An introduction to the control triple method for partial differential equations

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Abstract We give an introduction to the control triple method, a new type of noninvasive spatio-temporal feedback control. The notion of a control triple defines how we transform the output signal, space, and time in the control term. This Ansatz, especially well suited for the control of partial differential equations, does not exist in the literature so far. It incorporates the spatio-temporal patterns of the equilibria and periodic orbits into the control term. We give linear examples to demonstrate the success of the control triple method.

Dedicated to Bernold Fiedler on the occasion of his sixtieth birthday

1 Introduction

In this this chapter we give a short introduction to a recent extension of noninvasive time-delayed feedback control for partial differential equations: the control triple method.

The control triple method is based on Pyragas control [1] which is nowadays one of the most successful methods to control the stability of periodic orbits or equilibria in dynamical systems.

Consider an ordinary differential equation $\dot{z}(t) = f(z(t))$, $z \in \mathbb{R}^n$. Then the equation including Pyragas control is described by

$$\dot{z}(t) = f(z(t)) + k\big(z(t) - z(t-\tau)\big). \tag{1}$$

The weight $k \in \mathbb{R}^{n \times n}$ of the control term is called the *feedback gain*. The control term introduced by Kestutis Pyragas uses the difference between a delayed state

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$z(t-\tau)$ and the current state $z(t)$ of the system. If a periodic orbit $z^*(t)$ is stabilized, the time delay τ is chosen to be an integer multiple of its minimal period p. In this case, the control vanishes on the orbit itself, and $z^*(t)$ is also a solution of the equation including Pyragas control (1). Thus, the control does not change the periodic orbit itself, it only changes its stability properties. We call such a control term *noninvasive*. For the stabilization of equilibria, the time delay τ can be chosen arbitrarily.

In summary, Pyragas control is used to make unstable objects visible without changing them.

The Pyragas control method is one of the most used feedback control schemes today. The original paper from 1992 [1] has been cited more than 3500 times (as of February 2017). Its main advantage is given by the fact that one does not need to know anything about the periodic orbit besides its period. In particular, Pyragas control is a model-independent control scheme and no expensive calculations are needed for its implementation.

Even though many applications and extensions of Pyragas control have been proposed since 1992, surprisingly few publications consider the *spatial* properties of *partial* differential equations for control. A first attempt to use *space* as well as *time* was proposed by Lu et al. in 1996 [2], but there and in subsequent publications [3, 4] spatial modifications and time delay are only used separately.

Combinations of spatial and temporal delay have only been introduced recently, in form of the control triple method. These new noninvasive and *spatio-temporal* control terms have been developed and applied to scalar reaction-diffusion equations in the author's PhD thesis [5].

We give an introduction to the control triple method in this chapter with the purpose of illustrating the main concepts and why stabilization via the control triple method is successful. We follow [5].

This chapter is outlined as follows: In Section 2 we describe the setting in which we want to apply the control triple method. we propose our main goal and introduce the control triple. In Section 3 we introduce the concept of the control triple method and the corresponding control terms. Section 4 presents the main result of this chapter. Section 5 is devoted to an illustration of the method and the main theorem, showing how the control triple method works for the linear examples. In Section 6 we give a brief exposition on more general control triples and possible extensions. We conclude and discuss in Section 7.

2 Setting

In this chapter the main area of application of the new control terms are *scalar reaction-diffusion equations* including a linear advection term cu_x,

$$u_t = u_{xx} + f(u) - cu_x, \tag{2}$$

$u \in \mathbb{R}$, $x \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, $t > 0$, with periodic boundary conditions:

$$u(0,t) = u(2\pi,t),\ u_x(0,t) = u_x(2\pi,t) \text{ for all } t > 0. \tag{3}$$

The function $f : \mathbb{R} \to \mathbb{R}$ is real analytic and dissipative. These assumptions on f are not essential for control, but we restrict f in order not to lose ourselves in technical difficulties. The real parameter c is called the *wave speed*.

Note that the nonlinearity f does not depend explicitly on the space variable x. Therefore, equation (2) is is S^1-equivariant with respect to a shift R_θ in the x-variable,

$$R_\theta : \mathbb{X} \to \mathbb{X}, \quad (R_\theta u_0)(x) := u_0(x+\theta), \tag{4}$$

$\theta \in S^1$.

Equilibria $\mathscr{U}(x,t)$ of (2) are characterized by $\mathscr{U}_t \equiv 0$. Hence, they are 2π-periodic solutions of the ordinary differential equation

$$0 = \mathscr{U}_{xx} + f(\mathscr{U}) - c\,\mathscr{U}_x. \tag{5}$$

Those equilibria which additionally fulfill $R_\theta\,\mathscr{U} = \mathscr{U}$ for all $\theta \in S^1$ are called *homogeneous equilibria*. All other equilibria $\mathscr{U}$ are called *non-homogeneous equilibria* or *frozen waves*. Note that frozen waves can only occur if the wave speed $c = 0$.

Moreover, we find *relative equilibria* $\mathscr{U}(x,t)$ with respect to the group action of the equivariance group S^1. These relative equilibria are called *rotating waves of speed* $c \neq 0$ and they satisfy $\mathscr{U}(x-ct) = (R_{-ct}\,\mathscr{U})(x)$. Rotating waves $\mathscr{U}(x-ct)$ are 2π-periodic solutions of the ordinary differential equation

$$0 = \mathscr{U}_{zz} + f(\mathscr{U}), \tag{6}$$

in co-rotating coordinates $z = x - ct$. Note that the same equation also holds in the case $c = 0$, i.e., for frozen waves. Equation (6) is Hamiltonian, and we can therefore describe $\mathscr{U}$ as the motion of a point in a potential field with energy conservation. In theory, we can find the solutions with fixed energy E analytically via the relation

$$\mathscr{U}_z = \pm\sqrt{2(E - F(\mathscr{U}))}, \tag{7}$$

where F is the *potential*, $F'(\mathscr{U}) = f(\mathscr{U})$. Only for certain energy values E we find indeed periodic solutions with period 2π (where 2π is not necessarily the minimal period). A sketch of an arbitrary potential F and energy values which yield $2\pi/n$-periodic solutions, and hence rotating or frozen waves, can be found in Figure 1.

The rotating waves are *periodic orbits* unless the wave speed is $c = 0$, in which case they correspond to *frozen waves*, i.e., to a non-homogeneous equilibrium. Both rotating and frozen waves occur in circles given by the group orbits $\{R_\theta\,\mathscr{U} \mid \theta \in S^1\}$.

It was proven by Angenent and Fiedler [6] and by Matano [7] that all periodic orbits of (2) are indeed rotating waves. Moreover, all rotating waves are unstable [6]. We assume that the frozen or rotating waves are hyperbolic, in the sense that there exists no Floquet multiplier on the unit circle but the trivial one.

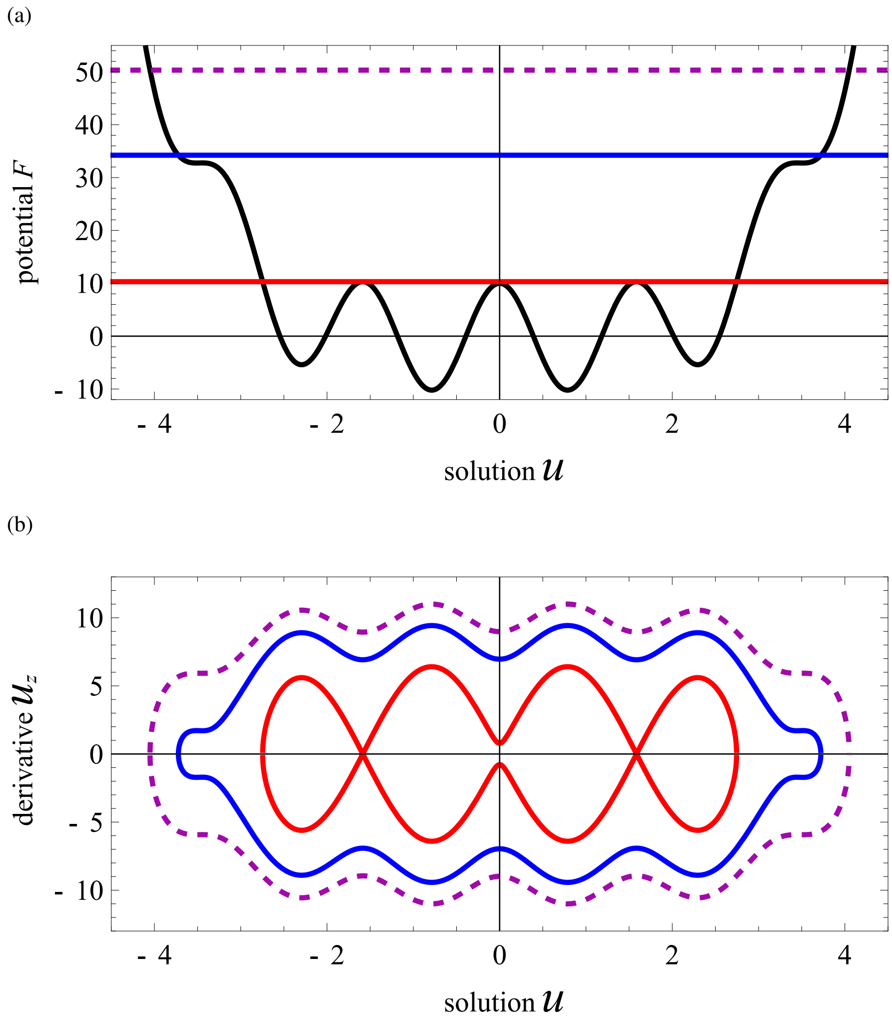

Fig. 1 (a) Hamiltonian potential $F(\mathscr{U})$ (black) for an odd nonlinearity $f(\mathscr{U}) = -f(-\mathscr{U})$ with energy levels corresponding to a 2π-periodic solution (red), a π-periodic solution (blue), and a $2\pi/3$-periodic solution (dashed violet). (b) Corresponding solutions in the phase-space $(\mathscr{U}, \mathscr{U}_z)$. This is a zoom-in to the interesting region of the hamiltonian potential; the higher order terms yielding dissipativity cannot be seen. This Figure has been published previously in [5].

Let us now consider odd nonlinearities f, i.e., $f(\mathscr{U}) = -f(-\mathscr{U})$ and rotating waves of minimal period $2\pi/n$, $n \in \mathbb{N}$. In this case, the potential $F(\mathscr{U})$, with $F'(\mathscr{U}) = f(\mathscr{U})$, is an even function. Therefore, if $\mathscr{U}(z)$ is a solution of equation (6), then $-\mathscr{U}(z)$ is also a solution of equation (6). These solutions may coincide as sets. If so, these solutions are phase-shifted by half the minimal period, i.e., we find

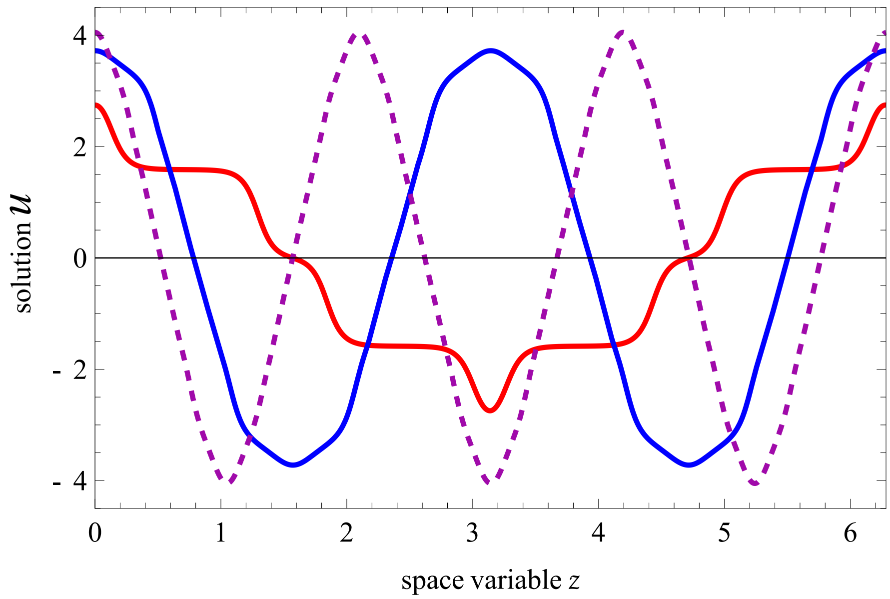

Fig. 2 Solutions $\mathscr{U}(z)$ from Fig. 1 (b), same color scheme. Note the rotational shift-symmetry for $2\pi/n$-periodic solutions, $n = 1, 2, 3$: $\mathscr{U}(z) = -\mathscr{U}(z - \pi/n)$. This Figure has been published previously in [5].

solutions of the form $\mathscr{U}(z) = -\mathscr{U}(z - \pi/n)$. See Figure 2 for example solutions with such rotational shift-symmetries.

In the following section we find new noninvasive control terms for the frozen and rotating waves, following the control triple method.

3 The control triple method

In this section, we introduce the control triple method, which is a new concept of noninvasive spatio-temporal feedback control for partial differential equations. The control triple method has been developed as a consequence of the failure of Pyragas control for the scalar reaction-diffusion equations [5]. Using the new concept of the control triple which we will describe below, we succeed in stabilizing certain periodic orbits and equilibria of the equation (2).

The general idea of the control triple method, as already used by Pyragas [1], is to use differences between output signals and “transformed” output signals. The resulting control must be noninvasive, i.e., vanish on the desired orbit. In the case of Pyragas control, “transformed” means “time-delayed”. Thus Pyragas uses the system parameter **time** for control. In the context of partial differential equations we can use the system parameter **space** x as well as **time** t for the construction of the new control terms. Also the **output signal** u of the system is an easily accessible system parameter, as it has been used previously in the context of equivariant Pyragas control [8, 9, 10, 11].

In total, we propose to introduce the notion of **control triples** to describe the transformation of the output signal:

$$\big(\textbf{output signal}, \textbf{space}, \textbf{time}\big)$$

We then construct the spatio-temporal feedback control as follows: We consider *noninvasive* differences of the current output signal $u(x,t)$ and the "transformed" output signal $\tilde{u}(\tilde{x},\tilde{t})$. The control triple indicates the precise transformation of each of the three system parameters: output signal $u \mapsto \tilde{u}$, space $x \mapsto \tilde{x}$, and time $t \mapsto \tilde{t}$.

We define a *control term* as a *fixed control triple* and a *variable feedback gain* k, where k is either a scalar or a matrix. A scalar feedback gain, as used here, decides the sign as well as the amplitude of the control.

Let us now find specific control terms for our model equation

$$u_t = u_{xx} + f(u) - cu_x, \tag{8}$$

following the control triple method.

In this introduction, we focus on the *control schemes of rotation type*: They combine a **scalar multiplication of the output signal**, **rotations in space**, and a **time delay**. We interpret the rotations in space as a spatial delay, and the controlled equation takes the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi u(x-\xi, t-\tau)\big), \tag{9}$$

where $k, \Psi, \in \mathbb{R}$, $\xi \in S^1$, and $\tau > 0$. As indicated above, we call the parameter ξ the *spatial delay*, and τ the *temporal delay*. All three parameters Ψ, ξ and τ are fixed and should be chosen a priori. The feedback gain is a variable parameter, it is chosen a posteriori to guarantee stabilization for a fixed control triple (Ψ, ξ, τ).

Let us now discuss the precise parameters for several special cases: In the previous section, we saw that all periodic orbits are indeed rotating waves of the form $u(x,t) = \mathscr{U}(x-ct)$. A time shift by $-\tau$ has then the same effect on the wave as a spatial rotation by $+c\tau$, and the controlled equation is of the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x-c\tau, t-\tau)\big). \tag{10}$$

Here we use an arbitrary *temporal delay* $\tau > 0$, and (only if the speed c of the wave is nonzero) a *spatial delay* $\xi = c\tau$. Furthermore, no transformation of the output is needed, i.e., $\Psi = 1$. The control term is clearly noninvasive on all rotating waves of speed $c \in \mathbb{R}$.

Note that control term proposed in equation (10) in fact contains the control of Pyragas type as a special case: The control terms of Pyragas and the control term as in (10) are equal if and only if $c\tau = 2\pi n$, $n \in \mathbb{N}$.

Consider next f odd and rotating or frozen waves with odd symmetry $\mathscr{U}(z) = -\mathscr{U}(z - m\pi/n)$, $m \in \mathbb{Z}$ is odd, and where $2\pi/n$ is the minimal spatial period. For such odd waves, the controlled equation can take the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - (-1)\,u(x - \xi, t - \tau)\big), \tag{11}$$

with the following condition relating the spatial delay ξ and the temporal delay τ:

$$\xi - c\tau = m\pi/n, \qquad m \in \mathbb{Z} \text{ odd}. \tag{12}$$

Here we introduce the parameter $m \in \mathbb{Z}$. It is clear that m needs to be odd, if the transformation of the output signal should be given by $\Psi = -1$. We will discuss this control term in detail in Section 5.

Next, consider homogeneous equilibria: For the application of the control triple method, we distinguish between those equilibria which take a fixed, non-zero value and those equilibria which take the value zero.

In the case of homogeneous non-zero equilibria, controlled equations are of the general form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x - \xi, t - \tau)\big). \tag{13}$$

The control-term is noninvasive on any homogeneous equilibrium for arbitrary spatial delay ξ and arbitrary temporal delay τ. The parameter Ψ is 1, similar to the case of rotating waves.

Homogeneous zero equilibria allow more general control triples: Any real parameter Ψ can be chosen for a noninvasive control, in addition to arbitrary spatial delay ξ and arbitrary temporal delay τ:

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi u(x - \xi, t - \tau)\big). \tag{14}$$

In this chapter, we focus on the control terms as given in equations (11), (12). The results concerning equations (10), (13), and (14) can be found in [5].

4 Main result

In this section we present our main result on the control equilibria and waves in scalar reaction-diffusion equations.

Theorem 1 (Successful stabilization of odd rotating and frozen waves, [5]). *Consider a rotating or frozen wave $\mathscr{U}(x - ct) = \mathscr{U}(z)$ with minimal spatial period $2\pi/n$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions. Additionally, assume $f(u) = -f(-u)$ and suppose that the rotating or frozen wave is odd, $\mathscr{U}(z) = -\mathscr{U}(z - \pi/n)$, with unstable dimension $2n - 1$.*

Then there exists a feedback gain $k^* \in \mathbb{R}$ such that the following holds:

For all $k < k^*$, there exists a time delay $\tau^* = \tau^*(k)$ such that the rotating or frozen wave $\mathscr{U}(x - ct) = \mathscr{U}(z)$ is stable in the controlled equation

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - (-1)u(x-\xi, t-\tau)\big), \tag{15}$$

where the spatial delay ξ *and the temporal delay* $\tau < \tau^*$ *are related via*

$$\xi - c\tau = m\pi/n, \tag{16}$$

where m is odd and co-prime to n.

In this chapter, our aim is to illustrate the conditions and the regions of stabilization as described in the theorem. We will do this by considering the simplest interesting examples, linear $f(u) = n^2u$, $n \in \mathbb{N}$. We will then be able to understand *why* the spatial delay ξ and the temporal delay $\tau < \tau^*$ are related via $\xi - c\tau = m\pi/n$, and also why m needs to be odd and co-prime to n. We will also briefly see why it is advantageous to use small time delays and see the region of stability in the feedback parameter k.

However, we do not prove this result, the complete mathematical details are far beyond the scope of this introductory chapter. Any interested reader will find the proof, as well as many related and more detailed results, in [5].

5 An illustration of the control triple method

In this section we illustrate the success and the main conditions on the control triple method, using the linear examples as a toy model. In Subsection 5.1, we present the example equations on which we will use the control triple method to stabilize its frozen waves. Next, we introduce the control triple method in Subsection 5.2. We start our investigation of stability for the case of zero time delay in Subsection 5.3. The case with time delay is divided into two parts: We calculate the real eigenvalues in Subsection 5.4 and find conditions on the complex conjugated eigenvalues in Subsection 5.5.

5.1 Linear reaction-diffusion equations

Let us consider the following linear reaction-diffusion equations:

$$u_t = u_{xx} + n^2u. \tag{17}$$

Throughout this section $n \in \mathbb{N}$ is fixed but arbitrary. No rotating waves exist since the waves speed $c = 0$. All frozen waves fulfill the ordinary differential equation

$$0 = u_{xx} + n^2u, \tag{18}$$

with 2π-periodic boundary conditions, and they are therefore of the form $\mathscr{U}(x) = A\sin(nx+\theta)$, $\theta \in S^1$, $A \in \mathbb{R}$.

Linearizing around the frozen waves yields again equation (17), since that equation is already linear. Let us solve that equation by separation of variables and an exponential Ansatz in time: $u(x,t) = g(x)e^{\lambda t}$. We obtain the ordinary differential equation

$$\lambda g = g_{xx} + n^2 g. \tag{19}$$

The 2π-periodic solutions of the linear equation (19) are called the *eigenfunctions*, $\lambda \in \mathbb{C}$ the corresponding *eigenvalues*. We calculate the eigenfunctions and eigenvalues using the exponential ansatz $g(x) = e^{\eta x}$, $\eta \in \mathbb{C}$. The function g is 2π-periodic if and only if $\eta = \pm iN$. We obtain 2π-periodic solutions for $\lambda = n^2 - N^2$, for $N \in \mathbb{N}$, where the eigenvalue is simple for $N = 0$ and double for $N \geq 1$. Note that we obtain exactly $2n-1$ positive eigenvalues, fulfilling the assumption of our main theorem.

5.2 *Using the control triple method*

Let us now invoke the control triple method to stabilize the frozen waves $\mathscr{U}(x)$:

$$u_t = u_{xx} + n^2 u + k\big(u - (-1)\, u(x - m\pi/n, t - \tau)\big), \tag{20}$$

where we use m odd and co-prime to n, and arbitrary time delay $\tau \geq 0$. It is straightforward to check that this control triple $(\Psi = 1, \xi = m\pi/n, \tau)$ is indeed noninvasive on all frozen waves $\mathscr{U}(x) = A\sin(nx + \theta)$, $\theta \in S^1$, $A \in \mathbb{R}$.

To understand the stabilization mechanism, let us compute the stability of these frozen waves in equation (20).

Again no linearization is needed, and we can solve equation (20) directly via separation of variables and an exponential Ansatz $u(x,t) = g(x)e^{\lambda t}$. We then obtain the following delay differential equation:

$$\lambda g = g_{xx} + n^2 g + k\left(g + e^{-\lambda\tau} g(x - m\pi/n)\right). \tag{21}$$

Note that the spatial and the temporal delay behave differently: The temporal delay gives an exponential term in λ. In contrast, the spatial delay results in a delay in equation (21).

Since equation (21) is again linear, we solve it via an exponential Ansatz, $g(x) = e^{\eta x}$, $\eta \in \mathbb{C}$. We search for periodic solutions of (not necessarily minimal) period 2π, since these solutions give us the eigenfunctions. We are interested in the question for which $\lambda \in \mathbb{C}$ there exist such 2π-periodic solutions, since the corresponding eigenvalues λ determine the stability of the frozen waves.

Solutions of period 2π exist if and only if $\eta = \pm iN$. As characteristic equations we obtain

$$\lambda = -N^2 + n^2 + k\left(1 + e^{-\lambda\tau \pm im\pi N/n}\right), \qquad N \in \mathbb{N} \tag{22}$$

We can split equation (22) into real and imaginary part, where we use the notation $\lambda = \mu + i\nu$:

$$\mu = -N^2 + n^2 + k\left(1 + e^{-\mu\tau}\cos(\nu\tau \pm m\pi N/n)\right), \qquad N \in \mathbb{N} \tag{23}$$
$$\nu = k e^{-\mu\tau}\sin(\nu\tau \pm m\pi N/n), \qquad N \in \mathbb{N}. \tag{24}$$

In the following three subsections, we will investigate these equations in detail to find the stabilization regions and understand the control mechanisms.

5.3 Stabilization for zero time delay

Let us consider zero time delay $\tau = 0$, first:

$$\mu = -N^2 + n^2 + k\left(1 + \cos(m\pi N/n)\right), \qquad N \in \mathbb{N}, \tag{25}$$
$$\nu = \pm k \sin(m\pi N/n), \qquad N \in \mathbb{N}. \tag{26}$$

In this case it is easy to check whether all eigenvalues have negative real part from equation (25):

$$0 < -N^2 + n^2 + k\left(1 + \cos(m\pi N/n)\right), \qquad N \in \mathbb{N}. \tag{27}$$

This condition is fulfilled for all feedback gains $k < k^*$ where

$$k^* = \min\left\{ \frac{N^2 - n^2}{1 + \cos(m\pi N/n)} \,\middle|\, N \in \mathbb{N},\ 0 \le N < n \right\}. \tag{28}$$

First, note that k^* is always negative. Second, note that k^* takes a finite value (i.e., control succeeds for all $k < k^*$) if and only if m is co-prime to n. This explains the condition on m in our main theorem (m is required to be odd because the control is supposed to be noninvasive).

Already at this stage, we can conclude successful stabilization for zero time delay.

5.4 Stabilization for nonzero time delay - Real eigenvalues

Let us next suppose $\tau > 0$. Here we distinguish between the real eigenvalues and the complex conjugated eigenvalues.

Let us start investigating the real eigenvalues. First, note that real eigenvalues only occur for such $N \in \mathbb{N}$ where $\sin(m\pi N/n) = 0$. Then either $\cos(m\pi N/n) = +1$ or $\cos(m\pi N/n) = -1$.

In both cases, we determine the real eigenvalues from the equation

$$\mu = -N^2 + n^2 + k\left(1 + e^{-\mu\tau}\cos(m\pi N/n)\right), \qquad N \in \mathbb{N}, \tag{29}$$

where we can solve for the feedback gain k, since it only occurs linearly.

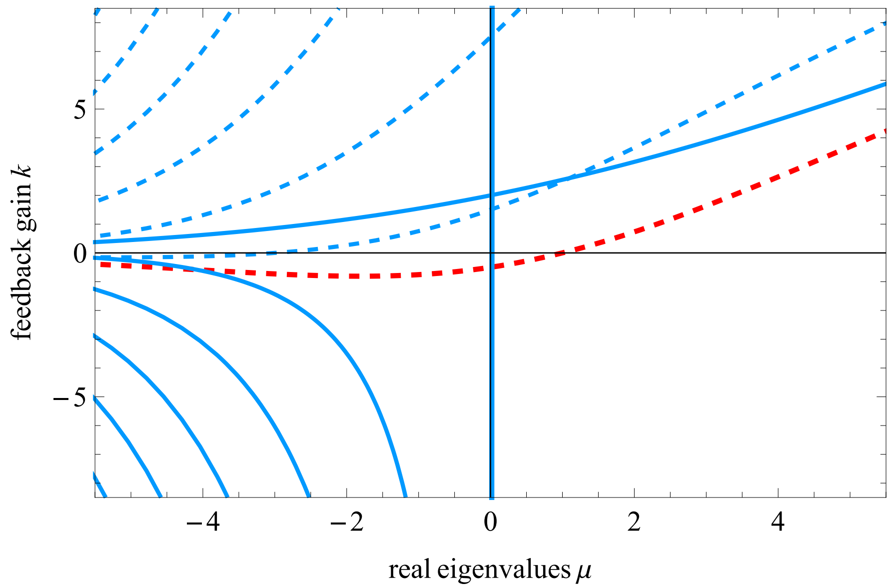

Fig. 3 The values of the feedback gain k (vertical axis), plotted versus the real eigenvalues μ (horizontal axis). The time delay is $\tau = 0.5$. Note that for $k < -0.5$ all nontrivial real eigenvalues are strictly negative. The curve for $N = 0$ is red, while all curves for $N \geq 1$ are blue. Curves corresponding to even N are dashed, to emphasize the difference between the two cases $\cos(m\pi N/n) = +1$ (dashed) and $\cos(m\pi N/n) = -1$ (solid). This Figure has been published previously in [5].

In the case $\cos(m\pi N/n) = +1$ we obtain

$$k_N(\mu) = \frac{\mu - n^2 + N^2}{1 + e^{-\mu\tau}}. \tag{30}$$

Note that this case only occurs for $N = 2\ell n$, $\ell \in \mathbb{N}$. This formula gives us the feedback gain k which has to be applied such that a real eigenvalue μ is reached, where we see N as a parameter. We conclude that the zero crossings of the real eigenvalues occur at

$$k_N(\mu) = \left(\mu - n^2 + N^2\right)/2 \quad \text{if} \quad \cos(m\pi N/n) = +1, \tag{31}$$

and thus does not depend on the time delay. The direction of the eigenvalue crossing is examined in detail in [5].

In the other case, $\cos(m\pi N/n) = +1$, we obtain

$$k_N(\mu) = \frac{\mu - n^2 + N^2}{1 - e^{-\mu\tau}}. \tag{32}$$

We conclude that no zero crossings, induced by the control, can occur. The eigenvalue curves (31) and (32) can be found in Figure 3 for the case $n = 1$.

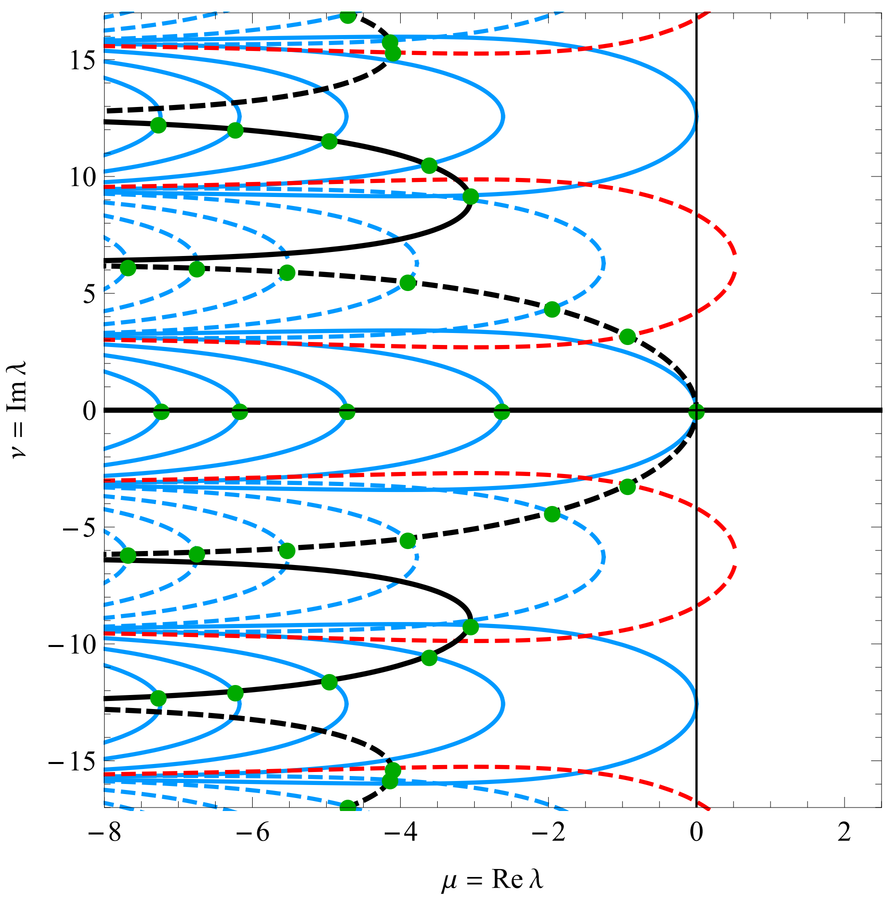

Fig. 4 Control triple method, success: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -2$. Here $Q = 1$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = \pi$, and $\tau = 0.5$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N > 0$. Curves for even N are dashed, curves for odd N are solid. This Figure has been published previously in [5].

Since none of the zero crossings depends on the time delay, we can directly conclude that the results from the case $\tau = 0$ also hold for the real eigenvalues with $\tau > 0$. It remains to check the complex conjugated eigenvalues for nonzero time delay to verify stabilization.

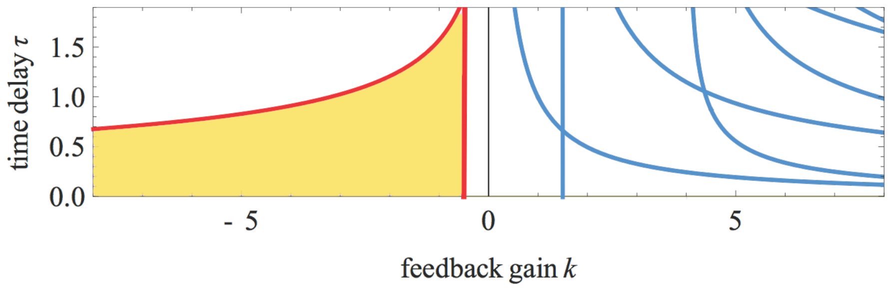

Fig. 5 Time delay τ^* versus feedback gain k for parameters $n = 1$, $\Psi = -1$, and $\xi = \pi$ for $N = 0,1,2,3$ (the curves are red for $N = 0$ and blue for $N = 1,2,3$). If the time delay τ is zero and $k < k^* = -1/2$, then control succeeds: In the yellow region in the background, all complex conjugated eigenvalues have negative real part, since $\tau < \tau^*(k)$. The vertical lines correspond to real eigenvalues, crossing zero at a specific feedback gain, they do not depend on the time delay.

5.5 Stabilization for nonzero time delay - Complex eigenvalues

From equations (23) and (24) it is straightforward to calculate that the complex conjugated eigenvalues are implicitly given by the crossings of the two curves

$$\nu(\mu) = \pm\frac{1}{\tau}\arccos\left(\frac{-\mu+Q+k-N^2}{k\Psi e^{-\mu\tau}}\right) \mp \frac{\varphi N - 2\pi n}{\tau},\ n \in \mathbb{N}_0, \tag{33}$$

$$\mu(\pm\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau \pm \varphi N)}\right), \tag{34}$$

in the complex plane (see [5] for the complete calculations). See also Figure 4.

We interested in the time delay $\tau^*(k)$ where the complex conjugated eigenvalues cross the imaginary axis, i.e., at the time delay where stability is lost (remember that stabilization is given for zero time delay). To this aim, let us look only for purely imaginary eigenvalues $\lambda = i\nu$:

$$0 = -N^2 + n^2 + k + k\cos(\nu\tau \mp m\pi N/n), \tag{35}$$

$$\nu = -k\sin(\nu\tau \mp m\pi N/n). \tag{36}$$

We square both equations, add them, and rearrange in the way that we obtain a quadratic equation in the imaginary part ν of the eigenvalues λ:

$$\nu^2 = k^2 - (-N^2 + n^2 + k)^2. \tag{37}$$

Going back to the first equation (35), we solve for τ:

$$\left(N^2 - n^2 - k\right)/k = \cos(\nu\tau \mp m\pi N/n), \tag{38}$$

$$\arccos\left(\left(N^2 - n^2 - k\right)/k\right) = \nu\tau \mp m\pi N/n, \tag{39}$$

and finally

$$\tau^*(k) = \frac{\arccos\left(\left(N^2 - n^2 - k\right)/k\right) \pm m\pi N/n}{\sqrt{k^2 - (-N^2 + n^2 + k)^2}}. \tag{40}$$

We can therefore conclude that control via the control triple method succeeds if m is co-prime to n, the feedback gain is chosen $k < k^*$, and additionally the time delay does not exceed the value calculate above, which is precisely the statement of the main result, if applied to a linear reaction diffusion equation.

6 More control triples

This chapter is intended as a gentle introduction to the control triple method. Therefore, we have so far focused on one specific form of the control triple for one particular type of equation, the reaction-diffusion equation. Since the control triple method is not limited to these specific control triples nor to this type of equation, we give a short overview on general control triples.

As a first example, consider an arbitrary *equilibrium* of some arbitrary partial differential equation. For such time-independent solutions, it is feasible to use differences of output signals at different moments of *time*. The transformations in the output signal u and the space x simplify to the identity transformation. Then

$$k\big(u(x,t) - u(x,t-\tau)\big), \tag{41}$$

is a noninvasive control term for all time delays $\tau > 0$. However, note that we are not limited to a fixed time delay τ; also state-dependent delay is a possibility here. In the case of time periodic orbits with minimal period p, the time delay is fixed to an integer multiple of the period p. While control terms of this type are the obvious application of Pyragas control to partial differential equations, it has recently been proven that they do not succeed in the case of scalar-reaction diffusion equations [5].

If the equilibrium has any additional structure, we can incorporate it into the control triple. For example, consider any *spatially periodic* equilibrium with period Φ and noninvasive control terms of the form

$$k\big(u(x,t) - u(x-\Phi,t-\tau)\big). \tag{42}$$

Next, consider the homogeneous zero equilibrium and noninvasive control terms of the form

$$k\big(u(x,t) - \Psi(u(\Phi(x),t-\tau))\big), \tag{43}$$

with arbitrary $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, $\Psi(0) = 0$, and arbitrary $\Phi : \Omega \to \Omega$, where Ω is the domain.

A nontrivial example is given by plane *waves* of the form $u(x,t) = A\exp(i\kappa\cdot x - ict)$, where $x \in \mathbb{R}^m$, $A \in \mathbb{R}$ is the amplitude, $\kappa \in \mathbb{R}^m$ is the wave vector and $c \in \mathbb{R}$ is the wave speed. Then

$$k\big(u(x,t) - \exp(i\kappa\cdot\xi - ic\tau)\,u(x-\xi,t-\tau)\big), \tag{44}$$

$a \in \mathbb{R}^n$, is a noninvasive control term with a control triple

$$(\Psi = \exp(i\kappa\cdot\xi - ic\tau), \xi = \xi, \tau = \tau)\,. \tag{45}$$

Many other examples such as spiral waves and traveling waves could be added to this list and be treated similarly.

A large class of systems to which the control triple method can be applied are the *equivariant systems*, where we find elaborate spatio-temporal patterns. Equivariance is usually described in terms of groups. Therefore, as a first step towards the construction of suitable control terms, it is necessary to find a description of the pattern in terms of group theory [12, 13, 14]. The transformations of the output signal, space, and time are interpreted as (linear) group actions in the equivariant setting.

We emphasize that all the described constructions of the control triple above do *not* depend on specific equations, they are model independent.

Nevertheless, let us go back to the scalar reaction-diffusion equations and see which other control triples could be used in addition to those which we have discussed already.

The *control schemes of reflection type* combine a **scalar multiplication of the output signal** and **reflections in space** with **time delay**. For such control terms, we only stabilize equilibria and we therefore restrict to the case $c = 0$.

More precisely, consider equilibria with the even reflection-symmetry $\mathscr{U}(x+\hat{x}) = \mathscr{U}(-x+\hat{x})$ around a reference point $\hat{x}$ (standing waves). We assume, without loss of generality, $\hat{x} = 0$. Then the controlled equation is of the general form

$$u_t = u_{xx} + f(u) + k\big(u - u(-x,t-\tau)\big), \tag{46}$$

i.e., we use a control triple of the form $(\Psi = 1, x \mapsto -x, \tau \geq 0)$.

Moreover, consider twisted standing waves, i.e., equilibria with odd reflection symmetry $\mathscr{U}(x) = -\mathscr{U}(-x)$. In this case, the controlled equation is of the form

$$u_t = u_{xx} + f(u) + k\big(u - \Psi u(-x,t-\tau)\big), \tag{47}$$

with $\Psi = -1$. If however, we want to stabilize the zero equilibrium (note that it is also a twisted standing wave), $\Psi \in \mathbb{R}$ can take any real value. Detailed results on the control of both standing waves and twisted standing waves can be found in [5].

For control schemes of reflection type, we do not consider rotating waves, since they would imply controls which combine rotations and reflections in space. Such **control schemes of mixed type** would then be of the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_1\Psi_2 u(-x-\xi,t-\tau)\big), \tag{48}$$

where both

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_1 u(x-\xi,t-\tau)\big), \tag{49}$$

and

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_2 u(-x, t-\tau)\big), \tag{50}$$

are valid equations of rotation and reflection type, respectively. At present, there are no results for control triples of mixed type.

Results which so far demonstrate the success of the control triple method are restricted to odd rotating waves. This is due to the fact that a non-identity transformation in the output signal is necessary for successful control. A great number of results could be obtained by using *non-constant* transformations of the output signal. We therefore propose to extend the control triple method by **control schemes of co-rotating type**:

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi(x - ct)\, u(x - \xi, t - \tau)\big). \tag{51}$$

Here both the spatial delay ξ and the time delay τ take arbitrary values and they do not need to be related in any way. Then Ψ is not necessarily a unique 2π-periodic function which guarantees noninvasiveness of the control triple.

In real life applications, distributed delays are a common feature. Therefore, we propose to include this phenomenon into the control triple method by **control schemes of distributed type**: We consider additive control terms of the form

$$k\left(u - \frac{1}{2\pi T}\int_0^T \int_0^{2\pi} \Xi(\xi)\Theta(\tau)\,\Psi(\xi,\tau)\, u(x-\xi, t-\tau)\, d\xi\, d\tau\right). \tag{52}$$

Note that we distribute the control both over space, with corresponding kernel $\Xi(\xi)$, as well as over time, with kernel $\Theta(\tau)$ and maximum time delay $0 \le T \le \infty$. Furthermore, note that the output transformations $\Psi(\xi,\tau)$ depend on the spatial delay ξ and the temporal delay τ. The kernels satisfy

$$\frac{1}{2\pi}\int_0^{2\pi} \Xi(\xi)\, d\xi = 1, \tag{53}$$

as well as

$$\frac{1}{T}\int_0^{T} \Theta(\tau)\, d\tau = 1, \tag{54}$$

to guarantee noninvasiveness. So far, only Dirac kernels Ξ and Θ have been discussed. This control scheme includes multiple discrete delays, as well as extended feedback control similar to [15], which has been proven very useful in the context of ordinary differential equations and should therefore be investigated in this general framework as well.

For similar reasons, we should also consider **control schemes of nonlinear type**,

$$u_t = u_{xx} + f(u) - cu_x + K(u(x,t), \Psi u(x-\xi, t-\tau)). \tag{55}$$

where $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is any (suitably smooth) function satisfying $K(y,y) = 0$. For ordinary differential equations, nonlinear control terms greatly enhance chances of stabilization [16, 17] and the question whether this also holds for partial differential equations should be the subject of further research.

7 Conclusion and discussion

In conclusion, we have presented a new approach to spatio-temporal feedback control of partial differential equations, namely the control triple method. In this short introduction into the topic, we have introduced the main concept, the control triples, which define how we transform output signal, space, and time in the control term such that the control term is noninvasive. We have also applied the control triple method directly to linear scalar reaction-diffusion equations. The example was chosen because it allows us to see the reason for the success of the control triple method as well as understand its main assumptions directly. The long and detailed proof for the general case is not included in this chapter, but can be found in [5].

The control triple method was inspired by Pyragas control, which it extends to a more general noninvasive control scheme. This step was necessary since it turns out that Pyragas control fails to stabilize equilibria and periodic orbits for scalar reaction-diffusion equations [5].

Let us now discuss and comment our results in the general framework of time-delayed feedback control.

In most situations, the presence of a time delay in a dynamical system is seen as a burden, as it greatly increases the dimensionality and the complexity of a dynamical system. Time delayed feedback control, and the control triple method in particular, use delay as a *tool* to achieve their goals. The control triple method has even introduced a *spatial delay*, thereby allowing stabilization to succeed. Stabilization is not possible if only time delay is used (Pyragas control). Our linear examples have shown us that a cleverly chosen combination of spatial and temporal delays renders stabilization possible and we were even able to get explicit results on the stabilization regions.

Such explicit results are rare: The combination of time delay, resulting in an infinite-dimensional equation, and the need for explicit and numerical results in control theory, are responsible for the fact that only few analytical results on Pyragas control and and its modifications have been obtained up to date. However, all the analytical results are extremely valuable, since they expand our knowledge on the mechanisms of time-delayed feedback control, which gives us the chance to design successful control terms.

Stepping away from the control aspect for a moment, let us interpret the spatial and temporal delays as additional parameters. This allows us to see interesting and in some sense unexpected dynamics for delay equations: Without delay, *all* rotating and frozen waves in scalar reaction-diffusion equations are unstable [6]. With spatio-temporal delay, however, we have indeed shown the existence of *stable waves* by explicit construction.

Let us end this chapter with a general outlook on future research and possible applications: The control triple method is designed to provide a tool for general partial differential equations. We have already seen how we can control arbitrary equilibria, equilibria with spatial patterns such as periodicity, and plane waves which occur in many physical systems. Equivariant systems also provide fruitful examples. A particular aspect of the control triple method is also that it *selects* orbits with desired properties by designing the control triple in such a way that it is noninvasive only on the desired orbit. In contrast, Pyragas control cannot distinguish between different equilibria, for example. For the reaction-diffusion equations, will be particularly interesting to apply the control triple method to higher dimensional domains, where many possibilities of spatial transformations arise, depending on the domain of the equation. Furthermore, systems of partial differential equations provide opportunities to use matrices as linear transformations of the output signal instead of scalar multiplications. For all these reasons, we encourage further investigations in the new research area of spatio-temporal feedback control for partial differential equations.

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