Delayed feedback control of three diffusively coupled Stuart-Landau oscillators: a case study in equivariant Hopf bifurcation

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Abstract

The modest aim of this case study is the noninvasive and pattern-selective stabilization of discrete rotating waves ("ponies on a merry-go-round") in a triangle of diffusively coupled Stuart-Landau oscillators. We work in a setting of symmetry-breaking equivariant Hopf bifurcation. Stabilization is achieved by delayed feedback control of Pyragas type, adapted to the selected spatio-temporal symmetry pattern. Pyragas controllability depends on the parameters for the diffusion coupling, the complex control amplitude and phase, the uncontrolled super-/sub-criticality of the individual oscillators and their soft/hard spring characteristics. We mathematically derive explicit conditions for Pyragas control to succeed.

1 Introduction

Following a broadly applicable idea of Pyragas [1], periodic orbits $z_*(t)$ of minimal period $p > 0$ can be stabilized by differences $z(t-\tau) - z(t)$ which involve a feedback term with time delay $\tau > 0$. The difference vanishes, i.e. the feedback becomes noninvasive, if $\tau = np$ is an integer multiple n of the minimal period p. For example this may happen in systems of the form

$$\dot{z} = F(z) + b[z(t-\tau) - z(t)] \tag{1.1}$$

with control matrix b and $z_*(t)$ solving $\dot{z} = F(z)$.

In the present paper we address the case of equivariant systems, i.e. of systems $\dot{z} = F(z)$ with a (linear) group action $z \mapsto gz$, such that $gz(t)$ is a solution whenever $z(t)$ is, for all elements g of an equivariance group G. The N-gon of diffusively coupled oscillators

$$\dot{z}_k = f(\lambda, z_k) + a(z_{k+1} - 2z_k + z_{k-1}), \tag{1.2}$$

k mod N, with parameters $\lambda \in \mathbb{R}$, $a > 0$ will serve as the principal example of our case study. The dihedral group $D_N = \langle \rho, \kappa \rangle$ of symmetries of a regular N-gon acts by index shift here. The rotations $\mathbb{Z}_N = \langle \rho \rangle$ are generated by $(\rho z)_k := z_{k-1}$ and κ is the reflection $(\kappa z)_k = z_{-k}$ with k mod N, $\kappa^2 = (\kappa\rho)^2 = \mathrm{Id}$. Frequently, the group $G = D_N$ itself is also described by its standard complex representation $\rho w = e^{2\pi \mathrm{i}/N} w$ and $\kappa w = \bar{w}$ on $w \in \mathbb{C}$ – not to be confused with the above action by index shift.

Following [2] the symmetry of a periodic orbit $z_*(t)$ of a G-equivariant system $\dot{z} = F(z)$ is given by triplets (H, K, Θ). Here $H \leq G$ leaves the orbit $\{z_*(t) : t \in \mathbb{R}\}$ fixed as a set. The possibly strict subgroup $K \leq H \leq G$ leaves each $z_*(t)$ fixed, pointwise, for each t. The group homomorphism $\Theta : H \to S^1 = \mathbb{R}/\mathbb{Z}$ is defined uniquely by time shift:

$$hz(t) = z(t + \Theta(h)p), \tag{1.3}$$

for all t. Note that Θ is well-defined because the normal subgroup of H is the kernel, $\Theta(K) = 0$. In local settings this construction has first been introduced by Golubitsky and Stewart; see for example [3, 4] .

Discrete *rotating waves* in oscillator N-gons (1.2), for example, are characterized by $H = \mathbb{Z}_N$ and $\Theta(e^{2\pi i/N}) = m/N$, usually with m coprime to N, i.e. $K = \{\mathrm{Id}\}$. *Standing waves*, in contrast, have $H = \langle \kappa \rangle$ and $\Theta(\kappa) = 0$. Another possibility, realized in human walking for example, is the alternating wave $H = \langle \kappa \rangle$ and $\Theta(\kappa) = 1/2$. Again $K = \{\mathrm{Id}\}$.

The main question of the present paper is the following: *How should one adapt the idea of delayed feedback control to selectively stabilize periodic orbits of prescribed symmetry type* (H, K, Θ)*?* In short: How to achieve noninvasive but pattern-selective feedback stabilization?

For the case of two oscillators, $N = 2$, $G = \langle \kappa \rangle$, this question has been addressed in [5], numerically. For a more detailed mathematical analysis see [6]. The crucial idea there were feedback terms of the form

$$hz(t - \tau) - z(t), \quad \text{with} \quad \tau := \Theta(h)p. \tag{1.4}$$

For $N = 2$, $H = \langle \kappa \rangle$ and $h = \kappa$ the delay τ therefore becomes the half period $p/2$, owing to $\Theta(h) = 1/2$, rather than the full period. Indeed control (1.4) becomes invasive on $z(t)$, unless

$$hz(t) = z(t + \Theta(h)p), \tag{1.5}$$

for all t; see (1.3).

In the present paper we pursue the same question in a more general equivariant context. In order not to get lost, neither in mathematical generalities nor in exuberant detail, we restrict ourselves to the modest paradigm of discrete rotating waves in a triangle of $N = 3$ oscillators (1.2). More precisely we study the specific control

$$\dot{z}_k = f(\lambda, z_k) + a(z_{k+1} - 2z_k + z_{k-1}) + b(z_{k+1}(t - \tau) - z_k(t)) \tag{1.6}$$

for k mod 3, real diffusion coupling $a > 0$, complex control b, and complex z_k. For mathematical convenience, only, we consider complex Stuart-Landau nonlinearities

$$f(\lambda, z_k) := (\lambda + \mathrm{i} + \gamma |z_k|^2) z_k \tag{1.7}$$

with real parameter λ and fixed complex γ. For some recent justification of this choice in a non-equivariant local center manifold and normal form setting see [7]. To further facilitate our computations we study only local symmetry-breaking Hopf bifurcation of (1.6), (1.7) in the two-parameter plane (λ, τ). But we invoke exchange of stability results in the full system. Thus our results are not restricted to any center manifold. See also [6, 8, 9, 10].

In section 2 we summarize our main results: stabilization of discrete rotating waves in the full system (1.6), (1.7) by noninvasive and pattern-selective delayed feedback. We distinguish sub-and supercritical cases, depending on the sign of $\mathrm{Re}\gamma$. We also distinguish the cases of soft and hard springs, depending on the sign of $\mathrm{Im}\gamma$. In section 3 we illustrate the control domains of successful stabilization of the trivial equilibrium $z = 0$ at Hopf bifurcation. Section 4 briefly sketches the proofs and section 5 summarizes our conclusions. All results are based on the Bachelor Thesis [11].

2 Main results

In this section we consider a triangle of coupled Stuart-Landau oscillators (1.6), (1.7), $N = 3$. We first study local Hopf bifurcation of discrete rotating waves

$$z_{k+1}(t - p/3) = z_k(t) \tag{2.1}$$

for k mod 3 and real t, with minimal period $p > 0$ and parameter λ, see proposition 1. Without feedback, $b = 0$, trivially, spatially homogeneous Hopf bifurcation with $z_0(t) \equiv z_1(t) \equiv z_2(t)$ occurs at $\lambda = 0$. This contributes real unstable dimension 2 to the spatially inhomogeneous Hopf bifurcation at $\lambda = 3a$, of real dimension 4, which is addressed in proposition 1. Theorems 1–3 summarize our main results on full stabilization of the bifurcating discrete rotating waves, at $\lambda = 3a$, by pattern-selective and noninvasive delayed feedback.

Proposition 1 *Consider the coupled oscillator triangle (1.6), (1.7). Hopf bifurcation of discrete rotating waves (2.1) occurs at the parameter value $\lambda = 3a$. The rotating waves are harmonic,*

$$z_k(t) = r\exp(2\pi \mathrm{i}(t/p + k/3)), \tag{2.2}$$

for $k = 0, 1, 2$, and are phase shifted by $2\pi/3$ between oscillators. Amplitude r and minimal period p are given explicitly by

$$\begin{aligned} r^2 &= (3a - \lambda)/\mathrm{Re}\gamma, \\ p &= 2\pi/(1 + r^2 \mathrm{Im}\gamma). \end{aligned} \tag{2.3}$$

In particular the Hopf bifurcation is supercritical, *i.e. towards $\lambda > 3a$, for* $\mathrm{Re}\gamma < 0$, *and* subcritical *for* $\mathrm{Re}\gamma > 0$. *The minimal period p grows with amplitude* (soft spring) *if* $\mathrm{Im}\gamma < 0$ *and decreases* (hard spring) *if* $\mathrm{Im}\gamma > 0$.

The proposition can be verified easily, by direct calculation. Note however that the imaginary eigenvalue $+\mathrm{i}$ of the linearization of (1.6), (1.7) is complex double at the Hopf bifurcation point $z = 0,\ \lambda = 3a$, rather than simple. This real dimension 4 of the Hopf eigenspace complicates the stability analysis. It also leads to the simultaneous bifurcation of further branches of periodic standing waves, $z_1(t) \equiv z_2(t)$, and of alternating waves, $z_0(t+p/2) = z_0(t)$ and $z_1(t+p/2) \equiv z_2(t)$, of possibly different periods p and bifurcation directions. In particular supercritical bifurcation need not lead to stable periodic orbits for two reasons. First, the 4-dimensional Hopf eigenspace can comfortably accomodate further unstable Floquet exponents. Second, the onset of homogeneous Hopf bifurcation $z_0(t) \equiv z_1(t) \equiv z_2(t)$ at $\lambda = 0$ forces the trivial equilibrium $z = 0$ itself to possess unstable dimension 2, for $0 < \lambda < 3a$. This is promptly inherited by the discrete rotating waves, and by all periodic orbits, which bifurcate at $\lambda = 3a$.

Nevertheless we obtain the following stability results, for small enough amplitudes $r > 0$ near $\lambda = 3a$, where we distinguish between the super- and subcritical cases and between the model for soft and hard springs. We recall that the hard spring model, $\mathrm{Im}\gamma > 0$, is characterized by decreasing period p as the amplitude r increases, whereas for the soft spring model, $\mathrm{Im}\gamma < 0$, the period p increases with amplitude.

Theorem 1 *Consider the* supercritical *case* $\mathrm{Re}\gamma < 0$ *for Hopf bifurcation of discrete rotating waves (2.1) of the Stuart-Landau triangle (1.6), (1.7), near* $\lambda = 3a$.

Then there exists a positive constant $a_+ \approx 0.0974$ *and continuous strictly monotone functions*

$$(0, a_+) \ni \quad a \longmapsto \underline{b}(a) < \bar{b}(a) \quad \in (0, \infty), \tag{2.4}$$

independent of γ*, such that the following conclusion holds for all real diffusion constants* a *and real control amplitudes* b *with*

$$0 < a < a_+, \qquad \underline{b}(a) < b < \bar{b}(a), \tag{2.5}$$

and for all sufficiently small $\lambda - 3a > 0$.

The delayed feedback control (1.5) under one-third period

$$(hz)_k = z_{k+1} \qquad \tau = \Theta(h)p = p/3 \tag{2.6}$$

stabilizes the discrete rotating wave solutions of proposition 1 for the above range of diffusion couplings a*, control amplitudes* b*, and parameters* λ*. The stabilization is noninvasive and pattern-selective. Moreover the following limits hold for the control range:*

$$\begin{aligned} 0 = \lim \underline{b}(a) < \lim \bar{b}(a) = \infty, \qquad & \text{for } a \searrow 0 \\ 0 < \lim \underline{b}(a) = \lim \bar{b}(a) \approx 0.4766, \qquad & \text{for } a \nearrow a_+. \end{aligned} \tag{2.7}$$

In the supercritical case $\mathrm{Re}\gamma < 0$, real controls b suffice. We comment on possible complex control variants in the supercritical case at the end of section 4.

The subcritical cases $\mathrm{Re}\gamma > 0$ necessarily require complex controls $b \in \mathbb{C}\backslash\mathbb{R}$. Moreover the amplitude dependence of period, as measured by $|\mathrm{Im}\gamma|$, has to be strong enough in proportion to subcriticality, as measured by $\mathrm{Re}\gamma$.

Theorem 2 *Consider the* subcritical *case* $\mathrm{Re}\gamma > 0$ *for* soft springs, $\mathrm{Im}\gamma < 0$, *with the setting and notation of proposition 1 and theorem 1 under delay feedback (2.6) of one third period.*

Then there exists a positive constant $a_+ \approx 0.0974$ *and a continuous, strictly monotone function*

$$(0, a_+) \ni \quad a \longmapsto \beta(a) \quad \in (0, \infty), \tag{2.8}$$

independent of γ, *such that the following conclusion holds for all diffusion constants* a *and spring nonlinearities* $\gamma \in \mathbb{C}$ *which satisfy*

$$0 < a < a_+, \qquad |\mathrm{Im}\gamma| > \beta(a)\ \mathrm{Re}\gamma > 0. \tag{2.9}$$

There exists an open region of complex controls $b \in \mathbb{C}\backslash\mathbb{R}$, *which depends on* a *and* γ, *such that the subcritically bifurcating discrete rotating waves are stabilized for all sufficiently small* $|\lambda - 3a|$. *Moreover*

$$\lim \beta(a) = \begin{cases} \bar{\beta} \approx 3.6397 & \text{for } a \searrow 0 \\ \infty & \text{for } a \nearrow a_+. \end{cases} \tag{2.10}$$

Theorem 3 *Consider the* subcritical *case* $\mathrm{Re}\gamma > 0$ *for* hard springs, $\mathrm{Im}\gamma > 0$, *with the above setting under delay feedback (2.6) of one third period.*

Then there exist positive constants $a_- \approx 0.2960$, $\bar{\beta} \approx 3.6397$ *independent of* γ, *such that the conclusion of theorem 2 holds for all diffusion constants* a *and spring nonlinearities* $\gamma \in \mathbb{C}\backslash\mathbb{R}$ *which satisfy*

$$0 < a < a_-, \qquad \mathrm{Im}\gamma > \bar{\beta}\ \mathrm{Re}\gamma > 0. \tag{2.11}$$

In summary, stabilization of discrete rotating waves at Hopf bifurcation succeeds for sufficiently weak diffusive coupling $a > 0$. In all three theorems the assumption $|\lambda - 3a|$ sufficiently small guarantees that the positive Floquet exponent originating from the homogeneous Hopf bifurcation is small enough for stabilization, see also [12].

3 Domains of stability

In this section we linearize the coupled oscillator triangle at the trivial equilibrium $z = 0$, and derive the characteristic equations. For complex controls b we also plot stability regions $b \in \Lambda_a$ of the remaining non-imaginary spectrum for the trivial equilibrium $z = 0$

at the Hopf point $\lambda = 3a$. We consider delay $\tau = p/3$ at the limiting minimal period $p = 2\pi$, associated to the complex double eigenvalue $+\mathrm{i}$. This is a necessary, but not sufficient, prerequisite for pattern-selective and noninvasive stabilization of the bifurcating discrete rotating wave by delayed feedback. The consequences for the bifurcating discrete rotating waves, which prove theorems 1–3, are considered in section 4.

The linearization of (1.6), (1.7) at the trivial equilibrium $z = 0$ commutes with the symmetry group $G = D_3$ of the oscillator triangle. By Schur's Lemma the linearization therefore diagonalizes in coordinates adapted to the irreducible representations of the linear D_3-action on (1.2), (1.6). The pertinent coordinates are

$$\begin{aligned} x_0 &= \frac{1}{3}(z_0 + z_1 + z_2) \\ x_1 &= \frac{1}{3}(z_0 + e^{2\pi \mathrm{i}/3} z_1 + e^{-2\pi \mathrm{i}/3} z_2) \\ x_2 &= \frac{1}{3}(z_0 + e^{-2\pi \mathrm{i}/3} z_1 + e^{2\pi \mathrm{i}/3} z_2). \end{aligned} \tag{3.1}$$

The diagonal form of the linearization, in these coordinates, decouples the characteristic equation $\chi(\eta) = 0$ for exponentials $x(t) = e^{\eta t}(x_0, x_1, x_2)$ into a product of three factors, $\chi = \chi_0 \cdot \chi_1 \cdot \chi_2 = 0$. The complex spectrum is therefore given by those $\eta \in \mathbb{C}$ which solve at least one of the three characteristic equations

$$\begin{aligned} \chi_0(\eta) &= \lambda + \mathrm{i} + b\left(e^{-\tau\eta} - 1\right) - \eta = 0 \\ \chi_1(\eta) &= \lambda - 3a + \mathrm{i} + b\left(e^{-2\pi \mathrm{i}/3 - \tau\eta} - 1\right) - \eta = 0 \\ \chi_2(\eta) &= \lambda - 3a + \mathrm{i} + b\left(e^{2\pi \mathrm{i}/3 - \tau\eta} - 1\right) - \eta = 0. \end{aligned} \tag{3.2}$$

Let $E(b)$ denote the strict unstable dimension of $z = 0$ at the Hopf point $\lambda = 3a$ with Pyragas delay $\tau = p/3 = 2\pi/3$, i.e. the total number of solutions η with $\mathrm{Re}\eta > 0$ for the characteristic equation $\chi(\eta) = 0$, counting algebraic multiplicities. Of course we seek domains where $E(b) = 0$ gives the bifurcating discrete rotating waves a chance to be born stable.

It is easy to determine the complex Hopf curves $b_j(\omega)$ where an additional Hopf eigenvalue $\eta = \mathrm{i}\tilde{\omega} = \mathrm{i}(1 + 3\omega)$ changes the strict unstable dimension $E(b)$ via $\chi_j(\mathrm{i}\tilde{\omega}) = 0$:

$$\begin{aligned} b_0(\omega) &= \frac{3}{2}\left(a - \mathrm{i}\omega - (\omega + \mathrm{i}a)\cot(\pi(\omega + 1/3))\right) \\ b_1(\omega) &= -\frac{3}{2}\omega\left(\cot(\pi(\omega + 2/3)) + \mathrm{i}\right) \\ b_2(\omega) &= -\frac{3}{2}\omega\left(\cot(\pi\omega) + \mathrm{i}\right). \end{aligned} \tag{3.3}$$

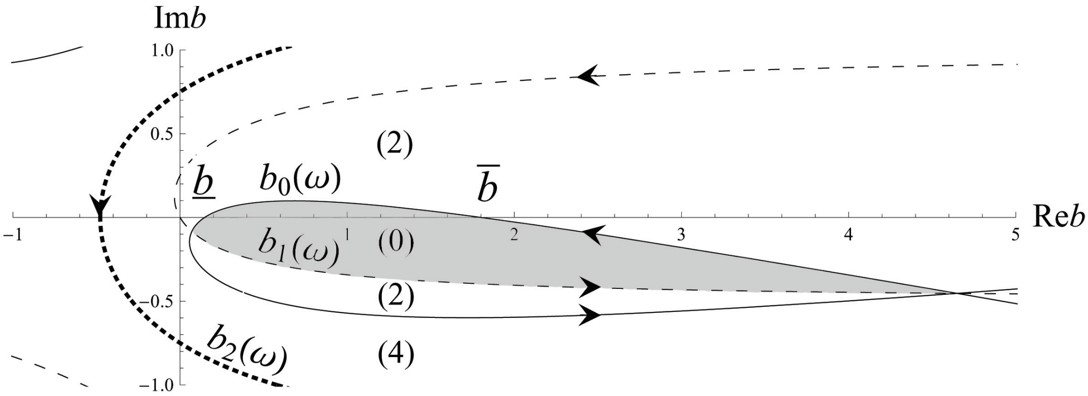

Figure 1: Hopf curves in the complex (Reb, Imb)-plane for $a = 0.06$. The numbers in the parentheses denote the strict unstable dimension $E(b)$ of $z = 0$ at the Hopf point $\lambda = 3a$ with Pyragas delay $\tau = p/3 = 2\pi/3$. The arrows indicate the orientation of the Hopf curves, i.e. they point towards increasing ω.

In figure 1 we illustrate the resulting strict unstable dimensions $E(b)$ of the trivial equilibrium $z = 0$ for $a = 0.06$. The stability domain

$$\Lambda_a := \{b \in \mathbb{C} \mid E(b) = 0 \text{ at } \lambda = 3a,\ \tau = 2\pi/3\} \tag{3.4}$$

disappears at the critical point

$$a = a_* \approx 0.2960, \qquad \omega = \omega_* \approx 0.0813 \tag{3.5}$$

where the complex derivative $b_0'(\omega)$ of the complex analytic function $\omega \mapsto b_0(\omega)$ vanishes. In fact it is easy to determine the stability domain. We know $E(0) = 2$, by direct analysis of the uncontrolled oscillators. Moreover, analyticity of $\mathbb{C} \ni \omega \mapsto b_j(\omega) \in \mathbb{C}$ implies that $E(b)$ is larger, by two, on the right side of the Hopf curves $b_j(\omega)$ when that curve is oriented towards increasing real ω, i.e. along the imaginary axis $\eta = \mathrm{i}\tilde{\omega} = \mathrm{i}(1 + 3\omega)$. See figure 1.

In fact the stability region Λ_a is bounded by $b_1(\omega)$ from below and by $b_0(\omega)$ from above, for $0 < a < a_*$. Note that Re$b > 0$ on Λ_a. For $a < a_*$ near a_*, we observe Im$b < 0$. Only when

$$a = a_+ \approx 0.0974 < a_*, \tag{3.6}$$

the loop of $b_0(\omega)$, which defines the relevant upper boundary of the stability domain Λ_a, touches the Reb axis and enables stabilization by real controls, as in theorem 1. For $0 < a < a_+$, the interval $(\underline{b}(a), \overline{b}(a)) = \Lambda_a \cap \mathbb{R}$ of (2.4) is given by the intersection of the loop of $b_0(\omega)$ with the real axis; see again figure 1. The monotonicity claims on $\underline{b}$, $\overline{b}$ can be derived from the fact that the map $\mathbb{R}^2 \ni (\omega, a) \mapsto b \in \mathbb{C}$ preserves local orientation. The limits $a \nearrow a_+$ in (2.7) indicate where $b_0(\omega)$ touches the real axis at $a = a_+$. The limits for $a \searrow 0$ follow by direct analysis of $b_0(\omega)$ at $a = 0$.

We emphasize that the above analysis only asserts linear stability $\mathrm{Re}\eta < 0$ of the remaining non-imaginary eigenvalues at the Hopf point $\lambda = 3a$, $\tau = p/3$, $p = 2\pi$ itself, for $b \in \Lambda_a$. In particular the nonlinearity coefficient $\gamma \in \mathbb{C}$ does not affect Λ_a.

4 Proof of theorems 1–3

We summarize the mathematical proofs of theorems 1–3. For complete details see also [11].

The basic strategy of proof is the same, for all three theorems. For given $0 < a < a_+, a_-$, respectively, we first fix the control parameter $b = B\exp(\mathrm{i}\varphi)$ in the region Λ_a of (3.4) where the characteristic equation (3.2) produces strictly stable non-Hopf eigenvalues $\mathrm{Re}\eta < 0$, only, at $\lambda = 3a$, $\tau = p/3$, $p = 2\pi$. The complex double Hopf eigenvalue $\eta = +\mathrm{i}$ is generated by a simple zero of each of the remaining factors χ_1, χ_2 of the characteristic equation $\chi(\eta) = \chi_0 \cdot \chi_1 \cdot \chi_2 = 0$; see (3.2). In particular χ_0 is strictly stable, at $\lambda = 3a$, $\tau = 2\pi/3$, and remains strictly stable, locally.

In the (λ, τ) plane, we then determine the Hopf bifurcation curves $\tau_j(\lambda)$ where $\chi_j(\eta) = 0$ for some purely imaginary $\eta = \mathrm{i}\omega$ and some remaining factor χ_j, $j = 1, 2$, of the characteristic equation.

Orientation considerations similar to section 3 determine the resulting total unstable dimensions $E(\lambda, \tau)$ of the trivial equilibrium $z = 0$ in the domains complementary to the Hopf curves. See figure 2 for numerical examples.

Locally near $\lambda = 3a$, we next restrict the control delay τ to the (smoothly extended) *Pyragas curve* $\tau_p(\lambda) = p(\lambda)/3$ of our proposed feedback, see (2.6). Explicitly,

$$\tau_p(\lambda) := \tfrac{1}{3}p(\lambda) = \tfrac{2\pi}{3}/\big(1 + (3a - \lambda)\mathrm{Im}\gamma/\mathrm{Re}\gamma\big) \tag{4.1}$$

by (2.3). Note how the Pyragas curve τ_p depends on the spring nonlinearity γ, but not on the control b. Conversely, the Hopf curves τ_j depend on b, but not on γ.

The additional assumptions of theorems 1–3 now guarantee that we only encounter a *standard supercritical Hopf bifurcation*, when we restrict all considerations to parameters (λ, τ) on the one-dimensional Pyragas curve $\tau_p(\lambda)$. "Standard" means that we only encounter transverse crossing of one simple imaginary eigenvalue, across the imaginary axis. The unique bifurcating Hopf branch then consists of exactly the original harmonic discrete rotating waves of proposition 1, due to the noninvasive property of delayed feedback control along the Pyragas curve. "Supercritical" means that the periodic orbits bifurcate to the part of the Pyragas curve $(\lambda, \tau_p(\lambda))$ in the domain where $E(\lambda, \tau) = 2$. Because $E(b) = 0$ and *because the standard Hopf bifurcation is guaranteed to be always supercritical along the Pyragas curve, standard exchange of stability therefore guarantees stability of the bifurcating discrete rotating waves.* See also [13].

To prove theorems 1–3 it only remains to check supercriticality along the Pyragas curve $\tau_p(\lambda)$. The Hopf curves $\tau_j(\lambda)$ at $b = B\exp(i\varphi)$ introduced above can be calculated explicitly to be

$$\begin{aligned}
\tau_1(\lambda) &= \frac{\pm\arccos\left(\cos\varphi - (\lambda-3a)/B\right) + \varphi - \frac{2\pi}{3} + 2\pi n}{1 - B\sin\varphi \mp \left(B^2\sin^2\varphi + (\lambda-3a)(2B\cos\varphi - (\lambda-3a))\right)^{1/2}} \\
\tau_2(\lambda) &= \frac{\pm\arccos\left(\cos\varphi - (\lambda-3a)/B\right) + \varphi + \frac{2\pi}{3} + 2\pi n}{1 - B\sin\varphi \mp \left(B^2\sin^2\varphi + (\lambda-3a)(2B\cos\varphi - (\lambda-3a))\right)^{1/2}}
\end{aligned} \tag{4.2}$$

with integer n.

From section 3, we already know the strict unstable dimension $E(b) = 0$ at $\lambda = 3a, \tau = p/3,\ p = 2\pi$ for our choice of $b \in \Lambda_a$. From (4.2) we conclude that $(\lambda, \tau_1(\lambda))$ does not enter a small neighborhood of $(\lambda, \tau) = (3a, 2\pi/3)$. Neither does $(\lambda, \tau_2(\lambda))$, unless $n = 0$ and the negative sign of $\pm$ is chosen. We also recall $\mathrm{Re}b > 0$ and hence $|\varphi| < \pi/2$. Therefore

$$\begin{aligned}
\tau_2(\lambda) &= \frac{-\arccos\left(\cos\varphi - (\lambda-3a)/B\right) + \varphi + \frac{2\pi}{3}}{1 - B\sin\varphi + \left(B^2\sin^2\varphi + (\lambda-3a)(2B\cos\varphi - (\lambda-3a))\right)^{1/2}} \\
\tau_2'(3a) &= -(3 + 2\pi\mathrm{Re}b)/(3\mathrm{Im}b).
\end{aligned} \tag{4.3}$$

4.1 Supercritical case

To address the supercritical case $\mathrm{Re}\gamma < 0$ of theorem 1, near $\lambda = 3a$, we observe that our real choice $\mathrm{Im}b = 0$ of the control b causes vertical slope of the Hopf curve $\tau_2(\lambda)$ at $\lambda = 3a$; see also figure 2(a). Moreover the region $E(\lambda, \tau) = 2$ is found towards larger λ, i.e. to the right of the Hopf curve τ_2. The original uncontrolled Hopf bifurcation was assumed supercritical, i.e. to the right. Therefore the controlled Hopf bifurcation remains supercritical, and theorem 1 is proved.

4.2 Subcritical soft spring case

To address the subcritical soft spring case $\mathrm{Re}\gamma > 0 > \mathrm{Im}\gamma$ of theorem 2, near $\lambda = 3a$, we also calculate the negative slope

$$\tau_p'(3a) = \tfrac{2}{3}\pi\mathrm{Im}\gamma/\mathrm{Re}\gamma \tag{4.4}$$

of the Pyragas curve, at $\lambda = 3a,\ \tau = 2\pi/3$.

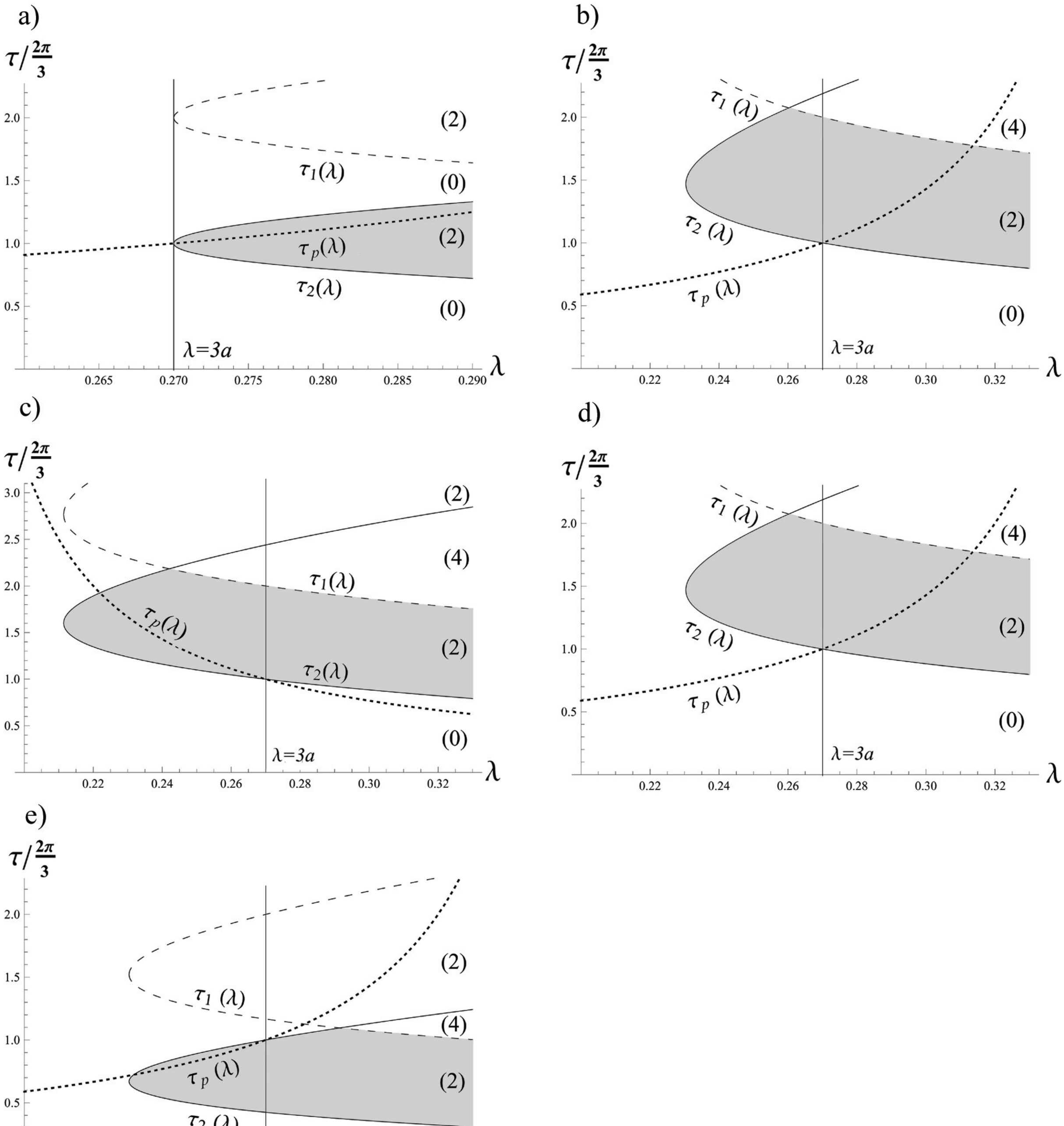

Figure 2: Hopf curves in the (λ, τ) plane for (a) the supercritical case with parameters $\mathrm{Re}\gamma = -1$, $\mathrm{Im}\gamma = -10$, $a = 0.09$, $B = 0.2$, $\varphi = 0$, (b) the subcritical soft spring case $\mathrm{Re}\gamma = 1$, $\mathrm{Im}\gamma = -10$, $a = 0.09$, $B = 0.4$, $\varphi = -\pi/7$, (c) the subcritical soft spring case with parameters $\mathrm{Re}\gamma = 1$, $\mathrm{Im}\gamma = -10$, $a = 0.09$, $B = 0.2$, $\varphi = \pi/4$, (d) for the subcritical hard spring case with parameters $\mathrm{Re}\gamma = 1$, $\mathrm{Im}\gamma = 10$, $a = 0.09$, $B = 0.4$, $\varphi = +\pi/7$, (e) for the subcritical hard spring case with parameters $\mathrm{Re}\gamma = 1$, $\mathrm{Im}\gamma = 10$, $a = 0.09$, $B = 0.4$, $\varphi = -\pi/7$.
The numbers in parentheses denote the strict unstable dimension $E(\lambda, \tau)$.

We first consider the case $\mathrm{Im}b < 0$ in the loop Λ_a. Then $\mathrm{Re}b > 0$ in Λ_a and (4.3), (4.4) imply $\tau_2'(3a) > 0 > \tau_p'(3a)$; see figure 2(b). In particular $E(\lambda, \tau_p(\lambda)) = 0$ for $\lambda < 3a$ and the Hopf bifurcation along the Pyragas curve is subcritical, rather than supercritical. Hence Pyragas stabilization fails for $\mathrm{Im}b < 0$.

Next consider the case $\mathrm{Im}b > 0$ in the loop Λ_a. Again the stability region $E(\lambda, \tau) = 2$ is located above the Hopf curve τ_2, locally. See figure 2(c). The subcritical Hopf bifurcation becomes supercritical, along the Pyragas line $\tau_p(\lambda)$, if and only if the region $\lambda < 3a$ of periodic orbits lies inside the dashed region $E(\lambda, \tau) = 2$, along the Pyragas curve. This, in turn, holds true if and only if the negative Pyragas slope $\tau_p'(\lambda)$ at $\lambda = 3a$ satisfies

$$\tfrac{2}{3}\pi \mathrm{Im}\gamma/\mathrm{Re}\gamma = \tau_p'(3a) < \tau_2'(3a) = -(3 + 2\pi \mathrm{Re}b)/(3\mathrm{Im}b). \tag{4.5}$$

This proves theorem 2 with the choice $\mathrm{Im}b > 0$ in the loop Λ_a for

$$\beta(a) := \min \left\{ \left(\frac{3}{2\pi} + \mathrm{Re}b \right) /\mathrm{Im}b \;\middle|\; b \in \Lambda_a,\ \mathrm{Im}b > 0 \right\}. \tag{4.6}$$

4.3 Subcritical hard spring case

We conclude with the case of subcritical hard springs, i.e. positive $\mathrm{Re}\gamma$, $\mathrm{Im}\gamma$ near $\lambda = 3a$, as addressed in theorem 3. If $\mathrm{Im}b > 0$ in Λ_a then $\mathrm{Re}b > 0$, (4.3), (4.4) imply $\tau_2'(3a) < 0 < \tau_p'(3a)$, this time, see figure 2(d). Hence Pyragas stabilization fails by arguments quite analogous to those given for theorem 2, $\mathrm{Im}b < 0$.

Next consider $\mathrm{Im}b < 0$ in Λ_a, see figure 2(e). Then it is sufficient to require

$$\tfrac{2}{3}\pi \mathrm{Im}\gamma/\mathrm{Re}\gamma = \tau_p'(3a) > \tau_2'(3a) = -(3 + 2\pi \mathrm{Re}b)/(3\mathrm{Im}b). \tag{4.7}$$

This proves theorem 3, (2.11) with

$$\bar{\beta} := \min \left\{ \left(\frac{3}{2\pi} + \mathrm{Re}b \right) /\mathrm{Im}b \;\middle|\; b \in \Lambda_a,\ \mathrm{Im}b < 0 \right\}. \tag{4.8}$$

In fact, $\bar{\beta}$ does not depend on a, due to the precise geometry of the curves $b_0(\omega)$ and $b_1(\omega)$ in figure 1.

This completes the proofs of theorems 1–3.

4.4 Remarks on the supercritical case

The above analysis for complex controls b also applies to the supercritical case of theorem 1, $\mathrm{Re}\gamma < 0$, of course. We consider the same complex control region Λ_a as for the subcritical cases, see also figure 1. We then obtain stabilization for an extended range

$$0 < a < a_* \approx 0.2960 \tag{4.9}$$

of diffusion couplings. The prize we have to pay, however, are restrictions on the nonlinearity quotient $\mathrm{Im}\gamma/\mathrm{Re}\gamma$, as follows:

First consider a control with $\mathrm{Im}b > 0$. Then stabilization is possible for

$$\mathrm{Im}\gamma < \beta(a)|\mathrm{Re}\gamma| \tag{4.10}$$

where $\beta(a)$ is given by

$$\beta(a) := \min\left\{ \left(\frac{3}{2\pi} + \mathrm{Re}b\right)/\mathrm{Im}b \,\middle|\, b \in \Lambda_a,\ \mathrm{Im}b > 0 \right\}. \tag{4.11}$$

In the soft spring case stabilization is always possible for $0 < a < a_+ \approx 0.0974$ and suitable $b \in \Lambda_a$, $\mathrm{Im}b > 0$. Specific restrictions on the nonlinearity only arise for hard springs.

Next consider a control with $\mathrm{Im}b < 0$. In this case, stabilization is possible for

$$\mathrm{Im}\gamma > \beta(a)|\mathrm{Re}\gamma| \tag{4.12}$$

where $\beta(a)$ is given by

$$\beta(a) := \max\left\{ -\left(\frac{3}{2\pi} + \mathrm{Re}b\right)/|\mathrm{Im}b| \,\middle|\, b \in \Lambda_a,\ \mathrm{Im}b > 0 \right\}. \tag{4.13}$$

It is the hard spring, this time, which is stabilizable for all $0 < a < a_- \approx 0.2960$ and suitable $b \in \Lambda_a$ with $\mathrm{Im}b < 0$. The soft spring case requires the above constraint.

5 Conclusions

The modest scope of our case study was the noninvasive and pattern-selective stabilization of discrete rotating waves in a triangle of diffusively coupled oscillators at symmetry-breaking Hopf bifurcation. The oscillators were assumed to be in Stuart-Landau normal form. Feedback was of delay type, reminiscent of Pyragas control but, adapted to be pattern-selective.

For explicit intervals of small enough diffusion coupling we have determined explicit domains of the control coefficient b, and explicit constraints on the spring nonlinearity, such that delayed feedback control succeeds.

In the supercritical case of theorem 1, real control b succeeded for diffusion coupling $0 < a < a_+ \approx 0.0974$ and arbitrary nonlinearities. Control also succeeds for $0 < a < a_* \approx 0.2960$ and suitable complex controls b, without constraints on the nonlinearities for hard springs, but with specific restrictions for soft springs.

In the subcritical cases of theorems 2, 3, complex controls b were necessary, along with constraints on the nonlinearities. These constraints depended on the diffusion coupling

$0 < a < a_+$, in the case of soft springs. For hard springs, the required constraints turned out to be independent of $0 < a < a_- \approx 0.2960$.

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