1

Symmetry-Breaking Control of Rotating Waves

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Abstract. Our aim is the stabilization of time-periodic spatio-temporal synchronization patterns. Our primary examples are coupled networks of Stuart-Landau oscillators. We work in the spirit of Pyragas control by noninvasive delayed feedback. In addition we take advantage of symmetry aspects. For simplicity of presentation we first focus on a ring of coupled oscillators. We show how symmetry-breaking controls succeed in selecting and stabilizing unstable periodic orbits of rotating wave type. Standard Pyragas control at minimal period fails in this selection task. Instead, we use arbitrarily small noninvasive time-delays. As a consequence we succeed in stabilizing rotating waves – for arbitrary coupling strengths, and far from equilibrium.

1.1 Introduction

In their 1990 publication "Controlling Chaos" [17], Ott, Grebogi and Yorke presented a first control scheme to stabilize unstable periodic orbits in chaotic systems. Another particularly successful method for stabilizing periodic orbits was introduced by Kestutis Pyragas in 1992, using time-delayed feedback [19] as follows. Consider any autonomous system

$$\dot{z}(t) = F(z(t)) \ , \tag{1.1}$$

say on a state space $z \in \mathbf{R}^N$ or $\mathbf{C}^N$. Suppose that there exists a solution $z^*(t)$ which is an unstable periodic orbit with minimal period $p > 0$. Pyragas suggests to stabilize the periodic orbit $z^*(t)$ by adding a delayed control term. It consists of the difference between the current state $z(t)$ and a delayed state $z(t-\tau)$ of the system (1.1). The resulting delayed feedback system takes the form

$$\dot{z}(t) = F(z(t)) + \mathbf{b}\big(z(t-mp) - z(t)\big) \ . \tag{1.2}$$

Here $\mathbf{b}$ is a scalar control parameter, or a control matrix. The time-delay $\tau = mp > 0$ is an integer multiple m of the minimal period p. By periodicity of $z(t)$ the control term in (1.2) vanishes on the periodic orbit and is therefore called *noninvasive*.

Many theoretical investigations and experimental implementations have shown the success of Pyragas control in stabilizing unstable periodic orbits. For an overview, see for example the survey paper by Pyragas [20]. Pyragas control can be applied without any explicit knowledge of the model (1.1) or its solutions $z(t)$. This is one of the main reasons for its widespread use in experiments.

Analytic results include the *odd-number-limitation*, which was formulated and proven by Nakajima in 1997 [15]. It states that in (generic) non-autonomous systems Pyragas control fails for periodic orbits with an odd number of unstable Floquet exponents, counting algebraic multiplicities of exponents with positive real part, see Just et al. [13]. Genericity basically requires the absence of a trivial zero Floquet exponent.

Autonomous systems do not depend on time, explicitly. Hence their non-stationary periodic solutions do possess a trivial Floquet exponent zero. For autonomous systems, the odd-number-limitation was in fact refuted by Fiedler et al. [8] in 2007. Subcritical Hopf bifurcation for the Stuart-Landau oscillator provided an analytically accessible counterexample.

Pyragas control for networks of coupled oscillators is a wide open subject of research. In fact the periodic solutions may exhibit various spatio-temporal symmetries, besides trivial complete synchrony. Indeed, we can prove that, in diffusively coupled networks with positive coupling strength, stabilization by *standard* Pyragas control is restricted to the fully synchronized periodic orbit.

Introducing symmetry-breaking control terms can overcome this limitation and target periodic orbits of prescribed spatio-temporal symmetry separately. Our control terms still follow the main idea of Pyragas [19] – they are non-invasive and time-delayed. However the new control terms are able to select specific prescribed *spatio-temporal patterns* of the periodic orbits, as we will describe in more detail in section 1.2.

A first step in the direction of using spatio-temporal symmetries was already proposed by Nakajima and Ueda in 1998 [16]. It was intended as a remedy to the odd-number-limitation [15], originally, which was believed to also hold for autonomous systems, at that time. Though the odd-number limitation has been refuted for autonomous systems, their approach remains a first paradigm for symmetric systems. For odd nonlinearities, $F(-z) = -F(z)$, p-periodic odd oscillations may arise which satisfy

$$z^*(t) = -z^*(t - p/2) \tag{1.3}$$

at half period. The delayed feedback system is of the form

$$\dot{z}(t) = F(z(t)) + \mathbf{b}\big(z(t) + z(t - p/2)\big) \; . \tag{1.4}$$

Note how the control term (1.2) is noninvasive on odd oscillations (1.3), even though the time-delay $\tau = p/2$ has been reduced to *half* the minimal period p.

Using the network structure, another control term with half-period delay τ has been applied to a system of two diffusively coupled Stuart-Landau oscillators. See Fiedler et al. in 2010 [10]. The controlled system is of the following form:

$$\dot{z}_0(t) = f(z_0(t)) + a(z_1 - z_0) + \mathbf{b}\big(z_0(t) - z_1(t - p/2)\big) \tag{1.5}$$

$$\dot{z}_1(t) = f(z_1(t)) + a(z_0 - z_1) + \mathbf{b}\big(z_1(t) - z_0(t - p/2)\big)\ . \tag{1.6}$$

Here the state vectors z_0 and z_1 denote the first and the second oscillator, respectively, for $z = (z_0, z_1)$. The parameter $a > 0$ denotes diffusive coupling. Note how this control scheme is noninvasive on p-periodic anti-phase oscillations

$$z_1^*(t) = z_0^*(t - p/2)\ , \tag{1.7}$$

where the two oscillators are phase-locked at half-period p. In 2013, slightly more general control schemes were used by Bosewitz [1] and Bubolz [3] for the same system, overcoming certain limitations. We will review their contribution in section 1.4.

Already for three equilaterally coupled Stuart-Landau oscillators, new challenges arise. Two different types of spatio-temporal symmetries arise, for p-periodic solutions (z_0, z_1, z_2) which are not fully synchronous. First we may encounter *discrete rotating waves*

$$z_1^*(t) = z_0^*(t - p/3), \qquad z_2^*(t) = z_0^*(t - 2p/3)\ . \tag{1.8}$$

Their reflected counterpart

$$z_2^*(t) = z_0^*(t - p/3), \qquad z_1^*(t) = z_0^*(t - 2p/3)\ , \tag{1.9}$$

rotates in the opposite direction. Another spatio-temporal symmetry type features double frequency oscillation of one node:

$$z_0^*(t) = z_0^*(t - p/2), \qquad z_2^*(t) = z_1^*(t - p/2)\ . \tag{1.10}$$

Here the oscillators z_1 and z_2 of minimal period p are phase-locked at half-periods, whereas z_0 oscillates at double frequency. Again permutation of indices produces related solutions of analogous spatio-temporal symmetry type. See [12] and Fiedler [7] for an in-depth discussion in local and global bifurcation settings, respectively.

This simple example already demonstrates why the control term should be able to *select* periodic solutions with the desired symmetry type. We call a control term *pattern-selective* or *symmetry-breaking*, if it is noninvasive on exactly *one* periodic solution with a prescribed spatio-temporal pattern. First successful controls in this sense were presented by Schneider in 2011 [21]; see also [22] and closely related work by Postlethwaite at al. [18].

Choe et al. considered delay-coupled networks in [4, 5]. They were able to show that by tuning the coupling phase it is possible to control the stability of synchronous periodic orbits. Later they generalized their results by using arbitrary and distributed time-delay as well as nonlinear coupling terms [6].

Unifying all control terms, a general equivariant formulation has been presented, and its success has been proven near equivariant Hopf bifurcation in [22, 23] as well as in [18] in 2013:

$$\dot{z}(t) = F(z(t)) + \mathbf{b}\left(- z(t) + hz(t - \Theta(h)\, p)\right) . \tag{1.11}$$

Here h and $\Theta(h)$ describe the spatio-temporal pattern of the periodic orbit such that the control term is again noninvasive on the periodic orbit. Again $\mathbf{b}$ denotes a suitably chosen control matrix. For details on the group theoretic formulation see Golubitsky, Stewart [11,12], Fiedler [7], and section 1.2 below.

For a ring of coupled oscillators, the most complete results on Pyragas stabilization, to date, have been presented in [24]. For rotationally symmetric oscillators, nonlinear controls have been constructed by Bosewitz [2]. Additionally, a sharp upper bound on the unstable Floquet multiplier allowing stabilization has been established in [2]. This upper bound depends on the time-delay τ, see also [9].

Our survey is organized as follows. We describe some general background on spatio-temporal patterns in section 1.2. Sections 1.3 and1.4 focus on a ring of n identical, diffusively coupled oscillators z_k, k mod n,

$$\dot{z}_k = f(z_k) + a(z_{k+1} - 2z_k + z_{k-1})\ , \tag{1.12}$$

where $a > 0$ is the diffusive coupling strength. The stabilization results for this model system will be discussed in section 1.4. We emphasize the aspect of symmetry-breaking and compare different control schemes. We show that standard Pyragas control is indeed not able to stabilize any but the totally synchronous periodic orbit in our model system. In section 1.5 we discuss results for general networks consisting of rotationally symmetric oscillators. In particular we establish that pattern-selective Pyragas control of rotating waves always succeeds, with sufficiently small delay τ and nonlinear control of order $1/\tau$. Section 1.6 summarizes our conclusions.

1.2 Spatio-Temporal Patterns: Theory

In this section we explain our concept of spatio-temporal symmetry patterns for time periodic solutions $z^*(t)$ of equivariant systems. We illustrate the abstract and rather general mathematical concept for the specific system (1.12) of a ring of diffusively coupled identical oscillators; see also 1.3. Our presentation follows [7].

For simplicity let us consider a linear action $z \mapsto g\, z$ of some group G of matrices g on $z \in \mathbf{R}^N$ or $\mathbf{C}^N$. We call the ODE system (1.1), $\dot{z} = F(z)$, *equivariant* under G, if $gz(t)$ is a solution of the ODE whenever $z(t)$ is, for any $g \in G$. Since the action of G is linear, this simply means that

$$F(gz) = g\,F(z) \tag{1.13}$$

for all elements $z \in \mathbf{R}^N$ or $\mathbf{C}^N$, and for all $g \in G$. We also call G an *equivariance group* of F in (1.1).

Consider our ring (1.12) of diffusively coupled oscillators, for example. Let D_n denote the dihedral group of n rotations and n reflections which leave the regular planar n-gon invariant. Then $G = D_n$ is an equivariance group of the oscillator ring (1.12). The n-gon is represented by the ring structure of the network. The vertices $k = 0, \dots, n-1$ indicate the oscillators z_k, and the edges define symmetric diffusion coupling. More precisely $D_n = \langle \rho, \kappa \rangle$ is generated by the rotation ρ over $2\pi/n$, and the reflection κ through the bisector of some fixed n-gon vertex angle. The linear action of $g \in D_n$ on $z = (z_0, \dots, z_{n-1})$ is given by index permutation:

$$(\rho z)_k = z_{k-1} \tag{1.14}$$

$$(\kappa z)_k = z_{-k} \tag{1.15}$$

for k mod n.

We now describe the spatio-temporal symmetry of any periodic solution $z^*(t)$ of any G-equivariant system $\dot{z} = F(z)$. Let $p > 0$ again denote the minimal period of $z^*(t)$, and let $\mathcal{O}^* = \{z^*(t) \,|\, 0 \le t < p\}$ denote the periodic orbit, as a set. We can then describe the *spatio-temporal symmetry* of $z^*(t)$ by a triplet (H, K, Θ), as illustrated in Fig. 1.1.

Let H denote the set of those group elements h in the equivariance group G which fix the orbit $\mathcal{O}^*$, as a set. In other words,

$$hz(t_0) = z(t_0 + \vartheta) \tag{1.16}$$

for some $t_0, \vartheta \in \mathbf{R}$. This definition makes sense because (1.16) holds for all $t_0 \in \mathbf{R}$, once it holds for any, by G-equivariance. Indeed, both $\tau \mapsto hz(t_0 + \tau)$ and $\tau \mapsto z(t_0 + \tau + \vartheta)$ are solutions of $\dot{z} = F(z)$ with the same initial condition at $\tau = 0$. Moreover the phase shift $\vartheta = \Theta(h)p$ is unique, for any given $h \in H$ and with normalized $\Theta(h)$ mod 1, by minimality of the period p of $z^*(t)$. This defines the *phase map*

$$\Theta : H \to S^1 = \mathbf{R}/\mathbf{Z}\,. \tag{1.17}$$

In fact G-equivariance implies that Θ is a group homomorphism:

$$\Theta(h_1 h_2) = \Theta(h_1) + \Theta(h_2) \bmod 1\,, \tag{1.18}$$

for all $h_1, h_2 \in H$. Finally, let $K = \ker \Theta = \{h \in H \,|\, \Theta(h) = 0\}$ denote the kernel of the homomorphism Θ. Then $K \le H$ is the set of group elements $h \in G$ which fix the periodic orbit $\mathcal{O}^*$, pointwise. In other words,

$$hz^*(t_0) = z(t_0) \tag{1.19}$$

for some (and hence for all) $t_0 \in \mathbf{R}$.

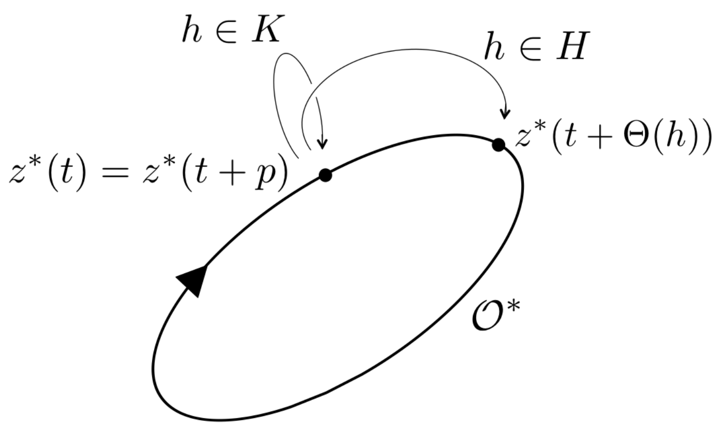

Fig. 1.1. Spatio-temporal symmetry $H^{\Theta} = \{(h, \Theta(h)) \in H \times S^1\}$ of any periodic orbit $z^*(t)$ with minimal period p. The group H fixes the periodic orbit as a set. The map $h \mapsto \Theta(h)$ indicates the (normalized) temporal phase shift on z^* effected by the spatial transformation $h \in H$. The kernel $K = \ker \Theta$ fixes any individual point on the periodic orbit, one by one.

To summarize, the spatio-temporal symmetry of a periodic orbit $z^*(t)$ with minimal period p is characterized by a triplet (H, K, Θ). The phase map homomorphism $\Theta : H \to S^1$ describes the normalized time shifts

$$hz(t) = z\big(t + \Theta(h)p\big) \tag{1.20}$$

for all $t \in \mathbf{R}$, $h \in H$, and the normal subgroup $K := \ker \Theta$ of H describes the purely spatial symmetry of any periodic point $z^*(t_0)$. We sometimes abbreviate the triplet by the *twisted symmetry*

$$H^{\Theta} := \{\big(h, \Theta(h)\big) \,|\, h \in H\}\,. \tag{1.21}$$

The range of the phase map Θ is a subgroup of S^1, and range $\Theta \cong H/K$ by the homomorphism theorem. For compact (and in particular for finite) subgroups H, continuity of Θ implies compactness of range Θ. Therefore range $\Theta \leq S^1$ is either finite, or else coincides with S^1. We call $z^*(t)$ a *discrete wave*, in the former case, and a *rotating wave*, in the latter. Of course, finite equivariance G cannot lead to rotating waves.

By equivariance, $gz^*(t)$ is time periodic of minimal period p, whenever $z^*(t)$ itself is, for any fixed $g \in G$. The spatio-temporal symmetry H_g^{Θ} of $gz^*(z)$ is conjugate to the twisted symmetry H^{Θ} of $z^*(t)$ itself, i.e.

$$H_g^{\Theta} = (gHg^{-1})^{\Theta}, \qquad \text{with} \quad \Theta(ghg^{-1}) := \Theta(h). \tag{1.22}$$

We also say that $z^*(t)$ and $gz^*(t)$ possess the same spatio-temporal *symmetry type*, differing only by conjugacy.

Let us now return to our example (1.12) of an oscillator ring with dihedral equivariance group $G = D_n$. The case $H = \langle\rho\rangle \cong \mathbf{Z}_n$ of the cyclic subgroup of rotations suggests $\Theta(\rho) := \pm 1/n \mod 1$ as a phase map with trivial kernel $K = \{\mathrm{id}\}$. Golubitsky and Stewart coined the term "ponies-on-a-merry-go-round" for such *discrete rotating waves*. These are spatially discrete analogues of the above rotating waves range $\Theta = S^1$. See (1.7) above, for the case $n = 2$, and (1.8), (1.9) for $n = 3$. Note how the cases $\Theta(\rho) = \pm 1/n$ are conjugate by the reflection $g = \kappa$; the right and left rotating discrete rotating waves belong to the same symmetry type.

More generally, we might observe discrete waves of the form

$$\Theta(\rho) = s/n \qquad \mod 1, \tag{1.23}$$

for any integer $0 \leq s < n$, and we do! Then Θ possesses trivial kernel, if and only if $s \in \mathbf{Z}_n^*$ is a multiplicative unit. In general $K \cong \mathbf{Z}(s,n)$, where (s,n) denotes the greatest common divisor of s and n. Again $\pm s$ belong to the same symmetry type, conjugated by the reflection κ.

Example (1.10) is of a different type: $H = \kappa \cong \mathbf{Z}_2$ and $\Theta(\kappa) = 1/2$. Of course the example generalizes to any n-ring, $n \geq 3$. We call $z^*(t)$ with this symmetry *standing waves*. For even n, only, the vertex $z_{n/2}(t)$ then oscillates at double frequency, as $z_0(t)$ always does. The reflection $\rho\kappa$ is nonconjugate to κ in $G = D_n$, for even n, and $H = \langle\rho\kappa\rangle \cong \mathbf{Z}_2$ with $\Theta(\rho\kappa) = 1/2$ does not feature any vertices with double frequency oscillations, in general.

Suppose next that $H = D_n$ arises in a spatio-temporal symmetry triplet (H, K, Θ) in the n-oscillator ring (1.12). Of course this is possible for $\Theta = 0$, i.e. for total synchrony $z_0 \equiv z_1 \equiv \ldots \equiv z_{n-1}$. We claim there is only one other possibility.

Indeed, $D_n/K = H/K \cong$ range $\Theta \leq S^1$ must be a nontrivial Abelian (in fact, cyclic) factor of D_n. Therefore K contains the commutator group $C(D_n) = [D_n, D_n]$ generated by all elements $g_1 g_2 g_1^{-1} g_2^{-2}$ with $g_1, g_2 \in D_n$. It is well-known that the commutator of D_n is generated by ρ^2. In particular the abeliarization $D_n/C(D_n)$ of D_n is $\mathbf{Z}_2$, for odd n, and the Klein 4-group $\mathbf{Z}_2 \times \mathbf{Z}_2$ for even n. Because $D_n/K = H/K \leq S^1$ is cyclic this implies range $\Theta = \{0, 1/2\}$. Moreover $K = \mathbf{Z}_n$ for odd n. This already implies $z_0 \equiv z_1 \equiv \ldots \equiv z_{n-1}$ and triviality $\Theta = 0$. Next suppose n is even. Then K corresponds to one of the $\mathbf{Z}_2$ factors in $D_n/C(D_n)$, i.e. $K = \mathbf{Z}_n$ or $K = D_{n/2}$. We can discard total synchrony $K = \mathbf{Z}_n$, as before.

The only remaining option, n even and $K = D_{n/2}$, corresponds to clustering into the two clusters $z_0 = z_2 = \ldots = z_{n-2}$ and $z_1 = z_3 = \ldots = z_{n-1}$, each of size $n/2$. By range $\Theta = \{0, 1/2\}$ the two clusters are half a period out of phase: $\Theta(\rho) = 1/2$. Note how our conclusions did not rely on any specific information about the underlying equations other than the symmetry aspect.

In sections 1.3 and 1.4 we focus on discrete right and left rotating waves of the type (1.23), as a target of Pyragas control. Rotating waves, i.e. range $\Theta = S^1$ are addressed in section 1.5. We plan to treat standing waves elsewhere.

1.3 Spatio-Temporal Patterns: Application to Rings of Oscillators

We have seen how, both, standing waves and discrete rotating waves may appear, hand-in-hand and at the same parameters. Hopf bifurcation in oscillator rings (1.12) provides an example. To be specific, and for simplicity of presentation, let us consider identical Stuart-Landau oscillators,

$$f(z) = (\lambda + \mathrm{i} + \gamma|z|^2)\, z\ , \tag{1.24}$$

for the individual dynamics in (1.12). Here λ is the real bifurcation parameter and the cubic term parameter γ is complex. In total, there are n types of discrete rotating waves and they all appear at equivariant Hopf bifurcations.

Proposition 1 (Equivariant Hopf bifurcations, [24]). *Consider the coupled oscillator ring (1.12), (1.24) of $n \geq 3$ identical, and identically diffusion coupled, Stuart-Landau oscillators. Hopf bifurcation occurs at the parameter values*

$$\lambda = \lambda_s = 2a\big(1 - \cos(2\pi s/n)\big)\ , \qquad s = 0, \ldots, n-1\ . \tag{1.25}$$

The purely imaginary eigenvalues are normalized to $\pm\mathrm{i}$, i.e. to unit frequency. They are of algebraic and geometric real multiplicity 4, for $s \notin \{0, n/2\}$. The associated eigenspace possesses complex dimension 2 and real dimension four. Both, standing waves and discrete rotating waves bifurcate, for each $s \notin \{0, n/2\}$. The discrete rotating waves are harmonic,

$$z_k(t) = r_s \exp\left(2\pi\mathrm{i}\left(\frac{t}{p_s} + s\frac{k}{n}\right)\right)\ , \tag{1.26}$$

for oscillators z_k, $k = 0, \ldots, n-1$, respectively, and are phase shifted by $2\pi s/n$ between adjacent oscillators. See (1.23). Amplitudes r_s and minimal periods p_s are given explicitly by

$$r_s^2 = (\lambda_s - \lambda)\,/\mathrm{Re}\,\gamma\ , \tag{1.27}$$
$$p_s = 2\pi/\left(1 + r_s^2\,\mathrm{Im}\,\gamma\right)\ . \tag{1.28}$$

In particular the discrete rotating waves bifurcate supercritically*, i.e. towards $\lambda > \lambda_s$, for* $\mathrm{Re}\,\gamma < 0$*, and* subcritically*, i.e. towards $\lambda < \lambda_s$, for* $\mathrm{Re}\,\gamma > 0$.

The minimal period p_s grows with amplitude (soft spring) *if* $\mathrm{Im}\,\gamma < 0$*, and decreases with amplitude* (hard spring) *if* $\mathrm{Im}\,\gamma > 0$.

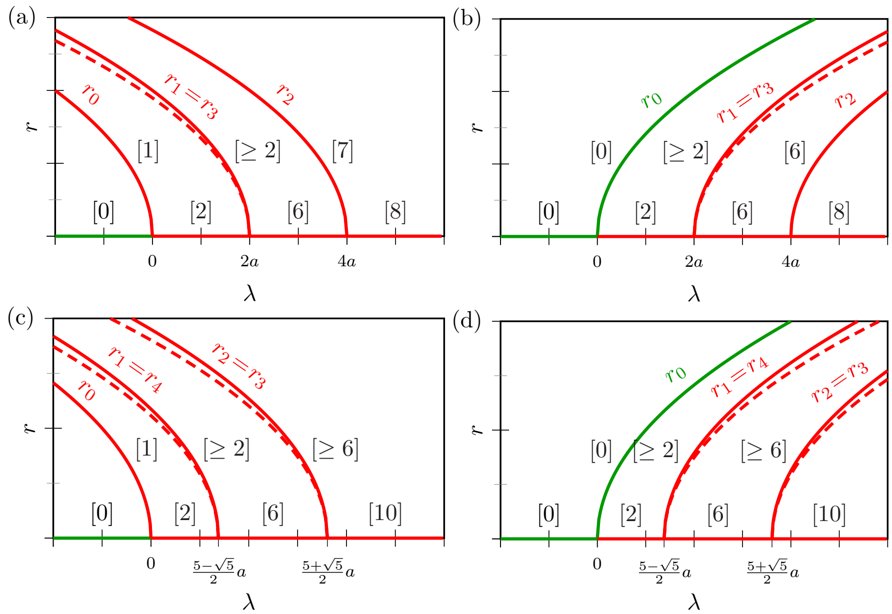

Fig. 1.2. Parameter-dependent stability of the equilibria and the bifurcating periodic orbits. The upper row shows the bifurcations for $n = 4$, while the lower one is for $n = 5$. In (**a**) and (**c**) one can see the subcritical bifurcations with $\mathrm{Re}\,\gamma = 0.1$ and in (**b**) and (**d**) the supercritical ones with $\mathrm{Re}\,\gamma = -0.1$. The coupling parameter was always chosen as $a = 0.2$. In brackets the number of unstable dimensions is denoted. Stable objects are colored in green, unstable ones in red. This figure, including its caption, has previously been published in [24].

Not surprisingly, the waves with index s and index $n - s$ bifurcate at the same point $\lambda_s = \lambda_{n-s}$. The resulting discrete left and right rotating waves are conjugate by reflection κ. See (1.23). See also Figure 1.2 for $n = 4$ and $n = 5$ oscillators. Standing waves are known to also bifurcate at $\lambda_s = \lambda_{n-s}$, for $s \notin \{0, n/2\}$; see [7,11]. They are not harmonic and their bifurcation direction and stability my differ from the discrete rotating waves. The Hopf bifurcation at $\lambda_0 = 0$ is a standard bifurcation with simple complex eigenvalues $\pm\mathrm{i}$ and real two-dimensional eigenspace. This leads to the bifurcation of fully synchronous periodic solutions.

An elementary proof of proposition 1 relies on the Ansatz (1.26). The harmonic character of the Ansatz (1.26) in time t is justified by the S^1-equivariance of the identical Stuart-Landau oscillators (1.24); see also section 1.5 below. The discrete harmonic character of the Ansatz (1.26) in the oscillator nodes k is justified by the imposed discrete rotating wave symmetry $\Theta : \mathbf{Z}^n \to S^1 = \mathbf{R}/\mathbf{Z}$ in (1.12). In fact, both aspects can also be unified by studying the ansatz (1.26) as the definition of a rotating wave under the full equivariance group $G = D_n \times S^1$ of (1.12), (1.24). Insertion of (1.26) into

(1.12) implies all other claims. For a more general analysis, in the framework of section 1.2, see again [7, 11].

The fully synchronous periodic solution is the only one which may be stable in the uncontrolled system, locally at Hopf bifurcation.

Proposition 2 (Stability of the periodic orbits, [24]). *For $s \neq 0$, the bifurcating discrete rotating waves (1.26), enumerated by s, are unstable, both in the sub- and the supercritical case. For $s = 0$, i.e. the synchronous case, the periodic solution is unstable in the subcritical and stable in the supercritical case.*

For a complete proof of Proposition 2 see [24]. The essential step of the proof is to study the n dynamically invariant complex irreducible representation subspaces

$$X_s = \big\{ (z_0, \ldots, z_{n-1}) \;\big|\; z_k = \mathrm{e}^{-2\pi \mathrm{i} s/n} z_{k-1} \quad \text{for all } k \bmod n \big\} \tag{1.29}$$

On each subspace $x_s \in X_s$ the system (1.12), (1.24) reduces to one single complex-valued equation

$$\dot{x}_s = f(x_s) - 2a\big((1 - \cos(2\pi s/n)\big)\, x_s \;. \tag{1.30}$$

Here $x_s \in X_s$. Hopf bifurcation occurs at $\lambda = \lambda_s = 2a(1 - \cos(2\pi s/n))$, with the spatio-temporal symmetry of the bifurcating periodic solution inheriting the symmetry of the invariant subspace X_s.

1.4 Pyragas Stabilization for a Ring of Coupled Stuart-Landau Oscillators

In this section we aim to stabilize unstable discrete rotating waves with prescribed spatio-temporal symmetry pattern

$$z_k(t) = z_{k-1}(t - sp_s/n) \;, \tag{1.31}$$

$k \bmod n$. In other words, an *index shift* ρ by 1 corresponds to a normalized *phase shift* $\Theta(\rho) = s/n$. The minimal period of the selected discrete rotating wave is denoted by p_s. The periodic orbit is uniquely determined by the choice of the parameter $s \in \{1, \ldots, n-1\} \bmod n$. We recall how the solutions for s and $-s$ are conjugate by reflection $\kappa : k \leftrightarrow -k$, mod n.

The stabilization problem has first been addressed for two coupled oscillators by Fiedler et al. [10], then for three coupled oscillators by Schneider [21,22] and Postlethwaite et al [18]. The general case of n oscillators has recently been discussed by Schneider and Bosewitz [24].

For an equivariant control term we choose

$$\mathbf{b}\big(- z(t) + h\, z(t - \tau)\big) \tag{1.32}$$

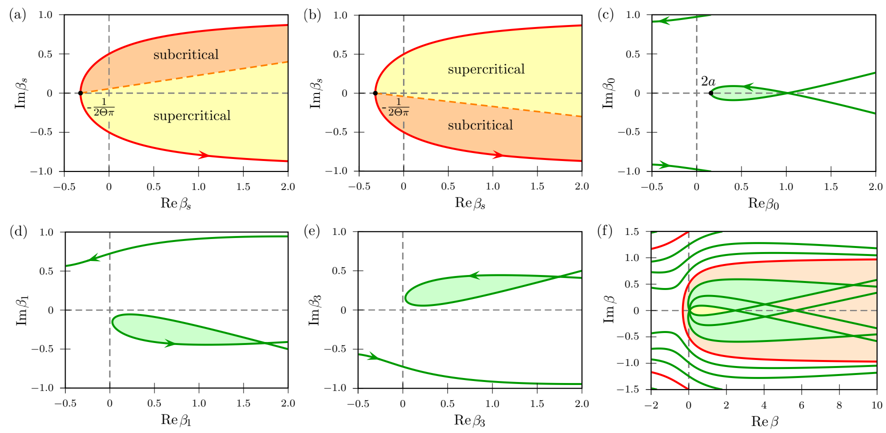

Fig. 1.3. Stabilization curves and regions for $n = 4$ coupled oscillators as in (1.12), (1.24) controlled as in (1.41) with index shift $m = 1$, for the third Hopf bifurcation (i.e. $s = 2$). The coupling constant was chosen as $a = 0.08$ in (**a**) to (**e**) and $a = 0.01$ in (**f**). The curve belonging to the parameter $\beta_s = \beta_2$ is drawn in red, while the curves corresponding to β_1, β_3 and β_4 are drawn in green. These figures have previously been published in [24].

with suitable $\mathbf{b}$, h and $\tau = \Theta(h)p_s$. Note that the control term is noninvasive on the periodic orbit (1.31); see also (1.20).

To achieve large stabilization regions, and preserve equivariance under $H = \mathbf{Z}_n$, it is suitable to employ complex circulant control matrices with constant diagonals, i.e.

$$\mathbf{b} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ b_{n-1} & b_0 & b_1 & \cdots & b_{n-2} \\ b_{n-2} & b_{n-1} & b_0 & \cdots & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_0 \end{pmatrix} \tag{1.33}$$

with constant diagonal coefficients $b_k \in \mathbf{C}$. Note the highly nonlocal character of (1.33) with all-to-all coupling. Equivalently to (1.33) we may employ n complex control parameters $\beta_0, \ldots, \beta_{n-1}$, one in each representation subspace X_s, $s = 0, \ldots, n-1$. The matrix $\mathbf{b}$ and the coefficients β are related via the invertible linear transformation

$$b_k := \sum_{s=0}^{n-1} \beta_s \exp\left(2\pi \mathrm{i} sk/n\right) . \tag{1.34}$$

The control parameters β_s diagonalize the $\mathbf{Z}_n$-equivariant circulant matrix $\mathbf{b}$ via the representations of $\mathbf{Z}_n = \langle \rho \rangle$ on X_s, but represent inherently nonlocal coupling.

In the control term (1.32) we are still free to choose $h = \rho^m$. Noninvasivity on the discrete rotating wave (1.32) of type s is guaranteed by the delay

$$\tau = \Theta(h) p_s = \Theta(\rho^m) p_s = msp_s/n, \qquad \text{mod } p_s \;, \tag{1.35}$$

and only by any such choice.

To be more precise suppose the control (1.32) with $h = \rho^m$, delay τ, and invertible circulant control matrix $\mathbf{b}$ is noninvasive on some solution $z^*(t)$ of the oscillator ring (1.12). Let ν denote the order of $h = \rho^m$ in $\mathbf{Z}_n$, i.e. $\nu > 0$ is minimal such that $\nu m \equiv 0$ mod n. Then noninvasivity implies

$$z^*(t) = h^\nu z(t) = z^*(t + \nu\tau) \;. \tag{1.36}$$

Therefore z^* is time-periodic with minimal period p dividing

$$\nu\tau = s' p \;, \tag{1.37}$$

for some positive integer s'. Let (H, K, Θ) denote the spatio-temporal symmetry of z^*. Of course $hz(t) = z(t - \tau)$ only guarantees $H \geq \langle h \rangle$. To ensure $\langle h \rangle \geq \mathbf{Z}_n$ let us choose m and n co-prime. In particular the order ν of h becomes n. Let $m'm = 1$ mod n denote the multiplicative inverse of m. Then

$$s := n\Theta(\rho) = n\Theta(\rho^{mm'}) = n\Theta(h^{m'}) = nm'\Theta(h) = nm'\tau/p = \nu m'\tau/p = m's' \tag{1.38}$$

mod n. We have used $mm' = 1$, $\rho^m = h$, the definition of the phase map Θ, $\nu = n$, and (1.37), successively, in this line. This readily identifies s in (1.32) from the Pyragas data $h = \rho^m$ and τ, via $(m', m) = 1$ and $s' = n\tau/p$.

In principle, it is possible to use a time-delay $\tau > p_s$. However, in [24] it was found that larger time-delay diminishes the stabilization regions.

The following theorem tells us that the constructed control is indeed stabilizing.

Theorem 1 (Successful stabilization of discrete rotating waves [24]). *Consider the Hopf bifurcation of discrete rotating waves*

$$z_k^*(t) = z_{k-1}^*(t - sp_s/n) \;, \tag{1.39}$$

of the Stuart-Landau ring

$$\dot{z}_k = (\lambda + \mathrm{i} + \gamma |z_k|^2)\, z_k + a(z_{k-1} - 2z_k + z_{k+1}) \;, \tag{1.40}$$

k mod n, with $\lambda \in \mathbf{R}$, $a > 0$ and $\gamma \in \mathbf{C} \setminus \mathbf{R}_+$.

Then for every combination of s and m, with $s, m \in \{1, \ldots, n-1\}$ and m co-prime to n, there exists a positive constant $a_{m,s}$ such that the following conclusion holds for all real coupling constants $0 < a < a_{m,s}$, and sufficiently near the selected Hopf bifurcation at $\lambda_s = 2a(1 - \cos(2\pi s/n))$.

There exist open regions of complex control parameters $\beta_0, \ldots, \beta_{n-1}$ such that in the delayed feedback system

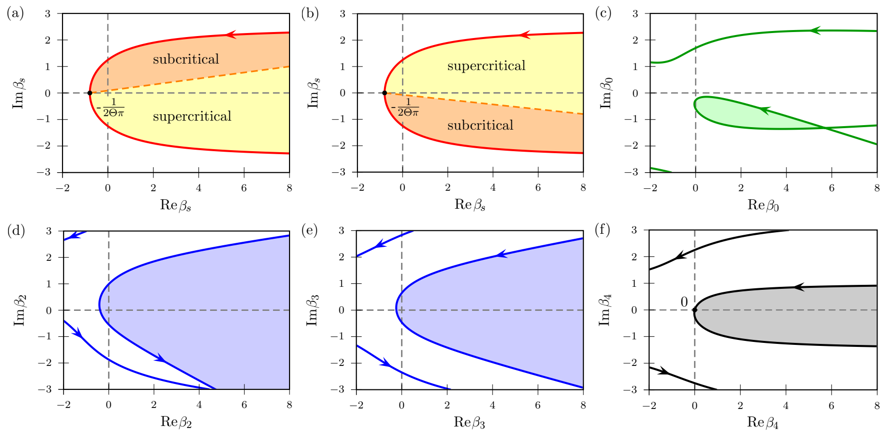

Fig. 1.4. Stabilization curves and regions for $n = 5$ coupled oscillators as in (1.12), (1.24) with coupling constant $a = 0.2$, controlled as in (1.41) with index shift $m = 1$, for $s = 1$. The curve belonging to the parameter $\beta_s = \beta_1$ is drawn in red, and the green curve corresponds to those parameters β_j for which $\lambda_j < \lambda_s = \lambda_1$ (compare also to Figure 1.3. Blue is used for the curves β_2 and β_3, which occur for $\lambda_2 = \lambda_3 > \lambda_s = \lambda_1$. For the black curve in (**f**), corresponding to β_4, we find $\lambda_4 = \lambda_s = \lambda_1$. Note that we only find four different types of stabilization regions for the parameters β_j, depending on the relative positions of λ_s and λ_j. See also Figure 1.2 (**c**) and (**d**). These figures have previously been published in [24].

$$\dot{z} = f(z) + a(\varrho z - 2z + \varrho^{-1} z) + \mathbf{b}\left(-z(t) + \varrho^m z(t-\tau)\right) \tag{1.41}$$

with circulant control matrix $\mathbf{b} = (b_{kl})$, $b_{kl} = \sum_{j=1}^{n-1} \beta_j \exp\left(2\pi \mathrm{i} j(l-k)/n\right)$, *the discrete rotating wave solution (1.39) is stabilized for a time-delay*

$$\tau = msp_s/n \mod p_s \,, \tag{1.42}$$

$0 < \tau \leq p_s$. *The stabilization is noninvasive and pattern-selective, i.e. the control vanishes only on the discrete rotating wave (1.39).*

If m is not co-prime to n, then the control term is noninvasive on more than one discrete rotating wave. This can obstruct stabilization, as we will see below for standard Pyragas control, $m = n$, $\tau = p_s$.

Theorem 1 is proved in [24]. Some examples for stabilization regions in the complex parameters $\beta_0, \ldots, \beta_{n-1}$ are depicted in Fig. 1.3, for $n = 4$ oscillators, and in Fig. 1.4, for $n = 5$ oscillators.

The constants $a_{m,s}$ limit the maximal coupling strength for which a stabilization region exists. They depend on the spatio-temporal pattern, identified by s, and the chosen control term, identified by the index shift m. The following theorem describes this upper bound $a_{m,s}$ as a solution to a system of trigonometric equations. For a graphical depiction see Figure 1.5.

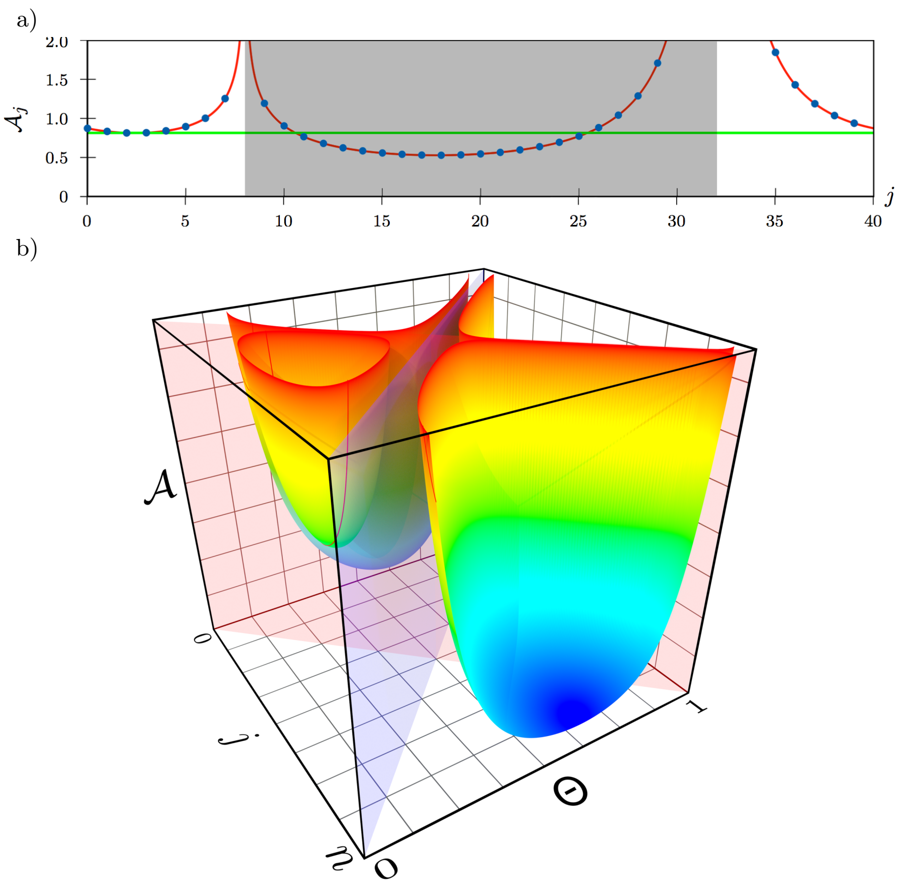

Fig. 1.5. Figure (**a**) shows the maximal coupling constant a for $n = 40$, $m = 1$, $s = 8$ and $\Theta = 1/5$. It is given by the minimum of $\mathcal{A}_j := A_j/|2\cos(2\pi j/n) - 2\cos(2\pi s/n)|$ for $j < 8$ and $j > 32$. The gray area given by $8 \leq j \leq 32$ is not relevant for the minimum. The maximal coupling constant a allowing stabilization of the system is marked as a horizontal (green) line. The solid (red) curve corresponds to the solutions for real $j \in [0, n]$. This red curve can also be seen in (**b**), where we see the general dependence of the threshold $\mathcal{A}_j$ on j as well as on Θ, for arbitrary n.

Theorem 2 (Maximal coupling constant for a given control scheme [24]). *Under the above conditions, the maximal coupling constant $a_{m,s}$ is given implicitly by the minimum of all*

$$A_j/|2\cos(2\pi j/n) - 2\cos(2\pi s/n)| \,, \tag{1.43}$$

with either $0 \leq j < s$ or $n - s < j < n$. *Here* (A_j, ω_j), $j = 0, \ldots, n-1$, *is the implicit solution of the system*

$$\sin \Omega_j \cos \Omega_j = -\omega_j \pi \Theta \tag{1.44}$$

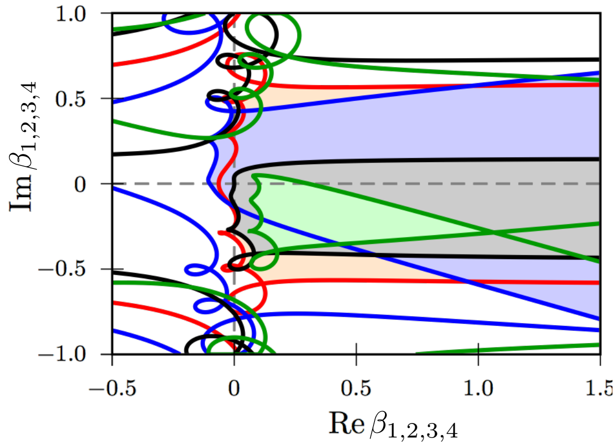

Fig. 1.6. Stabilization curves and regions for the sum of the delays as in with $n = 4$, $s = 1$, and $a = 0.2$ are shown. In accordance with the color coding of the other figures, β_0 is green, β_1 red, β_2 blue and β_3 black. This figure has previously been published in [24].

$$\sin^2 \Omega_j = A_j \pi \Theta \;, \tag{1.45}$$

with $\Omega_j := \pi(mj/n - \Theta(1+\omega_j))$, *and* $\Theta = ms/n \bmod 1,\ 0 < \Theta \leq 1$.

The upper bound $a_{m,s}$ for the coupling parameter a is strictly positive for the equivariant control type, and equal to zero for standard Pyragas control. For infinitesimal time-delay the threshold tends to infinity.

From Theorem 2, we can directly conclude that *standard* Pyragas control fails to stabilize any spatio-temporal pattern which is not completely synchronous. Indeed $h = \mathrm{id}$ and $\tau = p_s$ for standard Pyragas control. The system (1.44), (1.45) then simplifies to

$$\sin(\pi(1+\omega_j))\cos(\pi(1+\omega_j)) = \omega_j \pi \tag{1.46}$$

$$\sin^2(\pi(1+\omega_j)) = A_j \pi \;. \tag{1.47}$$

For any j, we only obtain the trivial solution $(A_j, \omega_j) = (0, 0)$. Hence we conclude $a_{0,s} = 0$ for all $s > 0$ and we obtain the following Corollary.

Corollary 1 (Failure of standard Pyragas control [24]). *The discrete rotating wave*

$$z_k(t) = z_{k-1}(t - sp_s/n) \;, \tag{1.48}$$

with $s \neq 0$ *cannot be stabilized by standard Pyragas control, i.e. by any delayed feedback system of the form*

$$\dot{z} = f(z) + a(\varrho z - 2z + \varrho^{-1} z) + \mathbf{b}\left(-z(t) + z(t - p_s)\right) \tag{1.49}$$

which preserves at least $\mathbf{Z}_n$-equivariance. In fact the solution (1.48), is unstable, sufficiently close to Hopf bifurcation for any complex circulant $n \times n$ control matrix $\mathbf{b}$ *as in (1.33), and any coupling constant $a > 0$.*

In the above control schemes, we have only used a single noninvasive control term. Linear combinations of such noninvasive control terms are an interesting extension. For example we may consider weighted sums of *all* noninvasive control terms,

$$\dot{z} = f(z) + a(\varrho z - 2z + \varrho^{-1} z) \tag{1.50}$$

$$+ \sum_{m=0}^{n-1} \mathbf{b}_m \left(- z(t) + \varrho^m z(t - \tau_m) \right) , \tag{1.51}$$

with appropriate time-delays $\tau_m = msp_s/n \bmod p_s$ and control matrices $\mathbf{b}_m$. Such control terms also yield stabilization regions, see Fig. 1.6. However, the existence of non-empty control regions for not necessarily small enough coupling constants a remains an open problem.

1.5 Rotating waves under free S^1-actions

From an abstract view point we return to the general G-equivariant dynamics

$$\dot{z} = F(z) \tag{1.52}$$

of section 1.2. We assume the group G to take the direct product form

$$G = \Gamma \times S^1 \tag{1.53}$$

$$g = (\gamma, \vartheta) \tag{1.54}$$

with elements $\gamma \in \Gamma$ and $\vartheta \in S^1 = \mathbf{R}/\mathbf{Z}$. We assume the linear action of the factor S^1 on z to be free; i.e. $\vartheta(z) = z$ only if $\vartheta = 0$ or $z = 0$. In other words, we can assume $z \in \mathbf{R}^{2n} \cong \mathbf{C}^n$ and

$$\vartheta(z) := \exp(2\pi \mathrm{i} \vartheta) z \; . \tag{1.55}$$

For simplicity we will assume the other factor Γ to be a finite group.

Our task, in the present section, will be the Pyragas stabilization of a hyperbolic rotating wave solution

$$hz^*(t) = z^*(t + \Theta(h)p) \tag{1.56}$$

of (1.52), in the sense of section 1.2. See in particular (1.16) – (1.21). For rotating waves z^* with spatio-temporal symmetry H^Θ, we recall that the homomorphism

$$\Theta : H \to S^1 \tag{1.57}$$

is surjective. We will first observe that

$$z^*(t) = \mathrm{e}^{\mathrm{i}\omega t} z^*(0) \tag{1.58}$$

is necessarily *harmonic* with frequency $\omega = \pm 2\pi/p$. We then show how *nonlinear Pyragas stabilization* of the form

$$\dot{z}(t) = F(z(t)) + B(\mathrm{e}^{\mathrm{i}\omega\tau} z(t-\tau), z(t)) \tag{1.59}$$

can always succeed, in this setting, for any small enough $\tau > 0$ and suitably chosen complex B of order $1/\tau$. Here $B(z_1, z_2)$ is a vector-valued nonlinear and nonlocal control which vanishes on the diagonal

$$B(z, z) = 0\ . \tag{1.60}$$

We will only sketch the relevant arguments; for further mathematical details we refer to [2, 9, 24].

Any finite graph of linearly or nonlinearly coupled, neither necessarily identical nor identically coupled, Stuart-Landau oscillators $z_0, \ldots z_{n-1}$ provides an example for our setting. Here Γ is the automorphism group of the coupled oscillator system; the elements γ of Γ are those permutations of the vertex indices $0, \ldots, n-1$ which leave the system of coupled oscillators invariant. Recall $\Gamma = D_n$ for the symmetric diffusion coupling in a ring (1.12) of identical and identically coupled oscillators. Slightly more generally than the Stuart-Landau choice (1.24), we only require the nonlinearities $f_k(z_k)$ to satisfy the S^1-equivariance

$$f_k(\mathrm{e}^{2\pi\mathrm{i}\vartheta}\zeta) = \mathrm{e}^{2\pi\mathrm{i}\vartheta} f_k(\zeta) \tag{1.61}$$

for all $\zeta \in \mathbf{C}$, $\vartheta \in S^1$.

To illustrate our abstract approach, we first consider the elementary planar case of a single S^1-equivariant oscillator

$$\dot{z}(t) = f(z(t)) + b(\mathrm{e}^{\mathrm{i}\omega\tau} z(t-\tau) - z(t))\ , \tag{1.62}$$

$z \in \mathbf{C}$. For absent complex scalar control $b = 0$, any nonstationary periodic solution $z = z^*(t)$ with minimal period $p > 0$ must be harmonic,

$$z^*(t) = \mathrm{e}^{\mathrm{i}\omega t} z^*(0)\ , \tag{1.63}$$

with amplitude $z^*(0) = r^* > 0$ and frequency $\omega = \pm 2\pi/p$. Note

$$\operatorname{Re} f(r^*) = 0\ , \tag{1.64}$$

$$\operatorname{Im} f(r^*) = \omega\ . \tag{1.65}$$

We do not assume small amplitude r^*, here or below. In co-rotating coordinates $\zeta(t) := \exp(-\mathrm{i}\omega t) z(t)$ we obtain

$$\dot{\zeta}(t) = f(\zeta(t)) - \mathrm{i}\omega\zeta(t) + b(\zeta(t-\tau) - \zeta(t)) \; . \tag{1.66}$$

Note how (1.66) remains autonomous, by S^1-equivariance of (1.62). The circle of stationary solutions $\zeta^* = \exp(\mathrm{i}\varphi)r^*$, $\varphi \in S^1$, testifies to noninvasivity of the Pyragas control scheme (1.62) on the harmonic rotating wave z^*, for all b.

Pyragas stabilization of z^* in (1.62) is equivalent to stabilization of the equilibrium $\zeta^* = r^*$ in (1.66), by some complex b. For small τ, we Taylor expand

$$\zeta(t-\tau) - \zeta(t) = -\tau\dot{\zeta}(t) + \ldots \tag{1.67}$$

with higher terms of order τ^2. In other words (1.66) reads

$$(1 + b\tau)\dot{\zeta} = f(\zeta) - \mathrm{i}\omega\zeta + \ldots \; . \tag{1.68}$$

Fix $\beta := b\tau$ and consider $\tau \to 0$. Then a remarkably early result by Kurzweil in 1971 indeed justifies (1.68) as an ODE approximation, even in the nonlinear case; see [14]. In modern language the ODE reduction (1.68) corresponds to a center manifold reduction with rapid attraction in the infinitely many remaining directions. For the characteristic equation of the linearization of (1.66) at $\zeta = \zeta^* = r^*$ the justification is trivial, by an exponential Ansatz. Note how the trivial zero eigenvalue of the linearization at ζ^* remains unaffected by (1.68). The only other eigenvalue μ, alias the nontrivial Floquet exponents of z^*, can be rotated at will by a suitably fixed chice of $\beta = b\tau \in \mathbf{C}$. This choice stabilizes z^*, for sufficiently small $\tau > 0$, because all omitted terms are of order $b\tau^2 = \beta\tau = O(\tau)$ or higher. *This completes noninvasive Pyragas stabilization of large rotating waves, in the planar case.*

The above result is in marked contrast with controls based on $z(t)$ and $z(t-p)$, only. In fact suppose the controls are even allowed to be of the nonlinear form (1.59), (1.60). In the general S^1-equivariant planar case, the nontrivial Floquet exponent μ must then satisfy a constraint

$$p \operatorname{Re} \mu < 9 \; , \tag{1.69}$$

in order to enable noninvasive control; see [9]. In [2] the analogous bound $\vartheta p \operatorname{Re}\mu < 9$ was established for controls based on $z(t)$ and $z(t - \vartheta p)$, in the S^1-equivariant case.

Let us now return to the general case of $G = \Gamma \times S^1$-equivariant systems $\dot{z} = F(z)$. Our task is to stabilize a rotating wave z^* of spatio-temporal symmetry H^Θ. We pursue the nonlinear Pyragas scheme (1.59) with small delays $\vartheta > 0$ and bounded control ϑB.

The spatio-temporal symmetry H^Θ of the rotating wave z^* possesses the following general structure:

$$H = H_0 \times S^1, \qquad \text{and} \tag{1.70}$$
$$\Theta(\gamma, \vartheta) = \Theta_0(\gamma) \pm \vartheta \tag{1.71}$$

with a suitable finite subgroup $H_0 \leq \Gamma$, and a homomorphism $\Theta_0 : H_0 \to S^1$. Here we use that $G = \Gamma \times S^1$ with Γ finite, and the free action $z \mapsto \mathrm{e}^{2\pi \mathrm{i}\vartheta} z$ of S^1 on $z \in \mathbf{C}$. We omit mere mathematical details. Fixing $\gamma = \mathrm{id}$, $\Theta_0(\gamma) = 0$ in (1.71) and letting $t = \pm\vartheta p$, we obtain

$$\begin{aligned} z^*(t) = z^*(\pm\vartheta p) = z^*(0 + \Theta(\mathrm{id}, \vartheta)p) &= (\mathrm{id}, \vartheta) z^*(0) = \\ &= \mathrm{e}^{2\pi \mathrm{i}\vartheta} z^*(0) = \mathrm{e}^{\mathrm{i}\omega t} z^*(0) \end{aligned} \tag{1.72}$$

with frequency $\omega = \pm 2\pi/p$. In particular $z^*(t)$ is harmonic, as claimed in (1.58), and the nonlinear Pyragas control scheme (1.59) is noninvasive on $z^*(t)$.

Noninvasive nonlinear Pyragas stabilization will be based on the pair

$$(z_1, z_2) = (hz(t - \Theta(h)p), z(t)) \tag{1.73}$$

for some suitable $h = (\gamma, \pm\vartheta) \in H = H_0 \times S^1$, $\vartheta > 0$. By (1.70) we may pick $\gamma = \mathrm{id}$. Then the pair (1.73) becomes

$$\big((\pm\vartheta) z(t - \vartheta p), z(t)\big) = \big(\mathrm{e}^{\mathrm{i}\omega\tau} z(t - \tau), z(t)\big) \tag{1.74}$$

with delay $\tau = \vartheta p > 0$. This shows that the noninvasive control scheme (1.59) is indeed based on the Pyragas difference (1.73).

To establish the success of (1.73) for small $\tau = \vartheta p > 0$ we proceed very much as in the planar case. In co-rotating coordinates $\zeta(t) := \mathrm{e}^{-\mathrm{i}\omega t} z(t)$, which freeze the harmonic rotation of $z^*(t)$, we obtain

$$\dot{\zeta}(t) = F(\zeta(t)) - \mathrm{i}\omega\zeta(t) + B(\zeta(t - \tau), \zeta(t)) \ . \tag{1.75}$$

Here we have replaced the complex scalar b by the complex vector nonlinearity B. We have assumed that B commutes with the S^1-action on $z \in \mathbf{C}^N$. The justifiable approximation (1.68) then reads

$$(\mathrm{id} + \beta)\dot{\zeta} = F(\zeta) - \mathrm{i}\omega\zeta + \dots \ . \tag{1.76}$$

with the complex nonlinearity $\beta := \tau \partial_1 B(\zeta, \zeta)$ chosen to be τ-independent.

Naively, at first, we might attempt to choose β to be a complex scalar multiple of an eigenprojection Q onto the strictly unstable eigenspace of the linearization $f'(\zeta^*) - \mathrm{i}\omega$ at $\zeta^* = z^*(0)$. Unfortunately the Jacobian $f'(\zeta^*)$ itself need not be *complex* linear, but will only be *real* linear in general. Indeed S^1-equivariance

$$f(\mathrm{e}^{\mathrm{i}\vartheta} z) = \mathrm{e}^{\mathrm{i}\vartheta} f(z) \ , \tag{1.77}$$

for all $\vartheta \in \mathbf{R}/2\pi\mathbf{Z}$ and all $z \in \mathbf{C}^n$, only implies conjugacy

$$f'(\mathrm{e}^{\mathrm{i}\vartheta} z) = \mathrm{e}^{\mathrm{i}\vartheta} f'(z) \mathrm{e}^{-\mathrm{i}\vartheta} \tag{1.78}$$

at $z \neq 0$, but not complex linearity. In particular, the Floquet eigenprojection $Q = Q(z^*(t))$ depends on the footpoint $z^*(t)$ of the linearization and satisfies the same conjugacy

$$Q(\mathrm{e}^{\mathrm{i}\vartheta}\zeta^*) = \mathrm{e}^{\mathrm{i}\vartheta} Q(\zeta^*) \mathrm{e}^{-\mathrm{i}\vartheta} \; . \tag{1.79}$$

However, by hyperbolicity of ζ^* the unstable eigenspace is transverse to the group orbit $z^*(t) = \exp(\mathrm{i}\omega t)\zeta^*$ of ζ^*; the tangent $\dot{z}^*(0) = \mathrm{i}\omega\zeta^*$ to that group orbit provides the trivial Floquet exponent $\mu = 0$ which remains unaffected by Q, B, and β. Therefore we can define a *nonlinear* control $B = B(z_1, z_2)$, which vanishes on the diagonal $z_1 = z_2$, stays S^1-equivariant, and provides the appropriate linearization β in a full neighborhood of the group orbit $z^*(t)$, for $z_1 = \exp(\mathrm{i}\omega\tau)z^*(t-\tau) = z^*(t) = z_2$.

In conclusion, *noninvasive nonlinear Pyragas stabilization of large rotating waves also succeeds in the general $\Gamma \times S^1$-equivariant case with free S^1-action.* In the general case, our result requires a nonlinear vector-valued control term of the form (1.59), (1.60).

In [23], Schneider has studied the above problem with complex linear control matrices $B = \mathbf{b}$ based on (1.59), (1.60), for rotating waves of small amplitudes near Hopf bifurcation form $z = 0$. In the supercritical case, control was successful in open regions of $\mathbf{b}$. The subcritical case required an additional condition: the minimal period had to depend sufficiently strongly on amplitude ζ^*. This condition is satisfied, essentially, for a sufficiently nonlinear soft spring or hard spring case.

1.6 Summary

In this chapter, we have summarized recent results which show how equivariant Pyragas control succeeds in networks where standard Pyragas control fails. For a ring of n diffusively coupled Stuart-Landau oscillators, explicit complex linear control terms were constructed for each periodic orbit of discrete rotating wave type. These control terms are nonlocal. They use an interplay between *index shifts* h and temporal *phase shifts* $\Theta(h)$ to select the spatio-temporal pattern, and they break the symmetry of the complete system. The control term is noninvasive only for exactly one type of discrete rotating waves. For full pattern-selectivity, the index shift m of $h = \rho^m$ must be chosen co-prime to the number n of oscillators coupled in the ring. For each discrete rotating wave, this provides several control terms for which a control is successful. In this case we have also provided an upper threshold on the maximally admissible coupling parameter, depending on the specific index shift.

For coupled Stuart-Landau oscillators, the additional S^1-equivariance of each oscillator allows additional conclusions. In this case, the existence of a noninvasive stabilizing control scheme on rotating waves can be guaranteed by choosing the time-delay small enough.

The control term will, however, become nonlocal and nonlinear in general. Only near Hopf bifurcation, results on stabilization by linear control have been obtained. The general results, on the other hand, are not limited to small amplitude. They do not require ring architecture. These results are based on equivariance under a free S^1-action as is provided by, but by no means limited to, the paradigm of coupled Stuart-Landau oscillators.

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