Bachelor Thesis

Oscillation suppression in nonlinear coupled oscillators with and without time delay

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Abstract

The aim of this thesis is to study analytically oscillation death in a system of two coupled Stuart-Landau oscillators. The coupling is chosen to break the rotational symmetry of the single oscillators. We consider both instantaneous and delayed coupling. A detailed bifurcation analysis of the system is done. In the instantaneous case we are able to show that oscillation death always appears if the coupling is strong enough. Explicit values are given. In the delayed case, we show for the first time that oscillation death can occur at all. We find that the region in which oscillation death occurs only varies slightly from the instantaneous case.

Contents

1 Introduction

The understanding and control of nonlinear oscillators is of great importance for nonlinear dynamics. Applications can be found in optics, electronic circuits, chemical reactions, biology and medicine. A good overview of oscillators in living organisms, such as neuron oscillations, heart beats or biochemical oscillations, is given by Goldbeter [7].

In this thesis, we consider Stuart-Landau oscillators. Among others, they belong to the so-called limit-cycle oscillators which frequently occur in biology [7]. They often serve as models of coupled neurons in the brain [17] or laser dynamics [13].

Mathematically, the study of Stuart-Landau oscillators is justified by normal form analysis, described in the textbook by Guckenheimer and Holmes [8].

Time delay is present in many physical systems. The two main reasons for time delay are the final signal propagation time, limited by the speed of light, and the signal processing time. It seems therefore natural to include time delay in the modelling of coupled oscillator systems.

The asymptotic stabilization of steady states in coupled oscillator systems is called oscillation suppression. There are two main mechanisms of oscillation suppression which have been investigated in the literature: amplitude death and oscillation death.

Amplitude death has been studied theoretically [2, 3, 12, 14] and experimentally [19]. By amplitude death one usually means the coupling-induced stabilization of an already existing steady state that is unstable in the absence of coupling. The main parameters leading to amplitude death are mismatches between the frequencies of the oscillators and coupling through dissimilar variables. It is well known that amplitude death can also appear in systems with time delay, see Ramana Reddy et al. [15, 16] for a detailed account.

Oscillation death, on the contrary, has not been studied very often so far [4, 9]. In contrast to amplitude death, oscillation death occurs through the stabilization of newly created steady states, which do not even exist in the uncoupled system. It can happen that these steady states are only locally stable and that an oscillation continues to coexist as a locally stable periodic orbit (for an example see [10], figure 2). In the paper of Zou et al. [20], which deals with Stuart-Landau oscillators, oscillation death is proven via numerical calculation of the largest Lyapunov exponent of the steady states. So far, there is no proof that oscillation death of coupled Stuart-Landau oscillators can also occur in systems with time delay.

The aim of this thesis is to give explicit analytic conditions for oscillation death of coupled Stuart-Landau oscillators for instantaneous coupling and then extend the results of the instantaneous coupling to delayed coupling. We show for the first time that oscillation death can also appear in delay-coupled Stuart-Landau oscillators.

This thesis is structured as follows: In chapter 2, we introduce the model of instantaneously coupled Stuart-Landau oscillators and discuss the newly created steady states with nonzero amplitude. In chapter 3, we perform a detailed bifurcation analysis for both the homogeneous and the inhomogeneous steady states. We can then derive explicit conditions for oscillation death, depending on the system parameters. Chapter 4 investigates the effects of time delay on oscillation death. In chapter 5, we summarize our results and discuss them.

2 Model of two coupled nonlinear oscillators

2.1 The single Stuart-Landau oscillator

Throughout this thesis, we consider planar Stuart-Landau oscillators, where the equation for one single oscillator is given by

$$\dot{z} = f(z) = \left(\lambda + \mathrm{i}\omega - |z|^2\right) z. \tag{2.1}$$

Here $z \in \mathbb{R}^2 \cong \mathbb{C}$ and $\lambda > 0$ is the bifurcation parameter. $\omega > 0$ is the so called Hopf-frequency. It is the normal form of a supercritical Hopf-bifurcation, truncated at third order [1, 8].

The system (2.1) can be written in polar coordinates $z(t) = r(t)\, e^{\mathrm{i}\theta(t)}$:

$$\begin{aligned} \dot{r} &= \left(\lambda - r^2\right) r \\ \dot{\theta} &= \omega \end{aligned}$$

There exists a steady state for $r = 0$, meaning $z \equiv 0$. The linearization δz of the system at the steady state $z \equiv 0$ is given by

$$\dot{\delta z} = (\lambda + \mathrm{i}\omega)\, \delta z.$$

The characteristic equation of the eigenvalue η,

$$\eta = \lambda + \mathrm{i}\omega,$$

which is linear here, can be obtained by an exponential ansatz of the linearized system. Here we deal with the complex notation. In real notation there are two complex conjugated eigenvalues $\eta = \lambda \pm \mathrm{i}\omega$. The real part of the eigenvalue η of the characteristic equation is equal to λ and therefore positive. Thus $z \equiv 0$ is an unstable steady state with two eigenvalues with strictly positive real part.

Besides this steady state, there is also a periodic orbit $z_p(t) = r \exp\left(\frac{2\pi \mathrm{i} t}{p}\right)$ whose radius r is given by

$$r = \sqrt{\lambda}$$

and whose minimal period p is

$$p = \frac{2\pi}{\omega},$$

independent of the value of r.

Note the rotational symmetry of the system (2.1): If $z_*(t)$ is a solution of equation (2.1), then also $e^{\mathrm{i}\phi} z_*(t)$ is a solution of (2.1) for all real ϕ.

2.2 Introducing suitable coupling

In this thesis, we want to study a system of two coupled Stuart-Landau oscillators. They cease to oscillate due to a suitable coupling term. By oscillation death we mean the stabilization of newly created isolated steady states with a non-zero amplitude. Coupling terms which keep the rotational symmetry of the single oscillator (2.1) never yield isolated new steady states, since there would always be a whole ring of steady states. Hence it is necessary that the coupling term breaks this rotational symmetry.

In this thesis we couple the real part of the two oscillators z_1 and z_2 with a real coupling parameter $\varepsilon > 0$ as follows:

$$\begin{aligned}\dot{z}_1 &= \left(\lambda + \mathrm{i}\omega - |z_1|^2\right) z_1 + \tfrac{1}{2}\varepsilon\left(z_2 + z_2^* - z_1 - z_1^*\right)\\ \dot{z}_2 &= \left(\lambda + \mathrm{i}\omega - |z_2|^2\right) z_2 + \tfrac{1}{2}\varepsilon\left(z_1 + z_1^* - z_2 - z_2^*\right).\end{aligned}$$

Here $z_{1,2}^* = x_{1,2} - \mathrm{i}y_{1,2}$ is the complex conjugate of $z_{1,2} = x_{1,2} + \mathrm{i}y_{1,2}$. This system can equivalently be written as

$$\begin{aligned}\dot{z}_1 &= \left(\lambda + \mathrm{i}\omega - |z_1|^2\right) z_1 + \varepsilon\left(\operatorname{Re} z_2 - \operatorname{Re} z_1\right)\\ \dot{z}_2 &= \left(\lambda + \mathrm{i}\omega - |z_2|^2\right) z_2 + \varepsilon\left(\operatorname{Re} z_1 - \operatorname{Re} z_2\right)\end{aligned} \tag{2.2}$$

where $\operatorname{Re} z_{1,2} = x_{1,2}$. It is worthwhile to remark that the coupling term and therefore the whole system is symmetric, meaning that z_1 and z_2 can be interchanged and the set of equations (2.2) remains the same. However, the rotational symmetry of the system is destroyed by this particular choice of coupling.

The next step is to find the new, inhomogeneous steady states created by the coupling term chosen as above. To simplify the search, we make a linear coordinate transformation

$$\begin{aligned}z_+ &= \tfrac{1}{2}\left(z_1 + z_2\right)\\ z_- &= \tfrac{1}{2}\left(z_1 - z_2\right).\end{aligned}$$

z_+ is sometimes called the *average*, and z_- the *asynchrony* of the two oscillators [6]. The system in the new coordinates is given by:

$$\begin{aligned}\dot{z}_+ &= \tfrac{1}{2}\left(f\left(z_+ + z_-\right) + f\left(z_+ - z_-\right)\right)\\ \dot{z}_- &= \tfrac{1}{2}\left(f\left(z_+ + z_-\right) - f\left(z_+ - z_-\right)\right) - 2\varepsilon \operatorname{Re} z_-\end{aligned} \tag{2.3}$$

Note that, owing to the symmetry, both $z_+ \equiv 0$ and $z_- \equiv 0$ are dynamically invariant subspaces. Similar to [6], in these subspaces the equations (2.3) can be simplified to be

$$\begin{aligned}\dot{z}_+ &= \left(\lambda + \mathrm{i}\omega - |z_+|^2\right) z_+\\ \dot{z}_- &= 0\end{aligned}$$

for the *in-phase-subspace* $Z_+ = \{(z_+, z_-) \mid z_- \equiv 0\}$, and

$$\begin{aligned}\dot{z}_+ &= 0\\ \dot{z}_- &= \left(\lambda + \mathrm{i}\omega - |z_-|^2\right) z_- - 2\varepsilon \operatorname{Re} z_-\end{aligned}$$

for the *anti-phase-subspace* $Z_- = \{(z_+, z_-) \mid z_+ \equiv 0\}$.

We have already studied the dynamics of the in-phase-subspace in the previous section, see equation (2.1). Therefore we know that there is a stable periodic orbit but no steady state besides the homogeneous steady state $(z_+, z_-) \equiv (0, 0)$ in this subspace.

The anti-phase-subspace, however, offers new dynamics and it is therefore justified to search for new steady states within this subspace.

Indeed, two different branches of steady states, the "+"-branch given by $(z_+, z_-) \equiv (0, z_+)$ and the "−"-branch given by $(z_+, z_-) \equiv (0, z_-)$ emerge from the trivial steady state $(z_+, z_-) \equiv (0, 0)$. Here $z_\pm = x_\pm + \mathrm{i}y_\pm$, where $x_\pm$ and $y_\pm$ are given by

$$x_\pm = \pm \frac{\mp\varepsilon + \sqrt{\varepsilon^2 - \omega^2}}{\varepsilon} \sqrt{\frac{\lambda\varepsilon - \omega^2 \pm \lambda\sqrt{\varepsilon^2 - \omega^2}}{2\varepsilon}}$$
$$y_\pm = \pm \sqrt{\frac{\lambda\varepsilon - \omega^2 \pm \lambda\sqrt{\varepsilon^2 - \omega^2}}{2\varepsilon}}.$$

As later on in chapters 3 and 4, we will have to solve quadratic eigenvalue-equations, which have their own $\pm, \mp$, it is reasonable to avoid confusion with the different branches of the steady states. We use red signs, i.e. $+$, $-$, $\pm$, $\mp$, instead, to clarify that we mean the "+"- and the "−"-branches of the steady states, or both, respectively. It is very important for the rest of this work that we distinguish clearly between the different branches, indicated by red signs, and all other signs which occur during the calculations. In particular, $z_+ \neq z_+$.

Note that each branch consists of two steady states lying opposite of each other. The characteristic equations for the eigenvalues are such that both steady states have the same eigenvalues and can therefore be treated at the same time. In other words, if oscillation death occurs on one of the steady states, it will automatically occur on the other steady state, too.

Both $x_\pm$ and $y_\pm$ are real. A necessary condition for the existence of the inhomogeneous steady states is therefore $\varepsilon \geq \omega$.

Additionally, it is also often useful to consider the anti-phase-subspace in polar coordinates $z_- = re^{\mathrm{i}\varphi}$:

$$\dot{r} = \left(\lambda - r^2 - 2\varepsilon\cos^2\varphi\right) r$$
$$\dot{\varphi} = \omega + 2\varepsilon\cos\varphi\sin\varphi.$$

Here, we find for the radius $r_\pm$ and the angle $\varphi_\pm$ of the new steady states

$$r_\pm = \sqrt{\lambda - \varepsilon \pm \sqrt{\varepsilon^2 - \omega^2}}$$
$$\varphi_\pm = \arccos\left(\sqrt{\frac{\varepsilon \pm \sqrt{\varepsilon^2 - \omega^2}}{2\varepsilon}}\right) \quad (+\pi)$$

By direct calculation one can check that indeed $x_\pm^2 + y_\pm^2 = r_\pm^2$. It is important to notice that the cases $\lambda < \omega$ and $\lambda > \omega$ lead to different behaviour of the steady states. From the equation for $r_\pm$, we can immediately see that the "−"-branch only exists for $\lambda > \omega$. It emerges from the homogeneous steady state at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ and bifurcates to the left first. There is a saddle-node bifurcation at $\varepsilon = \omega$, corresponding $\dot{\varphi} = 0$. The solution changes "direction" here, meaning that larger radius now corresponds to a larger coupling parameter ε, this corresponds to the "+"-branch. For $\lambda < \omega$, it is only the "+"-branch bifurcating directly to the right from the trivial steady state at the same bifurcation point.

In figure 2.1, the radius $r_\pm$ is plotted versus the coupling parameter ε.

a)

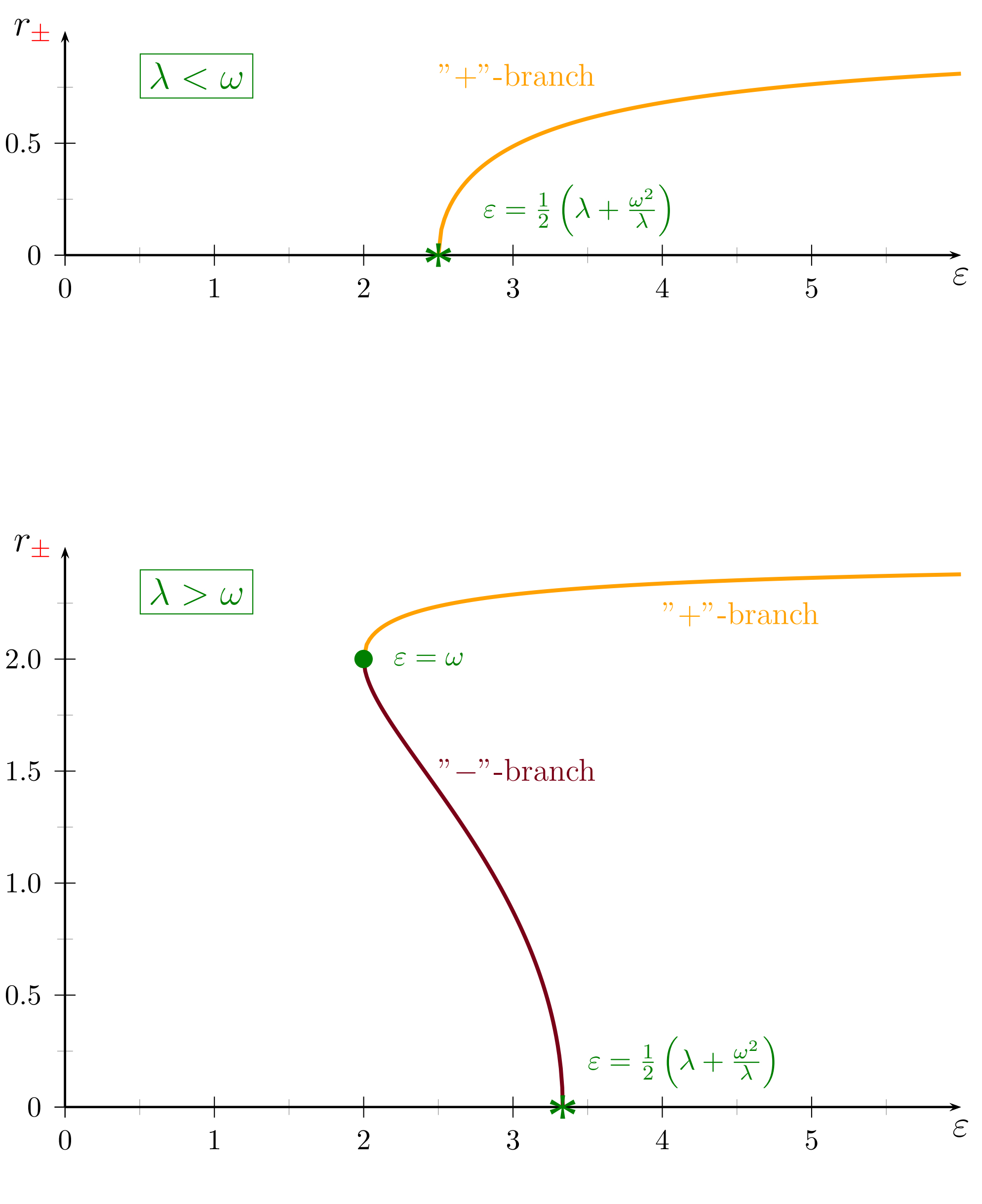


Figure 2.1: Radius $r_\pm$ of the inhomogeneous steady states plotted versus the coupling parameter ε. Figure a) shows the typical behaviour of the steady states if $\lambda < \omega$. Here $\lambda = 1$ and $\omega = 2$. Only the "+"-branch (orange) exists, it comes from a pitchfork bifurcation ($*$) at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. In figure b), we see the behaviour of the inhomogeneous steady states if $\lambda > \omega$. Here $\lambda = 6$ and $\omega = 2$. The "−"-branch (dark red) emanates from the homogeneous steady state at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ as a pitchfork bifurcation ($*$). At $\varepsilon = \omega$, there is a saddle-node bifurcation ($\bullet$), giving rise to the "+"-branch (orange).

3 Oscillation death without time delay

In this chapter, we carry out a detailed analytic bifurcation analysis of the system (2.1). Then we are able to show that for large enough coupling parameter ε oscillation death always appears. The critical value of ε as well as the mechanism underlying oscillation death depend strongly on the parameter choices for λ and ω.

3.1 Bifurcation analysis for the homogeneous steady state

The aim of this section is to do a rigorous bifurcation analysis of the trivial steady state $(z_1, z_2) \equiv (0, 0)$. This is simplified by the fact that the equations (2.3) decouple at the steady state $z \equiv 0$ if linearized. We obtain

$$\begin{aligned} \dot{\delta z_+} &= (\lambda + \mathrm{i}\omega)\, \delta z_+ \\ \dot{\delta z_-} &= (\lambda + \mathrm{i}\omega)\, \delta z_- - 2\varepsilon\delta \operatorname{Re} z_-. \end{aligned}$$

From the equation for z_+, we always get two complex conjugated eigenvalues with positive real part, i.e. $\lambda \pm \mathrm{i}\omega$, as λ was chosen to be positive (see also section 2.1).

To calculate the eigenvalues η for z_-, we express z_- in $\mathbb{R}^2$ instead of $\mathbb{C}$, $z_- = (x_-, y_-)$, leading to the following system:

$$\begin{aligned} \dot{\delta x_-} &= (\lambda - 2\varepsilon)\, \delta x_- - \omega \delta y_- \\ \dot{\delta y_-} &= \lambda \delta y_- + \omega \delta x_-. \end{aligned}$$

We then calculate the characteristic equation of the eigenvalues η, which is the determinant of the Jacobian:

$$\det \begin{pmatrix} \lambda - 2\varepsilon - \eta & -\omega \\ \omega & \lambda - \eta \end{pmatrix} = 0.$$

This yields a quadratic equation for the eigenvalues η,

$$\eta^2 + 2(\varepsilon - \lambda)\, \eta + \lambda^2 - 2\varepsilon\lambda + \omega^2 = 0,$$

which can be solved explicitly:

$$\eta = -\varepsilon + \lambda \pm \sqrt{\varepsilon^2 - \omega^2}.$$

The eigenvalues obtained by this equation are either complex conjugated or there are two distinct real eigenvalues. Note that real eigenvalues exist only for $\varepsilon \geq \omega$. At

$$\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$$

there is a simple eigenvalue $\eta = 0$, while the other eigenvalue is $\eta = \lambda - \frac{\omega^2}{\lambda}$. A double eigenvalue $\eta = 0$ exists if $\lambda = \omega$. We have to distinguish between $\lambda < \omega$ and $\lambda > \omega$: In the

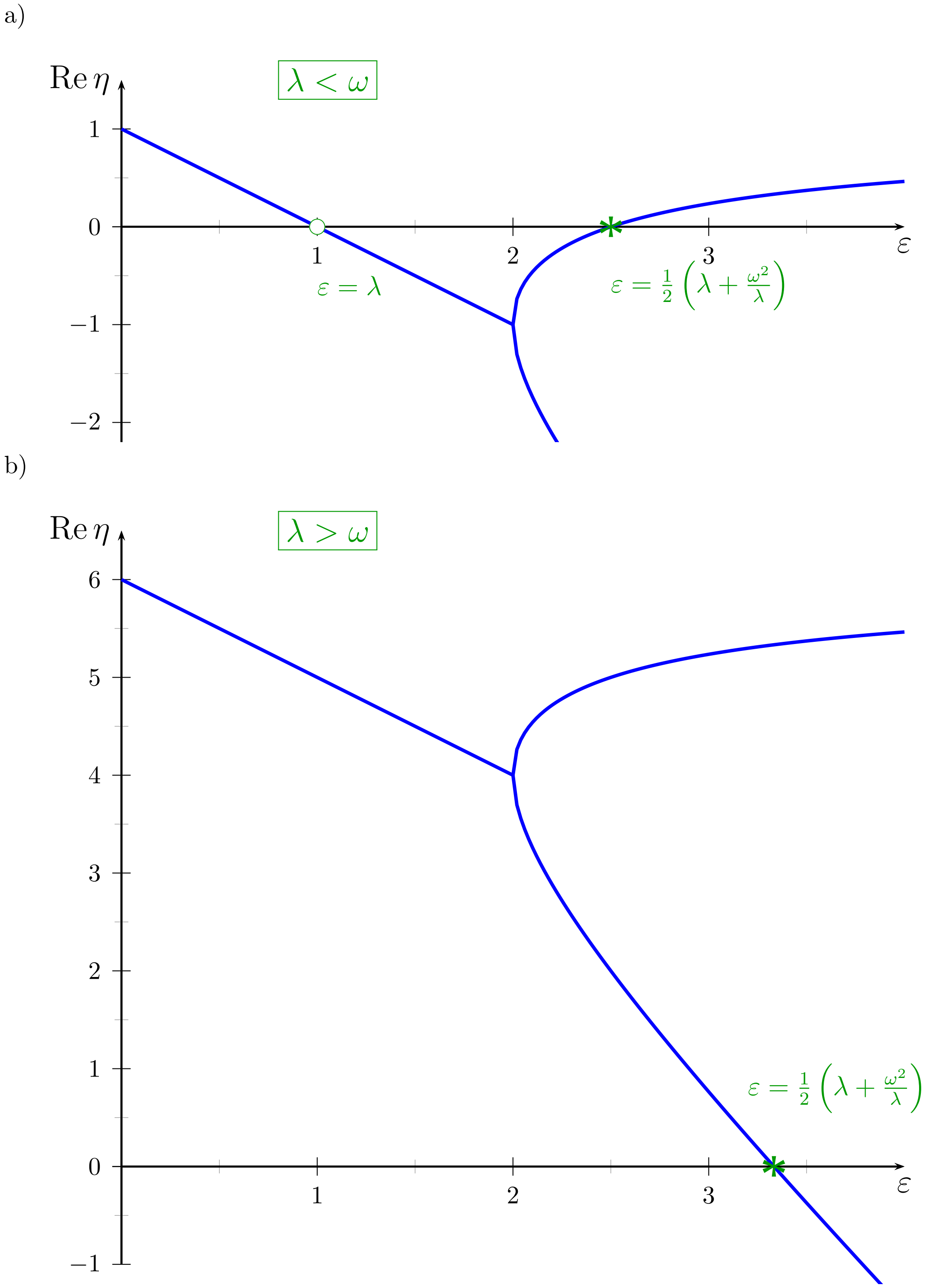


Figure 3.1: Real part of the eigenvalue η of the trivial steady state for $\lambda < \omega$ (figure a, $\lambda = 1,\ \omega = 2$) and $\lambda > \omega$ (figure b, $\lambda = 6,\ \omega = 2$) plotted versus the coupling parameter ε. For $0 < \varepsilon < \omega$ there is a pair of complex conjugated eigenvalues. For $\varepsilon > \omega$, there exist two distinct real eigenvalues. At $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ there is a pitchfork bifurcation ($*$). If $\lambda < \omega$, there is an additional Hopf bifurcation ($\circ$) at $\varepsilon = \lambda$.

first case, one of the eigenvalues η changes from negative real part to a positive real part at the bifurcation point $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ while in the latter case, this eigenvalue η changes from positive real part to a negative real part.

For a complete bifurcation analysis of the homogeneous steady state, we also need to ask whether there are any purely imaginary eigenvalues η which induce a Hopf-bifurcation. Since $\eta = -\varepsilon + \lambda \pm \sqrt{\varepsilon^2 - \omega^2}$, and ε and λ are both positive and real, there can only be purely imaginary eigenvalues if

$$\varepsilon = \lambda$$

and $\omega > \varepsilon$. So indeed, there is a Hopf-bifurcation of the trivial steady state at $\varepsilon = \lambda$ for $\lambda > \omega$, reducing the unstable dimension of the steady state by 2 for increasing ε.

By unstable dimension we mean the number of solutions with strictly positive real part of the characteristic equations of a steady state (z_1^*, z_2^*) of the system (2.2). We emphasize that the concept of counting unstable dimensions (see also [6]) is a very valuable one for this thesis because it gives very general and yet precise results, both in the instantaneous and the delayed case.

A summary of the bifurcation analysis for the homogeneous steady state can be seen in figure 3.1.

3.2 Bifurcation analysis for the inhomogeneous steady states

In addition to the bifurcation analysis for the trivial steady state $(z_1, z_2) \equiv (0, 0)$, we also perform a rigorous bifurcation analysis for the inhomogeneous steady states $(z_+, z_-) \equiv (0, z_\pm)$. Therefore, it is necessary to calculate the linearization of the system (2.2) along a steady state $(x_1, y_1, x_2, y_2) \equiv \pm\left(x_\pm, y_\pm, -x_\pm, -y_\pm\right)$. In real notation, this yields the following linearized system:

$$\begin{pmatrix} \dot{\delta x_1} \\ \dot{\delta y_1} \\ \dot{\delta x_2} \\ \dot{\delta y_2} \end{pmatrix} = \begin{pmatrix} \lambda - 3x_\pm^2 - y_\pm^2 - \varepsilon & -2x_\pm y_\pm - \omega & \varepsilon & 0 \\ -2x_\pm y_\pm + \omega & \lambda - 3y_\pm^2 - x_\pm^2 & 0 & 0 \\ \varepsilon & 0 & \lambda - 3x_\pm^2 - y_\pm^2 - \varepsilon & -2x_\pm y_\pm - \omega \\ 0 & 0 & -2x_\pm y_\pm + \omega & \lambda - 3y_\pm^2 - x_\pm^2 \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta y_1 \\ \delta x_2 \\ \delta y_2 \end{pmatrix}$$

Therefore the characteristic equation for the eigenvalues η can be calculated as

$$\det \begin{pmatrix} \lambda - 3x_\pm^2 - y_\pm^2 - \varepsilon - \eta & -2x_\pm y_\pm - \omega & \varepsilon & 0 \\ -2x_\pm y_\pm + \omega & \lambda - 3y_\pm^2 - x_\pm^2 - \eta & 0 & 0 \\ \varepsilon & 0 & \lambda - 3x_\pm^2 - y_\pm^2 - \varepsilon - \eta & -2x_\pm y_\pm - \omega \\ 0 & 0 & -2x_\pm y_\pm + \omega & \lambda - 3y_\pm^2 - x_\pm^2 - \eta \end{pmatrix} = 0.$$

Fortunately, the characteristic equation for the eigenvalue η can be factorized. Using the identities

$$x_\pm^2 + y_\pm^2 = r_\pm^2 = \lambda - \varepsilon \pm \sqrt{\varepsilon^2 - \omega^2}$$

and

$$\varepsilon\left(x_\pm^2 + 3y_\pm^2\right) = -\varepsilon^2 - \omega^2 \pm (\lambda + \varepsilon)\sqrt{\varepsilon^2 - \omega^2} + 2\varepsilon\lambda,$$

we find the two factors

$$\eta^2 + 2\left(2r_\pm^2 - \lambda\right)\eta + \omega^2 + \lambda^2 - 4\lambda r_\pm^2 + 3r_\pm^4 = 0 \tag{3.1}$$

and

$$\eta^2 + 2\left(2r_\pm^2 + \varepsilon - \lambda\right)\eta - \omega^2 + \lambda^2 - 4\lambda r_\pm^2 + 3r_\pm^4 - 2\varepsilon^2 + 2\varepsilon\lambda \pm 2\left(\lambda + \varepsilon\right)\sqrt{\varepsilon^2 - \omega^2} = 0. \tag{3.2}$$

We remark that both equations are quadratic in η and therefore again deliver either complex conjugated or real and distinct eigenvalues.

Equivalently, the same calculations could be done in the z_+ and z_- coordinates. We then obtain that the first factor (3.1) corresponds to the linearization of the equation for $\dot{z}_+$ and that the second factor (3.2) corresponds to the linearization of the equation for $\dot{z}_-$.

In the following two sections, we want to find the bifurcation points and the unstable dimensions of the inhomogeneous steady states. We will investigate the first and the second factor of the characteristic equation separately.

3.2.1 First factor of the characteristic equation (equation (3.1))

We recall that the first factor of the characteristic equation of the inhomogeneous steady states is given by eq. (3.1), i.e.

$$\eta^2 + 2\left(2r_\pm^2 - \lambda\right)\eta + \omega^2 + \lambda^2 - 4\lambda r_\pm^2 + 3r_\pm^4 = 0.$$

Solving this equation, which is quadratic in η, we obtain

$$\eta_{1,2} = \lambda - 2r_\pm^2 \pm \sqrt{r_\pm^4 - \omega^2}.$$

As we can express $r_\pm$ explicitly in terms of ε, λ and ω, also η only depends on these parameters. As is shown below, the eigenvalues show different kinds of behaviour depending on the ratio of the parameters λ and ω.

Let's start searching for purely imaginary eigenvalues. η can only be purely imaginary if the condition $\lambda = 2r_\pm^2$ is fulfilled. We can rewrite this condition and obtain a value for the coupling parameter ε for which a bifurcation occurs:

$$\begin{aligned} & \lambda = 2r_\pm^2 \\ \iff & \lambda = 2\left(\lambda - \varepsilon \pm \sqrt{\varepsilon^2 - \omega^2}\right). \end{aligned}$$

From this we calculate that there is a Hopf-bifurcation for the "$-$"-branch if the coupling parameter ε is

$$\varepsilon = \tfrac{1}{4}\left(\lambda + 4\tfrac{\omega^2}{\lambda}\right).$$

At this bifurcation point, two complex conjugated eigenvalues cross the imaginary axis transversely and for $\varepsilon > \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$, we have two eigenvalues with negative real part. Therefore, the unstable dimension decreases by two for increasing ε.

There is no eigenvalue-zero bifurcation for $\lambda < \sqrt{3}\omega$. However, for $\lambda > \sqrt{3}\omega$, we have zeros of the eigenvalue η for two values of the coupling parameter ε, namely at

$$\varepsilon = \tfrac{1}{3}\left(2\lambda \mp \sqrt{\lambda^2 - 3\omega^2}\right).$$

For $\sqrt{3}\omega < \lambda < 2\omega$, both zeros lie on the "+"-branch, while for $\lambda > 2\omega$, the first bifurcation lies on the "−"-branch and only the second on the "+"-branch. At the first bifurcation, there is always one real eigenvalue going from negative to positive sign, while at the second bifurcation, the eigenvalue changes sign from positive to negative.

We do not investigate whether these two eigenvalue-zero bifurcations are pitchfork-bifurcations of transcritical bifurcations. They cannot be saddle-node bifurcations because there exist steady states for all $\varepsilon > \omega$ on which the bifurcations occur. Numerical simulations [18] suggest that in both cases pitchfork-bifurcations occur.

Details of this bifurcation analysis can be seen in figures 3.2 and 3.3.

3.2.2 Second factor of the characteristic equation (equation (3.2))

In this subsection, we investigate the second factor of the characteristic equation for the inhomogeneous steady states. Recall that it is given by

$$\eta^2 + 2\left(2r_\pm^2 + \varepsilon - \lambda\right)\eta - \omega^2 + \lambda^2 - 4\lambda r_\pm^2 + 3r_\pm^4 - 2\varepsilon^2 + 2\varepsilon\lambda \pm 2\left(\lambda + \varepsilon\right)\sqrt{\varepsilon^2 - \omega^2} = 0.$$

Again, this is a quadratic equation in η which can be solved explicitly. Writing η only in terms of ε, λ and ω yields

$$\eta = -\lambda + \varepsilon \mp 2\sqrt{\varepsilon^2 - \omega^2} \pm \sqrt{-3\varepsilon^2 - 2\varepsilon\lambda + \lambda^2 + 4\omega^2 + 4\left(\varepsilon^2 - \omega^2\right)}$$

For the analysis of the second factor, we have to consider the cases $\lambda < \omega$ and $\lambda > \omega$ separately.

For $\lambda < \omega$ and $\varepsilon > \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ we only have eigenvalues with negative real part. Note that the pitchfork bifurcation point at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ is also visible in this set of characteristic equations of the inhomogeneous steady states, as a simple eigenvalue zero.

For $\lambda > \omega$ and $\varepsilon > \omega$, there are only negative real parts for the "+"-branch. At $\varepsilon = \omega$, we have a simple eigenvalue zero. Here a saddle-node bifurcation occurs and the "+"- and the "−"-branch meet. On the "−"-branch we have one positive real eigenvalue and one negative real eigenvalue. The latter indicates again the pitchfork-bifurcation at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ with a simple eigenvalue zero.

No Hopf-bifurcations occur for the second factor of the characteristic equation (3.2).

Bifurcation diagrams for the cases $\lambda < \omega$ and $\lambda > \omega$ can be seen in figure 3.4.

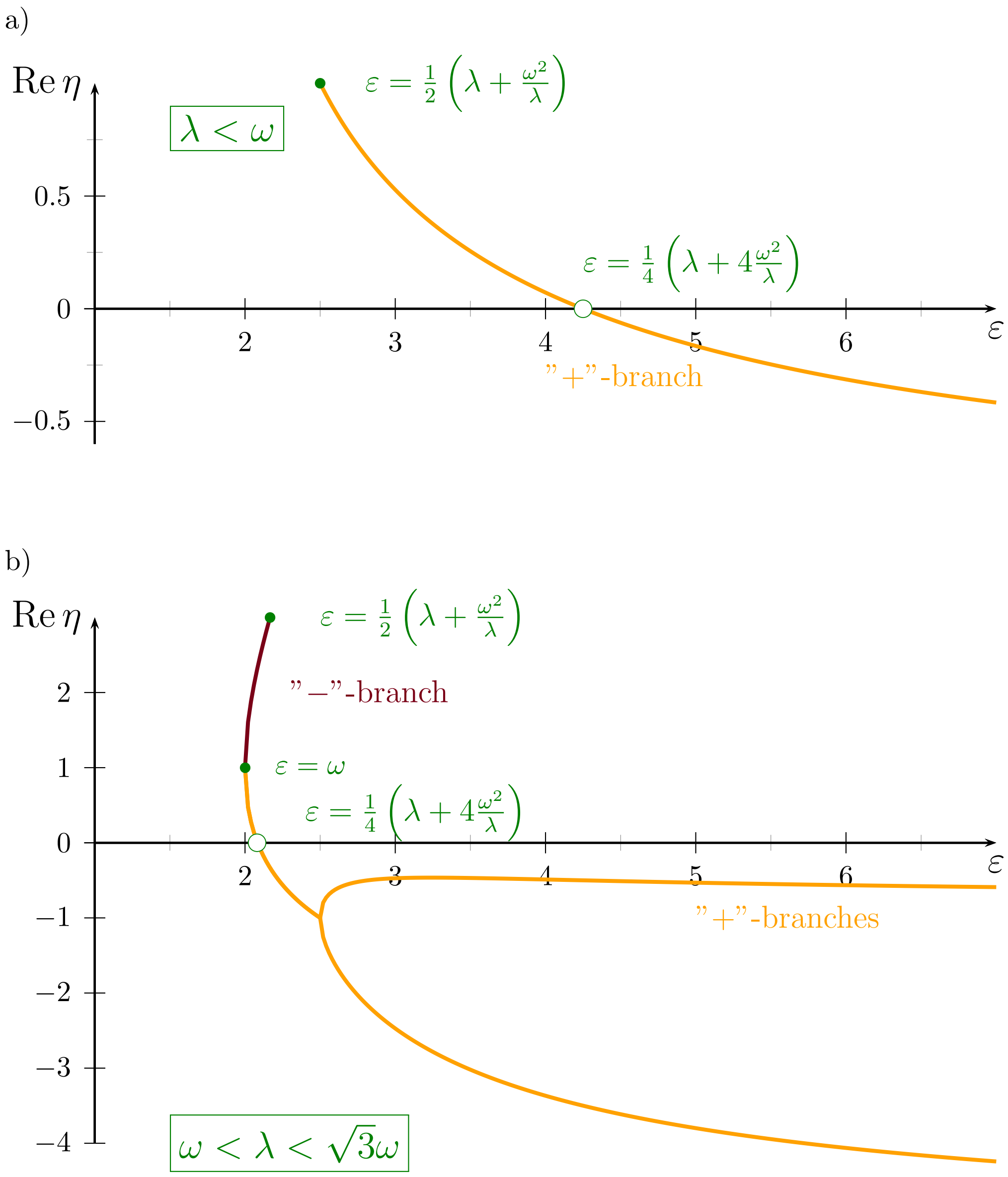
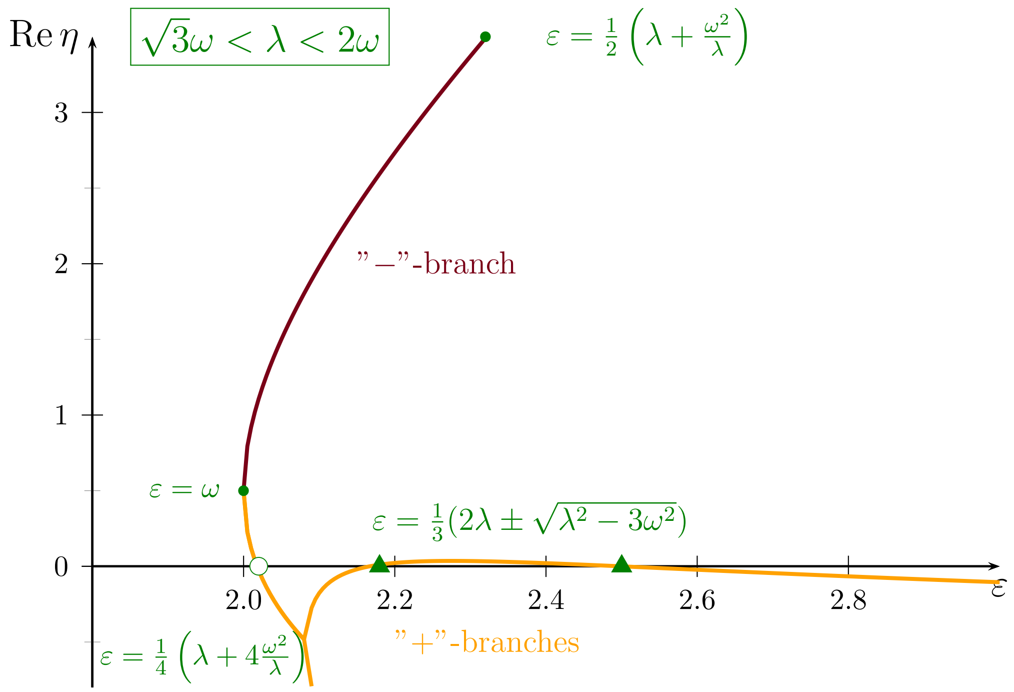


Figure 3.2: Real part of the eigenvalues η of the first factor of the characteristic equation for the inhomogeneous steady states plotted versus the coupling parameter ε. In the figure a) we have $\lambda < \omega$ (here $\lambda = 1$ and $\omega = 2$). There is a Hopf-bifurcation ($\circ$) of the the "+"-branch (orange) at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$. The "−"-branch does not exist for this choice of parameters. In figure b) we have $\omega < \lambda < \sqrt{3}\omega$ (here $\lambda = 3$ and $\omega = 2$). For this choice of parameters also the "−"-branch (dark red) exists for $\omega < \varepsilon < \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. Again there is a Hopf-bifurcation ($\circ$) of the the "+"-branch (orange) at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$.

a)



b)

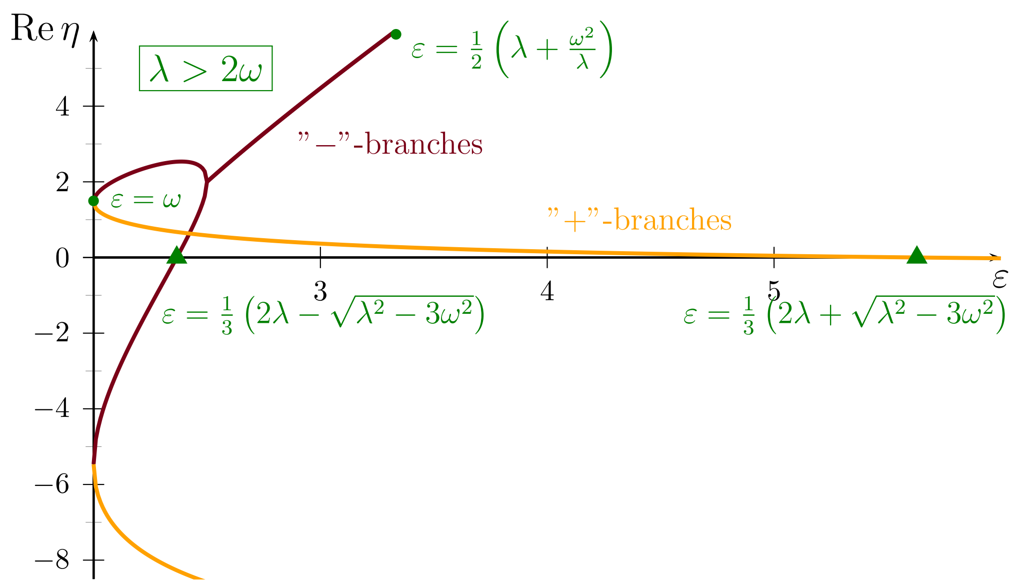
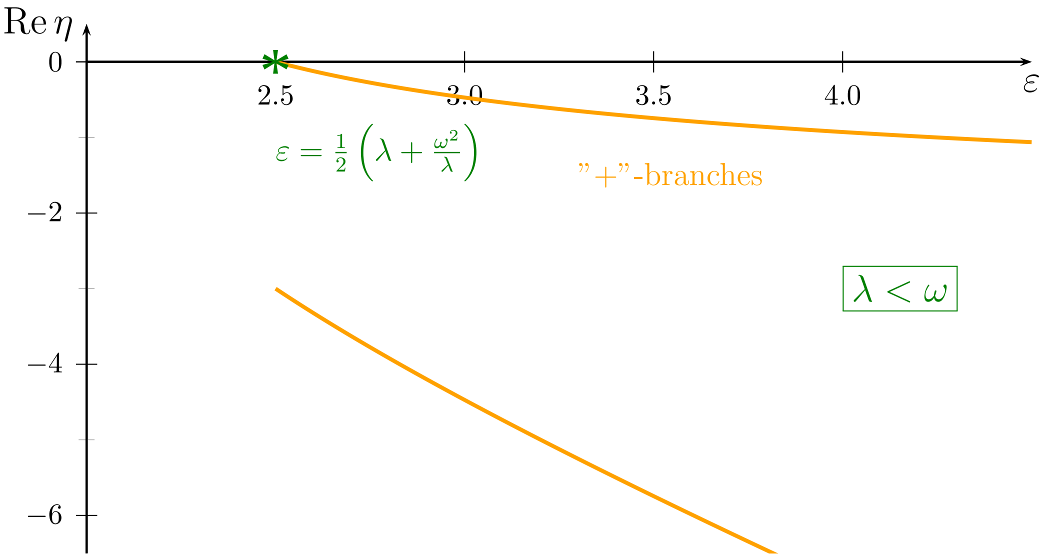


Figure 3.3: Real part of the eigenvalues η of the first factor of the characteristic equation for the inhomogeneous steady states plotted versus the coupling parameter ε. In figure a) $\sqrt{3}\omega < \lambda < 2\omega$ (here $\lambda = 3.5$ and $\omega = 2$). There is a Hopf-bifurcation (○) of the the "+"-branch (orange) at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$. There are also two eigenvalue-zero bifurcations (▲) of the "+"-branch (orange) at $\varepsilon = \frac{1}{3}(2\lambda \pm \sqrt{\lambda^2 - 3\omega^2})$. In figure b) $\lambda > 2\omega$ (here $\lambda = 6$ and $\omega = 2$). There is one eigenvalue-zero bifurcation (▲) of the "+"-branch (orange) at $\varepsilon = \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 + 3\omega^2}\right)$. The other eigenvalue-zero bifurcation (▲) is on the "−"-branch (dark red) and occurs for the coupling parameter $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 + 3\omega^2}\right)$.

a)



b)

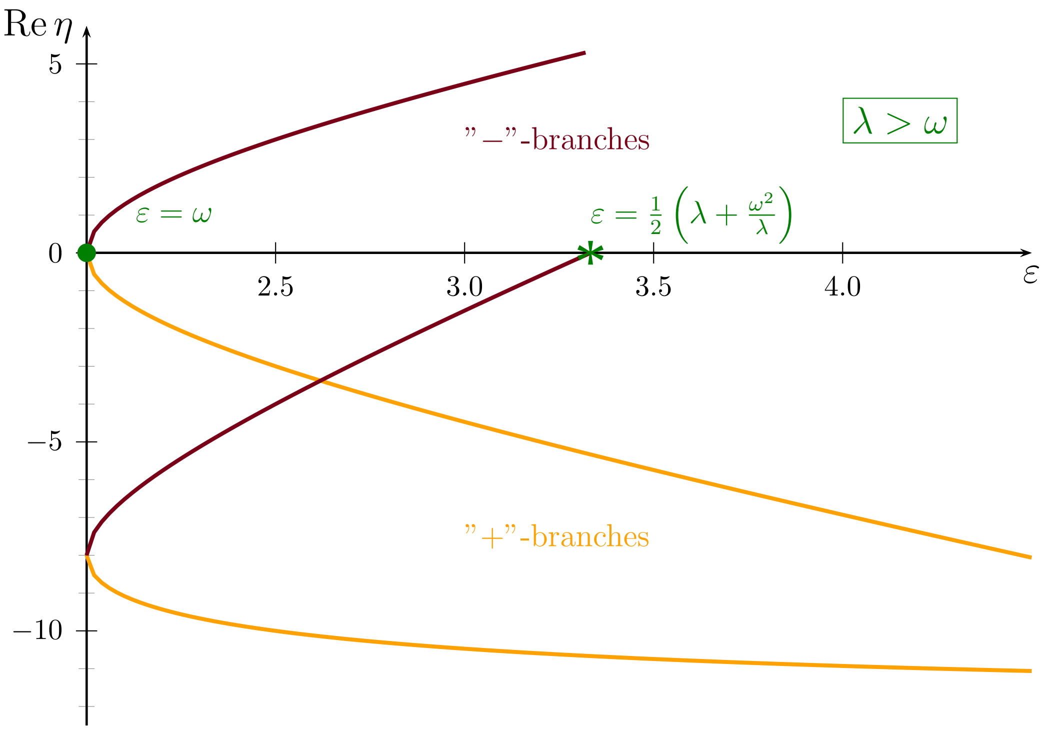


Figure 3.4: Real part of the eigenvalues η of the second factor of the characteristic equation for the inhomogeneous steady states plotted versus the coupling parameter ε. In figure a) $\lambda < \omega$ (here $\lambda = 1$ and $\omega = 2$) the "−"-branch does not exist. There is a pitchfork bifurcation ($*$) at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$, otherwise we only have eigenvalues with negative real part. In figure b) $\lambda > \omega$ (here $\lambda = 6$ and $\omega = 2$). For this choice of parameters also the "−"-branch (dark red) exists for $\omega < \varepsilon < \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. There is a saddle-node bifurcation ($\bullet$) at $\varepsilon = \omega$ and a pitchfork bifurcation ($*$) at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$.

3.3 Conclusion for oscillation death without time delay

Having done this bifurcation analysis, we are now able to determine directly the unstable dimension of the respective steady states, homogeneous as well as inhomogeneous. Note that the total unstable dimension is between zero and four and it is calculated by carefully summing up the respective unstable dimensions obtained in the previous sections. For the conclusion, we separate the four cases $\lambda < \omega$, $\omega < \lambda < \sqrt{3}\omega$, $\sqrt{3}\omega < \lambda < 2\omega$ and $\lambda > 2\omega$, which show significantly different behaviour (see sections 3.1 and 3.2).

Conclusion for $\lambda < \omega$

We start with the trivial fixed point $(z_1, z_2) \equiv (0, 0)$. For $0 < \varepsilon < \lambda$ the unstable dimension is four. At $\varepsilon = \lambda$ there is a Hopf-bifurcation and the unstable dimension is therefore 2 for $\lambda < \varepsilon < \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. At $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$, a pitchfork bifurcation occurs and the equilibrium changes its stability again, from 2 to 3 eigenvalues with strictly positive real part. The inhomogeneous steady states have unstable dimension 2 near $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. For increasing ε, there is another Hopf-bifurcation at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$ and the inhomogeneous steady state stabilizes, that is, the unstable dimension becomes zero.

This leads to $\varepsilon > \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$ as a necessary and sufficient condition for oscillation death.

Conclusion for $\omega < \lambda < \sqrt{3}\omega$

The trivial equilibrium has unstable dimension 4 for $0 < \varepsilon < \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. At $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$, a pitchfork bifurcation occurs and the unstable dimension reduces to 3. The inhomogeneous steady states have unstable dimension 3 near $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. We then follow first the "−"-branch, for decreasing ε. At the saddle-node bifurcation at $\varepsilon = \omega$ the stability of the fixed point reduces to 2. But now, we follow the "+"-branch for increasing ε, starting at $\varepsilon = \omega$. For increasing ε, there is a Hopf-bifurcation at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$ and the inhomogeneous steady state stabilizes, that is, the unstable dimension becomes zero.

Therefore we conclude that, also for $\omega < \lambda < \sqrt{3}\omega$, oscillation death occurs on the "+"-branch for $\varepsilon > \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$.

Conclusion for $\sqrt{3}\omega < \lambda < 2\omega$

For this parameter choice, all the bifurcations from the previous case also occur. However there is an additional bifurcation at $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$ with eigenvalue zero on the "+"-branch which increases the unstable dimension by 1 again. Therefore, oscillation death does occur at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$, but further increasing ε past $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$ destroys oscillation death again! Nevertheless, at the last bifurcation point at $\varepsilon = \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$, the unstable dimension reduces again to zero, leading to oscillation death.

Thus, oscillation death actually occurs on two different intervals on the "+"-branch: Once for ε between $\frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$ and $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$, and also once for $\varepsilon > \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$.

Conclusion for $\lambda > 2\omega$

Like in the two previous cases, the trivial equilibrium has unstable dimension 4 for $0 < \varepsilon < \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$ and again, at $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$, a pitchfork bifurcation occurs where the unstable dimension reduces to 3. Starting to follow the "−" branch, for decreasing ε, we find first an unstable dimension 3 near $\varepsilon = \frac{1}{2}\left(\lambda + \frac{\omega^2}{\lambda}\right)$. Further decreasing ε, there is a bifurcation at $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$ with eigenvalue zero, which decreases the unstable dimension once more. The saddle-node bifurcation at $\varepsilon = \omega$ reduces the unstable dimension by one again, which is then 1. Next, we follow the "+" branch for increasing ε, starting at $\varepsilon = \omega$. For increasing ε, there is another bifurcation with a simple eigenvalue zero at $\varepsilon = \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$ and the inhomogeneous steady state stabilizes, that is, the unstable dimension becomes zero.

In conclusion, oscillation death occurs on the "+"-branch if ε is chosen to be greater than $\frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$.

Summary

We conclude that for all cases oscillation death occurs for all parameter choices of λ and ω if the coupling parameter ε is chosen strong enough. We emphasize that there are different bifurcation mechanisms leading to oscillation death, depending strongly on the parameters λ and ω. If $\lambda < \sqrt{3}\omega$, oscillation death occurs for $\varepsilon > \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$. If $\sqrt{3}\omega < \lambda < 2\omega$, oscillation death occurs on two different intervals: once for $\frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right) < \varepsilon < \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$ and once for $\varepsilon > \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$. If $\lambda > 2\omega$, oscillation death occurs for $\varepsilon > \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$.

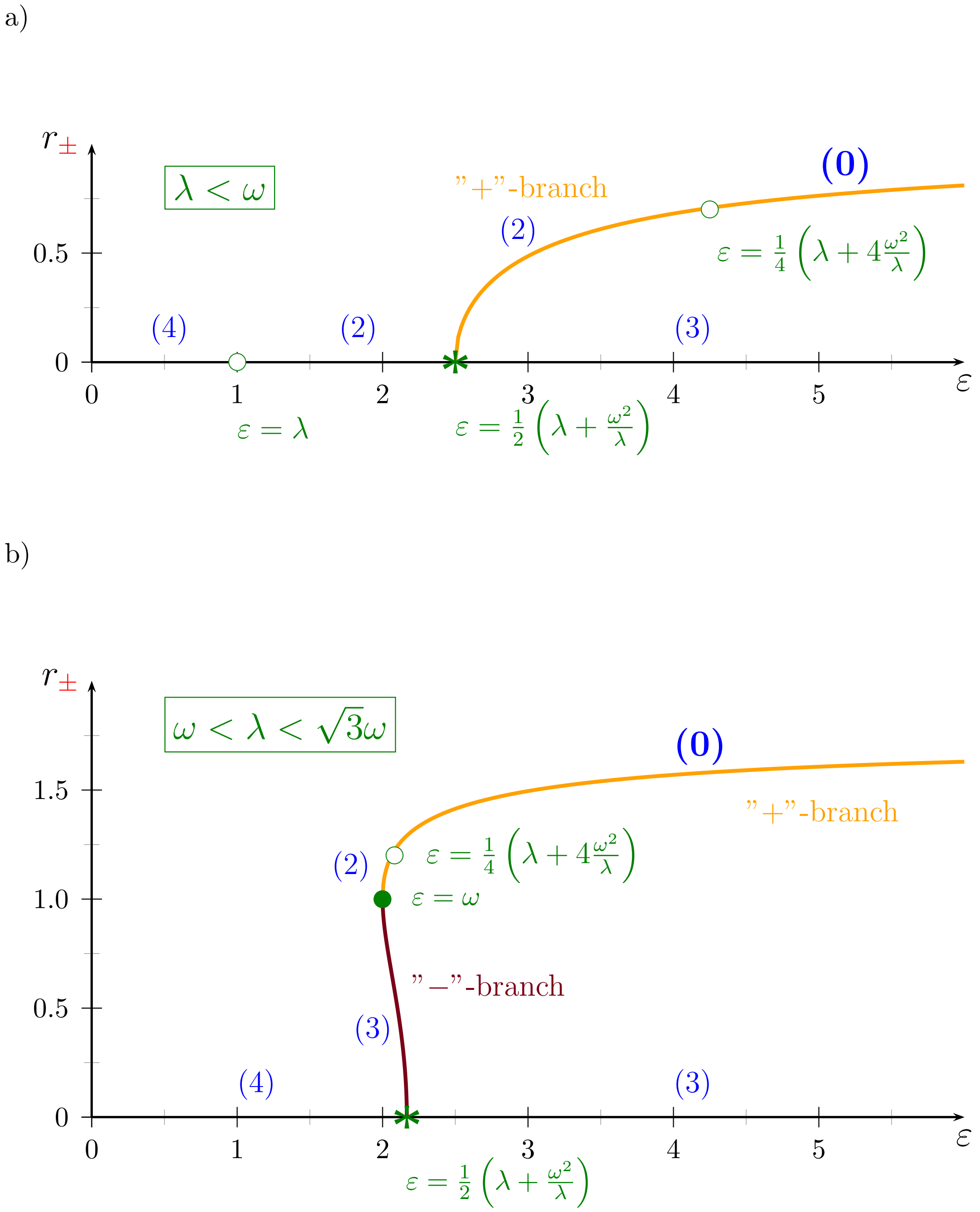
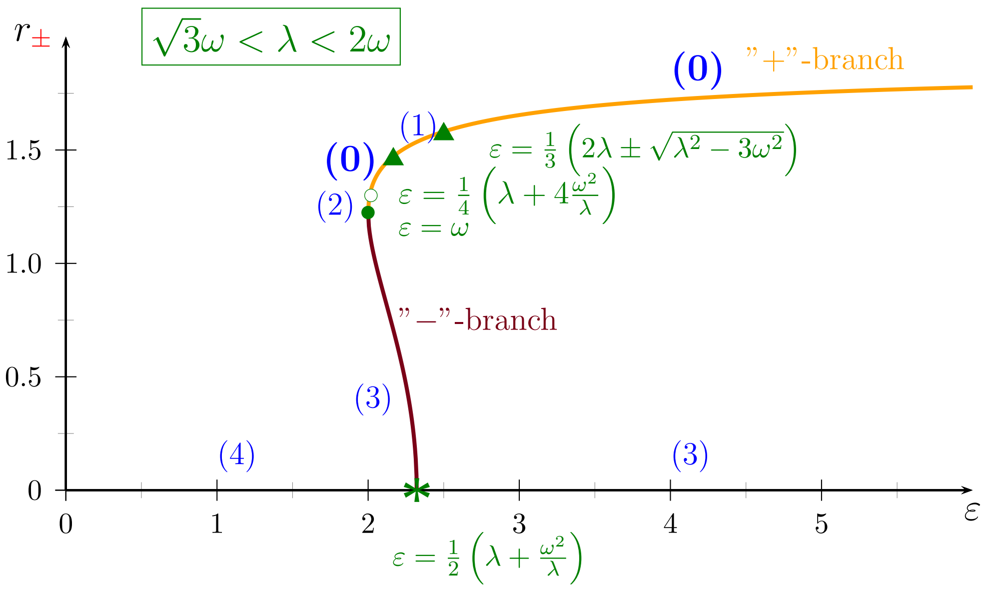


Figure 3.5: Conclusion of chapter 3 for the parameters $\lambda < \omega$ (here $\lambda = 1$ and $omega = 2$ in figure a) and $\omega < \lambda < \sqrt{3}\omega$ (here $\lambda = 6$ and $omega = 2$ in figure b). For detailed explanations see section 3.3. The numbers in parentheses (blue) denote the unstable dimensions of the respective steady states. There are Hopf-($\circ$), pitchfork-($*$) and saddle-node-($\bullet$) bifurcations.

a)



b)

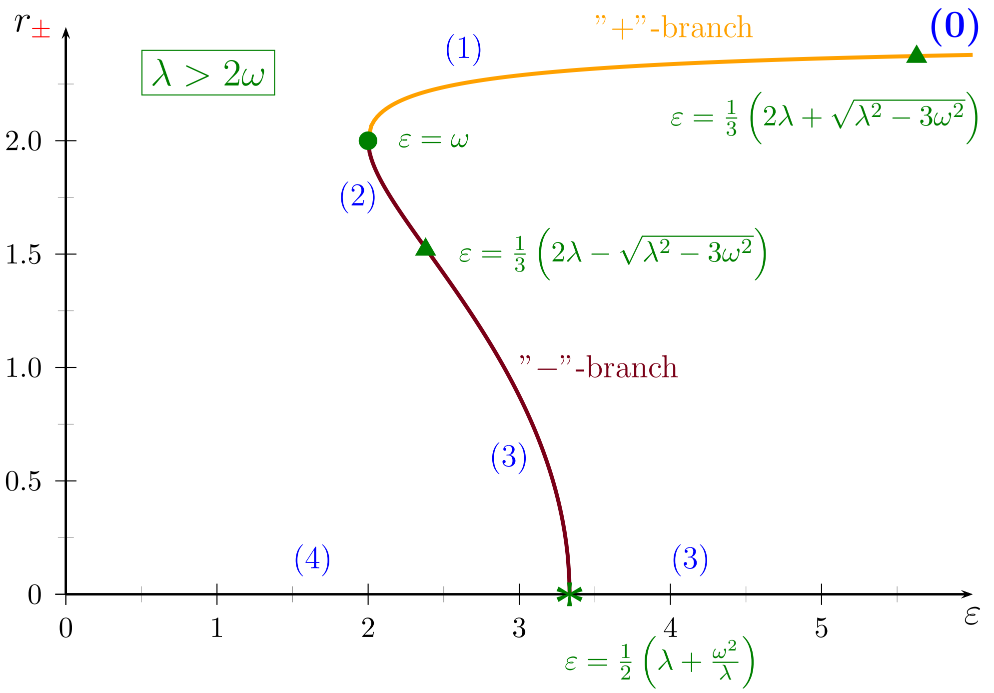


Figure 3.6: Conclusion of chapter 3 for the parameters $\sqrt{3} < \lambda < 2\omega$ (here $\lambda = 3.5$ and $\omega = 2$ in figure a) and $\lambda > 2\omega$ (here $\lambda = 6$ and $\omega = 2$ in figure b). For detailed explanations see section 3.3. The numbers in parentheses (blue) denote the unstable dimensions of the respective steady states. There are Hopf-($\circ$), pitchfork-($*$) and saddle-node-($\bullet$) bifurcations. There are also eigenvalue-zero bifurcations (▲), which can be either transcritical or pitchfork bifurcations.

4 Oscillation death with time delay

In this chapter, we want to find oscillation death also in systems with delay in the coupling term. In our model, we include the final signal propagation time between the two oscillators. We consider again Stuart-Landau oscillators,

$$\begin{aligned} \dot{z}_1(t) &= \left(\lambda + \mathrm{i}\omega - |z_1(t)|^2\right) z_1(t) + \varepsilon\left(\operatorname{Re} z_2(t-\tau) - \operatorname{Re} z_1(t)\right) \\ \dot{z}_2(t) &= \left(\lambda + \mathrm{i}\omega - |z_2(t)|^2\right) z_2(t) + \varepsilon\left(\operatorname{Re} z_1(t-\tau) - \operatorname{Re} z_2(t)\right), \end{aligned} \tag{4.1}$$

where $\tau > 0$ time delay. Some of the calculations done in the previous chapters 2 and 3 for the system with instantaneous coupling can also be used for the analysis of this system: The steady states, homogeneous as well as inhomogeneous, are of course the same, and also, eigenvalues zero do not change for any time delay τ.

It is worthwhile to remark that the inclusion of time delay into the modelling substantially alters the structure of the considered system: system (2.2) was four-dimensional, but we deal with infinitely many dimensions in system (4.1), see [5].

Therefore, the unstable dimension of a steady state is not restricted to 4 any more, but it can take arbitrary positive integer value. It follows that the characteristic equations, which are no longer polynomials, must be evaluated with great care and we keep in mind the infinitely many solutions they deliver.

Nevertheless, it is advantageous that we have already investigated the characteristic equations in detail for the instantaneous case. With this knowledge, we can continue our calculations analytically and obtain a deep insight into the involved mechanisms.

From the previous chapter, it is reasonable to expect that oscillation death in the delayed case also appears on the "+"-branch of the inhomogeneous steady states.

For a better overview and understanding, we first explain our procedure of detecting oscillation death in detail:

First, the system with time-delay is linearized around the "+"-branch of the inhomogeneous steady states. From the linearized system, we are able to calculate the characteristic equation, which again can be factorized into two factors which can be handled separately. Compared to the characteristic equation of the instantaneous system, these factors each contain additional exponential terms of the eigenvalues η and therefore provide infinitely many solutions.

The stability of the steady states can only change if either two complex conjugated eigenvalues cross the imaginary axis or if there is an eigenvalue-zero bifurcation. Eigenvalues which are zero are already known.

Therefore, it is our second step to search for purely imaginary eigenvalues $\eta = \mathrm{i}\Omega$. Ω only depends on ϵ, λ and ω and *not* on the time delay τ.

As is seen below, purely, imaginary, non-zero eigenvalues can be tracked with help of the $\tau(\varepsilon)$-curves parametrized by Ω ($\Omega \neq 0$).

Then we can plot both the zero eigenvalues and the purely imaginary eigenvalues in the $\varepsilon - \tau$-plane. Finally, we are able to count the unstable dimensions, starting from $\tau = 0$. We emphasize that only the analytical care from the two previous chapters allows us to draw conclusions from these curves without any numerical simulations.

4.1 Characteristic equation of the delayed system

We start by linearizing the system (4.1) along the "+"-branch of the inhomogeneous steady states. From this, we can calculate the characteristic equation, which is given by

$$\det \begin{pmatrix} \lambda - 3x_+^2 - y_+^2 - \varepsilon - \eta & -2x_+y_+ - \omega & \varepsilon e^{-\eta\tau} & 0 \\ -2x_+y_+ + \omega & \lambda - 3y_+^2 - x_+^2 - \eta & 0 & 0 \\ \varepsilon e^{-\eta\tau} & 0 & \lambda - 3x_+^2 - y_+^2 - \varepsilon - \eta & -2x_+y_+ - \omega \\ 0 & 0 & -2x_+y_+ + \omega & \lambda - 3y_+^2 - x_+^2 - \eta \end{pmatrix} = 0.$$

We find two factors of the form

$$\eta^2 + a\eta + b_{1,2}\,\eta e^{-\eta\tau} + c + d_{1,2}\,e^{-\eta\tau} = 0 \tag{4.2}$$

where the coefficients are given by

$$\begin{aligned} a &= 4r_+^2 - 2\lambda + \varepsilon \\ b_{1,2} &= \pm\varepsilon \\ c &= r_+^2\left(3r_+^2 - 4\lambda\right) + \lambda^2 + \varepsilon\lambda - \varepsilon^2 + (\lambda + \varepsilon)\sqrt{\varepsilon^2 - \omega^2} \\ d_{1,2} &= \pm\left(\varepsilon\lambda - \varepsilon^2 - \omega^2 + (\lambda + \varepsilon)\sqrt{\varepsilon^2 - \omega^2}\right) \end{aligned}$$

We recall that the radius r_+ is given by

$$r_+ = \sqrt{\lambda - \varepsilon + \sqrt{\varepsilon^2 - \omega^2}}.$$

Therefore, each of the coefficients a, $b_{1,2}$, c and $d_{1,2}$ only depends on ε, λ and ω, though we have chosen not to write the explicit dependence for brevity.

In the following section, we will carry out the calculation for both factors simultaneously. This is possible because, on the one hand, they have the same form, and on the other hand, they are independent of each other.

4.2 Calculation of the purely imaginary eigenvalues

We remind ourselves that eigenvalues zero do not change due to the additional time delay in the coupling term. However, we have to calculate the purely imaginary eigenvalues $\eta = \mathrm{i}\Omega$, $\Omega \neq 0$ for a renewed bifurcation analysis.

$\eta = \mathrm{i}\Omega$ can be plugged into the characteristic equation (4.2) which becomes

$$0 = -\Omega^2 + a\,\mathrm{i}\Omega + b_{1,2}\,\mathrm{i}\Omega e^{-\mathrm{i}\Omega\tau} + c + d_{1,2}\,e^{-\mathrm{i}\Omega\tau}.$$

Similar to [6], we separate real and imaginary part of this equation, yielding:

$$\begin{aligned} 0 &= -\Omega^2 + b_{1,2}\,\Omega\sin(\Omega\tau) + c + d_{1,2}\,\cos(\Omega\tau) \\ 0 &= a\,\Omega + b_{1,2}\,\Omega\cos(\Omega\tau) - d_{1,2}\,\sin(\Omega\tau)\,. \end{aligned}$$

We now want to eliminate τ, aiming for an equation which directly gives Ω. Therefore we carry out the following calculation:

$$\left(-\Omega^2 + c\right)^2 = \left(-b_{1,2}\,\Omega \sin\left(\Omega\tau\right) - d_{1,2}\,\cos\left(\Omega\tau\right)\right)^2 \tag{4.3}$$

$$a^2\,\Omega^2 = \left(-b_{1,2}\,\Omega \sin\left(\Omega\tau\right) + d_{1,2}\,\cos\left(\Omega\tau\right)\right)^2. \tag{4.4}$$

Adding the last two equations (4.3) and (4.4) and using the trigonometric identity $\cos^2\left(\Omega\tau\right) + \sin^2\left(\Omega\tau\right) = 1$ leads to the following expression

$$\Omega^4 + \Omega^2\left(a^2 - b_{1,2}^2 - 2c\right) + c^2 - d_{1,2}^2 = 0,$$

which is a biquadratic equation in Ω and can therefore be solved explicitly. Hence we know that the parametrization Ω of the purely imaginary eigenvalues, depending on ε, ω, and λ, is given by

$$\Omega^2 = \tfrac{1}{2}\left(\left(-a^2 + b_{1,2}^2 + 2c\right) \pm \sqrt{\left(-a^2 + b_{1,2}^2 + 2c\right)^2 - 4\left(c^2 - d_{1,2}^2\right)}\right).$$

Note that the expression for Ω only yields real values if Ω^2 is a real positive number. Therefore the condition for the existence of Ω results in

$$\left(-a^2 + b_{1,2}^2 + 2c\right)^2 > 4\left(c^2 - d_{1,2}^2\right).$$

Further note that the existence of Ω does not depend on the first or the second factor, because $b_{1,2}$ and $d_{1,2}$ only differ in sign, which is eliminated in the expression for Ω.

What does it mean for the stability analysis if there exists no real Ω?

If no real Ω exist for a certain parameter region of ε (and we will see that this situation does indeed occurs), then it means that for these coupling parameters the stability calculated in chapter 3 does not change for arbitrary time delay τ.

Now assume that $\Omega \neq 0$ is real. Then multiply equation (4.3) by $d_{1,2}$ and equation (4.4) by $b_{1,2}\Omega$ and add the two new equations. This procedure yields the following equation in Ω and τ:

$$\Omega^2\left(ab_{1,2} - d_{1,2}\right) + cd_{1,2} + \cos\left(\Omega\tau\right)\left(b_{1,2}^2\,\Omega^2 + d_{1,2}^2\right) = 0$$

Note that τ only occurs once in this equation and can easily be extracted from it. Finally, we get the wanted $\tau\left(\varepsilon\right)$-curve

$$\tau_{1,2}\left(\varepsilon\right) = \left(\arccos\left(\frac{\Omega^2\left(d_{1,2} - ab_{1,2}\right) - cd_{1,2}}{b_{1,2}^2\,\Omega^2 + d_{1,2}^2}\right) + 2\pi n\right)\Big/\,\Omega. \tag{4.5}$$

We remind ourselves once again that Ω as well as τ and the coefficients a, $b_{1,2}$, c and $d_{1,2}$ only depend on the coupling parameter ε and the two system parameters λ and ω. Also we remark that there are indeed infinitely many curves, one for each nonnegative integer n.

Although at first sight, this last equation of the $\tau\left(\varepsilon\right)$-curve looks very difficult, it is possible to do a rough sketch of its behaviour before even turning on the computer. First we identify its main structure, with is given by the well known arccos, extended to all possible solutions by the term $2\pi n$. This structure is only slightly distorted by the argument of the arccos-function as well as the division by Ω, which was nonzero by assumption. Second we notice that the first and the second factor are not so different: We remember that $b_{1,2}$ and $d_{1,2}$ only differ in sign and that Ω does not depend on the respective factor at all. In the denominator, the different signs cancel due to the sqare. The numerator only differs in one sign.

Writing $b = b_1 = -b_2$ and $d = d_1 = -d_2$, it seems therefore more appropriate to rewrite equation (4.5) such that the structure becomes visible:

$$\tau_{1,2(n)}(\varepsilon) = \left(\arccos\left(\pm\frac{\Omega^2(d-ab)-cd}{b^2\,\Omega^2+d^2}\right)\right)\Big/\,\Omega + \frac{2\pi n}{\Omega}.$$

4.3 Stability analysis

After all the calculations carried out in the previous sections, we are now able to plot the $\tau(\varepsilon)$-curves for different parameter values λ and ω. These curves give us the stability changes via Hopf-bifurcation. Each time we cross one of these curves by changing the parameter τ or ε, the stability of the inhomogeneous steady state changes by 2. Stability can also change via eigenvalue-zero bifurcations. As eigenvalues zero do not change because of the additional time delay, we can draw the respective $\tau(\varepsilon)$-curves without any additional investigation. They are just vertical lines at the respective ε-values calculated in section 3.2.

Stability analysis for $\lambda < \sqrt{3}\omega$

The $\tau(\varepsilon)$-curves coming from the both factors of the characteristic equation are drawn in figure 4.1. The unstable dimensions are known for $\tau = 0$, as we did detailed investigations in chapter 3. Furthermore, we know that the stability can only change either via Hopf-bifurcation or via an eigenvalue-zero bifurcation (which does not occur in this case). Note that the Hopf-bifurcation point at $\varepsilon = \frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$ can clearly be identified for $\tau = 0$. We know already, from chapter 3, on which side of the curve the unstable dimension is larger by two. Therefore we can identify easily the region with has unstable dimension zero. Here oscillation death occurs. There exists a threshold of the coupling parameter ε, given by the condition

$$\left(-a^2+b^2+2c\right)^2 < 4\left(c^2-d^2\right).$$

If this condition is fulfilled, then oscillation death actually occurs for all time delays τ. If the condition is not fulfilled, on the contrary, then the occurrence of oscillation death depends strongly on the time delay. We remark that for a given time delay τ the critical ε-value where the stabilizing Hopf-bifurcation occurs can as well decrease as increase.

Stability analysis for $\sqrt{3}\omega < \lambda < 2\omega$

We have the same Hopf-bifurcation curves as in the previous case, in a different ratio of course. Again, we can identify easily where the stable region lies. Additionally, the first factor of the characteristic equation gives two eigenvalue-zero bifurcations at $\varepsilon = \frac{1}{3}\left(2\lambda \pm \sqrt{\lambda^2 - 3\omega^2}\right)$, where the unstable dimension changes by one. They are described by the vertical lines. Between those two bifurcation lines, we also have Hopf-curves, but they can never decrease the unstable dimension to zero. This is due to the fact that the unstable dimension is 1 for $\tau = 0$ and the Hopf-curves can only decrease or increase the stability by two. This means that oscillation death can never occur between $\varepsilon = \frac{1}{3}\left(2\lambda - \sqrt{\lambda^2 - 3\omega^2}\right)$ and $\varepsilon =$

$\frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$. But if ε is chosen greater than $\frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$, we will again find oscillation death arbitrary time delays τ.

Stability analysis for $\lambda > 2\omega$

In this case, we only find one eigenvalue-zero bifurcation at $\varepsilon = \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$, because the other bifurcation wandered to the "−"-branch. Consequently the only Hopf-bifurcations on the "+"-branch are asymptotic to this eigenvalue-zero bifurcation. If on the one hand, $\varepsilon < \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$, no oscillation death can occur at all. But if on the other hand, $\varepsilon > \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$, then oscillation death occurs for all time delays τ.

4.4 Conclusion for oscillation death with time delay

For ratios of the parameters λ and ω, we are able to find regions of oscillation death of the coupling parameter ε. We have found that the region in which oscillation death occurs varies slightly from the instantaneous case if the stabilizing bifurcation is a Hopf-bifurcation. The threshold value for ε can both decrease and increase. If the stabilizing bifurcation is an eigenvalue-zero bifurcation, however, the ε-region of oscillation death does not change at all.

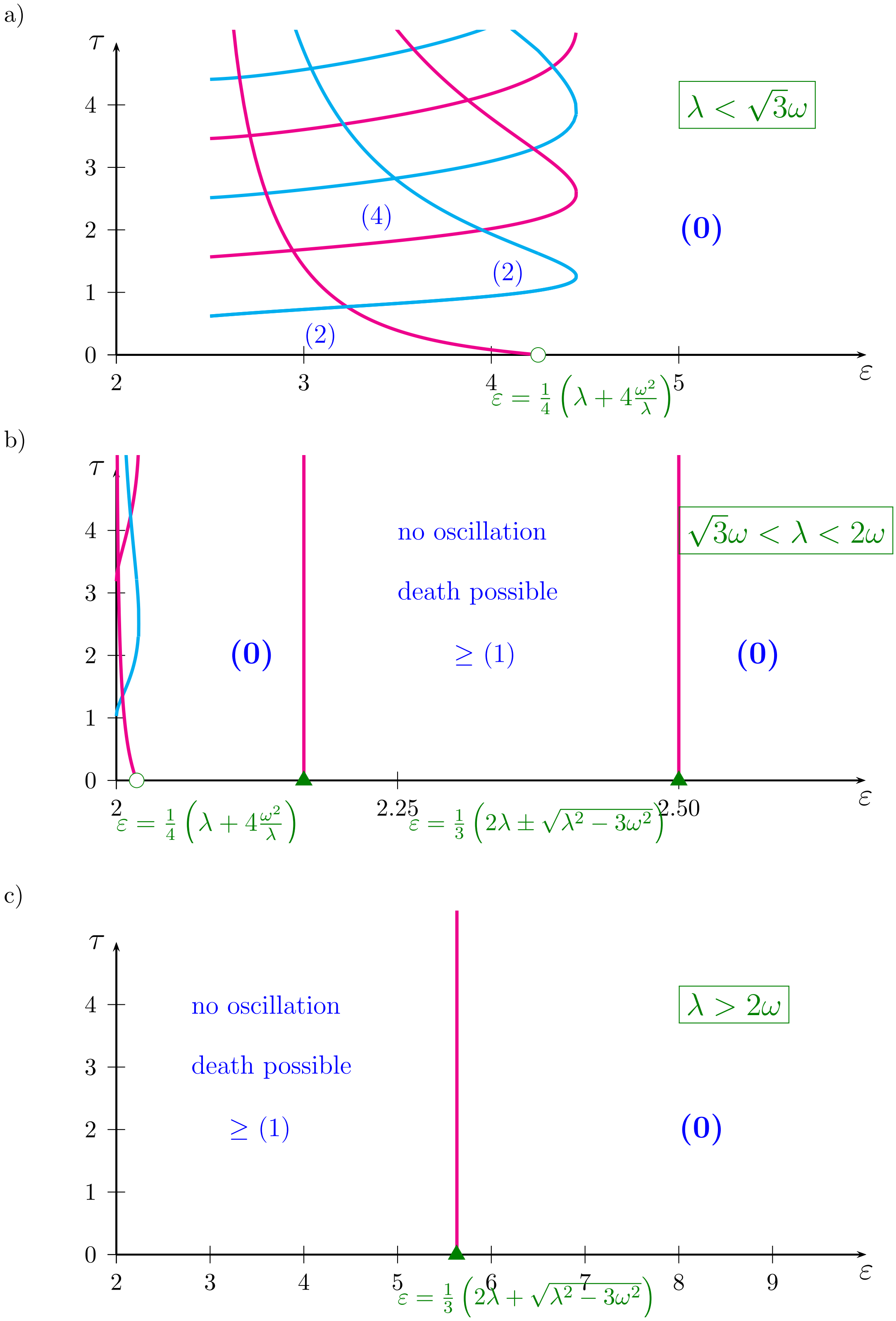


Figure 4.1: Conclusion for oscillation death with time delay. In figure a), we have $\lambda = 1$, in figure b), $\lambda = 3.5$, and in figure c) we have $\lambda = 6$. In all figures, λ is chosen to be 2. The pink curves represent the first factor and the blue curves the second factor of the characteristic equation with time delay. The numbers in parentheses denote the unstable dimension of the inhomogeneous steady states.

5 Conclusion and Discussion

We studied two coupled Stuart-Landau oscillators. The coupling was chosen in such a way that it breaks the rotational symmetry of the single oscillators. Due to this coupling term, new isolated inhomogeneous steady states with non-zero amplitude can be found. We considered both instantaneous and time-delayed coupling. For the instantaneous case, we did a detailed bifurcation analysis of both the homogeneous steady state and the coupling-induced inhomogeneous steady states. In the next step it was possible to extend the bifurcation analysis also to the time-delayed case.

In conclusion, we found that oscillation death occurs both in the instantaneous and the time-delayed case if the coupling parameter ε is chosen appropriately. We emphasize that the bifurcation mechanisms leading to oscillation death depend strongly on the parameters λ and ω where the eigenvalues of the single Stuart-Landau oscillators are given by $\lambda \pm \mathrm{i}\omega$.

For the *instantaneous coupling*, we have found the following results: If $\lambda < \sqrt{3}\omega$, oscillation death occurs if the coupling parameter ε is chosen to be greater than $\frac{1}{4}\left(\lambda + 4\frac{\omega^2}{\lambda}\right)$. The bifurcation leading to the stabilization of the inhomogeneous steady states is a Hopf-bifurcation. The same mechanism also occurs for $\sqrt{3}\omega < \lambda < 2\omega$. However, in this case, further increasing the coupling parameter ε first destabilizes and then again stabilizes the steady states. Both bifurcations have a single eigenvalue zero and they occur at $\varepsilon = \frac{1}{3}\left(2\lambda \pm \sqrt{\lambda^2 - 3\omega^2}\right)$. In the next case, which is $\lambda > 2\omega$, oscillation death occurs for $\varepsilon > \frac{1}{3}\left(2\lambda + \sqrt{\lambda^2 - 3\omega^2}\right)$. The stabilizing bifurcation is also an eigenvalue-zero bifurcation.

For all *time delays* τ and all parameters λ and ω, we are able to find oscillation death if the coupling parameter ε can be chosen accordingly. We have found that the region in which oscillation death occurs only varies slightly from the instantaneous case.

Further investigations of oscillation death of coupled Stuart-Landau oscillators could concentrate on more general forms of coupling and on non-identical oscillators. Since this thesis reveals that there are rich dynamics due to the symmetry-breaking coupling, it would be interesting to know if there are any other bifurcation mechanisms leading to oscillation death.

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Selbstständigkeitserklärung

Hiermit bestätige ich, Isabelle Schneider, dass ich die vorgelegte Bachelorarbeit mit dem

Thema

Oscillation suppression
in nonlinear coupled oscillators
with and without time delay

selbstständig angefertigt und nur die erwähnten Quellen und Hilfen verwendet habe.

Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

Berlin, den 12. April 2013