Eliminating restrictions of time-delayed feedback control using equivariance

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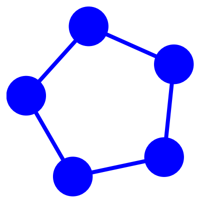

Pyragas control is a widely used time delayed feedback control for the stabilization of periodic orbits in dynamical systems. In this paper we investigate how we can use equivariance to eliminate restrictions of Pyragas control, both to select periodic orbits for stabilization by their spatio-temporal pattern and to render Pyragas control possible at all for those orbits. Another important aspect is the optimization of equivariant Pyragas control, i.e. to construct larger control regions. The ring of n identical Stuart-Landau oscillators coupled diffusively in a bidirectional ring serves as our model.

1 Introduction

A particularly successful method of time-delayed feedback control has been introduced in 1992 by Pyragas [10]. A short summary of the huge amount of experimental and theoretical results following the original publication can be found in [11]. A widely open subject of research is Pyragas control for networks of coupled oscillators, where the solutions have different spatio-temporal symmetries besides synchrony.

The time delayed feedback control as introduced by Pyragas is noninvasive on the periodic orbit. This is established by using a time delay which is an integer multiple of the minimal period p, i.e. for general systems $\dot{z}(t) = F(z(t))$, $z \in \mathbb{C}^n$ a control of the form

$$\dot{z}(t) = F(z(t)) + B\left[-z(t) + z(t - Np)\right]$$

is applied, where $N \in \mathbb{N}$, and B is either a complex control parameter or a matrix.

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For the network type presented in this paper, the only periodic orbit which can be stabilized by standard Pyragas control is the synchronized one. Hence, the modification to equivariant Pyragas control is necessary to eliminate restrictions of time-delayed feedback control.

Therefore, we use a modification of Pyragas control for stabilizing periodic orbits with prescribed spatio-temporal symmetries on networks. This modification has been discussed previously in [13, 9] for general equivariant systems near Hopf bifurcation. For general F, the control term is introduced as follows:

$$\dot{z}(t) = F(z(t)) + B\left[-z(t) + hz(t - \Theta(h)\,p)\right] \tag{1}$$

Here h and $\Theta(h)$ describe the spatio-temporal symmetry of the periodic orbit, for details see Fiedler [3] and section 2 below. The new control term is also noninvasive on the periodic orbit. For a detailed discussion on how to choose h and $\Theta(h)$, see sections 3 and 6.

In first works on small networks consisting of two and three Stuart-Landau oscillators, such as [5, 12], it was shown that the control method (1) can indeed stabilize unstable periodic orbit with prescribed symmetry near equivariant Hopf bifurcation. Also, an important aspect is, that the control, if chosen correctly, can select the desired periodic orbit, even if several periodic orbits exist with the same period. However, many restrictions remained: For example stabilization is impossible if the coupling between oscillators is to strong, or if the cubic term of the Hopf normal form expansion does not fulfill certain requirements. Successful attempts to overcome these restrictions for two coupled oscillators have been presented in [1, 2].

In the present work, we want to apply equivariant Pyragas control (1) to a specific network type, which consists of n identical Stuart-Landau oscillators coupled in a ring. Specifically, we want to select a periodic orbit with prescribed symmetry and stabilize it – we will see that, for this network type, all but one of the periodic orbits are unstable. The main task consists of finding a suitable description h and $\Theta(h)$ of the required symmetry type. Secondly, we want to overcome previously existing restrictions to the application of equivariant Pyragas control. A third, closely related aspect is the optimization of the control region, i.e. to construct larger control regions.

All three tasks lead back to the following important questions: *How should we describe the symmetry of a periodic orbit and how can it be utilized to optimize stabilization?*

This paper is organized as follows: In section 2, we present the model of n identical oscillators coupled diffusively in a ring. We state our main results, introducing equivariant Pyragas control adapted to the equivariant case in section 3. The proof of the main theorem follows in sections 4, where we discuss the stabilization regions, and 5, where we finalize the proof. The full rotational symmetry of the system is used in section 6 to further improve the stabilization results, i.e. to obtain even larger control regions and to use arbitrary time delay for the control. We include multiple time delays into the control term and discuss the consequences in section 7. Finally, we recapitulate the paper in section 8.

2 Model and periodic solutions with spatio-temporal symmetry

We consider n Stuart-Landau oscillators diffusively coupled in a ring: Each oscillator z_k is symmetrically coupled to its two nearest neighbors z_{k-1} and z_{k+1}, i.e. we consider local coupling

$$\dot{z}_k = f(z_k) + a(z_{k-1} - 2z_k + z_{k+1}), \qquad k = 0, \ldots, n-1, \tag{2}$$

where $z_k \in \mathbb{C} \cong \mathbb{R}^2$, $a > 0$ is the positive coupling parameter and

$$f(z_k) = (\lambda + \mathrm{i} + \gamma |z_k|^2)\, z_k, \tag{3}$$

i.e. the normal form of Hopf bifurcation truncated at third order, with the Hopf frequency scaled to unity. This can always be achieved by rescaling time. $\lambda \in \mathbb{R}$ is a real bifurcation parameter, $\gamma \in \mathbb{C}$ is fixed.

We define $z_{-k} = z_{n-k}$ and $z_n = z_0$ in order to cope with the indices.

The system (2), (3) is equivariant with respect to the group $D_n \times S^1$, where $D_n = \langle \rho, \kappa \rangle$ is the dihedral group symmetry induced by the coupling between the single oscillators. The rotations ρ are generated by the index shift $(\rho z)_k = z_{k-1}$, and the reflection κ is given by $(\kappa z)_k = z_{-k}$ with $k \mod n$ and $\kappa^2 = (\kappa\rho)^2 = \mathrm{Id}$. S^1 is the rotational symmetry of the truncated Hopf normal form, i.e. $e^{\mathrm{i}\theta} f(z) = f(e^{\mathrm{i}\theta} z)$ for all angles $\theta \in [0, 2\pi]$.

Following [3], we describe the symmetry of periodic orbits $z_*(t)$ of the $D_n \times S^1$-equivariant system (2), (3) by triplets (H, K, Θ). The isotropy subgroup $H \leq D_n \times S^1$ leaves the periodic orbit $\{z_*(t) : t \in \mathbb{R}\}$ fixed *as a set*, while $K \leq H$ leaves it fixed *pointwise*. The isotropy subgroups of $D_n \times S^1$ can for example be found in [7]. Θ is a group homomorphism which is defined uniquely by time-shift for all t:

$$z(t) = hz(t - \Theta(h)p)$$

For our system, we have $H = \mathbb{Z}^n$ and $\Theta(e^{2\pi \mathrm{i}\, m/n}) = ms/n$, where $s \in \mathbb{N}$, which corresponds to discrete rotating waves.

Applying this description to the above system (2), (3), we find n different discrete rotating waves ("ponies on a merry-go-round"[8]). They appear at equivariant Hopf bifurcations, which can be calculated directly:

Proposition 1. *Consider the coupled oscillator ring* (2), (3)*. Equivariant Hopf bifurcation of discrete rotating waves occurs at the parameter values* $\lambda_j = 2a(1 - \cos(2\pi j/n))$, $j = 0, \ldots, n-1$*. The rotating waves are harmonic,*

$$z_k(t) = r_j \exp\left(2\pi \mathrm{i}\left(\frac{t}{p_j} + j\frac{k}{n}\right)\right), \tag{4}$$

for oscillators z_k, $k = 0, \ldots, n-1$*, respectively, and are phase shifted by* $2\pi j/n$ *between oscillators. Amplitude* r_j *and minimal period* p_j *are given explicitly by*

$$r_j^2 = \frac{2a(1 - \cos(2\pi j/n)) - \lambda}{\operatorname{Re}\gamma},$$

$$p_j = \frac{2\pi}{1 + r_j^2 \operatorname{Im}\gamma}.$$

In particular the Hopf bifurcation is supercritical, *i.e. towards* $\lambda > 2a(1 - \cos(2\pi j/n))$*, for* $\operatorname{Re}\gamma < 0$*, and* subcritical *for* $\operatorname{Re}\gamma > 0$*. The minimal period* p_j *grows with amplitude* (soft spring) *if* $\operatorname{Im}\gamma < 0$ *and decreases* (hard spring) *if* $\operatorname{Im}\gamma > 0$.

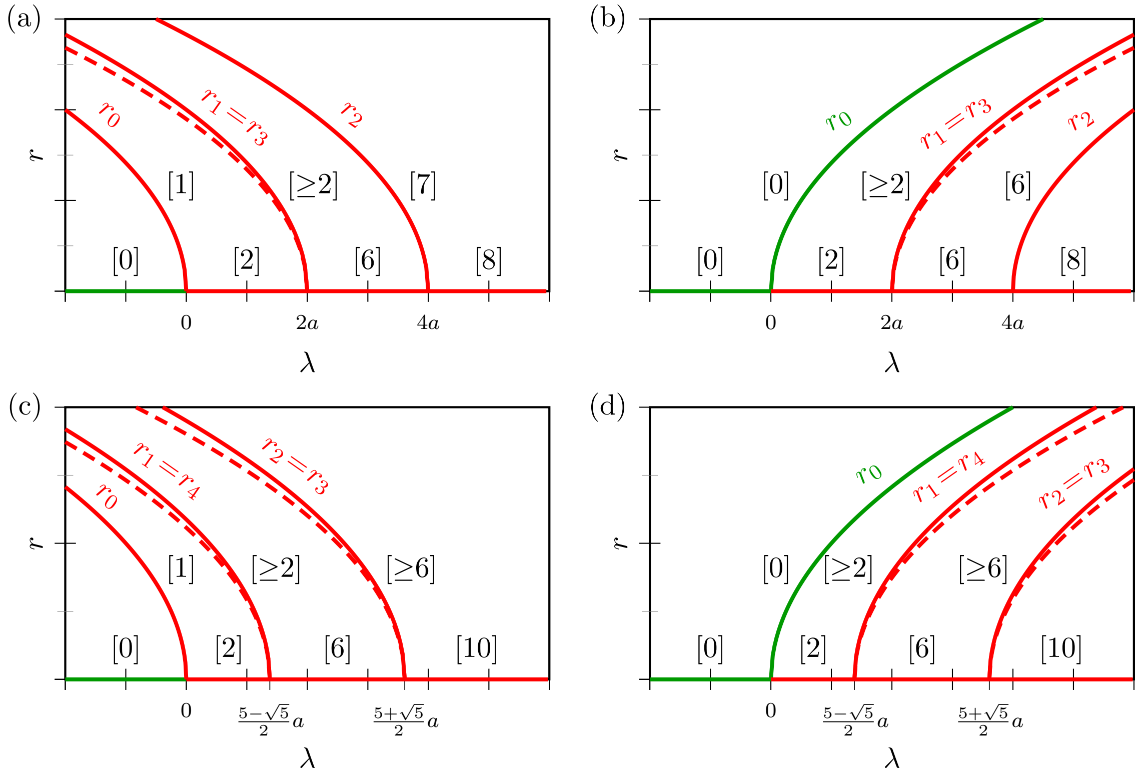

Figure 1: Parameter dependent stability of the equilibria and the bifurcating periodic orbits. The upper row shows the bifurcations for $n = 4$, while the lower one is for $n = 5$. In (a) and (c) one can see the subcritical bifurcations with $\mathrm{Re}\,\gamma = 0.1$ and in (b) and (d) the supercritical ones with $\mathrm{Re}\,\gamma = -0.1$. The coupling parameter was always chosen as $a = 0.2$. In brackets the number of unstable dimensions is denoted. Stable objects are colored in green / black, unstable ones in red / gray.

Remark. The wave with index j bifurcates at the same point λ_j as the wave with index $n-j$, see also figure 1 for $n = 4$ and $n = 5$ oscillators.

If all oscillators are identical, the diffusive coupling term $a(z_{k-1} - 2z_k + z_{k+1})$ vanishes and we can observe standard Hopf bifurcation at $\lambda_0 = 0$ leading to the synchronized periodic orbit.

Most periodic orbits of the system (2), (3) are unstable, making it suitable for the investigations of equivariant Pyragas control:

Proposition 2. *For $j \neq 0$, the bifurcating discrete rotating waves (4), enumerated by j, are unstable, both in the sub- and the supercritical case. For $j = 0$, i.e. the synchronized case, the periodic solution is unstable in the subcritical and stable in the supercritical case.*

Both propositions can be verified by direct calculation:

We choose appropriate coordinates $x_0, x_1, \ldots, x_{n-1} \in \mathbb{C}$, adapted to the equivariant nature of the coupled oscillator system, in which the linearization of the system at the trivial equilibrium $z_0 \equiv \ldots \equiv z_{n-1} \equiv 0$ decouples:

$$x_j = \frac{1}{n}\left(z_0 + e^{j\sigma\,\mathrm{i}} z_1 + e^{2j\sigma\,\mathrm{i}} z_2 + \cdots + e^{(n-1)j\sigma\,\mathrm{i}} z_{n-1}\right),$$

where we use the abbreviation $\sigma = 2\pi/n$. This leads to the inverse coordinate transformation

$$z_k = x_0 + e^{-k\sigma\,\mathrm{i}}x_1 + e^{-2k\sigma\,\mathrm{i}}x_2 + \cdots + e^{-(n-1)k\sigma\,\mathrm{i}}x_{n-1}.$$

The most important step of this analysis is to find n dynamically invariant subspaces

$$\begin{aligned} X_j &= \{(x_0,\ldots,x_{n-1}) \mid x_0 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_{n-1} = 0\} \\ &= \{(z_0,\ldots,z_{n-1}) \mid z_0 = e^{-2\pi\,\mathrm{i}\,j/n}z_j \quad \forall j = 0,\ldots,n-1\}. \end{aligned}$$

In the subspace X_j the system (2), (3) can be reduced to the two-dimensional equation

$$\dot{x}_j = f(x_j) - 2a(1-\cos(2\pi j/n))x_j,$$

which corresponds to a shifted Hopf normal form. Therefore, we can conclude that a Hopf bifurcation occurs at $\lambda_j = 2a(1-\cos(2\pi j/n))$, where the symmetry of the bifurcation periodic orbit mirrors the symmetry of the subspace X_j.

Conveniently, the linearization of the system (2), (3) decouples in the new coordinates X_j, yielding the same Hopf bifurcation points.

We have thus proven Proposition 1. Note that a simple Hopf bifurcation occuring at $\lambda_0 = 0$, corresponds to the synchronized periodic orbit. For n even there is another simple Hopf bifurcation at $\lambda_{n/2} = 2a(1-\cos\pi) = 4a$, corresponding to an antisymmetric periodic orbit. All other Hopf bifurcations are double.

The periodic orbit of standard Hopf bifurcation is stable in the supercritical and unstable in the subcritical case. In both cases, the trivial equilibrium becomes unstable for $\lambda > 0$. Hence none of the periodic orbits for $j > 0$ can be stable:
$2a(1-\cos(2\pi j/n)) > 0$ for $j > 0$, which proves Proposition 2.

3 Equivariant Pyragas control – main result

Equivariant Pyragas control can, for general F, be introduced as follows [13]:

$$\dot{z}(t) = F(z(t)) + B\left[-z(t) + hz(t-\Theta(h)\,p)\right]$$

Here $B \in \mathbb{C}^{n\times n}$ is a complex feedback parameter or matrix. For our concrete example (2), (3), the group element h is given by an index shift $(hz)_k = (\rho z)_k = z_{k-m}$ between oscillators z_k, and Θ corresponds to the phase shift of the given discrete rotating wave.

The discrete rotating waves, as discussed in section 2, are numbered by the index j. We now select one of these waves for equivariant Pyragas stabilization and denote it by the index $j = s$ (for "selected"). Thus we aim at stabilizing the unstable periodic orbits with spatio-temporal symmetry

$$z_k(t) = z_{k-m}(t - msp_s/n), \tag{5}$$

i.e. an index shift by m corresponds to a phase shift $\Theta = (ms)/n \mod 1$. Here p_s is the minimal period of the selected rotating wave. For the delayed control term as above, it is therefore suitable to use a delay time

$$\tau = (msp_s)/n \mod p_s. \tag{6}$$

Note that using a delay time larger than one period would be possible. However, we find that a larger time delay leads to smaller or even vanishing control regions, see section 6 for a detailed discussion.

In the present publication, we aim at stabilizing the discrete rotating waves (5) for $s > 0$, which are always unstable. For $s = 0$, see [4].

Moreover we require that our control is only noninvasive on our selected periodic orbit. Therefore we choose m coprime to n, to identify the periodic orbit uniquely by its symmetry.

In contrast to previous publications on Pyragas control of Stuart-Landau oscillators, we use a complex control matrix B which commutes with the index shift $z_k \mapsto z_{k-m}$. Such matrices have the form

$$B = \begin{pmatrix} B_0 & B_1 & B_2 & \ddots & B_{n-1} \\ B_{n-1} & B_0 & B_1 & \ddots & B_{n-2} \\ B_{n-2} & B_{n-1} & B_0 & \ddots & B_{n-3} \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ B_1 & B_2 & B_3 & \ddots & B_0 \end{pmatrix}$$

with coefficients $B_k \in \mathbb{C}$, i.e. matrices with constant diagonals. We can also define the matrix elements $B_0, \ldots, B_{n-1}$, via n control parameters $b_0, \ldots, b_{n-1}$, which will be helpful for later analysis:

$$B_k := \sum_{j=0}^{n-1} b_j \exp\left(jk\sigma\,\mathrm{i}\right)$$

In conclusion, we apply the control term as follows, with $\tau = (msp_s)/n \mod p_s$:

$$\dot{z} = f(z) + a(\rho z - 2z + \rho^{-1}z) + B\left[-z(t) + \rho^m z(t-\tau)\right] \tag{7}$$

where we again use the shift-representation $\rho z_k = z_{k-1}$. The main stabilization result then reads:

Main Theorem. *Consider the Hopf bifurcation of discrete rotating waves*

$$z_k(t) = z_{k-m}(t - msp_s/n), \tag{5}$$

of the Stuart-Landau ring

$$\dot{z}_k = (\lambda + \mathrm{i} + \gamma|z_k|^2)\, z_k + a(z_{k-1} - 2z_k + z_{k+1}), \qquad k = 0, \ldots, n-1, \tag{2, 3}$$

with $\lambda \in \mathbb{R}$, $a > 0$ and $\gamma \in \mathbb{C} \setminus \mathbb{R}_+$.

Then for every combination of s and m, with $s, m \in \{1, \ldots, n-1\}$ and m coprime to n, there exists a positive constant $a_{m,s}$ such that the following conclusion holds for all real diffusion constants $0 < a < a_{m,s}$, and near $\lambda_s = 2a(1 - \cos(2\pi s/n))$:

There exist open regions of complex control parameters $b_0, \ldots, b_{n-1}$ such that the delayed feedback control

$$\dot{z} = f(z) + a(\rho z - 2z + \rho^{-1}z) + B\left[-z(t) + \rho^m z(t-\tau)\right]$$

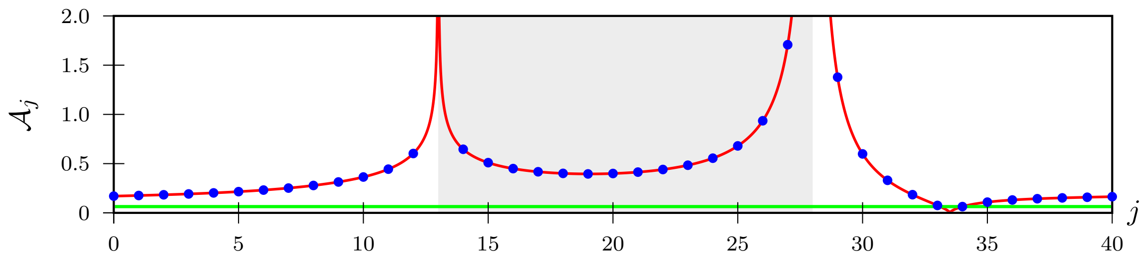

Figure 2: Maximal a for which a loop in $b_j(\omega)$ exists for $n = 41$, $m = 2$ and $s = 13$. $\mathcal{A}_j := A_j/|2\cos(2\pi j/n) - 2\cos(2\pi s/n)|$. The solid (red) line corresponds to the solutions for real $j \in [0, n]$. The maximal a allowing stabilization of the system is marked as a horizontal (green) line. The gray area is not relevant for the minimum.

with $B = (B_{kl})$, $B_{kl} = \sum_{j=1}^{n-1} b_j \exp(jl\sigma\,\mathrm{i} - jk\sigma\,\mathrm{i})$ *stabilizes the discrete rotating wave solution* (5) *for a time delay*

$$\tau = (msp_s)/n \mod p_s. \tag{6}$$

The stabilization is noninvasive and pattern-selective.

Remark. Note that the results hold for any combination s and m with m coprime to n. However, the constant $a_{m,s}$ depends on the choice of m and s. In other words, for prescribed s and fixed a, the existence of the control regions as well as their size depends strongly on the choice of m.

$a_{m,s}$ can be described more precisely as a solution to a system of equations:

Theorem 1. *Under the above conditions, the maximal coupling constant is given by*

$$a_{m,s} = \min\left\{\frac{A_j}{|2\cos(2\pi j/n) - 2\cos(2\pi s/n)|} : 0 < j < s \text{ or } n - s < j < n\right\},$$

where (A_j, ω_j), $j = 0, \dots, n-1$, *is the solution of the system*

$$\sin\Omega_j \cos\Omega_j = -\omega_j \pi\Theta$$
$$\sin^2\Omega_j = A_j \pi\Theta,$$

with $\Omega_j = \pi(mj/n - \Theta(1 + \omega_j))$, $\Theta = ms/n \mod 1$.

The proof of the theorem can be found at the end of section 4.

Remark. For $m = 1$ it is possible to find the minimum by considering $j \in [0, n] \subset \mathbb{R}$. Then differentiating $A_j/|2\cos(2\pi j/n) - 2\cos(2\pi s/n)|$ by j gives an expression on the (real) j for which the derivative is zero. By rounding we find the $j \in 0, \dots, n-1$ for which the minimum is obtained.

Theorem 1 implies that, indeed, *equivariant* Pyragas control is necessary to stabilize the discrete rotating waves for $s \neq 0$:

Corollary 1. *The discrete rotating wave*

$$z_k(t) = z_{k-m}(t - msp_s/n),$$

with $s \neq 0$ cannot be stabilized by standard Pyragas control, i.e. a control of the form

$$\dot{z} = f(z) + a(\rho z - 2z + \rho^{-1} z) + B\left[-z(t) + z(t - p_s)\right]$$

for any $B \in \mathbb{C}^{n \times n}$ and any $a > 0$.

4 Parameter regions for stabilization

By introducing equivariant Pyragas control, we select exactly one of the bifurcating discrete rotating waves. Thus equivariant Pyragas control is noninvasive only on the selected periodic orbit. This is the most important property of equivariant Pyragas control, and also the reason why it can succeed for the ring of coupled oscillators, where standard Pyragas control fails (except for the synchronized solution $z_0 = z_1 = \cdots = z_{n-1}$).

The goal of the proof is to reduce the problem to standard Hopf bifurcation. This facilitates the following stability analysis: It is possible to determine the stability of the selected bifurcating periodic orbit using standard exchange of stability in a two dimensional center manifold [6].

The proof consists of two parts: In the first part, we find the stabilization region, i.e. the region where the trivial equilibrium is stable at the selected Hopf bifurcation point. In the second part (see section 5), we prove that under the given conditions, the selected Hopf bifurcation is supercritical and thus stable.

Using characteristic equations, we can determine the stability of the trivial equilibrium. We linearize the coupled oscillator system, including the control term, in the new coordinates (2):

$$\dot{x}_j = \big(\lambda + \mathrm{i} - 2a(1 - \cos(2\pi j/n))\big)x_j + b_j\left[-x_j(t) + e^{2\pi \mathrm{i}\, mj/n} x_j(t-\tau)\right], \qquad j = 0, \ldots, n-1$$

Note that the linearized equations decouple corresponding to the invariant subspaces X_j, even when adding the delayed control term. By an exponential ansatz we obtain the characteristic equations for complex eigenvalues η:

$$0 = \chi_j(\eta) = \lambda + \mathrm{i} - 2a(1 - \cos(2\pi j/n)) + b_j\left[-1 + e^{2\pi \mathrm{i}\, mj/n - \tau\eta}\right] - \eta \tag{8}$$

The control matrix B is chosen carefully in such a way that every characteristic equation χ_j contains its own control parameter b_j, which makes it possible to choose them individually. Similar to [5] we define the *unstable dimension* $E(b_0, \ldots, b_{n-1})$ at the selected Hopf bifurcation point λ_s as the number of eigenvalues η with strictly positive real part, depending on the control parameters $b_0, \ldots, b_{n-1}$, counting multiplicity. Each characteristic equation χ_j contributes its unstable dimensions $E_j(b_j)$ to the total unstable dimension $E(b_0, \ldots, b_{n-1}) = \sum_{j=0}^{n-1} E_j(b_j)$ independently.

The unstable dimension of the uncontrolled system $E(b_0 = \cdots = b_{n-1} = 0)$ is given by

$$E(b_0, \ldots, b_{n-1} = 0) = \begin{cases} 2s & \text{for } s \leq n/2 \\ 2(n-s) & \text{for } s \geq n/2 \end{cases}.$$

Now consider $b_j \neq 0$. We search for purely imaginary eigenvalues η, which we parametrize by ω, more precisely $\eta = \mathrm{i}(1+\omega)$. We denote the corresponding curves on which the purely imaginary eigenvalues lie by $b_j(\omega)$. They are obtained by evaluating the characteristic equations at the Hopf bifurcation point $\lambda_s = 2a(1-\cos(2\pi s/n))$ where $p_s(\lambda_s) = 2\pi$:

$$\begin{aligned} b_j(\omega) &= \frac{\mathrm{i}\,\omega - (\lambda_s - \lambda_j)}{-1 + \exp\left(2\pi\,\mathrm{i}(mj/n - \Theta(1+\omega))\right)} \\ &= -\frac{1}{2}\Big(-(\lambda_s - \lambda_j) - \omega \cot\left(\pi(mj/n - \Theta(1+\omega))\right) \\ &\qquad + \mathrm{i}\,\omega - \mathrm{i}(\lambda_s - \lambda_j)\cot\left(\pi(mj/n - \Theta(1+\omega))\right)\Big). \end{aligned}$$

These curves denote the stability changes by Hopf bifurcation of the trivial equilibrium, i.e. the unstable dimension increases or decreases by 2 if the control parameter b_j crosses the corresponding line $b_j(\omega)$.

Note that the double eigenvalue $\eta = 0$ can only occur for $\omega = 0$ and thus lies on the already determined Hopf curves.

Further note that the curve $b_s(\omega)$, which corresponds to the selected Hopf bifurcation, is symmetric with respect to the real axis, crossing it only at $b_s(0) = 1/(2\pi\Theta)$. Moreover we find that the curve $b_{n-s}(\omega)$ goes through the origin: $b_{n-s}(0) = 0$.

Wherever the Hopf curves b_j, $j = 0, \ldots, n-1$, are complex differentiable they preserve complex orientation.

This enables us to identify regions with different unstable dimensions, as to the right of the oriented curve $b_j(\omega)$ the dimension will be bigger by 2 than to the left of the curve. For $b_j = 0$ the unstable dimension $E_j(b_j)$ is known and is either 0 ($\lambda_j \geq \lambda_s$) or 2 ($\lambda_j < \lambda_s$).

Next we determine the region with $E_j(b_j) = 0$. For $\lambda_j \geq \lambda_s$ the existence is trivial, as the origin is included.

For $\lambda_j < \lambda_s$, as the curves are all given explicitly, we find that the curve b_j forms a "loop". Following the curve for increasing ω, we find that the region inside this loop has no unstable eigenvalues, see also figures 4 and 5. This follows from complex differentiability which implies orientation preservation.

The existence of the loop can be seen as follows:

$$b_j(0) = \frac{\lambda_s - \lambda_j}{2}\Big(1 - \mathrm{i}\cot\left(\pi(mj/n - \Theta)\right)\Big)$$

lies to the right of the imaginary axis. Also the tangent to $b_j(0)$ limits where the curve extends:

$$\begin{aligned} b_j'(0) = \frac{1}{2}\Big(&\cot\left(\pi(mj/n - \Theta)\right) - \omega(\lambda_s - \lambda_j)\pi\Theta \sin^{-2}\left(\pi(mj/n - \Theta)\right) \\ &- \mathrm{i} + \mathrm{i}(\lambda_s - \lambda_j)\pi\Theta \sin^{-2}\left(\pi(mj/n - \Theta)\right)\Big). \end{aligned}$$

and the curvature of $b_j(\omega)$ is strictly negative for all ω between the poles:

$$\begin{aligned} \text{curvature } b_j(\omega) &= \left(\operatorname{Re} b_j'(\omega) \operatorname{Im} b_j''(\omega) - \operatorname{Re} b_j''(\omega) \operatorname{Im} b_j'(\omega)\right) \,/\, |b_j'|^3 \\ &= \frac{\Theta\pi\left(\omega\cos(\Omega_j)\sin(\Omega_j) - (\lambda_s - \lambda_j)\sin^2(\Omega_j)\right) + \sin^2(\Omega_j)}{8\Theta\pi|b'(\omega)|^3 \sin^4(\Omega_j)} > 0 \end{aligned}$$

if a is small enough for the loop to exist. For Ω_j tending to the pole, $b_j(\omega)$ tends to infinity in the real as well as in the imaginary part. Therefore, the area inside must have $E_j(b_j) = 0$.

However complex differentiability is not given for all coupling parameters a. If a becomes too large, the loop shrinks and finally disappears. For the parameter a where the loop disappears, the curve b_j is not complex differentiable.

Therefore we seek the point where the complex derivative of $b_j(\omega)$ vanishes. By defining $\Omega_j := \pi(mj/n - \Theta(1+\omega))$ and replacing $\lambda_s - \lambda_j$ with A_j, we find that

$$\begin{aligned} \operatorname{Re} b_j(\omega_j) &= \tfrac{1}{2}\left(A_j + \omega_j \cot \Omega_j\right) \\ \operatorname{Im} b_j(\omega_j) &= \tfrac{1}{2}\left(A_j \cot \Omega_j - \omega_j\right) \end{aligned}$$

Differentiating this expression with respect to ω, setting the derivatives to zero and rearranging the equations yields the two equations

$$\begin{aligned} \sin\left(\pi(mj/n - \Theta(1+\omega_j))\right) \cos\left(\pi(mj/n - \Theta(1+\omega_j))\right) &= -\omega_j \pi \Theta \\ \sin^2\left(\pi(mj/n - \Theta(1+\omega_j))\right) &= A_j \pi \Theta \end{aligned} \tag{9}$$

If $\lambda_s - \lambda_j > A_j$ for some j, no loop exists and therefore no stabilization is possible near the selected Hopf bifurcation.

In particular, it follows that we cannot use standard Pyragas control for the stabilization of the selected wave with index s. The equations in standard Pyragas control with $\Theta = 1$, and $m = 0$ read:

$$\begin{aligned} \sin\left(-\pi(1+\omega_j)\right) \cos\left(-\pi(1+\omega_j)\right) &= -\omega_j \pi \\ \sin^2\left(-\pi(1+\omega_j)\right) &= A_j \pi \end{aligned}$$

from which follows immediately that $A_j = 0$ for all j and therefore the stabilization is impossible.

For further analysis, we will call the regions where $E_j(b_j) = 0$ holds $\mathcal{B}_j$.

5 Proof of the main stabilization theorem

In section 4, we have achieved linear stability $E(b_0, \ldots, b_{n-1}) = 0$ at the selected Hopf bifurcation point λ_s. We fix the complex control parameters b_j in the regions $\mathcal{B}_j$ where the characteristic equations (8) produce only eigenvalues with strictly negative real part (with exception of the pair of purely imaginary eigenvalues of the selected Hopf bifurcation), see section 4 for details.

In the last step of the proof of stabilization, we must guarantee that we only encounter standard *supercritical* Hopf bifurcation at the Hopf point. Standard Hopf bifurcation for nonzero control amplitude is ensured because we only encounter eigenvalues with nonzero real part.

Consequently, it remains to show that the selected periodic orbit lies on the side of the Hopf bifurcation where the trivial equilibrium has unstable dimension two.

This is achieved by counting the unstable dimensions in the (λ, τ)-plane for fixed control parameters $b_0, \ldots, b_{n-1}$. The *Hopf curves* in this plane again tell us where the stability changes. The *Pyragas curve* determines the position of the periodic orbit.

The Pyragas curve $\tau_P(\lambda)$ is given by

$$\tau_P(\lambda) := \Theta\, p(\lambda),$$

where $\Theta = (ms)/n \mod 1$. Note that the Pyragas curve τ_P does not depend on the control matrix B. By the normalized Hopf frequency, we know that $p(\lambda_s) = 2\pi$. Furthermore, the continuation of the Pyragas curve is differentiable at $\lambda = \lambda_s$:

$$\tau_P'(\lambda)\big|_{\lambda=\lambda_s} = \Theta\, p'(\lambda_s)$$

Next, we determine the Hopf bifurcation curves $\tau_j(\lambda)$, $j = 0, \ldots, n-1$, at $b_j = |b_j| \exp(\mathrm{i}\,\beta_j)$ from the characteristic equations $\chi_j(\eta) = 0$:

$$\tau_j(\lambda) = \frac{\pm \arccos\big(\cos\beta_j - (\lambda - \lambda_s)/|b_j|\big) + \beta_j + 2\pi m j/n + 2\pi N}{1 - |b_j| \sin\beta_j \mp \sqrt{|b_j|^2 \sin^2\beta_j + (\lambda - \lambda_s)\big(2|b_j|\cos\beta_j - (\lambda - \lambda_s)\big)}},$$

for $j = 0, \ldots, n-1$, with integer N, enumerating the solutions. These curves determine the Hopf bifurcations in the (λ, τ)-plane. Similar Hopf curves are obtained in [13], therefore we do not repeat the calculation here. Note that the Hopf curves τ_j depend on the respective control parameter b_j.

For further calculations, we linearize the characteristic equations

$$\chi_j(\eta) = \lambda + \mathrm{i} + b_j \left[-1 + \exp(2\pi\, \mathrm{i}\, mj/n - \tau\eta) \right] - \eta, \qquad j = 0, \ldots, n-1,$$

with respect to $\lambda = \lambda_s + \bar{\lambda}$, $\eta = \mathrm{i}\,\bar{\omega}$, and $\tau = 2\pi\Theta + \bar{\tau}$. Of particular interest is the linearization of the characteristic equation for $j = s$, i.e. corresponding to the selected Hopf bifurcation. Linearizing for $j = s$ and separating into real and imaginary part yields the following expression:

$$\begin{aligned} \bar{\lambda} &= -\operatorname{Im} b_s\, (2\pi\Theta\bar{\omega} + \bar{\tau}) \\ 0 &= \operatorname{Re} b_s\, (2\pi\Theta\bar{\omega} + \bar{\tau}) + \bar{\omega} \end{aligned}$$

By rearranging these equations, we obtain

$$\bar{\tau} = -\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Re} b_s}\, \bar{\omega} \qquad \text{and} \qquad \bar{\lambda} = \frac{\operatorname{Im} b_s}{\operatorname{Re} b_s}\, \bar{\omega}.$$

Therefore we can conclude that the derivative of τ_s with respect to λ at the selected Hopf bifurcation is given by

$$\tau_s'(\lambda)\big|_{\lambda=\lambda_s} = -\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Im} b_s}\,.$$

By the following orientation considerations we can determine the resulting total unstable dimensions $E(\lambda, \tau)$ of the trivial equilibrium $x_0 = \cdots = x_{n-1} = 0$ in the domains complementary to the Hopf curves.

We once more linearize the characteristic equation (8) for $j = s$, but now with respect to τ and λ,

$$\varphi(\lambda, \tau) = \lambda - \eta\, \tau\, b_s\, \exp(2\pi\,\mathrm{i}\, ms/n - \tau\eta) = \xi,$$

and also with respect to η,

$$\psi(\eta) = 1 + \eta\, \tau\, b_s\, \exp(2\pi\,\mathrm{i}\, ms/n - \tau\eta) = \xi.$$

Now we find the expression

$$(\lambda, \tau) = (\varphi^{-1} \circ \psi)(\eta),$$

where ψ is orientation preserving because it is holomorphic. To determine the orientation of φ, we need to calculate its determinant at $\eta = \mathrm{i}$, $\lambda = \lambda_s$, $\tau = 2\pi\Theta$:

$$\begin{aligned} \det \varphi &= -\operatorname{Im}\big(\eta\, b_s\, \exp(2\pi\,\mathrm{i}\, ms/n - \tau\eta)\big) \\ &= -\operatorname{Re} b_s \end{aligned}$$

Thus, depending on the control parameter b_s, we conclude that φ is either orientation reversing ($\operatorname{Re} b_s > 0$) or orientation preserving ($\operatorname{Re} b_s < 0$). The τ_s-curve is oriented downwards in both cases. In the orientation reversing case, it follows that the region with $E(\lambda, \tau) = 2$ can be found to the left of the Hopf-curve τ_s in the (λ, τ)-plane, while in the orientation preserving case, this region can be found to the right.

In the following, we will carry out the analysis for the orientation reversing case, the other case being analogous.

Whether the Pyragas curve exists to the right or to the left of $\lambda = \lambda_s$ depends on whether the original bifurcation (without control) is subcritical ($\lambda < \lambda_s$) or supercritical ($\lambda > \lambda_s$).

Supercritical case: The Pyragas curve exists for $\lambda > \lambda_s$. If $\tau_s'(\lambda_s) < 0$ then we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Im} b_s} < \Theta\, p'(\lambda_s).$$

On the other hand, if $\tau_s'(\lambda_s) > 0$ we will find that it enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Im} b_s} > \Theta\, p'(\lambda_s).$$

Note that the control parameter b_s can also be chosen to be real. Indeed, if the control parameter b_s is chosen on the real line, then the Hopf curve is oriented vertically downwards and we can stabilize for all possible values of $p'(\lambda_s)$.

Subcritical case: The Pyragas curve exists for $\lambda < \lambda_s$. In this case, if $\tau_s'(\lambda_s) < 0$ then we will find that τ_P enters the region with $E(\lambda, \tau) = 2$ whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Im} b_s} > \Theta\, p'(\lambda_s).$$

Note that the inequality sign changes, compared to the supercritical case. On the other hand, if $\tau_s'(\lambda_s) > 0$ we will find that τ_P enters the region with unstable dimension 2 whenever

$$-\frac{1 + 2\pi\Theta \operatorname{Re} b_s}{\operatorname{Im} b_s} < \Theta\, p'(\lambda_s).$$

This concludes the proof of the Main Theorem.

6 Taking advantage of the full rotational symmetry

The ring of coupled Stuart-Landau oscillators (2), (3) offers an additional rotational symmetry which has not been used yet for the construction of the equivariant control term. Including this rotating wave property, we may choose *arbitrary delay time* by introducing an additional rotational operator.

Again we select the wave with index $j = s$ and we aim to stabilize by equivariant Pyragas control as follows:

$$\dot{z} = f(z) + a(\rho z - 2z + \rho^{-1} z) + B\left[- z + e^{2\pi \mathrm{i}(\Theta - ms/n)} \rho^m z(t - \Theta p)\right].$$

This control term is also noninvasive on the selected rotating wave, since after time Θp, each individual oscillator has rotated by an angle of $2\pi\Theta$. The parameter Θ, and consequently also the time delay $\tau = \Theta p$, can be chosen arbitrary. Even choices of $\Theta > 1$ make sense and offer chances to investigate the system for larger time delays.

The analysis from sections 4 and 5 can be carried out in the same manner as before. We obtain the characteristic equations ($j = 0, \ldots, n-1$)

$$\chi_j(\eta) = \lambda + \mathrm{i} - 2a(1 - \cos(2\pi j/n)) + b_j \left[-1 + \exp\left(2\pi \mathrm{i}\, m(j-s)/n + 2\pi \mathrm{i}\, \Theta - 2\pi\Theta\eta\right)\right] - \eta.$$

The stability of the trivial equilibrium changes by two whenever one of the curves

$$b_j(\omega) = \frac{\mathrm{i}\,\omega - (\lambda_s - \lambda_j)}{-1 + \exp\left(2\pi \mathrm{i}\, m(j-s)/n - 2\pi \mathrm{i}\, \Theta\omega\right)}$$

is crossed with the complex control parameter b_j. Note that these equations also depend on the weighted time delay Θ. Moreover the investigation of the supercriticality of the Hopf bifurcation can be carried out analogously, and is therefore not repeated.

This control ansatz results in larger control regions for smaller time delay. As we can choose Θ arbitrarily small, there particulary is no upper bound on the real part of the maximal eigenvalue $\lambda_s - \lambda_j$, compare (9).

Theorem 2. *Consider the Hopf bifurcation of discrete rotating waves as in the Main Theorem with fixed a and s. Let m be coprime to n. Choose $\Theta > 0$ small enough.*

Then there exist open regions of complex control parameters $b_0, \ldots, b_{n-1}$ such that the delayed feedback control

$$\dot{z} = f(z) + a(\rho z - 2z + \rho^{-1} z) + B\left[- z + e^{2\pi \mathrm{i}(\Theta - ms/n)} \rho^m z(t - \Theta p)\right].$$

stabilizes the discrete rotating wave solution

$$z_k(t) = z_{k-m}(t - msp_s/n),$$

selectively and noninvasively for a time delay $\tau = \Theta p$.

To prove this theorem, we investigate again the vanishing point of the loop of the curve b_j. Analogously to section 4, we obtain the equations

$$\sin\left(\pi m(j-s)/n - \pi\Theta\omega_j\right) \cos\left(\pi m(j-s)/n - \pi\Theta\omega_j\right) = -\omega_j \pi\Theta$$
$$\sin^2\left(\pi m(j-s)/n - \pi\Theta\omega_j\right) = A_j \pi\Theta.$$

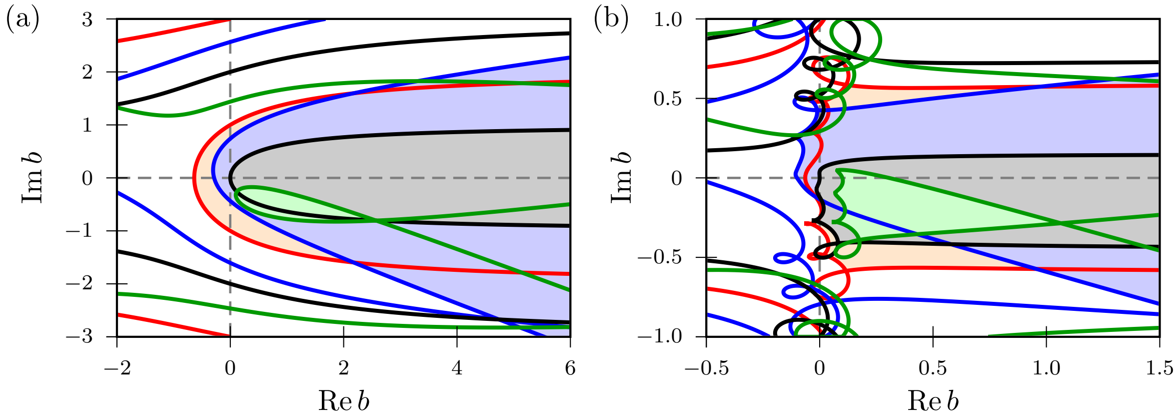

Figure 3: In (b) *the stabilization curves and regions for the sum of the delays as in* (10) *with* $n = 4$ *and* $a = 0.2$ *are shown. For comparison* (a) *shows the curves for only one-delay term with* $s = 1$ *as in* (7)*. In accordance with the color coding of the other figures,* b_0 *is green,* b_1 *red,* b_2 *blue and* b_3 *black in the colored version. In the black and white version they get lighter starting from the black* b_0*-curve.*

Since $j \neq s$ and m coprime to n, it follows that $\sin^2(\pi m(j-s)/n - \Theta\omega_j)$ converges to a fixed, nonzero value in the limit $\Theta \to 0$. The case $\sin^2(\pi m(j-s)/n - \Theta\omega_j) = 0$ corresponds to the poles, where the curve is not complex differentiable. These points are therefore not of interest here. Therefore it follows that $A_j \to \infty$ for $\Theta \to 0$.

7 Linear combinations of control terms

In addition to the control terms discussed in the previous sections, it is of course possible to include more than one noninvasive control term into the equivariant Pyragas control. However, arbitrary linear combinations of noninvasive equivariant control terms will in general not lead to non-empty control regions, even if the individual controls do separately. It hence seems to be a necessary condition for the existence of a control region is that the control is invasive on all the other bifurcations which occur for $\lambda \leq \lambda_s$, but up to date, no proof exists.

Another suggestion is to include the sum of all noninvasive control terms, with no weights:

$$\begin{aligned} \dot{z} &= f(z) + a(\rho z - 2z + \rho^{-1} z) \\ &\quad + \sum_{m=0}^{n-1} B\left[-z(t) + \rho^m z(t-\tau_m)\right]. \end{aligned} \tag{10}$$

For a single complex control parameter b and near equivariant Hopf bifurcation (i.e. $a = 0$), this has been explored in [13]. These results also hold for the control matrix B as introduced above and small coupling parameter a. Some examples can be seen in figure 3. Note that, for increasing n, the control region also grows in comparison with a single control term. It can therefore be suitable to introduce a control of this form.

8 Summary and Discussion

The main goal of this paper was to show how we can use equivariance to eliminate severe restrictions of Pyragas control. A network of n identical Stuart-Landau oscillators coupled in a bidirectional ring was investigated, whose symmetry group is given by $D_n \times S^1$, i.e. the direct product of dihedral group D_n and the circle group S^1. This coupled network system contains n periodic orbits, which can be distinguished by their different spatio-temporal symmetries, i.e. the interplay of index-shifts and phase-shifts between oscillators. Depending on the system parameters, only the synchronized periodic orbit, or even none of the orbits is stable. Due to these characteristic features, the ring of n coupled Stuart Landau oscillators is an ideal candidate for the application of equivariant Pyragas control.

Using equivariant control terms, we are able to *select* one of these unstable periodic orbits, and *stabilize* it. A stabilization with standard Pyragas control is restricted to the synchronized periodic orbit only, which is proven in section 5. Thus the main aim is to adapt Pyragas control for the non-synchronized periodic orbits. This is achieved by using equivariance of the periodic orbit and including the spatio-temporal symmetry in the time-delayed control term. As the main result, we show that, indeed, a control is now possible for *every* periodic orbit, regardless of its spatio-temporal symmetry. Also note that the cubic term of the Hopf normal form may now take *any* complex value except for $\gamma \in \mathbb{R}_+$. An upper bound, which is sharp, on the coupling parameter a is also established.

By including additionally the rotational symmetry into the control term, as demonstrated in section 6, the stabilization can also be achieved for arbitrary *strong coupling parameter* a. In fact, the rotational symmetry now also allows us to use *arbitrary time delay*, which can be expected to be useful in experimental realizations.

In contrast to previous publications concerning the Pyragas stabilization of coupled Stuart-Landau oscillators, we have included a complex *matrix* into the control term *which also incorporates the prescribed symmetry.* The case of a single control parameter is included in this control matrix. However, using only one control parameter diminishes the chances of stabilization drastically, since it is necessary to find an overlap of control regions $\mathcal{B}_j$. See figure 4 for an example where stabilization with a suitable control matrix succeeds but is not possible for any complex control parameter.

Linear combinations of noninvasive control terms were briefly discussed in section 7. Such linear combinations often provide control regions. However, necessary and sufficient conditions for the stabilization by linear combinations of noninvasive control terms are presently unknown.

The *sum of all possible control terms* does indeed give us large control regions for small coupling parameters a, see figure 3. An upper bound on a needs yet to be established.

In conclusion this publication shows that equivariant Pyragas control succeeds in situations where the well-established Pyragas control fails. Such situations include the equivariance of a system, too strong coupling parameters, restrictions on the cubic term in the Hopf normal form and a fixed time-delay. In the present setting of n diffusively coupled Stuart-Landau oscillators, we are able to give explicit analytic necessary and sufficient conditions leading to stabilization for different equivariant time-delayed control schemes.

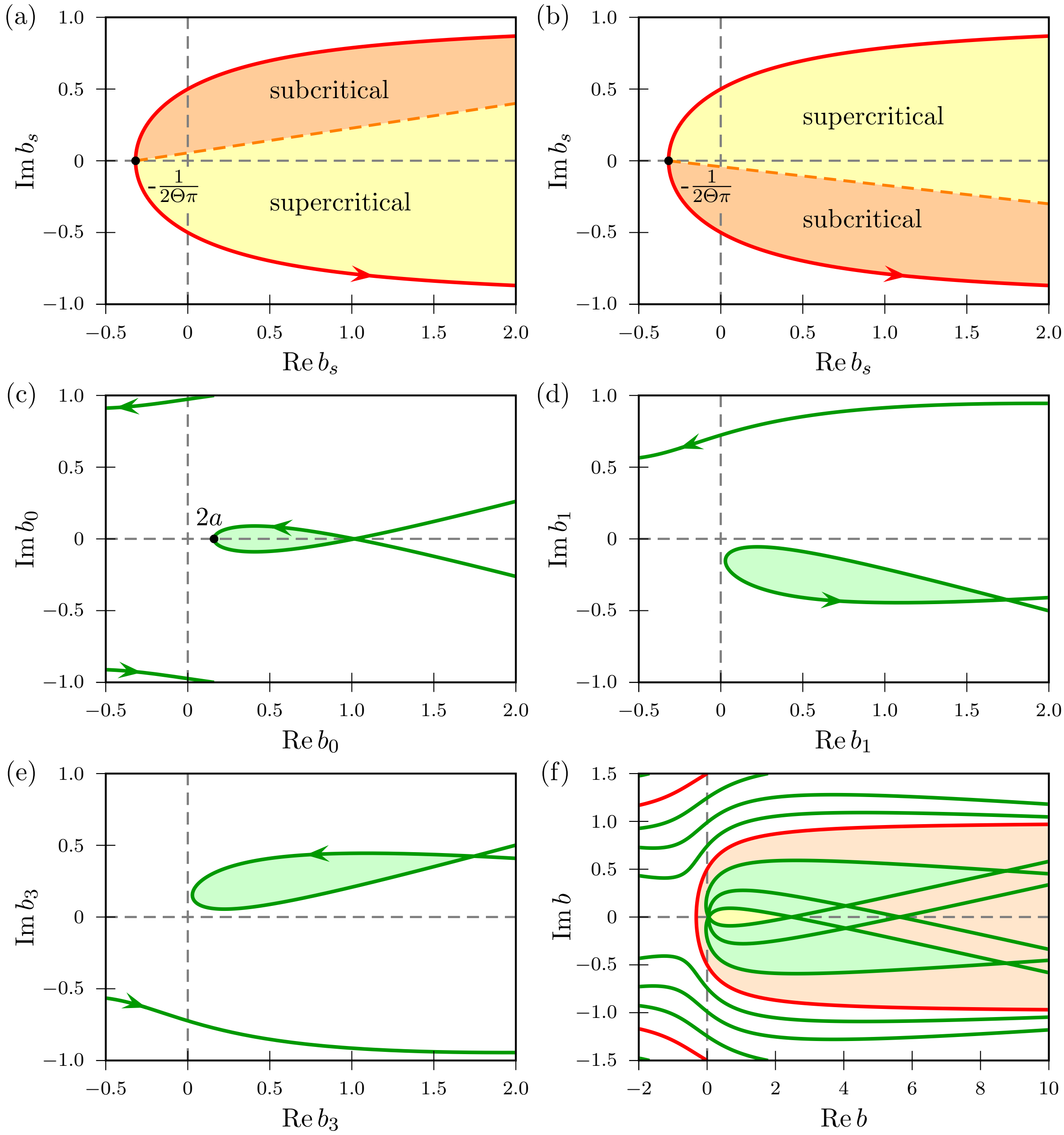

Figure 4: Stabilization curves and regions for $n = 4$ *coupled oscillators as in* (2), (3) *controlled as in* (7) *with index-shift* $m = 1$*, for the third hopf-bifurcation (i.e.* $s = 2$*). The coupling constant was chosen as* $a = 0.08$ *in* (a) *to* (e) *and* $a = 0.01$ *in* (f)*.*
The selected bifurcation ($j = s$*) is red while the green curves correspond to bifurcations before the selected bifurcation (i.e* $\lambda_j < \lambda_s$*) – see also* figure 1 (a) *and* (b)*.*

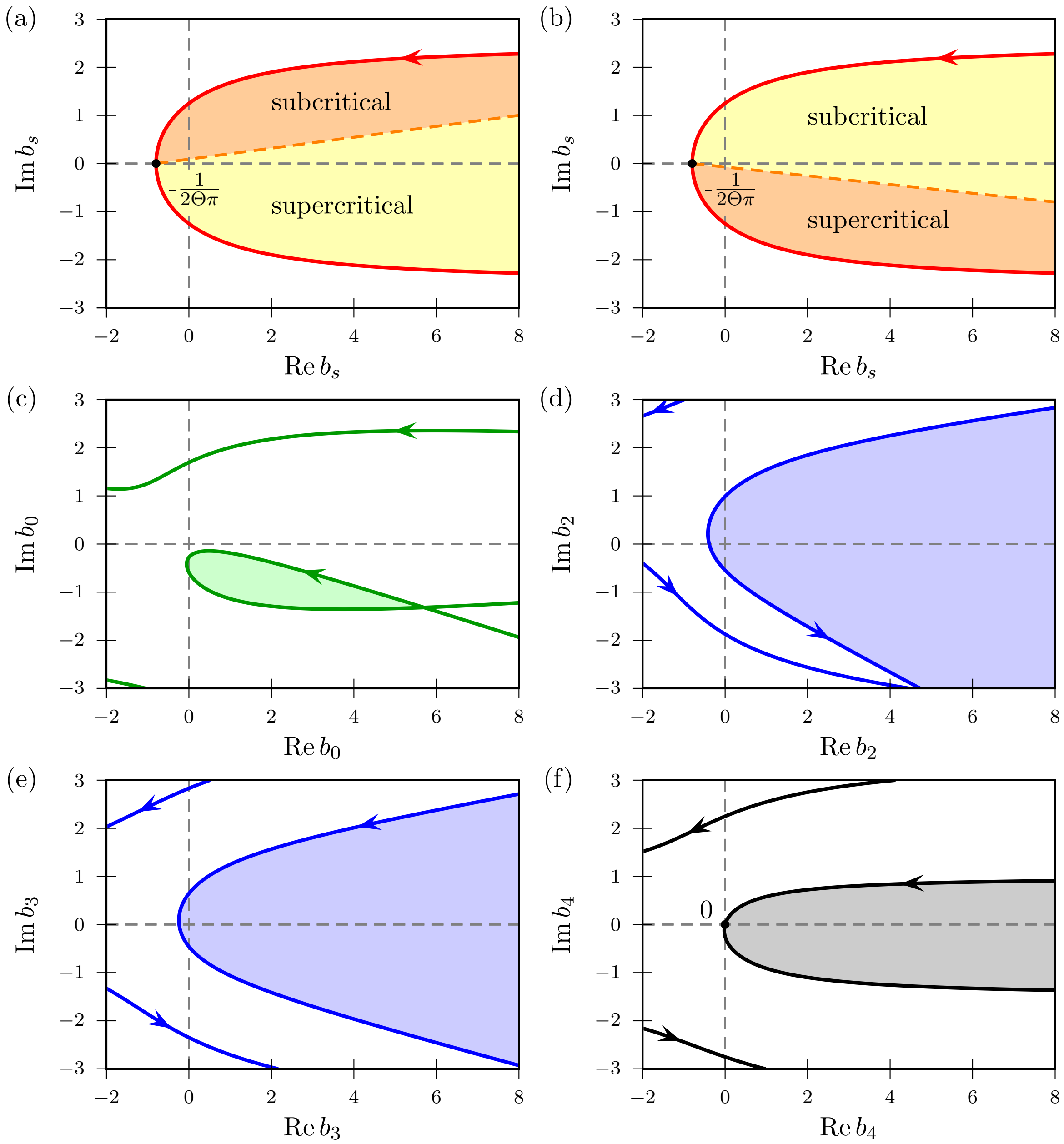

Figure 5: Stabilization curves and regions for $n = 5$ coupled oscillators as in (2), (3) *with coupling constant $a = 0.2$, controlled as in* (7) *with index-shift $m = 1$, for the second hopf-bifurcation (i.e. $s = 1$). The curve corresponding to the selected bifurcation ($j = s = 1$) is red and the green curve corresponds to the bifurcation before the selected one (i.e $\lambda_0 < \lambda_s$). Blue is used for bifurcations wich occur for $\lambda_j > \lambda_s$. The black curve in* (f) *is for the bifurcation that occurs simultaneously (i.e. $\lambda_j = \lambda_s$). See also* figure 1 (c) *and* (d).

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References

[1] M. Bosewitz, *Stabilisierung gekoppelter Oszillatoren durch verzögerte Rückkopplungskontrolle*, Bachelor Thesis, Freie Universität Berlin, 2013

[2] K. Bubolz, *Stabilisierung periodischer Orbits im System zweier gekoppelter Oszillatoren durch zeitverzögerte Rückkopplungskontrolle*, Bachelor Thesis, Freie Universität Berlin, 2013

[3] B. Fiedler, *"Global bifurcation of periodic solutions with symmetry"*, Lect. Notes Math. **1309**, Springer-Verlag, Heidelberg, 1988

[4] B. Fiedler, V. Flunkert, M. Georgi, P. Hövel, E. Schöll, *Refuting the odd number limitation of time-delayed feedback control*, Phys. Rev. Lett. **98** (2007), 114101

[5] B. Fiedler, V. Flunkert, P. Hövel, E. Schöll, *Delay stabilization of periodic orbits in coupled oscillator systems*, Phil. Trans. R. Soc. A **368** (2010), 319-341

[6] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, H.O. Walther, *"Delay Equations: Functional-, Complex-, and Nonlinear Analysis"*, Springer-Verlag, New York, Applied Mathematical Sciences Vol. **110**, 1995

[7] M. Golubitsky, I. Stewart, D. Schaeffer, *"Singularities and Groups in Bifurcation Theory"*, Vol. 2. AMS **69**, Springer-Verlag, New York, 1988

[8] M. Golubitsky, I. Stewart, *"The Symmetry Perspective"*, Progress in Mathematics, Volume **200**, Birkhäuser, Basel, 2002

[9] C. M. Postlethwaite, G. Brown, M. Silber, *Feedback control of unstable periodic orbits in equivariant Hopf bifurcation problems*, Phil. Trans. R. Soc. A **371** (2013), 20120467

[10] K. Pyragas, *Continous control of chaos by self-controlling feedback*, Phys. Lett. A **170** (1992), 421-428

[11] K. Pyragas, *A Twenty-Year Review of Time-Delay Feedback Control and Recent Developments*, Internationnal Symposium on Nonlinear Theory and its Applications (2012), Palma, Majorca

[12] I. Schneider, *Delayed feedback control of three diffusively coupled Stuart-Landau oscillators: a case study in equivariant Hopf bifurcation*, Phil. Trans. R. Soc. A **371** (2013), 20120472

[13] I. Schneider, *Equivariant Pyragas Control*, Master Thesis, Freie Universität Berlin, 2014

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