Master Thesis

Equivariant Pyragas control

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Abstract

In this thesis we show how the concept of Pyragas control can be adapted to equivariant dynamical systems near Hopf bifurcation. We introduce a control term which uses the spatio-temporal symmetry of the unstable periodic orbit. We give explicit necessary and sufficient conditions for the stabilization of the periodic orbit near equivariant Hopf bifurcation. For the remaining eigenvalues, we can find necessary conditions for the stabilization. We also give two variations of the original control scheme, aiming at a larger control region. In a detailed case study, we show how to apply equivariant Pyragas control.

Contents

Chapter 1

Introduction

In 1992, Pyragas [1] suggested a new method for the stabilization of unstable periodic orbits. This method, which is now known as *Pyragas control*, is a time-delayed feedback control. It is non-invasive on the periodic orbit itself, i.e. the control vanishes on the periodic orbit. This implies that the trajectory of the controlled periodic orbit is not altered due to Pyragas control, in contrast to its (linear) stability.

For example, if $z_*(t)$ is an unstable periodic solution of the dynamical system $\dot{z} = F(z)$, $z \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n \times n}$ is a control matrix, consider the following form of control [1]:

$$\dot{z}(t) = F(z(t)) + B\left(-z(t) + z(t-\tau)\right) \tag{1.1}$$

The periodic orbit $z_*(t)$ also solves the controlled system (1.1) using a time delay $\tau = np$, i.e. an integer multiple n of the minimal period p of $z_*(t)$. Then a stabilization of the unchanged periodic orbit might be possible if the control matrix B is chosen suitably.

The main advantage of Pyragas control is that it needs no knowledge of the system, specifically we do not need to know the periodic orbit. In the past 20 years, more than 1500 publications concerning Pyragas control have been published.

Fiedler et al. have proven that Pyragas control succeeds in dynamical systems near Hopf bifurcation [2, 3, 4].

We now want to study the effects of Pyragas control on periodic orbits with spatio-temporal symmetry emerging from equivariant Hopf bifurcation. Therefore, we consider G-equivariant dynamical systems near Hopf bifurcation, where G is the symmetry group.

Throughout this thesis, we consider a compact Lie group G which acts orthogonally on $\mathbb{R}^n$ by a linear representation, i.e. there is a group homomorphism

$$\begin{aligned} \rho\colon\ G &\to O(n) \\ g &\mapsto \rho(g). \end{aligned}$$

In a G-equivariant system $\dot{z} = F(z)$ with a linear group action $z \mapsto \rho(g)z$, $g \in G$, we find that $\rho(g)z(t)$ is a solution whenever $z(t)$ is a solution, for all elements g of the equivariance group G. We often abbreviate $gz := \rho(g)z$.

Following Fiedler [5], we describe the symmetry of a periodic orbit $z_*(t)$ of a G-equivariant system $\dot{z} = F(z)$ by triplets (H, K, Θ). Here H is a subgroup of G which leaves the periodic orbit $\{\, z_*(t) |\, t \in \mathbb{R}\}$ fixed *as a set*, while $K \leq H \leq G$ leaves $z_*(t)$ fixed for each t *pointwise*. The group homomorphism $\Theta \colon H \to S^1 = \mathbb{R}/\mathbb{Z}$ is defined uniquely by time shift

$$hz(t) = z(t + \Theta(h)\, p), \tag{1.2}$$

for all t. By definition, K is the kernel of Θ, $\Theta(K) = 0$. Θ is well-defined by the homomorphism theorem. In local settings this construction has first been introduced by Golubitsky and Stewart; see for example [6]. We call (1.2) the *spatio-temporal symmetry* of the periodic orbit.

The equivariant Hopf bifurcation theorem [6] (see appendix) states under which conditions periodic orbits with spatio-temporal symmetries such as (1.2) bifurcate from equilibria. It gives us a setting in which the effects of time-delayed feedback control such as Pyragas control can be analytically analysed.

The main question of this master thesis is the following: *How should one adapt the idea of delayed feedback control to selectively stabilize periodic orbits of prescribed symmetry type* (H, K, Θ)*?* In short: How can we achieve non-invasive but pattern-selective feedback stabilization? This question arised in [7], where a partial answer has been given.

In this present work we now attempt a more general answer: We show how Pyragas control can be adapted to equivariant dynamical systems and that the control indeed succeeds for systems near Hopf bifurcation. We discuss several possible control schemes which arise due to the different ways of describing the spatio-temporal symmetry of the periodic orbit.

We previously studied Pyragas control of equivariant systems near Hopf bifurcation [7, 8] in a system of three coupled Stuart-Landau oscillators. Postlethwaite et al. [9] considered equivariant systems near Hopf bifurcation in the center manifold. Additionally to [9], we also consider the remaining eigenvalues, possibly with positive real part, as well as two variations of the control scheme. Furthermore, we are now able to state explicit stabilization regions.

This thesis is structured as follows: We state and prove our main new theorems for Pyragas control near equivariant Hopf bifurcation in chapter 2, giving precise necessary and sufficient conditions for the stabilization of unstable periodic orbits. In chapter 3, we discuss two possible variations of the original control scheme and compare them to our original control scheme. The simplest interesting example – three symmetrically coupled Stuart-Landau oscillators – is discussed in a detailed case study in chapter 4, extending significantly the former results in [7, 8]. In chapter 5, we summarize our results and methods, compare them to previous results and discuss open questions concerning equivariant Pyragas control.

Chapter 2

Equivariant Pyragas control

As an answer to the main question of this thesis, we present in this chapter what we call *equivariant Pyragas control:*

$$\dot{z}(t) = F(\lambda, z(t)) + b\,(-z(t) + hz(t - \Theta(h)\,p)) \tag{2.1}$$

Here $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ is a bifurcation parameter. In this chapter, $b \in \mathbb{C}$ is a complex control parameter. Instead, we can also use a complex control matrix $B \in \mathbb{C}^{n\times n}$ to achieve larger control regions. The effects of linearly transformed feedback will be discussed in chapter 3.

Throughout, we assume that F is G-equivariant, i.e. $F(\lambda, gz) = gF(\lambda, z)$ for all $g \in G$. We pay special attention to group elements g of the form $g = (h, \Theta(h))$, $h \in H$, $\Theta(h) \in S^1$. H is a subgroup of G which leaves the periodic orbit fixed *as a set*, i.e.

$$H := \left\{h \in G \;\middle|\; \exists\, \Theta = \Theta(h) \in S^1 \;\text{ s. t. }\; z(t) = hz(t - \Theta(h)\,p)\right\}.$$

Similar to standard Pyragas control, the control term is noninvasive on the periodic orbit itself, using the identity

$$z(t) = hz(t - \Theta(h)\,p) \tag{2.2}$$

which holds on the periodic orbit. It describes its spatio-temporal symmetry given by the triplet (H, K, Θ). We often use $\mathrm{Stab}(e^{2\pi i\Theta(h)}h) = \{z(t) | z(t) = hz(t - \Theta(h)\,p)\}$.

In standard Pyragas control, $h = \mathrm{Id}$ and $\Theta(\mathrm{Id}) = 1$ are used, which reflects the periodicity of the orbit: $z(t) = z(t - p)$.

Of course, it is possible to include more than one group element h in the control term. For example, we can construct control terms which are linear combinations of single control terms. In particular, in chapter 3, we will consider

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)\,p)\,\mathrm{d}h\right),$$

where the integral over the group H denotes the Haar measure of the group H.

We will introduce equivariant Pyragas control in such a way that exactly one periodic orbit is left invariant by the control term. As each periodic orbit is uniquely identifiable by its symmetry, the equivariant Pyragas control therefore *selects* the periodic orbit.

In other words, if two or more periodic orbits with the same period exist, equivariant Pyragas control is noninvasive only on the periodic orbit of the prescribed symmetry type. This is in contrast to "standard" Pyragas control which is by construction noninvasive on all periodic orbits with the same period. It is therefore justified to interpret equivariant Pyragas control as a *selective stabilization* of periodic orbits by a suitable choice of h and Θ.

In this chapter we ask the question if we can find a combination of a group element h (with corresponding $\Theta(h)$) and complex control parameter b such that the control is successful and the unstable periodic orbit becomes stable.

In the first part of this chapter, consisting of sections 2.1 – 2.4, we investigate the effects of equivariant Pyragas control close to Hopf bifurcation. In section 2.1, we state our new results, using the control form (2.1). Methods of the proof are given in section 2.2, while the first part of the proof is presented in section 2.3. The proof of stabilization is completed in section 2.4.

In the second part of this chapter, which consists of section 2.5, we extend those results for dynamical systems for which not all eigenvalues are close to Hopf bifurcation, e.g. networks of coupled cells.

2.1 Control close to Hopf bifurcation

The setting in which we will investigate equivariant Pyragas control is given by the equivariant Hopf bifurcation Theorem ([6] and Theorem 8 in the appendix). The only additional assumption is that the equivariant Pyragas control is *noninvasive on exactly one periodic orbit* which emerges at the equivariant Hopf bifurcation. This plays a vital role in the proof of our claims.

This setting allows us to give explicit analytic conditions for the stabilization. For mathematical convenience, we also normalize the Hopf frequency to unity. This can always be achieved by rescaling time.

This section is divided into the sub- and the supercritical case, referring to the original Hopf bifurcation in the system without control.

We will see that if the Hopf bifurcation is originally *supercritical*, we always find an open and infinite region of control parameters for which the periodic orbit is stable. For the non-equivariant case, this result would be trivial. In the equivariant case, however, the stability of a periodic orbit is not given by its supercriticality alone [6], therefore this result indeed requires proof.

In the *subcritical* case we need an additional condition on the parameters of the uncontrolled system to guarantee the existence of an open (finite) control region. This is not surprising, since also in the non-equivariant case, a subcritical periodic orbit is unstable and a control region cannot always be found [2, 3].

The following theorem is the main result in the supercritical case, stating the existence of an open control region for equivariant Pyragas control (2.1).

Theorem 1 (Supercritical case) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let $A(\lambda) := D_z F(\lambda, 0)$ be the linearization with $\operatorname{spec} A(\lambda) = \lambda \pm \mathrm{i}$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$, where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

$h \in H$, $\Theta(h) \in S^1$ bifurcates with $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$.

Assume that the bifurcation is ***supercritical****.*

Then there exists a unique open and infinite region of control parameters b in the complex plane such that a control of the form

$$\dot{z}(t) = F(\lambda, z(t)) + b(-z(t) + hz(t - \Theta(h)p))$$

selectively stabilizes the periodic orbit with the above symmetry near Hopf bifurcation.

Note that the existence of the control region in b is guaranteed for all possible choices of the group element h and corresponding $\Theta(h)$.

The following two corollaries give us more information about the control region, they do not need any additional proof.

Corollary 1 (Control region versus minimal period) *If $-(1 + 2\pi\Theta \operatorname{Re} b)/\operatorname{Im} b < 0$ then the inequality*

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b}{\operatorname{Im} b} < \Theta\, p'(0)$$

holds for all b, $\operatorname{Im} b \neq 0$ which stabilize the unstable periodic orbit near Hopf bifurcation. On the other hand, if $-(1 + 2\pi\Theta \operatorname{Re} b)/\operatorname{Im} b > 0$, then the inequality

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b}{\operatorname{Im} b} > \Theta\, p'(0)$$

holds for all control parameters b, $\operatorname{Im} b \neq 0$ which stabilize the unstable periodic orbit near Hopf bifurcation.

Corollary 2 (Real control parameters) *In the* supercritical case*, real control parameters b suffice. In particular, all real and strictly positive control parameters are suitable for the stabilization.*

In the subcritical case the main theorem takes a very similar form to Theorem 1. Note however that there is a crucial difference, since this theorem needs an additional assumption to guarantee the existence of an open stabilization region.

Theorem 2 (Subcritical case) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let $A(\lambda) := D_z F(\lambda, 0)$ be the linearization with $\operatorname{spec} A(\lambda) = \lambda \pm \mathrm{i}$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$ where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

$h \in H$, $\Theta(h) \in S^1$ bifurcates with $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$.

*Assume that the bifurcation is **subcritical**.*

Then there exists a constant $\mathcal{C} \in \mathbb{R}$ such that if either

$$p'(\lambda)|_{\lambda=0} > \mathcal{C} > 0, \tag{2.3}$$

or

$$p'(\lambda)|_{\lambda=0} < \mathcal{C} < 0, \tag{2.4}$$

then *there exists a unique open and finite region of control parameters b in the complex plane such that a control of the form*

$$\dot{z}(t) = F(\lambda, z(t)) + b(-z(t) + hz(t - \Theta(h)p))$$

selectively stabilizes the periodic orbit with the above symmetry near Hopf bifurcation.

Remark: Due to the conditions (2.3), (2.4), i.e. $p'(0)$ bounded away from zero, real control parameters are excluded in the subcritical case, see details of the proof.

2.2 Stabilization mechanism and outline of the proof

If equivariant Pyragas control is introduced, exactly one of the bifurcating periodic orbits is selected by its symmetry. The control term in noninvasive only on the corresponding periodic orbit.

Therefore the original center manifold at $\lambda = 0$ splits and only the selected periodic orbit still bifurcates at $\lambda = 0$.

We emphasize that this is the most important observation, since we can now invoke standard exchange of stability in a two dimensional center manifold.

In other words, we have reduced the problem to standard Hopf bifurcation for which it is comparatively easy to determine the stability of the bifurcating periodic orbit.

Therefore it is necessary and sufficient for stabilization that the trivial equilibrium is stable and the Hopf bifurcation turns supercritical, see also [12].

This proof uses many ideas from [4], but needs new insights to work for equivariant dynamical systems. In section 2.3 we seek the domain for which the trivial equilibrium is stable at the Hopf bifurcation point. In section 2.4 we complete the proof of Theorems 1 and 2, respectively.

2.3 Domains of stability

Due to the equivariance of the linearization of the system (2.1) with respect to the group G, the assumptions of Schur's Lemma (see appendix) are fulfilled and the linearization of the system diagonalizes in the coordinates of the group representation of G.

Without loss of generality we assume that the system is given in those coordinates and that the selected Hopf bifurcation occurs in the first component z_1 of
$z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Furthermore, we also know that in these coordinates the group acts canonically, i.e.
$z_k \mapsto e^{2\pi \mathrm{i} M_k} z_k$ for each coordinate z_k, $k = 1, \ldots, n$, where $0 \leq M_k \leq 1$.

For notational convenience, we define $m_k := M_k - \Theta$.

The stability of the trivial equilibrium is governed by the eigenvalues of the linearization $A(\lambda)$ of the system with control (2.1) at $z \equiv 0$. The structure of the system is substantially altered by introducing time-delayed feedback control. Without control, the system is (complex) n-dimensional. Adding the feedback term makes the system infinite dimensional, yielding infinitely many eigenvalues, additionally depending on the control parameter b.

We will therefore not calculate the eigenvalues directly, but we aim at finding out for which complex control parameters b we can find stability changes of the trivial equilibrium. Let $E(b)$ denote the strict unstable dimension of the trivial equilibrium $z \equiv 0$ at the Hopf bifurcation point $\lambda = 0$ with time delay $\tau = p\,\Theta = 2\pi\,\Theta$, i.e. $E(b)$ counts the total number of eigenvalues η with strictly positive real part.

Proposition 1 (Changes of stability) *Under the assumptions of Theorem 1, the unstable dimension $E(b)$ changes by two if, and only if, the control parameter b crosses one of the curves*

$$b_k(\omega) = \frac{\mathrm{i}\omega}{-1+\exp(2\pi\mathrm{i}m_k - 2\pi\mathrm{i}\Theta\omega)} \tag{2.5}$$

$$= -\tfrac{1}{2}\,\omega\,(\cot(\pi m_k - \pi\Theta\omega) + \mathrm{i}) \tag{2.6}$$

for $k = 1, \ldots, n$. Specifically, in the subspace where the selected Hopf bifurcation occurs, we obtain

$$b_1(\omega) = \frac{\mathrm{i}\omega}{-1+\exp(2\pi\mathrm{i}\Theta\omega)}$$

$$= -\tfrac{1}{2}\,\omega\,(\cot(\pi\Theta\omega) + \mathrm{i}).$$

$E(b)$ never changes by one. Furthermore $E(0) = 0$.

First note that the assumptions of Theorem 1 are also assumptions of Theorem 2, thus the claims of Proposition 1 are valid in both settings. Further note that $E(0) = 0$ holds by assumption since we assume that the system without control, i.e. $b = 0$, has only purely imaginary eigenvalues at the Hopf bifurcation point.

As the linearization of the system diagonalizes, we can calculate the characteristic equation $\chi(\eta) = 0$ of the eigenvalue η. We use exponentials $z(t) = e^{\eta t}(z_1, \ldots, z_n)$ and obtain n separate factors. Therefore, we obtain a set of n characteristic equations $\chi_k(\eta)$:

$$\chi_k(\eta) = \lambda + \mathrm{i} + b\,(-1 + \exp(2\pi\mathrm{i}M_k - \tau\eta)) - \eta, \qquad k = 1, \ldots, n \tag{2.7}$$

From the characteristic equations, we can easily obtain the curves b_k, $k = 1, \ldots, n$, at the Hopf bifurcation point $\lambda = 0$, $\tau = 2\pi\,\Theta$. We look for purely imaginary eigenvalues $\eta = \mathrm{i}(1+\omega)$, where the curves are parametrized by $\omega \in \mathbb{R}$.

Since we calculate purely imaginary eigenvalues, we obtain stability changes by Hopf bifurcation, i.e. the unstable dimension $E(b)$ changes by two.

By direct calculation, we find that $\eta = 0$ can only occur for $M_k \in \{0, 1\}$, it occurs as a double eigenvalue zero and lies on the already determined Hopf curve.

For $k = 1$, by assumption of noninvasive control, we obtain that $M_k = \Theta$ and therefore $m_k = 0$, yielding the correct expression.

For better understanding of the following, it is useful to draw the curves $b_1, \ldots, b_n$ in the plane of the complex control parameter b.

Note that the curve b_1, which corresponds to the selected Hopf bifurcation, is symmetric with respect to the real axis, crossing it only at $b_1(0) = 1/(2\pi\Theta)$.

Furthermore note that all the other curves $b_2, \ldots, b_n$ go through the origin. Intuitively, this is obvious because we have chosen the equivariant Pyragas control such that it is noninvasive only on one selected periodic orbit. Thus, the stability of the trivial equilibrium changes for nonzero control which reflects the fact that the Hopf curves go through the origin. The curves $b_1, \ldots, b_n$ follow a U-turn with horizontal asymptotics.

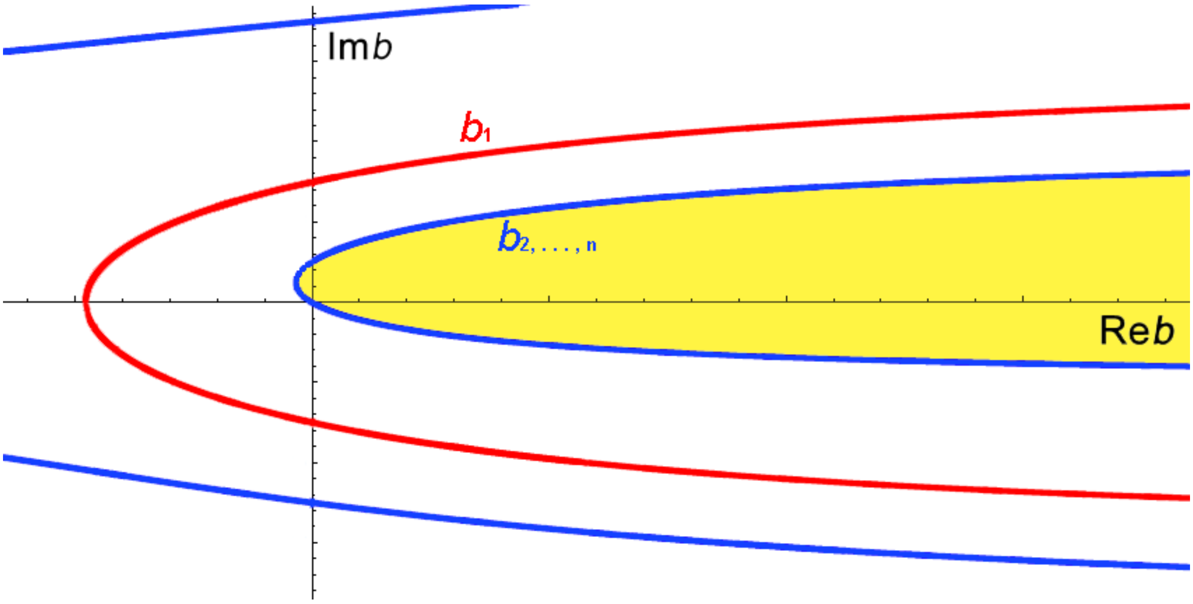

Figure 2.1: Sketch of the complex control region Λ in b. The curve b_1 is drawn in red, note that it is symmetric with respect to the Re b-axis. Only one of the curves $b_2, \ldots, b_n$ is sketched for simplicity (blue). Note that it goes through the origin. All curves are oriented downwards, the region with stable equilibrium at the Hopf point is shaded yellow.

All the curves $b_1, \ldots, b_n$ are complex differentiable (with exception of the poles) and therefore preserve complex orientation. Hence, we only need to know the orientation of the curves, because then we know that the region with smaller value of $E(b)$ can be found to the left of each curve.

The curves $b_1, \ldots, b_n$ are oriented downwards for increasing ω, due to the imaginary part which is given by $-\frac{1}{2}\omega\mathrm{i}$ for all curves $b_1, \ldots, b_n$.

$E(0) = 0$ by assumption. For the parameter b_1 we know that the region containing the origin has $E(b_1) = 0$. The curves $b_2, \ldots, b_n$ cross the origin, therefore we can conclude that there exists an open nonempty region to the right of the origin has $E(b_k) = 0$, $k = 1, \ldots, n$. The real line is included in all the regions $E(b_k) = 0$, $k = 1, \ldots, n$.

The region where $E(b) = 0$ consists of the overlap of all single regions $E(b_k) = 0$, $k = 1, \ldots n$. For further analysis, we will call the region where $E(b) = 0$ holds Λ. It lies to the right of the origin and includes the real positive axis. In particular, it is not empty.

We have now achieved linear stability $E(b) = 0$ at the Hopf point $\lambda = 0$, $\tau = 2\pi\,\Theta$ itself, for $b \in \Lambda$.

2.4 Proof of the main theorems

In this section we finish the proof of our main Theorems 1 and 2. We fix the complex control parameter $b = b_0 \exp(\mathrm{i}\beta)$ in the region Λ where the characteristic equations (2.7) produce only eigenvalues with strictly negative real part, at $\lambda = 0$, $\tau = p\,\Theta = 2\pi\,\Theta$ (with exception of the pair of purely imaginary eigenvalues of the selected Hopf bifurcation).

We now must guarantee that we only encounter standard *supercritical* Hopf bifurcation at the Hopf point. Standard Hopf bifurcation for nonzero control amplitude is ensured by assuming that $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi\mathrm{i}\Theta(h)}h)) = 2$.

It remains to show that the bifurcation is supercritical, i.e. that the selected periodic orbit lies at the side of the Hopf bifurcation where the trivial equilibrium has unstable dimension two.

We therefore count unstable dimensions in the (λ, τ)-plane. Then we compare them to the *Pyragas curve*, which determines the position of the periodic orbit, with the *Hopf curves*, which tells us where the stability of the trivial equilibrium changes.

The Pyragas curve $\tau_P(\lambda)$ is given by

$$\tau_P(\lambda) := \Theta\, p(\lambda)$$

and thus depends on the group homomorphism and on the particular function $F(\lambda, z)$. The Pyragas curve τ_P does not depend on the control parameter b. By the normalized Hopf frequency, we know that $p(0) = 2\pi$. Furthermore the continuation of the Pyragas curve is differentiable at $\lambda = 0$. We denote

$$\tau_P'(\lambda)|_{\lambda=0} = \Theta\, p'(0).$$

We also determine the Hopf bifurcation curves $\tau_k(\lambda)$, $k = 1, \ldots, n$, at $b = b_0 \exp(\mathrm{i}\beta)$ where $\chi_k(\eta) = 0$ for purely imaginary eigenvalues $\eta = \mathrm{i}\bar{\omega}$:

$$\tau_k(\lambda) = \frac{\pm \arccos\left(\cos\beta - \lambda/b_0\right) + \beta + 2\pi M_k + 2\pi N}{1 - b_0 \sin\beta \mp \sqrt{b_0^2 \sin^2\beta + \lambda(2b_0\cos\beta - \lambda)}}$$

for $k = 1, \ldots, n$, with integer N. This formula is obtained by separating the characteristic equations $\chi_k(\mathrm{i}\bar{\omega}) = 0$ into real and imaginary part:

$$0 = \lambda + b_0\left(\cos(\beta + 2\pi M_k - \bar{\omega}\tau) - \cos\beta\right) \tag{2.8}$$
$$\bar{\omega} - 1 = b_0\left(\sin(\beta + 2\pi M_k - \bar{\omega}\tau) - \sin\beta\right) \tag{2.9}$$

By rearranging the equations above, and taking the square, we obtain:

$$(\cos\beta - \lambda/b_0)^2 = \cos^2(\beta + 2\pi M_k - \bar{\omega}\tau)$$
$$(\sin\beta + (\bar{\omega} - 1)/b_0)^2 = \sin^2(\beta + 2\pi M_k - \bar{\omega}\tau)$$

Adding up both equations gives a quadratic equation in $\bar{\omega}$, which can be solved explicitly:

$$\bar{\omega} = 1 - b_0 \sin\beta \pm \sqrt{b_0^2 \sin^2\beta + \lambda(2b_0\cos\beta - \lambda)}$$

The result can be substituted into (2.8), yielding the above expression.

In contrast to the Pyragas curve τ_P, the Hopf curves τ_k depend on the control parameter b, but they do not depend on any nonlinearity of the original system.

For further calculations, we linearize

$$\chi_k(\eta) = \lambda + \mathrm{i} + b\,(-1 + \exp(2\pi \mathrm{i} M_k - \tau\eta)) - \eta, \qquad k = 1, \ldots, n,$$

with respect to $\lambda = 0 + \bar{\lambda}$, $\eta = 0 + \mathrm{i}\bar{\omega}$, and $\tau = \Theta 2\pi + \bar{\tau}$. Of particular interest is the linearization of the above equation for $k = 1$, i.e. $M_1 = \Theta$. In this case, we obtain:

$$\begin{aligned} \bar{\lambda} &= -\operatorname{Im} b\,(2\pi\Theta\bar{\omega} + \bar{\tau}) \\ 0 &= \quad \operatorname{Re} b\,(2\pi\Theta\bar{\omega} + \bar{\tau}) + \bar{\omega} \end{aligned}$$

Rearranging yields

$$\bar{\tau} = -\frac{1 + 2\pi\Theta \operatorname{Re} b}{\operatorname{Re} b}\,\bar{\omega} \qquad \text{and} \qquad \bar{\lambda} = \frac{\operatorname{Im} b}{\operatorname{Re} b}\,\bar{\omega}.$$

Therefore we can conclude that

$$\tau_1'(\lambda)\big|_{\lambda=0} = -\frac{1 + 2\pi\Theta \operatorname{Re} b}{\operatorname{Im} b}\,.$$

By orientation considerations we can determine the resulting total unstable dimensions $E(\lambda, \tau)$ of the trivial equilibrium $z = 0$ in the domains complementary to the Hopf curves:

We consider once more the characteristic equations (2.7) and linearize with respect to τ and λ:

$$\varphi(\lambda, \tau) = \lambda - \eta\,\tau\,b\,\exp(2\pi \mathrm{i} M_k - \tau\eta) = \xi$$

and also with respect to η:

$$\psi(\eta) = 1 + \eta\,\tau\,b\,\exp(2\pi \mathrm{i} M_k - \tau\eta) = \xi.$$

Now it is possible to write

$$(\lambda, \tau) = (\varphi^{-1} \circ \psi)(\eta),$$

where ψ is orientation preserving because it is holomorphic. Furthermore we need to calculate the determinant $(\det\varphi)$ at $\eta = \mathrm{i}\omega$, $\omega = 1$, $\lambda = 0$, $\tau = 2\pi\Theta$ to find its orientation behaviour:

$$\det\varphi = -\operatorname{Im}\big(\eta\,b\,\exp(2\pi \mathrm{i} M_k - \tau\eta)\big)$$

The only curve which is relevant here is the one with $k = 1$, i.e. the one corresponding to the selected Hopf bifurcation. For this curve, it is possible to simplify the above expression for the determinant and we obtain

$$\det\varphi = -\operatorname{Re} b. \tag{2.10}$$

Since we have fixed the control parameter b in a region where its real part is always positive, we can now conclude that φ is orientation reversing. Hence it follows that the region with $E(\lambda,\tau) = 2$ can be found at the left side of the Hopf-curve τ_1 in the (λ,τ)-plane. The τ_1-curve is oriented downwards.

Supercritical case: The Pyragas curve exists for $\lambda > 0$. If $\tau_1'(0) < 0$ then we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta\,\mathrm{Re}\,b}{\mathrm{Im}\,b} < \Theta\, p'(0).$$

On the other hand, if $\tau_1'(0) > 0$ then we will find that it enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta\,\mathrm{Re}\,b}{\mathrm{Im}\,b} > \Theta\, p'(0).$$

Note that the real axis can always be used for control. Indeed, if the control parameter b is chosen on the real line, then the Hopf curve is oriented vertically downwards and we can stabilize for all possible values of $p'(0)$. Also note that we have hereby proven Corollaries 1 and 2.

Subcritical case: The Pyragas curve exists for negative λ. In this case, if $\tau_1'(0) < 0$ then we will find that τ_P enters the region with $E(\lambda,\tau) = 2$ whenever

$$-\frac{1 + 2\pi\Theta\,\mathrm{Re}\,b}{\mathrm{Im}\,b} > \Theta\, p'(0),$$

note that the inequality sign changes compared to the supercritical case. On the other hand, if $\tau_1'(0) > 0$ then we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta\,\mathrm{Re}\,b}{\mathrm{Im}\,b} < \Theta\, p'(0).$$

In contrast to the supercritical case, we cannot guarantee the existence of control parameters b for which the stabilization is possible. This is due to the fact that there might be no choices for b which are in the region determined in section 2.4 and satisfy the above inequalities at the same time. The constant $\mathcal{C}$ is then given by the respective minimum/ maximum-value which $\tau_1'(0)$ can take for $b \in \Lambda$. For example, real control parameters b are excluded in the subcritical case.

This proves Theorems 1 and 2. $\square$

In summary, the assumption that $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$ of Theorems 1 and 2 guarantees that we only encounter a standard Hopf bifurcation, when we restrict the time delay to the Pyragas curve $\tau_P(\lambda)$. The unique bifurcating Hopf branch then consists of exactly the selected periodic orbit unchanged by the control term. Because $E(b) = 0$ and because we have proven that under the above conditions standard Hopf bifurcation is always supercritical along the Pyragas curve, we can invoke standard exchange of stability. Therefore we have proven that, near the Hopf bifurcation point $\lambda = 0$, the selected periodic orbit can be stabilized.

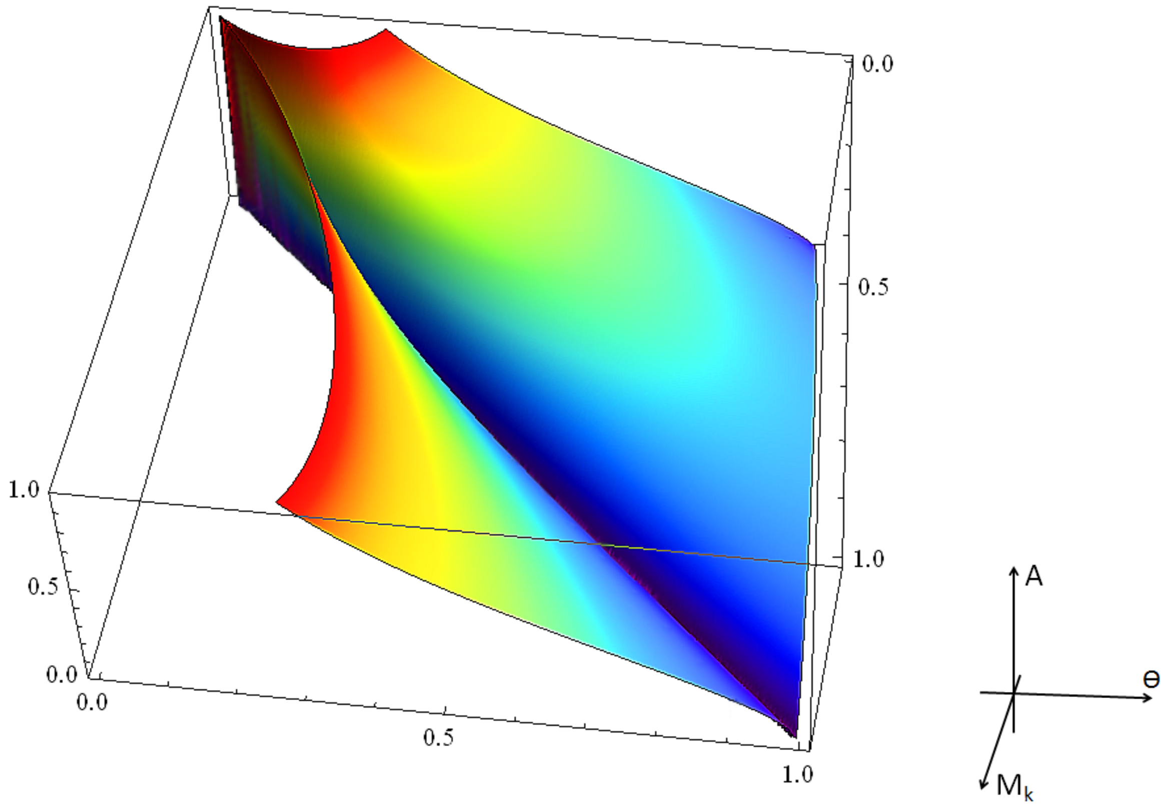

Figure 2.2: Upper threshold value A of the real part of the remaining eigenvalues (z-axis) for $c_k = 1$, depending on M_k and Θ

2.5 Control of the remaining eigenvalues

In physical or biological applications, we can often find more eigenvalues than the purely imaginary ones at the Hopf bifurcation point. If these eigenvalues have positive real part, their instability is inherited by the bifurcating Hopf-branches, thus making them unstable. The question whether Pyragas control can still succeed in this case is therefore an important one.

This question is motivated by [4, 7, 8, 10, 13], where two and three symmetrically and diffusively coupled oscillators are discussed. In both cases, an upper threshold on the coupling strength was found, which is linearly correlated with the real part of the positive eigenvalue.

Whether Pyragas control still succeeds in the presence of additional eigenvalues with positive real part depends crucially on the particular system. Therefore we do not attempt to give a general and yet precise answer. However, we are able to give upper threshold values, in other words, necessary conditions for the stabilization. Interestingly enough, these depend not only on the largest eigenvalue itself, but also on the choice of the group element h and correspondingly on the time-delay $\Theta(h)p$ used for control.

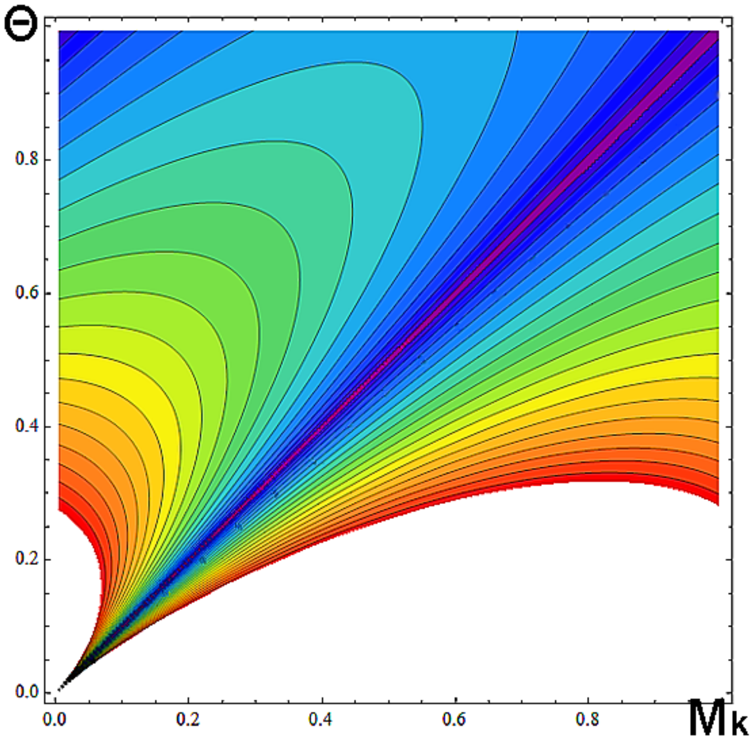

Figure 2.3: Figure 2.2 as seen from above. Only values of A which satisfy $0 \leq A \leq 1$ are included. Note the periodicity in M_k-direction.

Theorem 3 (Upper threshold on the remaining eigenvalues) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^d \cong \mathbb{R}^{2d}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let

$$\dot{z}_k = (\lambda + a_k + \mathrm{i}\, c_k)\, z_k$$

be the linearization of the system with $a_k = 0$ for $k = 1, \ldots, n$ and $a_k \neq 0$ for $k = n+1, \ldots, d$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$, where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

$h \in H$, $\Theta(h) \in S^1$ bifurcates with $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$. Consider a control term of the form

$$\dot{z}(t) = F(\lambda, z(t)) + b(-z(t) + hz(t - \Theta(h)p)).$$

If for some (a_k, c_k), $k = n+1, \ldots, d$, we have $a_k > A$ where (A, ω) is the solution of

$$\begin{aligned} \sin \mathcal{N} \cos \mathcal{N} &= \omega \pi \Theta \\ \sin^2 \mathcal{N} &= A \pi \Theta, \end{aligned} \tag{2.11}$$

with $\mathcal{N} = \pi(M_k - \Theta(c_k + \omega))$, $\Theta = \Theta(h)$, then there is no stabilization possible for this particular control scheme $(h, \Theta(h))$ near equivariant Hopf bifurcation.

This implies the following Corollary:

Corollary 3 (A is unbounded) *There is no general limit on the maximal real part of the remaining eigenvalues which equivariant Pyragas control can stabilize. In other words, A is unbounded if Θ and h can be varied.*

Remark 1: We emphasize once again that the above thresholds only constitute necessary conditions. We will see in chapter 3 that we are able to give a variation of equivariant Pyragas control such that the conditions in Theorem 3 are both necessary and sufficient.

Remark 2: Corollary 3 follows directly from the new Theorem 3 above. However, in [10], we can already find an example which demonstrates Corollary 3, without proving Theorem 3.

We briefly resume the example in [10]: Here $\Theta(h) - M_k = \frac{1}{2}$. We find a solution $(A, 0)$ of equation (2.11) with

$$A = \frac{1}{\Theta(h)\pi}.$$

Therefore, if $\Theta(h)$ is chosen arbitrarily small, which is indeed possible in the example in [10], the maximal eigenvalue A can be chosen arbitrarily large.

It follows that A is unbounded on $\Theta \in [0, 1]$, $M_k \in [0, 1]$ and the Corollary follows directly.

Proof:

As before, we start our investigations with determining the changes of stability in the plane of the complex control parameter b:

$$b_k(\omega) = \frac{\mathrm{i}\omega - A}{-1 + \exp(2\pi\mathrm{i}M_k - 2\pi\mathrm{i}c_k\Theta - 2\pi\mathrm{i}\Theta\omega)}$$

If $A = 0$ (and $c_k = 1$), we find the equations for the center manifold calculated in section 2.3 with normalized Hopf frequency.

However these new equations are not always complex differentiable, and at this non-differentiable point a loop appears for decreasing A. This can be verified by directly analysing the curves, which take a simple twisted cotangent-form, chapter 4. Within this loop, the unstable dimension at the Hopf point is zero (only for each characteristic equation separately). At the origin, the unstable dimension is 2, for positive A. If the loop disappears for increasing A then also the region with $E(b_k) = 0$ disappears and no stabilization of the equilibrium is possible.

Therefore we seek for the point where the complex derivative of $b_k(\omega)$ vanishes. By defining $\mathcal{N} := \pi(M_k - \Theta(c_k + \omega))$, we find that

$$\begin{aligned}
\operatorname{Re} b_k(\omega) &= -\tfrac{1}{2}\left(\omega \cot \mathcal{N} - A\right) \\
\operatorname{Im} b_k(\omega) &= -\tfrac{1}{2}\left(\omega + A \cot \mathcal{N}\right)
\end{aligned}$$

By differentiating with respect to A, ω and rearranging the equations we obtain the two equations

$$\begin{aligned} \sin\mathcal{N}\cos\mathcal{N} &= \omega\pi\Theta \\ \sin^2\mathcal{N} &= A\pi\Theta, \end{aligned}$$

as claimed in Theorem 3. If $a_k > A$ for some k, no loop exists and therefore no stabilization is possible near the Hopf bifurcation. The instability of the trivial equilibrium is inherited by the bifurcation Hopf branch, i.e. the selected periodic orbit.

This completes the proof of Theorem 3. □

Note that we only consider each characteristic equation separately, we do not know if there is an overlap of all control regions. Therefore we cannot guarantee stabilization. However, for concrete examples, it is possible to either guarantee or exclude stabilization numerically.

Remark 3: Note that Theorem 3 also tells us in which cases standard Pyragas control does not work. We fix for example $c_k = 1\ \forall k$. Then there is a zero $A = 0$ of equation (2.11) at $(\Theta, M_k) = (1, 1)$ which corresponds to standard Pyragas control. Now consider n identical Stuart-Landau oscillators, symmetrically and diffusively coupled in a bidirectional ring:

$$\dot{z}_k = (\lambda + \mathrm{i} + \gamma|z_k|^2)z_k + a(z_{k_1} - 2z_k + z_{k+1})$$

Assume that the coupling constant a is positive. There is a spatially homogeneous Hopf bifurcation at $\lambda = 0$ and all other Hopf bifurcations are equivariant and occur for $\lambda > 0$. In [2], it has been proven that the homogeneous Hopf bifurcation can be stabilized by standard Pyragas control. Theorem 3 now tells us that we cannot stabilize any of the periodic orbits emerging from equivariant Hopf bifurcation.

Chapter 3

Two variations of equivariant Pyragas control

In this chapter, we introduce two variations of the control scheme proposed in chapter 2. In the first part, we study linearly transformed feedback, i.e. we use a control matrix $B \in \mathbb{C}^{n\times n}$ instead of a scalar control parameter $b \in \mathbb{C}^n$ as in chapter 2:

$$\dot{z}(t) = F(\lambda, z(t)) + B\left(-z(t) + hz(t - \Theta(h)\, p)\right)$$

In the second part of this chapter, we consider controls of the form

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)\, p)\,\mathrm{d}h\right),$$

where the integral over the group H denotes the Haar measure of the group H.

3.1 Pyragas control with linearly transformed feedback

The method introduced in this section aims at lifting several restrictions which exist for the control in chapter 2. It is a generalization of the method introduced by Matthias Bosewitz [10] for two diffusively coupled oscillators in Hopf normal form.

In this section, we extend his method to general (complex) n-dimensional dynamical systems near Hopf bifurcation. We aim, similar as in [10], at decoupling the characteristic equations completely. The main idea is to use different control parameters b_k, $k = 1, \ldots, n$, for each factor of the characteristic equation.

As a consequence, the condition of zero eigenvalues with real part strictly greater than zero only needs to be fulfilled for each factor of the characteristic equation separately. It is clear that this is a great relaxation of the conditions. In particular, it means that there is no condition on the nonlinear term except for a nondegeneracy condition. It also implies that Theorem 3 gives not only a necessary but also a sufficient condition for stabilization.

Theorem 4 (Linearly transformed feedback) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let $A(\lambda) := D_z F(\lambda, 0)$ be the linearization with $\operatorname{spec} A(\lambda) = \lambda \pm \mathrm{i}$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$ where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

$h \in H$, $\Theta(h) \in S^1$ bifurcates with $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$.

Consider a control of the form

$$\dot{z}(t) = F(\lambda, z(t)) + B(-z(t) + hz(t - \Theta(h)p))$$

where $B \in \mathbb{C}^{n\times n}$. Suppose that B commutes with the group action h, i.e. $hB = Bh$. Then there exists an open region of control matrices B such that a control of the above form selectively stabilizes the periodic orbit with the above symmetry near Hopf bifurcation.

Note that the use of a control matrix B guarantees the existence of a stabilizing feedback control in the super- as well as the subcritical case. The idea of decoupling the characteristic equations completely is “hidden” in the condition that B and h commute. Therefore, they can be simultaneously diagonalized [14] and the characteristic equation can be factorized into n separate equations.

Proof:

The proof is again divided into two parts: Finding a control region and guaranteeing supercritical bifurcation.

In contrast to theorems 1 and 2, we have now used a complex control matrix B instead of a complex control parameter b. We have assumed that $hB = Bh$ and therefore [14] we can simultaneously diagonalize the linearization of the controlled system. As a consequence (compare with section 2.3) we find n factors of the characteristic equation

$$\chi_k(\eta) = \lambda + \mathrm{i} + \tilde{b}_k \left(-1 + \exp(2\pi \mathrm{i} M_k - \tau\eta)\right) - \eta, \qquad k = 1, \ldots, n, \tag{3.1}$$

where $\tilde{b}_k \in \mathbb{C}$ are the diagonal elements of the diagonalized matrix B.

We can now conclude stability in almost the same way as in chapter 2: If for each of the characteristic equations we have zero eigenvalues with positive real part, the unstable dimension of the Hopf point at $\lambda = 0$ is zero. We can use the same curves which give the changes of stability (2.5). Note however, that we can use different $\tilde{b}_k$ for each characteristic equation because B is a matrix. As a result we do not require the $\tilde{b}_k$ to lie in the overlap of the simple control regions. If we choose all $\tilde{b}_k \neq 0$ and we require $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$, we have a simple Hopf bifurcation at $\lambda = 0$.

The fact that we can use different control parameters $\tilde{b}_k$ for each characteristic equation is particularly useful for proving that the $\tilde{b}_k$ can be chosen such that the bifurcation is supercritical.

In particular, we can now choose $\tilde{b}_1$ independent of $\tilde{b}_2, \ldots, \tilde{b}_n$, which allows us to choose b_1 with negative real part. In the case that $\tilde{b}_1$ is chosen with positive real part, we can continue the proof as in chapter 2, ending up with the same conditions on the supercriticality as before.

In the case that $\operatorname{Re} \tilde{b}_1 < 0$, however, we know that the τ_1-curve is orientation preserving (2.10), and that the region with $E(B) = 2$ can now be found at the right side of the Hopf-curve τ_1 in the (λ, τ)-plane.

In the **supercritical** case, the Pyragas curve exists for $\lambda > 0$. If $\tau_1'(0) > 0$ then we will find that τ_P enters the region with $E(\lambda, \tau) = 2$ whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} \tilde{b}_1}{\operatorname{Im} \tilde{b}_1} < \Theta\, p'(0),$$

i.e. we can take the absolute value of $\operatorname{Re} \tilde{b}_1$ and reverse the direction of the inequality, which gives us the same condition as before. On the other hand, if $\tau_1'(0) < 0$ then we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} \tilde{b}_1}{\operatorname{Im} \tilde{b}_1} > \Theta\, p'(0).$$

In the **subcritical** case, the Pyragas curve exists for negative λ. In this case, if $\tau_1'(0) < 0$ then we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} \tilde{b}_1}{\operatorname{Im} \tilde{b}_1} < \Theta\, p'(0).$$

On the other hand, if $\tau_1'(0) > 0$ then we will find that it enters the region $E(\lambda, \tau) = 2$ whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} \tilde{b}_1}{\operatorname{Im} \tilde{b}_1} > \Theta\, p'(0).$$

This completes the proof of Theorem 4. $\square$

Remark: In the case of additional unstable eigenvalues $a_k,\ k = n+1, \ldots, d$ in a complex d-dimensional system, it is now possible to stabilize the selected unstable periodic orbit whenever additionally the following condition is fulfilled: If for all $a_k,\ k = n+1, \ldots, d$, we have $a_k < A$ where (A, ω) is the solution of

$$\begin{aligned} \sin \mathcal{N} \cos \mathcal{N} &= \omega\pi\Theta \\ \sin^2 \mathcal{N} &= A\pi\Theta, \end{aligned}$$

with $\mathcal{N} = \pi(M_k - \Theta(c_k + \omega))$, $\Theta = \Theta(h)$, then there exists an open region of control matrices B such that the periodic orbit with the above symmetry near Hopf bifurcation can be stabilized.

3.2 Pyragas control with multiple time delays

The second variation of equivariant Pyragas control was brought to my attention by Prof. Fiedler. Until now, a proof of stabilization is still missing in the literature. In this master thesis, we attempt the proof of stabilization near equivariant Hopf bifurcation.

The main idea is to include all elements of the isotropy subgroup H into the control term, and not only a single one as in equation (2.1).

Specifically, we propose a new control term of the form:

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)p)\,\mathrm{d}h\right)$$

This new control term raises the following question: Can we guarantee the existence of a stabilization region?

Once again, we find that the situation is different in the subcritical and in the supercritical case. The results can be found in Theorems 5 and 6 below. As before, we can guarantee the existence of a stabilization region in the supercritical case. In the subcritical case, we once again need an additional condition to guarantee stabilization – which is the same as for one control term; compare with chapter 2, Theorem 2. One difference of the results therefore lies in the shape of the stabilization region, compare figure 3.1 with figure 2.1.

Furthermore, we can now apply equivariant Pyragas control also in the case where $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) \neq 2$. This is due to the assumption of equivariant Hopf bifurcation – the dimension fixed point subspace of the isotropy subgroup H must be 2, see appendix.

Theorem 5 (Multiple time delays – supercritical case) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let $A(\lambda) := D_z F(\lambda, 0)$ be the linearization with $\operatorname{spec} A(\lambda) = \lambda \pm \mathrm{i}$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$ where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

*$h \in H$, $\Theta(h) \in S^1$ bifurcates **supercritically**.*

Then there exists an open region of control parameters b in the complex plane such that a control of the form

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)p)\,\mathrm{d}h\right)$$

selectively stabilizes the periodic orbit with the above symmetry near Hopf bifurcation.

Theorem 6 (Multiple time delays – subcritical case) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t))$$

with G-equivariant F and $F(\lambda, 0) \equiv 0$, where F is k-times real differentiable, $k \geq 2$. Let $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}$ be a bifurcation parameter. Let $A(\lambda) := D_z F(\lambda, 0)$ be the linearization with $\operatorname{spec} A(\lambda) = \lambda \pm \mathrm{i}$.

Let there occur an equivariant Hopf bifurcation at $\lambda = 0$ where a periodic orbit with spatio-temporal symmetry

$$z(t) = hz(t - \Theta(h)p),$$

$h \in H$, $\Theta(h) \in S^1$ bifurcates ***subcritically****.*

Then there exists a constant $\mathcal{C} \in \mathbb{R}$ such that if either

$$p'(\lambda)|_{\lambda=0} > \mathcal{C} > 0,$$

or

$$p'(\lambda)|_{\lambda=0} < \mathcal{C} < 0,$$

then *there exists an open region of control parameters b in the complex plane such that a control of the form*

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)p)\,\mathrm{d}h\right)$$

selectively stabilizes the periodic orbit with the above symmetry near Hopf bifurcation.

Again, real control parameters are excluded in the subcritical case.

The **Proof** of Theorems 5 and 6 uses the same ideas as before, therefore we only draw attention to the points which differ from the original proof of Theorems 1 and 2.

First we calculate the characteristic equations for the complex eigenvalues η at the trivial equilibrium z_0:

$$\eta z_0 = z_0(\lambda + \mathrm{i}) + b\left(-z_0 + \int_H \exp\big(-\Theta(h)\tau\eta\big) hz_0\,\mathrm{d}h\right)$$

In the subspace of the selected Hopf bifurcation, which is again without loss of generality given by z_1, we have $hz_0 = \exp(2\pi\mathrm{i}\Theta(h))z_0$, therefore we can eliminate z_0, obtaining:

$$\eta = \lambda + \mathrm{i} + b\left(-1 + \int_H \exp\big(-\Theta(h)(\tau\eta + 2\pi\mathrm{i})\big)\,\mathrm{d}h\right) \tag{3.2}$$

We calculate the b_1-curve in the complex plane of the control parameter b for purely imaginary eigenvalues $\eta = \mathrm{i}(1 + \omega)$:

$$b_1(\omega) = \frac{\mathrm{i}\omega}{-1 + \int_H \exp(-2\pi\mathrm{i}\Theta\omega)\,\mathrm{d}h}$$

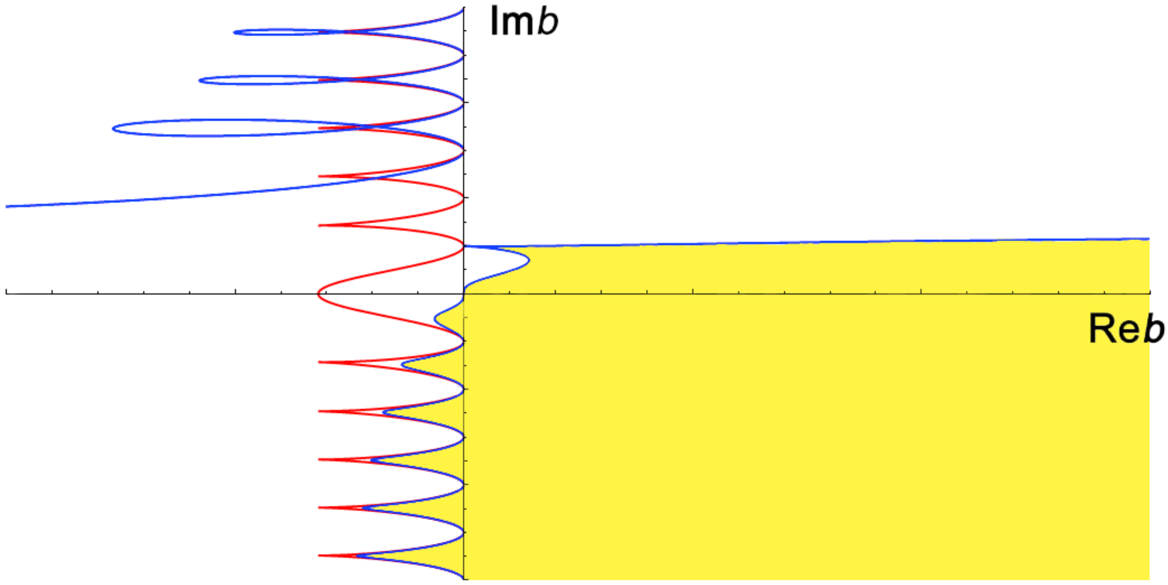

Figure 3.1: Example of a control region for $O(2)$-symmetry if we use the integral over all group elements for control. In the yellow region, the selected periodic orbit is born stable. Once again, the blue curve, corresponding to a non-selected Hopf bifurcation goes through the origin, while the red one, which lies in the subspace of the selected Hopf bifurcation, does not go through the origin, compare to figure 2.1.

We do not calculate the curves $b_2, \ldots, b_n$. Note however, that they go through the origin, since the equivariant Pyragas control is noninvasive only on the selected periodic orbit.

We can linearize the characteristic equation (3.2) with respect to λ, ω and τ, obtaining

$$0 = \lambda - b \int_H 2\pi \mathrm{i}\omega\, \Theta(h)\, \mathrm{d}h - \mathrm{i}\omega - b \int_H \mathrm{i}\tau\, \Theta(h)\, \mathrm{d}h.$$

This we can simplify to

$$0 = \lambda - \mathrm{i}\omega + b\,(2\pi \mathrm{i}\omega + \mathrm{i}\tau) \int_H \Theta(h)\, \mathrm{d}h.$$

Since the Haar measure is normalized to unity, i.e. $\int_H \Theta(h)\, \mathrm{d}h = 1$, we now have:

$$0 = \lambda - \mathrm{i}\omega + b\,(2\pi \mathrm{i}\omega + \mathrm{i}\tau)$$

From here on, we can continue the proof in the same way as the proof of Theorems 1 and 2 as in chapter 2. $\square$

In figure 3.1 we find an example of the region where the selected periodic orbit is born stable, here we considered $O_2 \times S^1$-symmetry in a 4-dimensional center-manifold.

Chapter 4

A case study: Coupled oscillators in a triangular symmetry

In this chapter, we reconsider the example of three symmetrically coupled oscillators, which has already been addressed in [7, 8]. We give a review and we show additionally how the choice of the group element influences the control region and how the two variations discussed in chapter 3 can be applied for this particular example.

4.1 Model and periodic solutions

Throughout this chapter we consider the following model system of three symmetrically and diffusively coupled *Stuart-Landau oscillators:*

$$\begin{aligned}
\dot{z}_1 &= f(z_1) + a(z_2 - z_1) + a(z_3 - z_1) \\
\dot{z}_2 &= f(z_2) + a(z_1 - z_2) + a(z_3 - z_2) \\
\dot{z}_3 &= f(z_3) + a(z_1 - z_3) + a(z_2 - z_3)
\end{aligned} \tag{4.1}$$

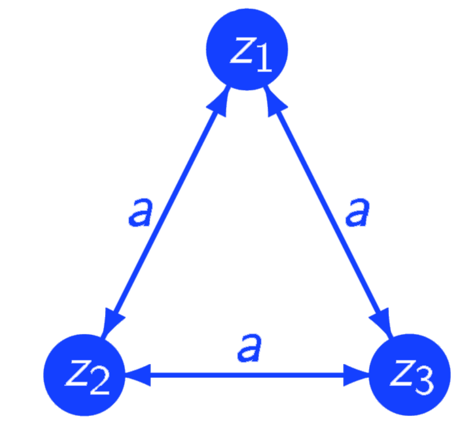

Figure 4.1: Schematic sketch of the network structure of system (4.1).

where

$$f(z_k) = (\lambda + \mathrm{i} + \gamma |z_k|^2) z_k, \quad k = 1, 2, 3. \tag{4.2}$$

Stuart-Landau oscillators are widely used as models for oscillating systems in physics and biology and they correspond to the Hopf normal form truncated at third order.

The system is $D_3 \times S^1$-equivariant, where D_3 is the dihedral group including all possible permutations of three elements, and S^1 is the rotational symmetry of every single oscillator.

As in the previous chapters, $\lambda \in \mathbb{R}$ is used as the bifurcation parameter. The Hopf-frequency is normalized to unity, this can always be achieved through rescaling of time. γ is a fixed complex number and the coupling parameter a is positive.

Spatially homogeneous Hopf bifurcation with $z_1 \equiv z_2 \equiv z_3$ occurs at $\lambda = 0$, i.e. all oscillators have the same frequency, amplitude and phase.

At $\lambda = 3a$, there occurs an equivariant Hopf bifurcation of two discrete rotating waves. Here the oscillators are phase-shifted by one third of the minimal period p with respect to each other, the two waves can only be distinguished in the numbering of oscillators:

$$z_k(t) = z_{k+1}\left(t - \tfrac{p}{3}\right) \qquad \text{and} \tag{4.3}$$

$$z_k(t) = z_{k-1}\left(t - \tfrac{p}{3}\right) \tag{4.4}$$

for k mod 3, with minimal period $p > 0$, see proposition 2 and [7].

Proposition 2 (Discrete rotating waves, [7]) *Consider the coupled oscillator triangle* (4.1), (4.2). *Equivariant Hopf bifurcation of discrete rotating waves* (4.3), (4.4) *occurs at the parameter value $\lambda = 3a$. The rotating waves are harmonic,*

$$z_k(t) = r \exp\left(2\pi \mathrm{i}\left(\frac{t}{p} + \frac{k}{3}\right)\right) \qquad \textit{and}$$

$$z_k(t) = r \exp\left(2\pi \mathrm{i}\left(\frac{t}{p} + 2\frac{k}{3}\right)\right),$$

for $k = 1, 2, 3$, respectively, and are phase shifted by $p/3$ between oscillators. Amplitude r and minimal period p are given explicitly by

$$r^2 = \frac{3a - \lambda}{\operatorname{Re}\gamma},$$

$$p = \frac{2\pi}{1 + r^2 \operatorname{Im}\gamma}.$$

In particular the Hopf bifurcation is supercritical, *i.e. towards $\lambda > 3a$, for $\operatorname{Re}\gamma < 0$, and* subcritical *for $\operatorname{Re}\gamma > 0$. The minimal period p grows with amplitude* (soft spring) *if $\operatorname{Im}\gamma < 0$ and decreases* (hard spring) *if $\operatorname{Im}\gamma > 0$.*

By direct calculation, this proposition can be verified easily, therefore we skip its proof.

For planar Hopf normal form (see appendix) the periodic orbit is unstable if the bifurcation is subcritical, and stable in the supercritical case. For the three coupled oscillators however, the discrete rotating waves are unstable both in the sub- and the supercritical case. They inherit their instability from the spatially homogeneous Hopf bifurcation at $\lambda = 0$.

In this chapter we select one of the discrete rotating waves (4.3) and stabilize its periodic orbit.

4.2 Equivariant Pyragas control with one time delay

In this section we choose the discrete rotating wave of the form (4.3) and apply equivariant Pyragas control as proposed in (2.1):

$$\dot{z}_k(t) = \left(\lambda + \mathrm{i} + \gamma|z_k(t)|^2\right) z_k(t) + b\left(-z_k(t) + z_{k+1}\left(t - p/3\right)\right) \qquad k = 1, 2, 3$$

Here, the group element h_1 describing the spatial symmetry is given by $h_1 z_k = z_{k+1}$, i.e. an index shift between the oscillators. The time delay τ is given by $\tau = \Theta(h_1)p = p/3$. $\Theta(h_1)$ describes the temporal symmetry, which is a time shift by one third of the period.

By Theorems 1 and 2, we can find the conditions for the stabilization within the center manifold. Additionally, we can use Theorem 3 to find the largest possible coupling parameter a for which a stabilization is possible. We will see that for the oscillator triangle the necessary condition given in Theorem 3 is also sufficient.

In a first step it is useful to decouple the linearization of the system (4.1), (4.2) at the trivial equilibrium $z \equiv 0$ which is possible due to Schur's Lemma (see appendix). The new coordinates are given by:

$$\begin{aligned} x_1 &= \tfrac{1}{3}(z_1 + z_2 + z_3) \\ x_2 &= \tfrac{1}{3}(z_1 + e^{+2\pi\mathrm{i}/3} z_2 + e^{-2\pi\mathrm{i}/3} z_3) \\ x_3 &= \tfrac{1}{3}(z_1 + e^{-2\pi\mathrm{i}/3} z_2 + e^{+2\pi\mathrm{i}/3} z_3) \end{aligned} \tag{4.5}$$

The diagonal form of the linearization, in these new coordinates (4.5), including the control term as given above, is then

$$\begin{aligned} \dot{x}_1(t) &= (\lambda + \mathrm{i})\, x_1(t) + b\,(-x_1(t) + x_1(t-\tau)) \\ \dot{x}_2(t) &= (\lambda - 3a + \mathrm{i})\, x_2(t) + b\left(-x_2(t) + e^{-2\pi\mathrm{i}/3} x_2(t-\tau)\right) \\ \dot{x}_3(t) &= (\lambda - 3a + \mathrm{i})\, x_3(t) + b\left(-x_3(t) + e^{+2\pi\mathrm{i}/3} x_3(t-\tau)\right). \end{aligned}$$

Note that without control, i.e. $b = 0$, the linearization of the second and the third coordinate coincide, both giving a Hopf bifurcation at $\lambda = 3a$, which corresponds to the equivariant Hopf bifurcation of discrete rotating waves as discussed before. With control, i.e. $b \neq 0$, however, the linearization of the second and the third coordinate do not coincide anymore. This reflects the fact that equivariant Pyragas control, as introduced above, is noninvasive only on exactly one periodic orbit (4.3), corresponding to the coordinate x_3.

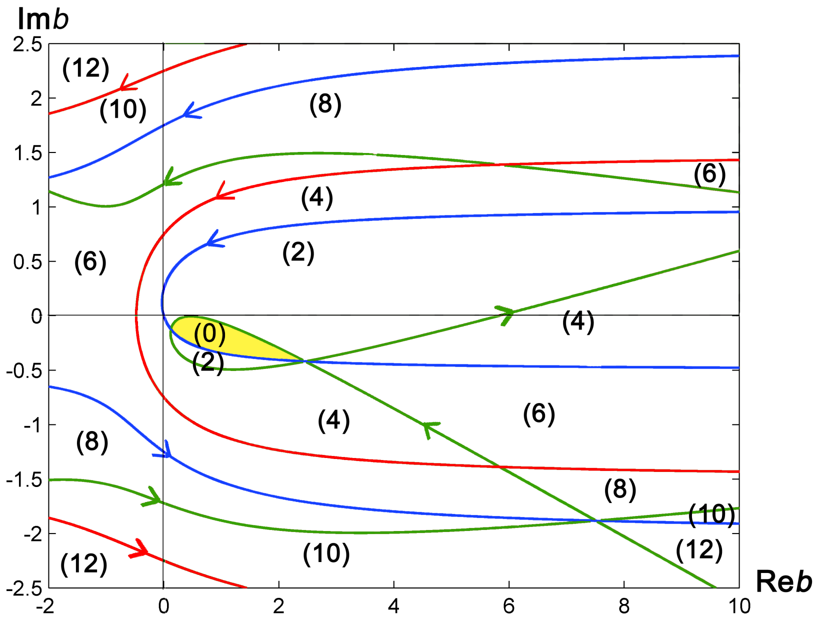

Figure 4.2: b-curves in the complex plane for $a = 0.1$. The numbers in parentheses give the values $E(b)$. The region with zero eigenvalues with positive real part is shaded yellow.

The characteristic equation $\chi(\eta) = 0$ for exponentials $x(t) = e^{\eta t}(x_1, x_2, x_3)$ decouples into a product of three factors, $\chi = \chi_1 \cdot \chi_2 \cdot \chi_3 = 0$ which are given by

$$\begin{aligned}
\chi_1(\eta) &= \lambda \qquad\quad + \mathrm{i} + b\left(-1 + e^{-\tau\eta} \qquad\quad\right) - \eta \\
\chi_2(\eta) &= \lambda - 3a + \mathrm{i} + b\left(-1 + e^{-2\pi \mathrm{i}/3 - \tau\eta}\right) - \eta \\
\chi_3(\eta) &= \lambda - 3a + \mathrm{i} + b\left(-1 + e^{+2\pi \mathrm{i}/3 - \tau\eta}\right) - \eta.
\end{aligned}$$

From these characteristic equations, it is now easy to calculate the stability changes in the complex plane of the control parameter b, compare with Proposition 1:

$$\begin{aligned}
b_1(\omega) &= \tfrac{3}{2}\left(a - \mathrm{i}\omega - (\omega + \mathrm{i}a)\cot\left(\pi\left(\omega + \tfrac{1}{3}\right)\right)\right) && \textbf{(green)} \\
b_2(\omega) &= -\tfrac{3}{2}\,\omega\left(\cot\left(\pi\left(\omega + \tfrac{2}{3}\right)\right) + \mathrm{i}\right) && \textbf{(blue)} \\
b_3(\omega) &= -\tfrac{3}{2}\,\omega\left(\cot\left(\pi\omega\right) + \mathrm{i}\right) && \textbf{(red)}.
\end{aligned}$$

Two examples of the b-curves in the complex plane are shown in figures 4.2 ($a = 0.1$) and 4.3 ($a = 0.035$), the region with zero eigenvalues with real part greater than zero, i.e. the region where the controlled discrete rotating wave (4.3) is born stable, is marked (0) and shaded yellow.

We can observe that the yellow region shrinks if the coupling parameter a is increased, compare figure 4.2 with 4.3. The yellow region disappears if $3a$ is increased beyond the solution of (2.11), in agreement with Theorem 3. The b-curves preserve complex orientation, the orientation of the curves is indicated by arrows in figures 4.2 and 4.3. Also note that only the green curve, corresponding to coordinate x_1, depends on the

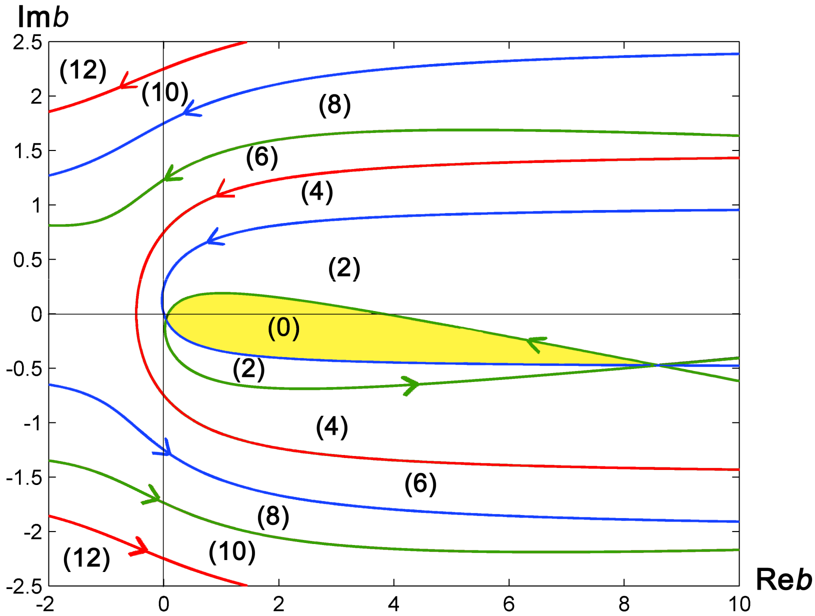

Figure 4.3: b-curves in the complex plane for $a = 0.035$. The numbers in parentheses give the values $E(b)$. The region with zero eigenvalues with positive real part is shaded yellow.

coupling parameter a. The red and the blue curve, which give the equivariant Hopf bifurcation in the uncontrolled system, show exactly the behaviour sketched in figure 2.1.

Furthermore, we can give an explicit value for the condition for supercriticality: In the case of the soft spring $\operatorname{Im}\gamma > 0$, we can define the function $\beta(a)$ as

$$\beta(a) := \min\left\{ -\frac{\operatorname{Re} b + 1/(2\pi\Theta)}{\operatorname{Im} b} \;\middle|\; b \in \Lambda, \operatorname{Im} b > 0 \right\}.$$

In the subcritical case, we can then guarantee the existence of a stabilization region if the condition

$$|\operatorname{Im}\gamma| > \beta(a) \operatorname{Re}\gamma > 0$$

is fulfilled. In the hard spring case $\operatorname{Im}\gamma < 0$, we define

$$\beta(a) := \min\left\{ \frac{\operatorname{Re} b + 1/(2\pi\Theta)}{\operatorname{Im} b} \;\middle|\; b \in \Lambda, \operatorname{Im} b < 0 \right\}.$$

Note that in this case, as explained in [7, 8], $\beta(a) = \bar\beta$ is a constant. Here, we can guarantee the existence of a stabilization region if the condition

$$\operatorname{Im}\gamma > \bar\beta \operatorname{Re}\gamma > 0$$

can be fulfilled.

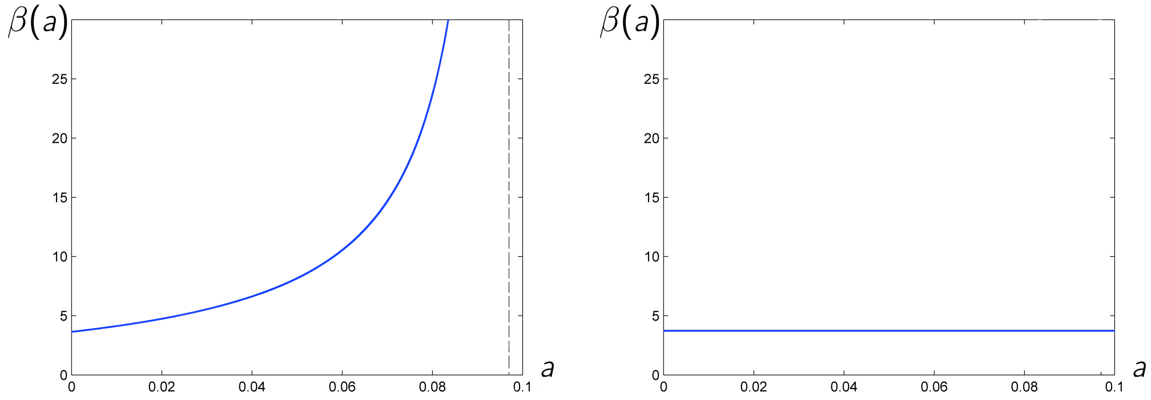

Figure 4.4: Left: Sketch of the function $\beta(a)$ in the soft spring case $\operatorname{Im}\gamma < 0$ if the group element $h_1 z_k = z_{k+1}$ is used, or in the hard spring case $\operatorname{Im}\gamma > 0$ for the group element $h_2 z_k = z_{k-1}$. Right: Sketch of the function $\beta(a)$ in the soft spring case $\operatorname{Im}\gamma < 0$ if the group element $h_2 z_k = z_{k-1}$ is used or in the hard spring case $\operatorname{Im}\gamma > 0$ for the group element $h_1 z_k = z_{k+1}$.

4.3 Choice of the group element

In the previous section, we chose the group element $h_1 z_k = z_{k+1}$ to be the index shift by one oscillator with corresponding time delay $\tau = \Theta(h_1)p = \frac{p}{3}$. It is also possible to choose the index shift by two oscillators $h_2 z_k = z_{k+2} = z_{k-1}$, resulting in a time delay of $\tau = \Theta(h_2)p = \frac{2p}{3}$. We can then compare the characteristic equations and the region of stabilization, see figure 4.5 for $a = 0.01$.

Note that the choice of group element does not only influence the shape of the stabilization region, but also its existence, particularly in the subcritical case.

Since Λ depends on the group elements h_1, h_2, as well as Θ, we can see that the function $\beta(a)$ can behave differently for different h_1, h_2. In the best case, it can be a constant, in the worst case it diverges for $a \to A$ where A is the value where the loop disappears.

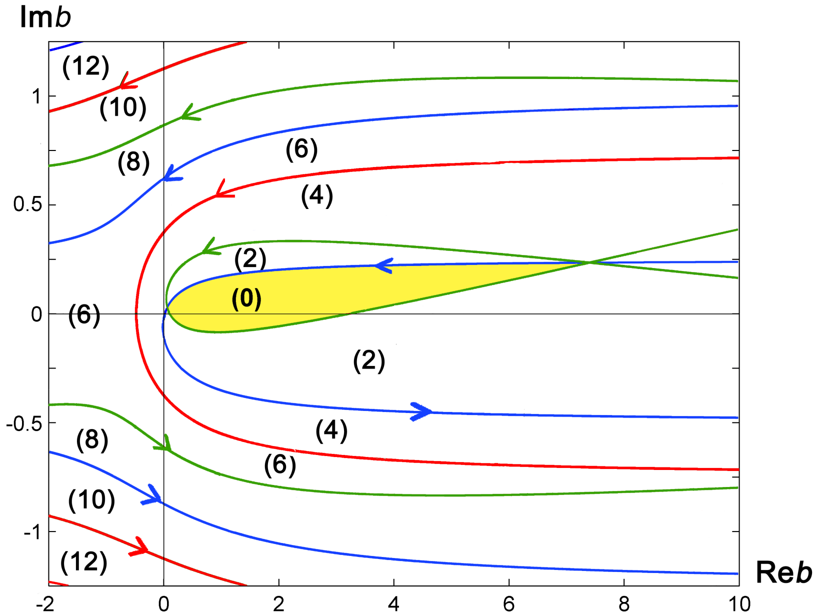

Figure 4.5: Choice of the group element $h_2 z_k = z_{k+2}$, $\Theta(h_2) = 2/3$: b-curves in the complex plane for $a = 0.01$. The numbers in parentheses give the values $E(b)$. The region with zero eigenvalues with positive real part is shaded yellow. Note the different scaling of the Im b-axis in comparison with figures 4.2 and 4.3.

4.4 Controlling with the sum over all group elements

In the previous section, we have seen that the choice of group elements greatly influences the shape and the existence of a control region. Now we want to examine what happens, if we include all group elements to the coupling, i.e. we introduce a control of the form

$$\begin{aligned}\dot{z}_k(t) = &\left(\lambda + \mathrm{i} + \gamma |z_k(t)|^2\right) z_k(t)\\ &+ b\left(-z_k(t) + \frac{1}{3}\left(z_k(t) + z_{k+1}\left(t - \frac{p}{3}\right) + z_{k+2}\left(t - \frac{2p}{3}\right)\right)\right)\end{aligned}$$

for $k = 1, 2, 3$. We can again use the coordinate transformation (4.5), and we obtain the characteristic equations

$$\begin{aligned}\chi_1(\eta) &= \lambda \qquad\quad + \mathrm{i} + \frac{b}{3}\Big(-2 + e^{-\tau\eta} \qquad\quad + e^{-2\tau\eta} \qquad\qquad\Big) - \eta\\ \chi_2(\eta) &= \lambda - 3a + \mathrm{i} + \frac{b}{3}\Big(-2 + e^{-2\pi\mathrm{i}/3 - \tau\eta} + e^{+2\pi\mathrm{i}/3 - 2\tau\eta}\Big) - \eta\\ \chi_3(\eta) &= \lambda - 3a + \mathrm{i} + \frac{b}{3}\Big(-2 + e^{+2\pi\mathrm{i}/3 - \tau\eta} + e^{-2\pi\mathrm{i}/3 - 2\tau\eta}\Big) - \eta.\end{aligned}$$

The corresponding b-curves where the stability changes, are drawn in figure 4.6 for $a = 0.1$. Note that their shape changes.

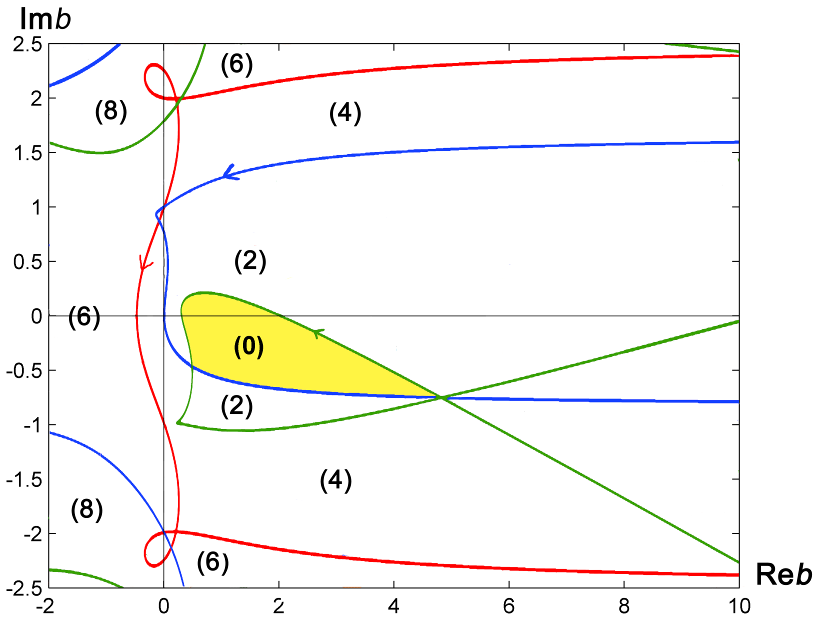

Figure 4.6: Sum over all group elements: b-curves in the complex plane for $a = 0.1$. The numbers in parentheses give the values $E(b)$. The region with zero eigenvalues with positive real part is shaded yellow.

Also the existence of the yellow region corresponding to $E(b) = 0$ is now given for $a \lesssim 0.350$, as compared to $a \lesssim 0.296$ for the use of group element h_1 alone and $a \lesssim 0.148$ compared to h_2.

4.5 Combining a control matrix and a single delay term

At the end of this case study, let us shortly consider the case of a control matrix B instead of a complex control parameter b. Matrices which commute with the group element $h_1 z_k = z_{k+1}$ take the form

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_3 & b_1 & b_2 \\ b_2 & b_3 & b_1 \end{pmatrix}$$

where b_1, b_2 and b_3 are complex each and can be chosen independently. This can be checked directly since we can write h_1 in its matrix form:

$$h_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The characteristic equations are

$$\begin{aligned}
\chi_1(\eta) &= \lambda \qquad\quad + \mathrm{i} + b_1 \left(e^{-\tau\eta} \qquad\quad - 1\right) - \eta \\
\chi_2(\eta) &= \lambda - 3a + \mathrm{i} + b_2 \left(e^{-2\pi\mathrm{i}/3-\tau\eta} - 1\right) - \eta \\
\chi_3(\eta) &= \lambda - 3a + \mathrm{i} + b_3 \left(e^{+2\pi\mathrm{i}/3-\tau\eta} - 1\right) - \eta.
\end{aligned}$$

As described in section 3.1, the three equations are now completely decoupled, meaning that we can now find a control region in the parameter b_1 independent from the regions in b_2 and b_3 and vice versa. Therefore, the control region is larger than just the overlap of the three regions, as required for one single complex control parameter b. This gives us much more freedom for the choice of control.

Furthermore, the condition for a supercritical bifurcation must only be fulfilled in the b_3 parameter, not for b_1 or b_2, as investigated in section 3.1. Therefore the only condition which remains for the existence of a stabilization region, even in the subcritical case, is $a \lesssim 0.296$, in agreement with Theorem 3 with $a_1 = 3a$.

Chapter 5

Summary and Discussion

Summary

In the present thesis, we modified the well-known Pyragas control to include the spatio-temporal symmetry of the unstable periodic orbit as follows:

$$\dot{z}(t) = F(\lambda, z(t)) + b\,(-z(t) + hz(t - \Theta(h)\,p)) \tag{5.1}$$

The method, which we call *equivariant Pyragas control*, can be applied to all periodic orbits with a two-dimensional fixed-point space of the isotropy group. In particular, we applied it near G-equivariant Hopf bifurcation, where G is the symmetry group, and gave explicit necessary and sufficient conditions for the stabilization. It is most important that the modified Pyragas control only leaves a two-dimensional dynamical subspace invariant. This subspace includes the selected periodic orbit.

We were able to give an explicit region of control parameters b where the stabilization is successful. The proof of stabilization of the selected periodic orbit goes via stabilization of the trivial equilibrium and establishing supercriticality of the Hopf-bifurcation.

For the remaining eigenvalues with positive real part, we found upper thresholds depending on the element which describes the spatio-temporal symmetry in the control term.

We also discussed two variations of equivariant Pyragas control: In the first variation, we used a control matrix B versus a control parameter b. Apart from a nondegeneracy condition, the existence of a control region can now always be guaranteed. In the second variation, we considered control terms of the form

$$\dot{z}(t) = F(\lambda, z(t)) + b\left(-z(t) + \int_H hz(t - \Theta(h)\,p)\,\mathrm{d}h\right). \tag{5.2}$$

This results in different and sometimes larger stabilization regions compared to the form of control discussed above.

As a detailed case study, we considered three diffusively coupled Stuart-Landau oscillators coupled in a symmetric triangle, i.e. a system with $D_3 \times S^1$-symmetry. We were able to apply equivariant Pyragas control and show that the concrete results are in complete agreement with the general results developed in chapters 2 and 3. Additionally we also discussed the choice of the group element for equivariant Pyragas control, a question which can up to date not be answered in general.

Discussion

The introduction of equivariant Pyragas control (5.1) and (5.2) raises several questions, especially in comparison to standard Pyragas control.

The main advantage of Pyragas control is that it is model-independent – it can be applied to practically any periodic orbit without specific knowledge of the dynamical system. Equivariant Pyragas control requires more knowledge – the knowledge of a spatio-temporal pattern. However there is a new advantage: We are now able to select unstable periodic orbits and stabilize them.

This example is realized in our case study in chapter 4 – three diffusively coupled Stuart-Landau oscillators. There, two discrete rotating waves of the same period and the same radius bifurcate at the same bifurcation point. In fact, they are only distinguishable in the numbering of oscillators. Only with equivariant Pyragas control we can select one of the two waves and stabilize it. This is in contrast to standard Pyragas control which would be noninvasive on both periodic orbits.

But also in the case where we do not need to select a periodic orbit, because it is already sufficiently described by its period, Pyragas control might fail where equivariant Pyragas control succeeds. For example, think of two diffusively coupled Stuart-Landau oscillators, as considered formerly in [4, 10, 13]. Here a time delay of half a period was used, because a stabilization region does not exist for standard Pyragas control. This is in agreement with Theorem 3. The same phenomenon occurs in rings of coupled oscillators with positive diffusive coupling. This is an indication that equivariant Pyragas control can be applied in coupled oscillator systems where Pyragas control fails.

A similar problem to the one in this thesis has been discussed by Postlethwaite et al. [9]. The authors establish only the existence of a stabilization region within the center manifold. In our thesis, we are now able to give explicit regions of stabilization. Additionally, we have also overcome the condition $\dim_{\mathbb{R}}(\mathrm{Stab}(e^{2\pi \mathrm{i}\Theta(h)}h)) = 2$ for a specific group element by introducing a control of the form (5.2).

The control of the form (5.1) or (5.2) can be introduced to general equivariant dynamical systems. In this thesis we have considered unstable periodic orbits close to equivariant Hopf bifurcation. However, equivariant Pyragas control does not require any knowledge of the particular system except for the spatio-temporal symmetry of the periodic orbit. Hence, it should in principle also be possible to apply the control of the form (5.1) to any periodic orbit with symmetry.

Appendix A

Schur's Lemma

Theorem 7 (Schur's Lemma, [6]) *Let ρ be an irreducible complex representation of a group G on X where X is a vector space. Let $A\colon X \to X$ be linear and equivariant, i.e. A commutes with all $\rho(g)$, $g \in G$.*

Then there exists $\lambda \in \mathbb{C}$ such that

$$A = \lambda \,\mathrm{Id}\,.$$

Equivariant Hopf bifurcation

Theorem 8 (Equivariant Hopf bifurcation, [6]) *Consider the differential equation*

$$\dot{z}(t) = F(\lambda, z(t)) \tag{A.1}$$

with $F(\lambda, 0) \equiv 0$, where $F \in C^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $k \geq 2$. Let F be G-equivariant. Let $A(\lambda) := D_z F(\lambda, 0)$.

Assume furthermore:

- $\pm\mathrm{i} \in \operatorname{spec} A(0)$, *and its algebraic multiplicity equals its geometric multiplicity*
- *the real dimension of the fixed point subspace of the group H^Θ*

  $$\begin{aligned} H^\Theta &:= \left\{(h, \Theta) \in H \times S^1 \mid \Theta = \Theta(h) \quad (\forall h \in H)\right\} \\ H &:= \left\{h \in G \mid \exists\, \Theta = \Theta(h) \ \in S^1 \ \text{ s. t. } \ hz(t) = z\left(t + p\,\Theta\right) \quad (\forall t)\right\}, \end{aligned}$$

  $p > 0$ minimal period, in $\mathrm{Eig}(\pm\mathrm{i}, A(0))$ *is two*
- $\pm n\mathrm{i} \notin \operatorname{spec} A(0)$
- *the continuation $\mu(\lambda) \in \operatorname{spec} A(\lambda)$ of $\mu(0) = \pm\mathrm{i}$ crosses the imaginary axis transversally at $\lambda = 0$, i.e.*

  $$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \operatorname{Re}\mu(\lambda) \neq 0$$

Then follows the local bifurcation of a branch of periodic solutions

$$s \mapsto (\lambda(s), p(s), z(t,s))$$

of equation (A.1) in $\lambda = 0,\ z = 0$ *with symmetry* H^{Θ} *and* $\lambda(0) = 0,\ p(0) = 2\pi,\ z(t,0) = 0$ *and* $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} z(t,s) \in \mathrm{Eig}(\pm \mathrm{i}, A(0))\backslash\{0\}$.
The minimal period of $t \mapsto z(t,s)$ *for* $s > 0$ *is* $p(s)$.

Planar Hopf normal form

One oscillator in Hopf normal form, also known as Stuart-Landau oscillator in physics, is given by

$$\dot{z} = f(z) = (\lambda + \mathrm{i} + \gamma |z|^2) z.$$

Here $z \in \mathbb{C}$, $\lambda \in \mathbb{R}$, $\gamma \in \mathbb{C}$. This system can be rewritten into polar coordinates $z(t) = r(t)\, e^{\mathrm{i}\varphi(t)}$:

$$\begin{aligned} \dot{r} &= (\lambda + \mathrm{Re}\,\gamma\, r^2)\, r \\ \dot{\varphi} &= 1 + \mathrm{Im}\,\gamma\, r^2. \end{aligned}$$

We see that there is an equilibrium for $r \equiv 0$ (equivalently $z \equiv 0$). It is useful to linearize the system at $z = 0$ to find out the local stability of the equilibrium:

$$\dot{z} = (\lambda + \mathrm{i}) z.$$

For $\lambda > 0$ we have therefore two eigenvalues with positive real part and the equilibrium is strictly unstable. We say that the *unstable dimension*, counting the number of eigenvalues with strictly positive real part, is two. For $\lambda < 0$ we have two eigenvalues with negative real part, resulting in asymptotic stability of the equilibrium and unstable dimension zero. The standard Hopf bifurcation occurs at $\lambda = 0$. Here a pair of complex conjugated eigenvalues $\pm\mathrm{i}$ crosses the imaginary axis and a periodic orbit emerges from the trivial equilibrium. For $(\lambda\, \mathrm{Re}\,\gamma) < 0$ there exists a periodic orbit $z(t) = r \exp(\frac{2\pi \mathrm{i} t}{p})$, whose radius r and minimal period p are given by

$$\begin{aligned} r^2 &= \frac{-\lambda}{\mathrm{Re}\,\gamma} \\ p &= \frac{2\pi}{1 + r^2\, \mathrm{Im}\,\gamma}. \end{aligned}$$

Note that the radius as well as the period depend on the bifurcation parameter λ. We call the bifurcation *subcritical* if the periodic orbit exists for $\lambda < 0$ and *supercritical* if it exists for $\lambda > 0$. In this special case, the periodic orbit is unstable if the bifurcation is subcritical, and stable in the supercritical case.

A *soft spring* is given if the minimal period p increases with growing amplitude, in contrast to the *hard spring* where p decreases with amplitude.

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Selbstständigkeitserklärung

Hiermit bestätige ich, Isabelle Schneider, dass ich die vorgelegte Masterarbeit mit dem Thema

Equivariant Pyragas control

selbstständig angefertigt und nur die erwähnten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

Berlin, den 3. Februar 2014