$ or if $z_1(u)$ doesn't exist. Under those conditions s_1 is **below** s_2 at u for the $u-z$ system.*

Note that in both systems a change in above/below status of two solutions means that they intersected each other in the corresponding system.

Suppose that two solutions of the $u-W$ system do cross each other. Then the order in which those solutions reach the intersection point is dictated by their initial condition.

Lemma 4.2. *Let s_1 and s_2, $u_1(0) < u_2(0)$, be two intersecting solutions of the $u-W$ system, first intersecting at (u_m, W_m) with $u_m > 0$. Then: $R_1(u_m) > R_2(u_m)$.*

Proof. Examine the solutions at the crossing point (u_m, W_m). Before the crossing point, s_1 is below s_2, while after it they change roles. It means, that at $R_1(u_m), R_2(u_m)$:

$$\frac{W_1'}{u_1'}(R_1(u_m)) \leq \frac{W_2'}{u_2'}(R_2(u_m)). \tag{4.1}$$

Plugging into $\frac{\partial W}{\partial u}$ the values of W' and u' yields

$$\frac{\partial W}{\partial u} = -\frac{3[u^3 - u - dW]}{4RW^{1/3}}, \tag{4.2}$$

which is positive and bigger for smaller values of R. Combining all of that shows that $R_1(u_m) \geq R_2(u_m)$. Moreover, equality is impossible since the uniqueness theorem would then show that the two solutions were identical. □

Define the variable $z = R^{3d/4}W$. Plugging that in system (3.12) yields

$$\begin{array}{r} u'(R) = -R^{-d/4}z(R)^{1/3} \\ z'(R) = \frac{3}{4}R^{\frac{3d}{4}-1}[u^3 - u] \end{array} . \tag{4.3}$$

$R^{\frac{3d}{4}-1}$ is an increasing function of R for $d \geq 2$, while $R^{-d/4}$ decreases.

Lemma 4.3. ***(Intersection for the*** $u-z$ ***system precedes intersection for the*** $u-W$ ***system)*** *Let* s_1, s_2 *be two intersecting solutions in the* $u-W$ *system, such that their first intersection happens at* (u_m, W_m) *with* $u_m > 0$. *Then there is an intersection in the* $u-z$ *system at* (u_{m_z}, z_{m_z}), *where* $u_{m_z} > u_m$.

Proof. Assume, without losing generality, that $u_1(0) < u_2(0)$, and therefore s_1 is below s_2 in the $u-z$ coordinates at $R=0$.

By Lemma 4.2, $R_1(u_m) > R_2(u_m)$. Since $z = R^{3d/4}W$ this means that $z_1(u_m) > z_2(u_m)$. By definition 4.1 s_1 is above s_2 at the $u-z$ system at u_m, meaning that there was an intersection at the $u-z$ system at a point (u_{m_z}, z_{m_z}) where $u_{m_z} > u_m$. □

Definition 4.4. *Let* s_1, s_2 *be two solutions.*
1. s_1 *and* s_2, *with* s_1 *below* s_2, *are in prior to u-equality at* u *if* $R_1(u) < R_2(u)$.
2. s_1 *and* s_2 *are in u-equality at* u, *if* $R_1(u) = R_2(u)$.
3. s_1 *and* s_2 *are in* W*-equality at time* R, *if* $W_1(R) = W_2(R)$.

Remarks: 1. When using the first definition we have to specify which system the below status refers to.
2. Every pair of solutions are in prior to u-equality at $R=0$.
3. If two solutions are in u-equality, then they are in this status for both $u-W$ and $u-z$ systems.
4. Note that the first two definitions depend on u, while the last one depends on R. It is possible to define the 'prior to u-equality' and u-equality using R as well. It is not possible to define W-equality using u.

Lemma 4.5. *Let* s_1, s_2 *be two solutions that are, at some* u, *in prior to u-equality status in the* $u-W$ *system, with* s_1 *below* s_2.
1. *If the solutions cross each other for the first time after passing* u *at* (u_m, W_m), *then there exists a unique* u_e *in* (u_m, u) *at which* $R_1(u_e) = R_2(u_e)$ *and at least one* R_W *satisfying* $R_2(u_m) > R_W > R_1(u_e) = R_2(u_e) > R_2(u)$ *at which* $W_1(R_W) = W_2(R_W)$.
2. *If* W*-equality holds at* R_W, *then there is some* $R < R_w$ *such that the solutions are in u-equality at* $u_1(R)(= u_2(R))$.

Proof. **1.** u_e existence: Since the solutions are prior to u-equality, and s_1 is below s_2, then $R_1(u) < R_2(u)$. On the other hand, at u_m, $R_1(u_m) > R_2(u_m)$ by Lemma 4.2. By the continuity of $R(u)$ there exists $u_e < u$ such that $R_1(u_e) = R_2(u_e)$.

R_W existence: note that $W_1(R_2(u)) < W_1(R_1(u)) < W_2(R_2(u))$ since W is decreasing, $R_2(u) > R_1(u)$ by definition of the prior-to-u-equality state, and s_1 is below s_2 at u. But $W_1(R_2(u_m)) > W_1(R_1(u_m)) = W_2(R_2(u_m))$ since W monotonically decreases and $R_1(u_m) > R_2(u_m)$ by Lemma 4.2. W continuity then shows that R_W exists as requested.

u_e is unique: assume that s_1 is below s_2 at u_+, in which they are in u-equality status. The differential equation for u can be written as $\frac{dR}{du} = -\frac{1}{W^{1/3}}$, which shows that $R_1(u) > R_2(u)$ on some interval $(u_+ - \delta, u_+)$. Hence if $u_- < u_+$ is the first point of u-equality to the left of u_+ then $R_1(u) > R_2(u)$ on the entire interval (u_-, u_+). But if s_1 is still below s_2 at u_-, then the same ODE shows that $R_1(u) < R_2(u)$ on some interval $(u_-, u_- + \delta)$. This contradiction shows that there can be no second u-equality as long as s_1 remains below s_2.

$R_W > R_1(u_e) = R_2(u_e)$: assume by contradiction the opposite: $R_W < R_2(u_e)$. Since $R_W < R_2(u_m)$ we know that $u_2(R_W)$ must be smaller than $u_1(R_W)$. Otherwise, from simple geometrical reasons, R_W would occur after an intersection point. The continuity of u would then show that there is another point of u-equality at some time between $R_1(u)$ and R_w, which would contradict u_e uniqueness.

2. If that there is a crossing in the u-W axis before R_W, then by the first part of the lemma there is a u-equality before that. If no such crossing occurs then $R_2(u_1(R_W)) < R_W = R_1(u_1(R_W))$ but $R_2(u_1(0)) > 0 = R_1(u_1(0))$ which by continuity shows that u-equality occurred for $u > u_1(R_W)$. □

To get results about order of arrival to an intersection point for the $u - z$ system, similar to those of Lemma 4.2, the region $0 < u < u_0$ must be divided into two parts: $u_0 > u > 1$ and $0 < u < 1$. The line $u = 1$ will be dealt with later on.

Lemma 4.6. *Assume that two solutions, s_1 and s_2, cross each other in the $u - z$ system at (u_m, z_m) with $u_m > 0$. Assume that s_1 is below s_2 for u in some interval $(u_m, u_m + \delta)$. Then*
If $u_m < 1$ then $R_1(u_m) < R_2(u_m)$.
If $u_m > 1$ then $R_1(u_m) > R_2(u_m)$.

Proof. Before the crossing point, s_1 is below s_2, while after it they change roles. Therefore the following inequality must take place

$$\frac{z_1'}{u_1'}(R_1(u_m)) > \frac{z_2'}{u_2'}(R_2(u_m)). \tag{4.4}$$

Plugging u' and z' in (4.4) yields

$$\frac{z'}{u'}(R) = \frac{3(u - u^3)}{4z^{1/3}} R^{d-1}. \tag{4.5}$$

The factor $u - u^3$ is negative for $u > 1$ and positive for $u < 1$. Therefore for (4.4) to be true, $R_1(u_m)$ must be smaller than $R_2(u_m)$ if $u_m < 1$, and bigger than $R_2(u_m)$ is $u_m > 1$. □

Using the results of Lemma 4.6, it is possible to show that intersection for $u > 0$ is not possible in the $u - z$ system if there is no u-equality. That result is divided into three lemmas, the first of which handles the line $u = 1$.

Lemma 4.7. ***(no crossing in u-z system at** $u = 1$**)** *Let s_1 and s_2 be two intersecting solutions of the $u - z$ system. Assume that s_1 is below s_2 for u in some interval $(1, 1 + \delta)$. Then their first intersection point can't be on the line $u = 1$.*

Proof. Assume by contradiction that s_1 and s_2 intersect at $u = 1$. Examine the derivative of z by u

$$\frac{dz}{du} = -\frac{3R(u)^{d-1}[u^3 - u]}{4z^{1/3}}. \tag{4.6}$$

In order to get a derivative that does not depend on z, consider the system with $z^{4/3}$ instead of z:

$$\frac{dz^{4/3}}{du} = -4R(u)^{d-1}[u^3 - u]. \tag{4.7}$$

The function from z to $z^{4/3}$ is an injection in the region $z > 0$, and therefore a crossing in the $u - z$ plane is also a crossing in the $u - z^{4/3}$ plane. s_1 is under s_2 before $u = 1$ in the $u - z^{4/3}$ plane, and above it after that line. Therefore there exists some sequence of values $u_j^+ > 1$ and $u_j^- < 1$ such that the following inequality holds for all $u_j^\pm$

$$\frac{dz_1^{4/3}}{du_1}(u_j^\pm) \leq \frac{dz_2^{4/3}}{du_2}(u_j^\pm). \tag{4.8}$$

If that was not true than $z_1(u) - z_2(u)$ would be monotonically increasing in $(u - \delta, u + \delta)$ for some $\delta > 0$, therefore contradicting the intersection.

The right side of (4.7) depends only on u and $R(u)$. $u^3 - u$ changes sign at $u = 1$, and therefore $R_1(u) \geq R_2(u)$ on the sequence u_j^+ to the right of one, and the opposite holds to the left. This shows from continuity that $R_1(1) = R_2(1)$, which contradicts the uniqueness theorem for solutions of ODEs. □

Lemma 4.8. ***(no crossing in u-z system for** $u > 1$**)*** *Let s_1 and s_2 be two intersecting solutions in the $u - z$ system. If $d \leq 4$ then they don't intersect in the region $u > 1$.*

Proof. Define

$$f(u) = u^3 - u, \tag{4.9}$$

and

$$F(u) = u^4/4 - u^2/2 + \frac{1}{4}. \tag{4.10}$$

Note that $F(u)$ is, up to a constant, integral of $f(u)$ with regard to u.

Define $\mathscr{E}$ to be

$$\mathscr{E} = \frac{z^{4/3}}{R^{d-1}} + F(u). \tag{4.11}$$

Up to an additive constant, this is the same expression defined to be $\mathscr{E}$ in the previous section, only expressed in terms of z rather than W. Since both u and z are functions of R, we may also consider $\mathscr{E}$ to be a function of R; differentiating $\mathscr{E}$ with respect to R and using (4.3) yields

$$\frac{d\mathscr{E}}{dR} = (1 - d)z^{4/3}R^{-d}. \tag{4.12}$$

Note that this calculation, which is essentially the same as the calculation in (3.17), gives the full derivative even though calculating the partial derivative with respect to R would yield the same result. Let $K(u) = \frac{F(u)}{f(u)}$, and define the function

$$P = R^{3d/4}\mathscr{E} - dzK(u). \tag{4.13}$$

Calculate P's full derivative with respect to R:

$$\begin{array}{l}\frac{dP}{dR} = \frac{3d}{4}R^{\frac{3d}{4}-1}\mathscr{E} + R^{\frac{3d}{4}}\mathscr{E}' - dz'K(u) - dzu'\frac{\partial K(u)}{\partial u} = \\ (1-\frac{d}{4})R^{-\frac{d}{4}}z^{4/3} + dR^{-d/4}z^{4/3}\frac{\partial K(u)}{\partial u} = z^{\frac{4}{3}}R^{-\frac{d}{4}}L(u) = -zL(u)u',\end{array} \tag{4.14}$$

where

$$L(u) = 1 - \frac{d}{4} + d\frac{\partial K(u)}{\partial u}. \tag{4.15}$$

$L(u)$ is positive for every $u > 1$ and every d, since

$$\frac{\partial K(u)}{\partial u} = \frac{1}{4}(1 + \frac{1}{u^2}), \tag{4.16}$$

which equals $\frac{1}{2}$ at $u = 1$, monotonically decreases for $u > 1$ and has the limit at infinity: $\lim_{u\to\infty}\frac{\partial K(u)}{\partial u} = \frac{1}{4}$.

Integrating (4.14) in view of the facts that $z(0) = 0$ and hence also $P = 0$ at $R = 0$ yields

$$P = \int_0^R \frac{dP}{dR}dR = -\int_{u_0}^{u(R)} zL(u)du = \int_{u(R)}^{u_0} zL(u)du. \tag{4.17}$$

Let s_1, s_2, such that $u_1(0) < u_2(0)$, be two solutions that intersect at (u_*, z_*) with $u_* < 1$. Let $P_j = P(R, u_j, z_j)$. Using (4.17) yields

$$P(R_1, u_*, z_*) - P(R_2, u_*, z_*) = \int_{u(R)}^{u_0} (z_1 - z_2)L(u)du, \tag{4.18}$$

where z_1 is defined to equal zero for $u > u_1$. Since $L(u) > 0$ and $z_1(u) < z_2(u)$ for every u before intersection, the right side of (4.18) is negative.

Plugging the left side into (4.13) yield

$$\begin{array}{l}P(R_1, u_*, z_*) - P(R_2, u_*, z_*) \\ = R_1^{1-\frac{d}{4}}z^{4/3} + R_1^{3d/4}F(u) - R_2^{1-\frac{d}{4}}z^{4/3} - R_2^{3d/4}F(u) = \\ (R_1^{1-\frac{d}{4}} - R_2^{1-\frac{d}{4}})z^{4/3} + (R_1^{3d/4} - R_2^{3d/4})F(u),\end{array} \tag{4.19}$$

which is positive for every $d \leq 4$ since by Lemma 4.6 the fact that $u_* > 1$ implies that $R_1 > R_2$. This contradiction shows that no intersection is possible in the region $u > 1$. $\square$

The last lemma is more restricted, since we will assume in it that u-equality occurs somewhere in the region $u > 0$.

Lemma 4.9. ***(no crossing in u-z system for** $u > 0$**)*** *Let s_1 and s_2 be two solutions of the $u - z$ system with $d \leq 4$. If the solutions are in u-equality at some point $u_e > 0$, then they don't intersect in the region $u > 0$.*

Proof. Assume by contradiction that s_1, s_2 intersect for the first time at (u_m, z_m), and let s_1 be below s_2 at $u_1(0)$ ($u_1(0) < u_2(0)$). Since $d \leq 4$, lemmas 4.7 and 4.8 says that there is no intersection in the $u - z$ system for $u \geq 1$, which means that $u_m < 1$.

There can be at most one u-equality for $u > u_m$: this is true exactly from the same reasons as the part about u-equality uniqueness in the proof of lemma 4.5, where the differential equation for W is replaced by that of z.

Lemma 4.6 shows that s_1 reaches u_m before s_2. Since there's at most one u-equality for $u > u_m$, and since after u-equality s_1 reach every point u after s_2, we get that s_1 reaches u_m before it reaches u-equality, and therefore $u_e < u_m < 1$.

There cannot be a second intersection between u_m and the point u_e of u-equality closest to u_m, since that would mean that s_1 reaches the second intersection point first, even though it is above s_2 before that, which contradicts lemma 4.6. Therefore, s_1 reaches u_e when it is above s_2 when viewed in the u-z coordinates.

However, this is impossible since by lemma 4.5 there cannot be any $u - W$ intersection to the right of u_e, and therefore $W_2(u) > W_1(u)$ and $R_2(u) > R_1(u)$ for every $u > u_e$. Combining this with the definition $z = R^{\frac{3d}{4}} W$ shows that $z_2(u) > z_1(u)$ in that region, and therefore s_1 is below s_2 when $u > u_e$. This contradiction shows that there is no intersection point in the $u - z$ system. $\square$

Combining Lemmas 4.3, 4.5, and 4.9 yield two conclusions:

Corollary 4.10. *For $2 \le d \le 4$:*
1. *If two solutions have u-equality, then they don't intersect for $u > 0$ in the $u - z$ system.*
2. *There is no intersection of solutions for $u > 0$ in the $u - W$ system.*

Proof. The first conclusion is simply a restatement of Lemma 4.9. By Lemmas 4.3 and 4.5, if there exists an intersection in the $u - W$ system, then there must also exist both a point of u equality and a crossing in the $u - z$ system, which is impossible. □

Theorem 4.11. *Let s_1 be a solution that reaches the origin at R_1. Then, for $2 \le d \le 4$, any other solution that reaches the origin must do so at the same time R_1.*

Proof. Let s_2 be another solution that reaches the origin, and does that at $R_2 > R_1$. By corollary 4.10, if $u_2(0) > u_1(0)$ then s_2 is above s_1 in the $u-W$ system at $u_2(R_1)$ (u position of s_2 when s_1 reach the origin). On the other hand if $u_2(0) < u_1(0)$, then u continuity shows that there is u-equality for some $u_e > 0$, which means, by corollary 4.10, that s_2 is below s_1 not only in the $u - W$ system, but also in the $u - z$ system.

In the first case $u_2(0) > u_1(0)$ and s_2 is above s_1 at R_1 in the $u - W$ system. There is no u-equality for $0 < u < u_1(0)$: s_2 is above s_1 for every $0 < u < u_1(0)$ so the same logic as in the proof of 4.5 shows that there is at most one u-equality for $0 < u < u_1(0)$. The possibility of a single u-equality is canceled by the fact $u_2(0) > u_1(0)$ and $u_2(R_1) > u_1(R_1)$. Since there is no u-equality, then there is also no W-equality for $0 < u < u_1(0)$ by the second part of Lemma 4.5. Hence $R_2(W) > R_1(W)$ and $u_2(W) < u_1(W)$ for all $0 < W < W_1(0)$.

It is possible to write $\frac{du}{dW}$ as a product of two functions

$$\frac{du}{dW} = \frac{1}{3[u(W) - u(W)^3 + dW]} 4R(W)W^{1/3} = f(W, u(W))g(W, R(W)), \tag{4.20}$$

where

$$\begin{array}{lll} f(W,u) = \frac{1}{3[u(W)-u(W)^3+dW]} > 0 & \forall R > 0 \text{ and } (u,W) \in \Omega \\ g(W,R) = 4R(W)W^{1/3} > 0 & \forall R > 0 \text{ and } (u,W) \in \Omega. \end{array} \tag{4.21}$$

We want to work in a region in which $u(W) - u(W^3)$ is an increasing function, and that happens for $u < \sqrt{1/3}$. Hence, choose W_* such that $u_1(W_*) < \sqrt{1/3}$ (remember that $u_1(W) > u_2(W)$ for every $0 < W < W_1(0)$). Now $u_1(W) - u_1(W)^3 > u_2(W) - u_2(W)^3$ in $0 < W \leq W_*$, and that implies that $f_2(W, u(W)) > f_1(W, u(W))$. We also have that $g_2(W, R(W)) > g_1(W, R(W))$ since $R_2(W) > R_1(W)$ for every $0 < W \leq W_*$. Using that in $\frac{du}{dW}$ yields

$$\frac{du_2}{dW} > \frac{du_1}{dW}. \tag{4.22}$$

Integrate the difference of $\frac{du_2}{dW}$ and $\frac{du_1}{dW}$

$$\int_0^{W_*} (\frac{du_2}{dW} - \frac{du_1}{dW})\, du = [u_2(W) - u_1(W)] - [u_2(0) - u_1(0)]. \tag{4.23}$$

We already know that $[u_2(W) - u_1(W)] < 0$ for every $0 < W < W_*$, while $(\frac{dW_2}{du} - \frac{dW_1}{du}) > 0$ in that region due to (4.22), which shows that $[u_2(0) - u_1(0)]$ can't be zero and s_2 cannot reach the origin.

In the second case $u_2(0) < u_1(0)$ and s_2 is below s_1 at R_1 in both the $u - W$ and the $u - z$ systems. Remember that the solutions have u-equality at some u_e. Hence $R_1(u) < R_2(u)$ for $0 < u < u_e$. Since s_2 is below s_1 for every $0 < u \leq u_2(R_1)$, then $z_1(u) > z_2(u)$ for $u < u_2(R_1)$. Choose $u_* = \min\{u_e, u_2(R_1), 1\}$ since we want the above inequalities for $R_j(u)$ and $z_j(u)$ to hold, and also for $u - u^3$ to be non-negative.

Examine $\frac{dz}{du}$:

$$\frac{dz}{du} = \frac{3R(u)^{d-1}[u - u^3]}{z(u)^{1/3}}. \tag{4.24}$$

The relations for R, Z that we found show that $\frac{dz_1}{du} < \frac{dz_2}{du}$.

Integrate the difference between $\frac{dz_2}{du}$ and $\frac{dz_1}{du}$ to get

$$\int_0^{u_*} (\frac{dz_2}{du} - \frac{dz_1}{du}) = [z_2(u) - z_1(u)] - [z_2(0) - z_1(0)]. \tag{4.25}$$

This time $[z_2(u) - z_1(u)]$ is negative for every $0 < u < u_*$, while $(\frac{dz_2}{du} - \frac{dz_1}{du})$ is non negative. This shows that $[z_2(0) - z_1(0)]$ does not equal zero and s_2 doesn't reach the origin.

Finally, let s_3 be a solution that reaches the origin at $R_3 < R_1$. Then as was shown in this proof s_1 cannot reach the origin at R_1, which is a contradiction to the theorem assertion. □

5 Uniqueness Result Part II: Uniqueness of solutions reaching the origin at a given time

In this section we prove that only one solution can reach the origin at a given time. Combining this result with Theorem 4.11 of the previous section proves Theorem 4.2 which is, as we already noted, equivalent to Theorem 4.1. It is very important to emphasize that the result presented in this section is a local one, i.e. no assumptions are made about the solution except at the origin. The proof itself is also local, and based on analysis of a neighborhood of the origin. This is a big difference from a global existence result, that takes into consideration the initial values of solution, or of proof that analyses solutions' values all over the plane. Such a result and proof can be found in Pucii and Serrin solution that is presented in the next section.

Let R_* be the time in which a solution reaches the origin. Define $t = R - R_*$ to transfer system (3.12) to

$$\begin{array}{l} u' = -W^{\frac{1}{3}} \\ W' = \frac{3}{4(t+R_*)}[u^3 - u - dW] \end{array} . \tag{5.1}$$

This shifts the time at which the solution reaches the origin to $t = 0$; note that we are then interested in times $t \leq 0$.

In order to find estimations on the sizes of u, W, we switch to the $u - z$ system (4.3) from the previous section. Using the t variable z becomes $z = (t + R_*)^{\frac{3d}{4}} W$, and the system in this variable is

$$\begin{array}{l} u' = -(t + R_*)^{-\frac{d}{4}} z^{\frac{1}{3}} \\ z' = \frac{3}{4}(t + R_*)^{\frac{3d}{4}-1}[u^3 - u] \end{array} . \tag{5.2}$$

Using (5.2) and z's monotonity, which holds for t close enough to zero so that $u < 1$, it is possible to estimate u in a bounded area of the origin by

$$|u'| \leq A|z^{\frac{1}{3}}| \rightarrow |u(t)| \leq A|\int_0^t z^{\frac{1}{3}}(s)|ds \leq At|z(t)|^{\frac{1}{3}}. \tag{5.3}$$

This result can be used in z's equation to obtain

$$\begin{array}{l} |z'| = |\frac{3}{4}(t+R_*)^{\frac{3d}{4}-1}[u-u^3]| \leq B|u-u^3| \leq B|u| \leq ABt|z|^{\frac{1}{3}} \Rightarrow \\ \Rightarrow |(z^{\frac{2}{3}})'| \leq \frac{2}{3}|z'||z^{-\frac{1}{3}}| \leq ABt \Rightarrow |z|^{\frac{2}{3}} \leq Ct^2 \Rightarrow |z| \leq Ct^3 \end{array}. \tag{5.4}$$

Equations (5.3) and (5.4) imply that $|z| \leq Ct^3$ and $|u| \leq Ct^2$. A similar bound can be achieved for W, Since $z = (t+R_*)^{\frac{3d}{4}}W$

$$|z| = |(t+R_*)^{\frac{3d}{4}}W| \leq Ct^3 \Rightarrow |W| \leq (t+R_*)^{-\frac{3d}{4}}Ct^3 \leq Dt^3, \tag{5.5}$$

in every bounded region of the origin.

These bounds for u and W leads us to define

$$\begin{array}{l} u = t^2P(t) \\ W = -t^3Q(t). \end{array} \tag{5.6}$$

Remember that $t \leq 0, u, W \geq 0$ and hence $0 < P, Q \leq C$ for $-R_* < t < 0$ (not necessarily the same C as before). Plug this into (5.1)

$$\begin{array}{l} 2tP + t^2P' = tQ^{\frac{1}{3}} \qquad \Rightarrow P' = \frac{1}{t}[Q^{\frac{1}{3}} - 2P] \\ 3t^2Q + t^3Q' = \frac{3}{4(t+R_*)}[t^2P - t^6P^3 - dt^3Q] \\ \Rightarrow Q' = \frac{1}{t}[\frac{3P}{4(t+R_*)} - 3Q] - \frac{3}{4(t+R_*)}[dQ + t^3P^3]. \end{array} \tag{5.7}$$

It is more convenient to write the Q' equation differently:

$$\begin{array}{l} Q' = \frac{1}{t}[\frac{3P}{4R_*} - 3Q] - \frac{3}{4(t+R_*)}[dQ + t^3P^3] + \frac{3P}{t}[\frac{1}{4(t+R_*)} - \frac{1}{4R_*}] = \\ \frac{3}{t}[MP - Q] - \frac{3}{4(t+R_*)}[dQ - 4MP + t^3P^3], \end{array} \tag{5.8}$$

where $M = \frac{1}{4R_*}$. This system can be written in vector form upon defining

$$\begin{array}{r} A = \begin{pmatrix} Q^{\frac{1}{3}} - 2P \\ -3Q + 3MP \end{pmatrix}, \\ B = B(t,P,Q) = \begin{pmatrix} 0 \\ -\frac{3}{4(t+R_*)}[dQ - 4MP + t^3P^3] \end{pmatrix}. \end{array} \tag{5.9}$$

Note that $B = O(P+Q)$.

Using those vectors system (5.7) becomes

$$\begin{pmatrix} P \\ Q \end{pmatrix}' = \frac{1}{t}A + B, \tag{5.10}$$

where A and B are both bounded in some neighborhood of the origin. It is convenient to make the change of variables

$$\theta = -\ln(-t) \Rightarrow t = -e^{-\theta}. \tag{5.11}$$

The following transformation is taking place

$$-t \begin{pmatrix} P \\ Q \end{pmatrix}' = -A - tB \Rightarrow \begin{pmatrix} P \\ Q \end{pmatrix}' = -A + \tilde{B}, \tag{5.12}$$

where the derivative is now taken with respect to θ, and

$$\tilde{B} = e^{-\theta} B(-e^{-\theta}, P, Q). \tag{5.13}$$

Note that $\theta \to +\infty$ when t goes to 0^- and that the transformed system tends to an autonomous system as $\theta \to +\infty$. **Note that** $\tilde{B} = O(e^{-\theta}(P+Q))$, so $\lim_{\theta\to\infty} \tilde{B} = 0$.

From this point and in the rest of the section we will be working on equation (5.12).

Remember that $0 < P, Q \leq C$ for all θ in some interval (θ_0, ∞). A has two rest points in the first quadrant: the origin, and $Q_* = (\frac{M}{2})^{\frac{3}{2}}, P_* = \frac{\sqrt{M}}{2\sqrt{2}}$.

Lemma 5.1. *If* $0 < P, Q < C$ *holds for all* θ *in an interval* (θ_0, ∞)*, then* (P, Q) *tends either to the origin or to* (P_*, Q_*) *as* $\theta \to \infty$*.*

Proof. For convenience, denote by D the square $0 < P < C$, $0 < Q < C$. Divide D into four regions:
1. $R_1 = D \cap \{P > \frac{1}{2}Q^{\frac{1}{3}}, Q < MP\}$, in which $P' > 0$.
2. $R_2 = D \cap \{P < \frac{1}{2}Q^{\frac{1}{3}}, Q > MP\}$, in which $P' < 0$.
3. $R_3 = D \cap \{P > \frac{1}{2}Q^{\frac{1}{3}}, Q > MP\}$, in which $P' > 0$.
4. $R_4 = D \cap \{P < \frac{1}{2}Q^{\frac{1}{3}}, Q < MP\}$, in which $P' < 0$.

Finally, define

$$\widetilde{R}_i := R_i \setminus S \text{ for } 1 \leq i \leq 4,$$

where S is a small neighborhood of (P_*, Q_*) to be defined later.

Only one component, namely, the lower component (vertical component) of the vector $\tilde{B}(-e^{-\theta}, P, Q)$, is nonzero. Hence the sign of P' mentioned above is not influenced by $\tilde{B}$ and is fixed for each of the regions.

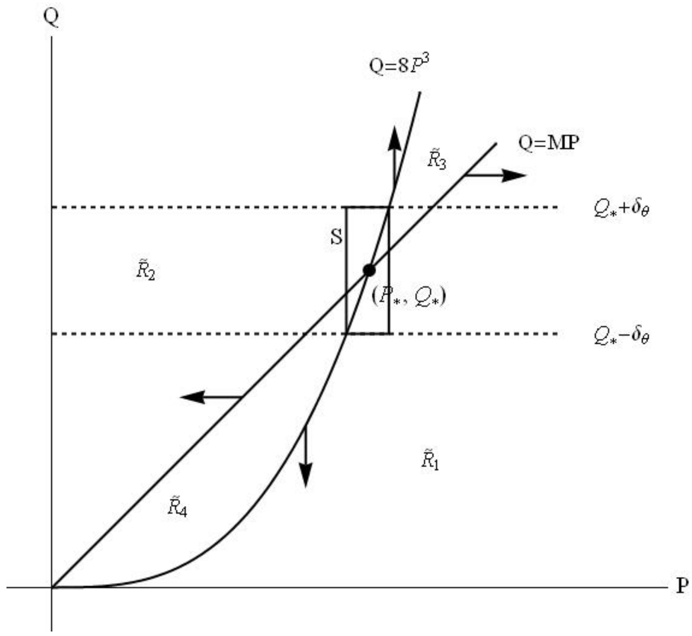

Figure 5.1: Phase plane of First Quadrant of P-Q

We would like to choose a region S that isolates (P_*, Q_*), i.e., such that solutions that start outside S won't enter it, and such that in addition the sign of Q' is guaranteed to be the same as that of $-A_Q$, where A_Q is the lower component of A, on the portion of the curve $Q = MP^3$ lying outside of S. For that we need a function, $\delta = \delta(\theta)$, large enough such that $Q' < 0$ in $R_1 \cap \{Q \leq Q_* - \delta(\theta)\}$, $Q' > 0$ in $R_2 \cap \{Q \geq Q_* + \delta(\theta)\}$, and yet satisfies $\delta \to 0$ as $\theta \to \infty$.

In the region $R_1 \cap \{Q \leq Q_* - \delta\}$ it is sufficient to check that Q' is negative on a set $R_1 \cap \{Q_* - \delta_1 \leq Q \leq Q_* - \delta(\theta)\}$ for some fixed $0 < \delta_1 < Q_*$. This is sufficient since in $R_1 \cap \{\delta_2 \leq Q \leq Q_* - \delta_1\}$ the function $-A_Q$ is bounded away from zero, which implies that Q' is negative for θ sufficiently large, while we show later on that for δ_2 sufficiently small, Q' is also negative on $R_1 \cap \{Q \leq \delta_2\}$.

Before doing so, denote by $M_{\tilde{B}} = M_{\tilde{B}}(\theta)$ the maximum of $\tilde{B}(\theta, P, Q)$ in $P, Q \in D$, which exists since D is bounded and $\tilde{B}$ continuous. Remember that $\lim_{\theta\to\infty} \tilde{B}(\theta, P,) \to 0$, so $M_{\tilde{B}}(\theta)$ also goes to zero as $\theta \to \infty$.

On the set $R_1 \cap \{Q_* - \delta_1 \leq Q \leq Q_* - \delta(\theta)\}$, the maximum of $-A_Q$

occurs on the line $Q = Q_* - \delta(\theta)$. Hence it is sufficient to check that $Q' = -A_Q + M_{\tilde{B}} < 0$ on that line. $-A_Q$ is a decreasing function of P that is negative in R_1, so its maximum on the line is achieved at the intersection point of $Q = 8P^3$ and $Q = Q_* - \delta$. Plug $Q_* - \delta$ into the equation of $Q = 8P^3$ to achieve $8P^3 = Q_* - \delta \Rightarrow P = \frac{1}{2}(Q_* - \delta)^{1/3}$, which means that the intersection point is at $N := (\frac{1}{2}(Q_* - \delta)^{1/3}, Q_* - \delta)$. Plug N into $-A_Q$ to get: $-A_Q(N) = -c(\delta)$ where $c(\delta) = \frac{3M}{2}(Q_* - \delta)^{1/3} - 3(Q_* - \delta)$. Differentiate $c(\delta)$ to obtain $c'(\delta) = 3 - \frac{M}{2(Q_* - \delta)^{2/3}}$. $c(0) = 0$ and $c'(0) > 0$, which imply that c is positive and invertible in some interval $(0, \delta_0)$.

Now we can finally define δ, and we'll do that by choosing 'good' values of c.

Choose $\delta(\theta) := \delta(c(\theta) = 2M_B(\theta))$. By construction this function satisfies our requirement of Q' on $R_1 \cap \{Q \leq Q_* - \delta(\theta)\}$. To see that $\lim_{\theta \to \infty} \delta \to 0$ note that $\lim_{\theta \to \infty} 2M_B(\theta) \to 0$, and $\delta(c = 0) = 0$.

Doing the exact same process for R_2 and $Q_* + \delta(\theta)$, and choosing the maximum of both $\delta(\theta)$'s for every θ gives the desired function.

Using $\delta(\theta)$, define a region, S, around (P_*, Q_*) to be a rectangle with it's bottom left corner and upper right corner on the line $Q = 8P^3$, and the upper left corner and bottom right corner to lay on the lines $Q_* + \delta(\theta)$ and $Q_* - \delta(\theta)$ correspondingly (see figure 5.1). We claim that solutions can't enter from $\widetilde{R}_i$, $1 \leq i \leq 4$ to S.

On both vertical boundaries P' points away from S, and since $\tilde{B}$ doesn't have a horizontal component solutions can't enter S across those boundaries. Only the vertical boundaries of S intersect regions $\widetilde{R}_3, \widetilde{R}_4$, so solutions can't enter S from those regions. However, regions $\widetilde{R}_1, \widetilde{R}_2$ intersect also the horizontal boundaries of S, but by the construction of S the vector field points out of it along those horizontal boundaries, and therefore the whole vector field points out of S in those regions.

We now determine the regions $\widetilde{R}_j$ and $\widetilde{R}_k$ for which the solution cannot move from $\widetilde{R}_j$ to $\widetilde{R}_k$. First, for sufficiently large θ solutions cannot move from $\widetilde{R}_1$ to $\widetilde{R}_4$. To see this, note that the vertical component of $-A$ is negative on the boundary between those two regions and tends to zero either as P does, or as we P, Q move towards (P_*, Q_*) as $\theta \to \infty$. The latter case is canceled by the above construction of S, that guarantees that we cannot move towards (P_*, Q_*) quickly enough from within $\widetilde{R}_1$ to have Q' change sign. Remembering that $P' > 0$ in $\widetilde{R}_1$, it is suffices to show that solutions cannot cross the curve near the origin.

This crossing doesn't happen due to the fact that the on the curve $Q = 8P^3$, the vertical component of A is less than $-c_1 P$ for some c_1, and for θ big enough its sign is not influenced by $\tilde{B}$, since $|\tilde{B}| \leq ce^{-\theta}(|P| + |Q|)$, which is less than $c_1 P/2$ for θ sufficiently large.

A similar analysis, without the part about a neighborhood of the origin, shows that solutions cannot move from $\widetilde{R}_2$ to $\widetilde{R}_3$.

The vector field points out of the first quadrant along the boundary of the first quadrant(except the origin). Hence solutions that reach the boundary leave the first quadrant and are not in D for all time. Solutions in region R_2 and R_4 can reach the origin at finite time or tend to the origin as $\theta \to \infty$. Otherwise they either leave to R_1, reach the boundary, or leave D from within R_2.

The fact that solutions cannot move from $\widetilde{R}_1$ to $\widetilde{R}_4$ nor from $\widetilde{R}_2$ to $\widetilde{R}_3$ implies that any solution that remains in D but not in S (which would imply convergence to (P_*, Q_*)), eventually stays either in the union of $\widetilde{R}_1$ and $\widetilde{R}_3$ (plus their boundary) or else within the union of $\widetilde{R}_2$ and $\widetilde{R}_4$, In the former region $P' > 0$ so the solutions cannot tend to the origin and must leave D in a finite time, while in the latter region $P' \leq 0$ and is bounded away from zero except in any neighborhood of the origin, so the solutions must either tend to the origin or else leave D in finite time.

To conclude it all, we showed that a solution can only stay in the square $0 < P < C$, $0 < Q < C$ for all time if it tends to the origin at $\theta \to \infty$ or stays forever in region S. But this region shrinks to the fixed point (P_*, Q_*) as $\theta \to \infty$, which concludes the proof.

□

Next, we will show that any solution in the region $P > 0, Q > 0$, that tends to the origin must in fact reach the origin at a finite value of θ. This shows that it is not a solution that reaches the origin at t=0.

Begin with the simpler equation:

$$\begin{pmatrix} P \\ Q \end{pmatrix}' = -A. \tag{5.14}$$

The next few lemmas discuss properties of that equation for a solution that tends to the origin as $\theta \to \infty$.

Lemma 5.2. *A solution of (5.14) that lies in the first quadrant and tends to the origin as $\theta \to \infty$, must satisfy: $Q' < 0, P' < 0$, and stay in the region: $8P^3 < Q < MP$.*

Proof. We use the definitions of R_i, $1 \le i \le 4$ from the previous proof, and note that in the simpler system the sign of Q' is fixed as well in each region. Namely, $Q' > 0$ in R_2, R_3 and $Q' < 0$ in R_1, R_4. Since we discuss only solutions that tend to the origin, we may assume that the solution lies in the union of R_1, R_2, R_4 plus the boundaries separating them.

In region R_1, $P' > 0$ and therefore solutions can't reach the origin. In region R_2, $Q' > 0$ and hence solutions don't reach the origin as well. Therefore, the only possibility for a solution to tend to the origin as $\theta \to \infty$ is that it lies in region R_4, which means that: $8P^3 < Q < MP$. □

Define: $S_1 = \frac{Q}{P}$.

Lemma 5.3. *A solution of (5.14) that lies in the first quadrant and tends to the origin must satisfy $Q \le (3M)^{\frac{3}{4}} P^{\frac{3}{2}}$.*

Proof. Calculate S_1':

$$S_1' = \frac{Q'}{P} - \frac{P'Q}{P^2} = 3S_1 - 3M - 2S_1 + \frac{Q^{\frac{4}{3}}}{P^2} = S_1 + S_1^2 Q^{-\frac{2}{3}} - 3M. \tag{5.15}$$

If

$$S_1^2 Q^{-\frac{2}{3}} > 3M, \tag{5.16}$$

then $S_1' > S_1 > 0$. From $S_1' > 0$ and $Q' < 0$, it is possible to see that if (5.16) takes place at some θ_1, it will take place for all $\theta > \theta_1$.

If $S_1' > S_1$ at some θ_1, then it will hold for all $\theta > \theta_1$, and $S_1 \to \infty$ as $\theta \to \infty$. But, by Lemma 5.2, for every solution that goes to the origin, $S_1 < M$ This means that (5.16) never takes place. Replacing S_1 for $\frac{Q}{P}$ in (5.16), and isolating Q, gives the desired result. □

Define: $S_2 = \frac{Q}{P^{\frac{3}{2}}}$.

Lemma 5.4. *For any solution of (5.14) that lies in the first quadrant and tends to the origin, $\exists K > 0$ such that $S_2 > K$.*

Proof. It suffices to obtain the conclusion when P is smaller than some arbitrary positive value ε, since for P larger than that value the result follows from Lemma 5.2. Now

$$\begin{array}{l} S_2' = \frac{Q'}{P^{\frac{3}{2}}} - \frac{3}{2}\frac{1}{P^{\frac{5}{2}}}P'Q = \frac{3Q}{P^{\frac{3}{2}}} - \frac{3M}{P^{\frac{1}{2}}} - \frac{3Q}{2P^{\frac{3}{2}}} + \frac{3}{2}\frac{1}{P^{\frac{5}{2}}}Q^{\frac{4}{3}} \\ = \frac{3Q}{2P^{\frac{3}{2}}} + \frac{1}{P^{\frac{1}{2}}}[\frac{3}{2}(\frac{Q}{P^{\frac{3}{2}}})^{\frac{4}{3}} - 3M] < \frac{3}{2}(3M)^{3/4} + \frac{1}{P^{\frac{1}{2}}}[\frac{3}{2}S_2^{\frac{4}{3}} - 3M] \end{array}, \tag{5.17}$$

where we have used Lemma 5.3.

Assume, by contradiction, that the assertion of the theorem is false. Then in particular $S_2 < (2M)^{\frac{3}{4}}$ at some time θ_1. This means that the expression in brackets in the last line in (5.17) is negative. Since we may assume that P is as small as we want at that time, we can obtain that $S_2' < \delta < 0$ at θ_1. Since P is decreasing, as long as S_2' is negative the bound on the right of (5.17) is decreasing, which shows that $S_2' < \delta < 0$ for all $\theta \geq \theta_1$. This implies that S_2 becomes negative after a finite time, which contradicts the non-negativity of P and Q. □

Define: $S_3 = P + Q^{\frac{2}{3}}$. The bounds from Lemmas 5.3 and 5.4 imply that $c_1 Q^{\frac{2}{3}} < P < c_2 Q^{\frac{2}{3}}$. Which means that the terms in S_3 are of comparable size. We'll use S_3 to show that the solution reaches the origin at a finite value of θ.

Lemma 5.5. *A solution for system (5.14) that lies in the first quadrant and reaches the origin, does it in a finite value of θ.*

Proof.

$$S_3' = P' + \frac{2}{3}Q^{-\frac{1}{3}}Q' = 2P - Q^{\frac{1}{3}} + 2Q^{\frac{2}{3}} - 2MPQ^{-\frac{1}{3}} < 2P + 2Q^{\frac{2}{3}} - Q^{\frac{1}{3}} - 2MC_1Q^{\frac{1}{3}}. \tag{5.18}$$

The last inequality follows the fact that $c_1 Q^{\frac{2}{3}} < P$. $Q^{\frac{1}{3}}$ is of a bigger magnitude than $P, Q^{\frac{2}{3}}$, which means that eventually $S_3' < -dQ^{\frac{1}{3}}$.

Examine the expression $(S_3^{\frac{1}{2}})'$

$$(S_3^{\frac{1}{2}})' = \frac{1}{2}S_3^{-\frac{1}{2}}S_3' < \frac{-dQ^{\frac{1}{3}}}{2(P+Q^{\frac{2}{3}})^{\frac{1}{2}}} \leq \frac{-dQ^{\frac{1}{3}}}{2[(c_1+1)Q^{\frac{2}{3}}]^{\frac{1}{2}}} = -\frac{d}{2(c_1+1)^{\frac{1}{2}}}. \quad (5.19)$$

The last inequality follows again, from the fact the $c_1 Q^{\frac{2}{3}} < P$. Therefore: $(S_3^{\frac{1}{2}})' < -\delta$ for some $\delta > 0$. This shows that S_3, and hence both P and Q, reach zero at a finite value of theta. □

Theorem 5.6. *A solution of (5.12) that that lies in the first quadrant and tends to the origin, reaches the origin at a finite time.*

Proof. Since the only change from equations (5.12) to (5.14) is in Q', and the change is bounded by for $\epsilon(P+Q)$ some arbitrary small ϵ, the equation for Q' can be written as:

$$Q' = c_1(\theta)Q - c_2(\theta)P,$$

where $3-\epsilon < c_1(\theta) < 3+\epsilon$, $3M-\epsilon < c_2(\theta) < 3M+\epsilon$. The case with just the original A corresponds to taking $c_1(\theta) = 3$ and $c_2(\theta) = 3M$, while the full case including $\tilde{B}$ is included by an appropriate choice of the c_j.

It is possible to re-evaluate the estimation of Lemmas 5.2 to 5.5, using c_j notation this time. In Theorem 5.2 the bounds change to: $8P^3 < Q < \max_{\theta>\theta_0}\{\frac{c_2(\theta)}{c_1(\theta)}\}P$. We used the maximum over time of $\frac{c_2(\theta)}{c_1(\theta)}$, because bounding only with $\frac{c_2(\theta)}{c_1(\theta)}$ possesses a problem since c_1 and c_2 depend on θ. Then if $\frac{c_2(\theta)}{c_1(\theta)}$ increases then a solution may be in the region where $Q > \frac{c_2(\theta)}{c_1(\theta)}P$ at some time and then in the region where $Q < \frac{c_2(\theta)}{c_1(\theta)}P$ at a later time, even though the motion of the solution tends to take it in the other direction, provided that the boundary moves faster than the solution. Since a solution can of course also move from the latter region to the former, it may move back and forth, and so would not have to satisfy the above bound even eventually.

In Lemma 5.3, the assertion will now be: $Q \leq (\max_{\theta>\theta_0} c_2(\theta))^{\frac{3}{4}} P^{\frac{3}{2}}$, with similar adjustments in the other lemmas. The proof of that lemma, and also the proofs of Lemmas 5.4 and 5.5 remain valid, since they merely use the fact that a bound exists, not its exact value.

Choosing c_1, c_2 appropriately, shows that a solution of (5.12) that tends to the origin, reaches the origin at a finite time. □

Corollary 5.7. *If a solution (P, Q) of (5.12) remains in the region $0 < P, Q < C$ for $\theta_0 \leq \theta < \infty$ then (P, Q) tends to (P_*, Q_*) as $\theta \to \infty$.*

Theorem 5.8. *There's a unique solution of (3.12) that reaches the origin at R_**

Proof. Equation (3.12) corresponds to (5.1), which corresponds to (5.7) which in turn corresponds to (5.12). Therefore it is sufficient to prove uniqueness to (5.12).

Define: $dP = P - P_*$, $dQ = Q - Q_*$. Let J denote the Jacobian of the mapping $(P, Q) \to A$ at the point (P_*, Q_*). J is $\begin{pmatrix} -2 & \frac{2}{3M} \\ 3M & -3 \end{pmatrix}$. J's eigenvalues, and their corresponding eigenvectors are: $\lambda_1 = -1, L_1 = \begin{pmatrix} 3M \\ 1 \end{pmatrix}$ and $\lambda_2 = -4, L_2 = \begin{pmatrix} -\frac{3M}{2} \\ 1 \end{pmatrix}$.

We can write the system (5.12) in the terms of dP, dQ as:

$$\begin{pmatrix} dP \\ dQ \end{pmatrix}' + J \begin{pmatrix} dP \\ dQ \end{pmatrix} = F(\theta, dP, dQ), \tag{5.20}$$

where $F(\theta, dP, dQ) = \tilde{B}(\theta, P_* + dP, Q_* + dQ) - A(P_* + dP, Q_* + dQ) + J \begin{pmatrix} dP \\ dQ \end{pmatrix}$.

Define the iteration $\begin{pmatrix} oldP \\ oldQ \end{pmatrix} \to \begin{pmatrix} dP \\ dQ \end{pmatrix}$ by

$$\begin{pmatrix} dP \\ dQ \end{pmatrix}' + J \begin{pmatrix} dP \\ dQ \end{pmatrix} = F(\theta, oldP, oldQ), \tag{5.21}$$

together with the condition that $dP, dQ \to 0$ as $\theta \to \infty$.

Multiply both side by J's eigenvector, L_j^T. Multiply the result by $e^{\lambda_J \theta}$, where λ_j is the corresponding eigenvalue, to get:

$$\left(L_j^T e^{\lambda_J \theta} \begin{pmatrix} dP \\ dQ \end{pmatrix} \right)' = L_j^T e^{\lambda_J \theta} F(\theta, oldP, oldQ). \tag{5.22}$$

We would like to integrate (5.22) from a variable point θ to infinity. The integral under consideration will converge whenever:

$$\int_\theta^\infty L_j^T e^{\lambda_J \Theta} F(\Theta, oldP, oldQ) d\Theta < \infty. \tag{5.23}$$

Examine F as $\theta \to \infty$. Since J is A's Jacobian at (P_*, Q_*): $-A(P_* + oldP, Q_* + oldQ) - J\begin{pmatrix} oldP \\ oldQ \end{pmatrix} = O\begin{pmatrix} oldP^2 \\ oldQ^2 \end{pmatrix}$, where the $O\begin{pmatrix} oldP^2 \\ oldQ^2 \end{pmatrix}$ term is $\begin{pmatrix} 0 \\ -\frac{2}{9}Q_*^{-\frac{5}{3}}Q^2 \end{pmatrix}$. Taking that with $|\tilde{B}(\theta, P_*+dP, Q_*+dQ)| = Ce^{-\theta} + O(e^{-\theta}(oldP + oldQ))$ at (P_*, Q_*), concludes in $F = C_1e^{-\theta} + O(e^{-\theta}(oldP + oldQ) + (oldP^2 + oldQ^2))$. $\lambda_j < 0$ for both $j = 1, 2$, which means that $L_j^T e^{\lambda_J \theta}$ converges, and the whole expression inside the integral of (5.23), goes to zero at least as fast as $O(e^{(\lambda_j - 1)\theta} + e^{\lambda_j\theta}(oldP^2 + oldQ^2))$. Therefore, (5.23) holds.

Integrate (5.22), Divide the result by $e^{\lambda_j \theta}$ to achieve:

$$\left(L_j^T \begin{matrix} dP \\ dQ \end{matrix}\right) = -e^{-\lambda_J \theta} \int_\theta^\infty L_j^T e^{\lambda_J \Theta} F(\Theta, oldP, oldQ) d\Theta. \tag{5.24}$$

Define the norm: $||f||_\theta = \max_{s \geq \theta} |f(s)|$. From now on, when using the norm notation, we will mean $|| \cdot ||_\theta$.

In order to show that the process for (dP,dQ) is a contraction mapping, we will use the next lemma:

Lemma 5.9. *Let $G(\theta, P, Q)$ be a function that goes to zero as $\theta \to \infty$. Then, for $\lambda > 0$:*

$$O(\int_\theta^\infty e^{-\lambda\Theta} G(\Theta, P, Q) d\Theta) = O(e^{-\lambda\Theta}) O(||G(\theta, P, Q)||_\theta). \tag{5.25}$$

Proof. G goes to zero as $\theta \to \infty$, so it is bounded, and its bound is its norm. Use that in the integral to get

$$\begin{aligned} &O(\textstyle\int_\theta^\infty e^{-\lambda\Theta} G(\Theta, P, Q) d\Theta) \leq O(||G(\theta, P, Q)||_\theta \int_\theta^\infty e^{-\lambda\Theta} d\Theta) = \\ &O(||G(\theta, P, Q)||_\theta \tfrac{1}{\lambda} e^{-\lambda\theta}) = O(e^{-\lambda\theta}) O(||G(\theta, P, Q)||_\theta) \end{aligned}$$

□

Let N be a matrix whose rows are $L_j, j = 1, 2$: $N = \begin{pmatrix} 3M & 1 \\ -\frac{3M}{2} & 1 \end{pmatrix}$, and $N^{-1} = \begin{pmatrix} \frac{2}{9M} & -\frac{2}{9M} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$. We can represent (dP, dQ) iterative equation by:

$$\begin{pmatrix} dP \\ dQ \end{pmatrix} = -N^{-1} \begin{pmatrix} e^{-\lambda_1\theta} \int_x^\infty L_1^T e^{\lambda_1\Theta} F(\Theta, oldP, oldQ) d\Theta \\ e^{-\lambda_2\theta} \int_x^\infty L_2^T e^{\lambda_2\Theta} F(\Theta, oldP, oldQ) d\Theta \end{pmatrix}. \tag{5.26}$$

Define: $\tilde{T}_i(\theta, oldP, oldQ) = e^{-\lambda_i\theta}\int_x^\infty L_i^T e^{\lambda_i\Theta}F(\Theta, oldP, oldQ)d\Theta$. Three remarks follows that definition:
1. $\tilde{T}_i = C_1e^{-\theta} + O(e^{-\theta}(oldP + oldQ) + O(oldP^2 + oldQ^2)$ by Lemma 5.9.
2. Using that definition, the iteration is represented as:

$$\begin{pmatrix} dP \\ dQ \end{pmatrix} = -\begin{pmatrix} N_{11}\tilde{T}_1 + N_{12}\tilde{T}_1 \\ N_{21}\tilde{T}_2 + N_{22}\tilde{T}_2 \end{pmatrix}. \tag{5.27}$$

3. It is also possible to estimate the iteration norm:

$$\left|\left| \begin{matrix} dP \\ dQ \end{matrix} \right|\right| = \left|\left| \begin{matrix} N_{11}\tilde{T}_1 + N_{12}\tilde{T}_1 \\ N_{21}\tilde{T}_2 + N_{22}\tilde{T}_2 \end{matrix} \right|\right| \leq 4D||\tilde{T}_{i_{max}}||, \tag{5.28}$$

where $D = Max\{N_{ij}\}$, $1 \leq i, j \leq 2$, and $\tilde{T}_{i_{max}} = max\{\tilde{T}_1, \tilde{T}_2\}$.

Define $S[\theta, \delta]$ to be the set of continuous functions defined on $[\theta, \infty)$, that converge to zero as $\theta \to \infty$, and takes their values in $[0, \delta]$. Every solution (dP, dQ) that converges to zero at infinity will be in $S[\theta, \delta]$ for some θ and δ. Also, if a solution is in one $S[\theta_1, \delta_1]$ then it is obviously also in $S[\theta_2, \delta_2]$ for all $\theta_2 \geq \theta_1$ and $\delta_2 \geq \delta_1$. Since we can choose θ_0 in (5.12) (and hence also in (5.20)) to be as large as we want, then we can consider only solutions that are in some specific $S[\theta, \delta]$ that we are free to choose. All of those show that uniqueness of the solution in every $S[\theta, \delta]$ with θ sufficiently large and δ sufficiently small implies uniqueness of all solutions (dP, dQ) of (5.20) that converges to zero at infinity.

In order to prove that the process is a contraction mapping for every θ sufficiently large and δ sufficiently small, and therefore has a unique solution, we need to show that it is a mapping from some set $S[\theta_*, \delta_*]$ into itself, and that it is contracting.

Mapping from S into itself. Using estimate (5.28)

$$\left|\left| \begin{matrix} dP \\ dQ \end{matrix} \right|\right| \leq 4D||\tilde{T}_{i_{max}}|| \leq 4D||C_1(e^{-\theta} + C_2(oldP^2 + oldQ^2))||. \tag{5.29}$$

Since $(oldP, oldQ) \to 0$, there exists θ_{1a} and $\delta_1 > 0$ such that $||DC_2(oldP^2 + oldQ^2)|| < \frac{1}{8}||oldP(\theta_{1a})||, ||oldQ(\theta_{1a})||$ for every $(oldP, oldQ) \in S[\theta_1, \delta_1)]$. Choose $\theta_1 \geq \theta_{1a}$ such that $C_1e^{-\theta} < \frac{1}{8}||oldP(\theta_{1a})||, ||oldQ(\theta_{1a})||$ for every

$\theta > \theta_1$. Adding those together shows that the process maps $S[\theta_1, \delta_1)]$ into itself.

Contraction mapping. Before we'll get an estimation on the size of a difference of two $\tilde{T}_i$, it's worth to get one on the size of a difference of two F functions:

$$F(\theta, oldP_1, oldQ_1) - F(\theta, oldP_2, oldQ_2) =$$
$$e^{-\theta}[B(\theta, oldP_1, OldQ_1) - B(\theta, oldP_2, OldQ_2)] - A(P_1 - P_2, Q_1 - Q_2) - J\begin{pmatrix} oldP_1 - oldP_2 \\ oldQ_1 - OldQ_2 \end{pmatrix} =$$
$$O(e^{-\theta}((oldP_1 - oldP_2) + (oldQ_1 - oldQ_2)) + (oldP_1 - oldP_2)^2 + (oldQ_1 - oldQ_2)^2). \tag{5.30}$$

This result is useful when calculating the order of a difference of two $\tilde{T}_i$:

$$||[\tilde{T}_i(oldP_1, oldQ_1) - \tilde{T}_i(oldP_2, oldQ_2)|| =$$
$$||e^{-\lambda_i\theta} \int_x^\infty L_i^T e^{\lambda_i\Theta}[F(\Theta, oldP_1, oldQ_1) - F(\Theta, oldP_2, oldQ_2)]d\Theta|| =$$
$$||O(e^{-\theta}((oldP_1 - oldP_2) + (oldQ_1 - oldQ_2)) + (oldP_1 - oldP_2)^2 + (oldQ_1 - oldQ_2)^2)|| \leq$$
$$\leq C||(e^{-\theta}(oldP_1 - oldP_2 + oldQ_1 - oldQ_2) + (oldP_1 - oldP_2)^2 + (oldQ_1 - oldQ_2)^2||, \tag{5.31}$$

for some $C > 0$. The difference of the iteration on two points is:

$$\left|\left|\begin{pmatrix} dP_1 \\ dQ_1 \end{pmatrix} - \begin{pmatrix} dP_2 \\ dQ_2 \end{pmatrix}\right|\right| \leq$$
$$\sum_{i=1,2.j=1,2} ||N_{ij}(\tilde{T}_i(oldP_1, oldQ_1) - \tilde{T}_i(oldP_2, oldQ_2))|| \leq$$
$$2D\sum_{i=1,2} ||(\tilde{T}_i(oldP_1, oldQ_1) - \tilde{T}_i(oldP_2, oldQ_2))|| \leq$$
$$4DC||(e^{-\theta}(oldP_1 - oldP_2 + oldQ_1 - oldQ_2) + (oldP_1^2 - oldP_2^2 + oldQ_1^2 - oldQ_2^2)||, \tag{5.32}$$

where D is again where $D = Max\{N_{ij}\}$, $1 \leq i, j \leq 2$. Since $(oldP, oldQ) \to 0$, then for some $0 < H < 1$, there exists $\delta_2 > 0$, and $\theta_2 > 0$ such that:

$$4DC||(e^{-\theta}(oldP_1 - oldP_1 + oldQ_1 - oldQ_1) + (oldP_1^2 - oldP_2^2 + oldQ_1^2 - oldQ_2^2)|| <$$
$$H\left|\left|\begin{pmatrix} oldP_1 \\ oldQ_1 \end{pmatrix} - \begin{pmatrix} oldP_2 \\ oldQ_2 \end{pmatrix}\right|\right|, \tag{5.33}$$

for every $(oldP, oldQ) \in S[\theta_2, \delta_2)]$.

Let $\delta_* = min\{\delta_1, \delta_2\}$, and $\theta_* = max\{\theta_1, \theta_2\}$. Then the process is a contraction mapping in $S[\theta_*, \delta_*)]$. This shows that there is a unique solution that tends to the (P_*, Q_*) as $\theta \to \infty$, which, in turn, shows that there's a unique solution of (3.12) that reaches the origin at R_*. □

6 Pucci-Serrin Method

This section introduces the solution of Pucci & Serrin (PS) [10] for the uniqueness of radial ground states of the general problem

$$div(A|Du|)Du) + f(u) = 0, \tag{6.1}$$

which are the radially symmetric solutions of class $C^1(\mathbb{R}^n)$ such that $u \geq 0, u \neq 0, u(x) \to 0$ as $|x| \to \infty$. The proof presented in this section is a little bit different that the original work of PS, and uses various notations and results from the dynamical systems approach that was presented in the previous sections.

To make the section more readable, define for $\rho > 0$

$$\Omega(\rho) = \rho A(\rho), \quad G(\rho) = \int_0^\rho \Omega(\rho) d\rho.$$

The proof is valid under some restrictions on the operator A and function f. The following conditions are supposed on the operator $A \in C^1(0, \infty)$:

1. $\Omega'(\rho) > 0$ for $\rho > 0$; $\Omega(\rho) \to 0$ as $\rho \to 0$,
2. $\Omega(\rho) \leq$ Const., and $\Omega(\rho) \leq \Omega'(\rho)$ for ρ near 0,
3. $\frac{\rho\Omega(\rho)}{G(\rho)}$ is (non-strictly) increasing for $\rho > 0$.

The function $f(u), 0 \leq u \leq \infty$ satisfies the following set of conditions:

(a) f is continuous on $[0, \infty)$, and $f(0) = 0$,

(b) f is continuously differentiable on $(0, \infty)$,

(c) there exists $a > 0$ such that $f(a) = 0$ and

$$\begin{array}{lll} f(u) < 0 & for & 0 < u < a, \\ f(u) > 0 & for & a < u < \infty. \end{array}$$

Define the following critical constant

$$m = \inf_{\rho > 0} \frac{\rho\Omega(\rho)}{G(\rho)}, \tag{6.2}$$

which will later be used to apply more restrictions on f. It can be seen easily that $G(\rho) < \rho\Omega(\rho)$ for $\rho > 0$, since Ω is an increasing function. That in turn means that $m \geq 1$.

Radial solutions $u(r)$ of (6.1) satisfy the ordinary differential equation

$$(A(|u'|)u')' + \frac{d-1}{r}A(|u'|)u' + f(u) = 0, \tag{6.3}$$

for $r > 0$ and $u'(0) = 0$. Equivalently, the last equation can be written as

$$[r^{d-1}A(|u'|)u']' + r^{d-1}f(u) = 0. \tag{6.4}$$

d here, can be considered a real parameter, with $d > 1$. In terms of our work from the previous chapters this means that the dimension don't have to be a natural number.

For the rest of the proof ρ will be regarded as in the next definition

Definition 6.1. $\rho(r) = |u'(r)|$

Using that definition, and assuming that $u' < 0$, which will be shown later, equation (6.4) can be written as

$$[r^{d-1}\Omega(\rho)]' = r^{d-1}f(u). \tag{6.5}$$

Another definition that would be helpful of us is that of $H(\rho)$

$$H(\rho) = \rho\Omega(\rho) - G(\rho), \rho \geq 0.$$

It is convenient to define $\Omega(0) = G(0) = 0$ so that $H(0) = 0$ as well. PS [10] notes that in an earlier paper they proved that solutions u of (6.3) satisfy

$$\frac{d}{dr}[H(\rho) + F(u)] = -(d-1)\frac{\rho\Omega(\rho)}{r}, \tag{6.6}$$

where

$$F(u) = \int_0^u f(t)dt.$$

Note that this is a generalization of (3.17) in section 3.

In order to better see the relationship between the method of PS and the dynamical systems approach, set $v = A(|u'|)u'$. This transfers the equation (6.3) into

$$\begin{aligned} u' &= -\Omega^{-1}(-v) \\ v' &+ \tfrac{d-1}{r}v + f(u) = 0. \end{aligned} \tag{6.7}$$

Setting $v = -rW$ in that equation yields

$$\begin{aligned} u' &= -\Omega^{-1}(rW) \\ W' &= \tfrac{1}{r}[f(u) - dW], \end{aligned} \tag{6.8}$$

The equation for W in (6.8) shows that a necessary condition for smoothness is

$$W(0) = \frac{f(u(0))}{d}, \tag{6.9}$$

or otherwise W' would be singular at $r = 0$.

To bound the size of $u(0)$ one must turn to the operator $H(\rho) + F(u)$. That decreasing operator, as can be see from (6.6), vanishes at the origin and therefore must be positive at $r = 0$. H is zero at $r = 0$, and hence the possible values for u are those at which $F(u) > 0$. Since $f(u) < 0$ for $u < a$, it is obvious that $F(u)$ is also negative (at least) in $u \leq a$. Hence $u(0) > a$, and (6.9) shows that $W(0) > 0$. To see that a ground state solution must stay above the W-axis, we note that $H(\rho) + F(u)$ is negative on the line segment $(0, a]$ of the u-axis, so a solution that reaches that segment cannot be a ground state. On the rest of the u axis, $(a, \infty]$, $W(r)'$ is positive for all r, so a solution can't cross it.

We conclude that ground states must start at $r = 0$ on the curve $W = \frac{f(u(0))}{d}$ with $u(0) > a$, and remain, for all time, at the region bounded by that line, the positive W-axis and the segment $(0, a]$ of the u-axis. The fact that the region is bounded by the W-axis, i.e. $u > 0$, is a matter of definition - namely that a ground state is non-negative.

The claim that u' is indeed negative follows from the equation $u' = -\Omega^{-1}(rW)$, after noticing that $r > 0$, W has been shown to remain positive until the solution reaches the origin, and Ω maps positive values to positive values. Note that in order to show the same conclusion on the sign of u', PS turned to FLS which uses a different approach.

The next definition is equivalent to Definition 4.4 of the special case. It comes in handy when phrasing the PS results in dynamical systems terms.

Definition 6.2. *Let s_1, s_2 be two solutions of (6.7) such that s_1 is below s_2.*
1. The solutions are said to be in 'prior to u-equality' at u, if $r_1(u) < r_2(u)$, or $r_2(u)$ doesn't exist.
2. The solutions are said to be in 'u-equality' at u if $r_1(u) = r_2(u)$.

Note that solutions begins at prior to u-equality state at $r = 0$. It is convenient for the rest of the chapter to define $\alpha = min\{u_1(0), u_2(0)\}$.

Remark: A ground state, $s = (u, v)$, of (6.7) is a ground state to (6.3) and hence also a radial ground state to (6.1). Those claims work in the other direction as well, so every definition that we make for ground state s of (6.7) can be seen as a definition for radial ground state u of (6.1). This equivalence is being used in proofs later on.

The key for the proof of PS is the function

$$P(r, u, \rho) = r^d[H(\rho) + F(u)] - dr^{d-1}\Omega(\rho)K(u), \tag{6.10}$$

defined for $r \geq 0, \rho \geq 0, u > 0 (\neq a)$, where

$$K(u) = \frac{F(u)}{f(u)} \quad and \quad F(u) = \int_0^u f(\tau)d\tau.$$

The following is Proposition 1. from PS [10]. For the special case when $A(\rho) = \rho^{m-2}$ it was proved earlier by Erbe and Tang [3]. The case of the main problem of this thesis is $m = 4$.

Proposition 6.1. *Let $u = u(r)$ be a non-negative solution of (6.4) with $u'(r) \leq 0$. Then*

$$\frac{d}{dr}P(r, u(r), \rho(r)) = dr^{d-1}\rho\Omega(\rho)\left\{\frac{dK(u)}{du} - \frac{G(\rho)}{\rho\Omega(\rho)} + \frac{1}{d}\right\},$$

for all $r > 0$ such that $u(r) \neq 0, a$.

PS proved that proposition by direct calculation.

A special case of P was defined in (4.13) and was used in the proof of Lemma 4.8.

The main focus of this section is the proof of

Theorem 6.1. *Assume that the above assumptions on A and f hold, and that*

$$\frac{d}{du}\left[\frac{F(u)}{f(u)}\right] \geq \frac{d-m}{dm} \quad for \quad u > 0, u \neq a. \tag{6.11}$$

Then Equation (6.1) with m define by (6.2), admits at most one radial ground state.

The proof for the theorem is given in two parts in subsections 6.2 and 6.3. The first of which, subsection 6.2 proves the following series of lemmas which end with the result that there is no u-equlity for $0 < u \leq a$, $r > 0$. All the lemmas in this part are used in later on in the second part to prove the uniqueness result. As full technical details for the second part are not provided, we encourage the reader to refer to PS original paper to see where the lemmas are used in. In order to stay consistent with the rest of this paper, the lemmas are phrased in dynamical systems terminology, even though originally PS used a different one. In specific, PS calls u-equality 'intersection' since it is intersection in the $u - r$ plane, whereas in the dynamical systems approach intersection refers to intersection in the phase plane.

We use in those lemmas the inverse function $r(u)$ of $u(r)$, defined for $0 < u < \alpha$.

Lemma 6.2.1. *Let $u_1(r), u_2(r)$ be two radial ground states of (6.1). Define the ratio of the derivative in the RHS of (6.5) for u_1, u_2 to be*

$$T_{12}(u) = \left(\frac{r_1(u)}{r_2(u)}\right)^{d-1} \frac{\Omega_1(u)}{\Omega_2(u)}, \;\; 0 < u < \alpha.$$

Then for $a < u < \alpha$ we have (with $' = d/du$)

$$T'_{12}(u) > 0 \quad if \quad and \quad only \quad if \quad r'_2(u) < r'_1(u),$$

while for $0 < u < a$

$$T'_{12}(u) > 0 \quad if \quad and \quad only \quad if \quad r'_2(u) > r'_1(u).$$

Lemma 6.2.2. *Let $u(r)$ be a radial ground state of (6.1). Then*

$$\lim_{r\to\infty} r^{d-1}\Omega(\rho) = finite \quad limit \quad \lambda \geq 0 \tag{6.12}$$

$$\liminf_{r\to\infty} r^d[H(\rho)+F(u)] = 0. \tag{6.13}$$

Lemma 6.2.3. *Let $u_1(r), u_2(r)$ be two radial ground states of (6.1). If $r_2(u) - r_1(u) > 0$ on an interval $I \subset (0,\alpha)$, then $r_2(u) - r_1(u)$ can have at most one critical point on I. If such a point occurs, it must be a strict maximum. Moreover, if $I = (0,n), n \leq a$, then $r_2'(u) - r_1'(u) < 0$ on I.*

Lemma 6.2.4. *Let $u_1(r), u_2(r)$ be two different ground states for (6.1). Then the ground states don't have u-equality in the set $0 < u \leq a$, $r > 0$.*

Subsection 6.3 completes the proof of the theorem. It is first shown there that there is no u-equality for any $0 < u < \alpha$. Then this result is used to prove Theorem 6.1.

6.1 Dynamical Systems Approach

This section discusses some general theorems which are easily phrased and proved under the dynamical systems terms. Those theorems will be used in our version of Pucci and Serrin proof, so as to show the strength of the dynamical systems approach.

Definition 6.1.1. *Let s_1, s_2 be two ground states of (6.7) starting on the curve $v(0) = 0$. s_1 is **below** s_2 at u in $u - v$ system (6.7) if $v_1(u) < v_2(u)$ or if $v_2(u)$ does not exist. The latter case means that $u > u_2(0)$ and the definition follows since s_1 is below s_2 at $u_2(0)$ (remember that v is negative). Under those conditions s_2 is **above** s_1 at u.*

Those definitions are valid for every system (i.e. $u - W$ system (6.8)) with a change of v in Definition 6.1.1 to the relevant variable. They are also equivalent to Definition 4.1 of the special case.

The first Lemma is equivalent to Lemma 4.2 of the special case.

Lemma 6.1.2. *Let $s_1(r), s_2(r)$ be two different ground states of (6.7). If they intersect in the $u - v$ system at (u_*, v_*) for $u_* > 0$, and s_1 is below s_2 before u_*, then $r_1(u_*) < r_2(u_*)$.*

Proof. The proof is very similar to those of Lemma 4.2, even though v' is not necessarily monotonic like W' was there.

s_1 is below s_2 before u_*, and above it afterwards. In order for that to happen, regardless to the question if v' is positive or negative, the following has to take place: $\frac{v_1'}{u_1'}(r_1(u_*)) < \frac{v_2'}{u_2'}(r_2(u_*))$. Dividing v' by u' in (6.7) yields

$$\frac{v'}{u'} = \frac{(1-n)v}{r(u)\Omega^{-1}(v)} - \frac{f(u)}{\Omega^{-1}(v)}.$$

which is an increasing function of r. Combining with $\frac{v_1'}{u_1'}(r_1(u_*)) < \frac{v_2'}{u_2'}(r_2(u_*))$ proves that $r_1(u_*) < r_2(u_*)$. □

The following lemma is similar to Lemma 4.5 of the special case. We have adjusted its conditions and conclusions so that we are better equipped later in this chapter.

Lemma 6.1.3. *Let $s_1(r), s_2(r)$ be two different ground states of (6.7) intersecting in the $u-v$ system at (u_*, v_*). Set u_{**} to be with the minimum $u > u_*$ in which there is an intersection point. If no such point exist, i.e. u_* is the first intersection point of the ground states, then set $u_{**} = \alpha$. Under those definitions there is exactly one u_e, $u_* < u_e < u_{**}$, in which the solutions are in u-equality.*

Proof. The ground states are prior to u-equality at u_*. Assume, without losing generality, that s_1 is above s_2 between u_* and u_{**}. Then $r_1(u_*) < r_2(u_*)$ by Lemma 6.1.2. This is true by definition of 'below' state even if $u_* = \alpha$. On the other hand at u_{**}, $r_1(u_{**}) > r_2(u_{**})$ by Lemma 6.1.2. The continuity of u and v shows that there exists $u_* < u_c < u_{**}$ such that the ground states are at u-equality at it.

To see that u_c is unique, remember that s_1 is above s_2 at u_c. Then since Ω^{-1} is an increasing function, the differential equation for u shows that while s_1 is above s_2 there must exist an interval $(u_c, u_c + \delta]$ with $\delta > 0$, such that $r_1(u) > r_2(u)$ for u in that interval. But $r_1(u) < r_2(u)$ for every u after u_c, which means that for a second u-equality to occur s_1 has to be below s_2. This can happen only after intersection, meaning: after u_*. □

6.2 Proof of Theorem 6.1. Part I

This subsection focuses on proving that there is no u-equality for $0 < u < a$. Our goal in this subsection is mostly to involve dynamical systems methods in the proof of the lemmas. The lemmas were already stated in the first subsection of this chapter. Below are given their proofs.

Proof of Lemma 6.2.1. Using the formula for the derivative of a logarithm yields

$$\begin{aligned}
&\tfrac{1}{T}\tfrac{dT}{du} = \tfrac{d}{du}logT = \tfrac{d}{du}log(r_1(u)^{d-1}\Omega_1(u)) - \tfrac{d}{du}log(r_2(u)^{d-1}\Omega_2(u)) \\
&= \tfrac{1}{r^{d-1}\Omega(\rho)}\tfrac{d}{dr}\big[r^{d-1}\Omega(\rho)\big]\tfrac{dr}{du}\Bigg|_{\rho=\rho_2(r),r=r_2(u)}^{\rho=\rho_1(r),r=r_1(u)} \\
&= \tfrac{1}{\rho\Omega(\rho)}\Bigg|_{\rho=\rho_1(r_1(u))}^{\rho=\rho_2(r_2(u))} \cdot f(u).
\end{aligned} \tag{6.14}$$

We know that whenever $0 < u < \alpha$, we have that $u'(r) < 0$. This together with equation (6.4) will give us the last step.

Note that $\rho\Omega(\rho)$ is an increasing function by assumption (1), and that

$$\rho_2(r_2(u)) = -\frac{1}{r_2'(u)}, \quad \rho_2(r_1(u)) = -\frac{1}{r_1'(u)},$$

which is true since $u'(r) < 0$ and $\rho(r) = |u'(r)| = -u'(r)$. Therefore $\rho_2(r_2(u)) > \rho_1(r_1(u))$ if and only if $r_2'(u) > r_1'(u)$, and this means that dT/du has the same sign as $f(u)$ if and only if $r_2'(u) < r_1'(u)$. Now the conditions that $f(u) > 0$ for $a < u < \alpha$ and $f(u) < 0$ for $0 < u < a$ complete the proof. □

Proof of Lemma 6.2.2. (6.12) can be proved from writing (6.4) in dynamical systems terms by setting $z = r^{d-1}\Omega(\rho)$

$$\begin{aligned} u' &= \Omega^{-1}(r^{1-d}) \\ z' &= -r^{d-1}f(u). \end{aligned}$$

z' is negative for r_0 large enough for which $u(r_0) < a$. Therefore z is non-increasing in (r_0, ∞). The definition of z plus condition (1) on the operator A shows that z is non-negative. Combining all of those together proves (6.12).

The second part is proven in PS and we will not bother to repeat the proof here.

□

Proof of Lemma 6.2.3. First we note that a critical point of $r_2(u) - r_1(u)$ is an intersection point of the $u - v$ system. This is true since $r_2'(u) = r_1'(u)$ iff $u_2'(r_2(u)) = u_1'(r_1(u))$, which by (6.7) holds iff $v_2 = v_1$ when $u_2 = u = u_1$.

The first part of the theorem says, in dynamical systems terms, that there is at most one intersection between two u-equalities. The second part says, in dynamical systems terms again, that for two ground states there is no intersection between the origin and the u-equality point. The first claim is a direct result of Lemma 6.1.3.

The second claim follows from noticing that if s_2 is below s_1, and $r_2(u) > r_1(u)$ then intersection is impossible as it will contradicts Lemma 6.1.2. A ground state s_2 cannot reach the origin and be below s_1 in the neighborhood of the origin due to the same reason as in the proof in Theorem 4.11. Hence a ground state that reaches the origin must be above s_1 for all times in which there is no u-equality. If s_2 is above s_1 then by definition of 'above' state $v_2(u) > v_1(u)$. The definition of v combined with the fact that $\Omega'(\rho) > 0$ by condition (1) on Ω shows that $u_2'(u) > u_1'(u)$. Taking that with $r'(u) = \frac{1}{u'(r)}$ shows that $r_2'(u) < r_1'(u)$ as in the second claim.

This proof is essentially different from Pucci and Serrin's proof, which invokes Lemmas 3.3.1 and 3.6.5 of [FLS] to prove the theorem. □

Lemma 6.2.4 is Lemma 3.4 from [10]. It's proof will not be supplied here as it doesn't use special dynamical systems tools.

remark: Theorem 4.11 essentially shows the same as Lemma 6.2.4. Since if the only option for two ground states is that they both reach the origin at the same time, then there can't be u-equality at any point $0 < u < \alpha$, and in specific in $0 < u \leq a$.

6.3 Proof of Theorem 6.1. Part II

This subsection finishes the proof of Theorem 6.1. Again, note that Pucci and Serrin call u-equality 'intersection'.

Invoking a few propositions from [FLS], with conditions (a),(b),(c) and (1),(2),(3), Pucci and Serrin showed that if u_1, u_2 are two different radial ground states, with $u_1(0) = \alpha_1, u_2(0) = \alpha_2$, then $\alpha_1 \neq \alpha_2$.

remark: Although the above result only claims that only one ground state can start at any given point on the initial curve, the proof in [FLS] seems to show that there is a unique solution starting from any point on the initial curve, whether it is a ground state or not. This was shown in section 3 for the particular case considered in sections 3-5.

In the rest of the subsection we always assume that $\alpha_1 < \alpha_2$, with, as usual, $\alpha_1 > a$.

We saw in part I that there can be no u-quality at (R, U) for which $R \geq 0, 0 < U \leq a$. In this part we first show that there can be at most one u-quality (R, U) where $U > a$. Say that there were two such points (r_I, u_I) and (r_{II}, u_{II}). Let $r_I < r_{II}, u_I > u_{II}$ and

$$u_1(r_I) = u_2(r_I) = u_I, \quad u_1(r_{II}) = u_2(r_{II}) = u_{II}.$$

By definition s_1 is above s_2 for $u > \alpha$. Since there is no u-quality for $\alpha > u > u_I$, s_1 must remain above s_2 in that area. Again, by definition of above/below state, this means that $0 > v_1(u) > v_2(u)$ in $\alpha > u > u_I$, and in addition system (6.7) shows $u_1'(u) > u_2'(u)$. Hence

$$r_1'(u) < r_2'(u) \quad for \quad u_I < u < \alpha. \tag{6.15}$$

Lemma 6.1.3 shows that there is exactly one point $u_c > a$, i.e. the u-quality point in $u - v$ system, such that $r_1'(u_c) = r_2'(u_c)$ and

$$\begin{array}{ll} r_2'(u) - r_1'(u) > 0 \quad for \quad u_c < u < u_I \\ r_2'(u) - r_1'(u) < 0 \quad for \quad u_{II} < u < u_c. \end{array}$$

combining with (6.15) shows that

$$r_2'(u) > r_1'(u) \quad for \quad u_c < u < \alpha. \tag{6.16}$$

Let $C = T_{12}(u_c)$. By (b) we have $f(u) > 0$ for $u \geq u_c$. Then from (6.16) and Lemma 3.1 we see that $T_{12}'(u) < 0$ for $u_c < u < \alpha$. Therefore

$$C > T_{12}(u), \quad u_c < u < \alpha. \tag{6.17}$$

The last part of the proof invokes Proposition 6.1 twice, each time showing that the expression

$$P(R_1, u_1(R_1), \rho_1(R_1)) - Const * P(R_2, u_2, \rho_1(R_2))$$

is non-positive by that proposition, but must be positive by definition (6.10) of P. The first usage, in which R_1 and R_2 were chosen to be R_{1c}, R_{2c} correspondingly, shows that there can be at most one u-equality in the region $r \geq 0, u > 0$, and that this can occur only when $u > a$.

In the second usage R_1 and R_2 are chosen to be the times in which each radial ground state reach the point $u = \epsilon$ for $\epsilon < 0$ arbitrary small. This case in specific involves a difficulty because of the singularity of $P(r, u, \rho)$ at $u = a$. However, this singularity is shown to be removable, and a contradiction is achieved as before. This implies that there is no u-equality for $r \geq 0, u > 0$, and hence

$$r_2(u) > r_1(u), \quad 0 < u < \alpha.$$

This in turn is impossible since it means by Lemma 6.2.3 that

$$r_2'(u) - r_1'(u) < 0 \quad 0 < u < \alpha.$$

But this cannot happen since it was mentioned before that $r_1'(u) \to \infty$ as $u \to \alpha^-$, while $r_2'(\alpha)$ is infinite. This completes the proof the Theorem 6.1.

Remark: The last part here can also be seen as a shorter way to get the result of section 5. However, it is not a local result as it uses the initial conditions of the solutions, and therefore is a less general result than the proof in section 5.

As a last note we must mention that Pucci and Serrin also proved, in Theorem 2 in their article, that functions of the form $f(u) = -u^p + u^q$, with certain limits on p and q, satisfy the conditions of Theorem 6.1, and therefore have a unique radial ground state. Therefore, there is a unique radial ground state for our problem with $q = 3, p = 1$ and $A(\rho) = \rho^2$.

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