Structural Stability of Travelling Waves in an Integrodifferential Equation

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Abstract

In this article we study the structural stability of travelling waves of an integrodifferential equation, which can be viewed as the nonlocal analogue to the usual reaction-diffusion system $\dot{u} = u_{xx} + f(u)$. More precisely, we are interested in the question whether a travelling wave solution persists under small perturbations of the equation. Since the travelling wave equation is a functional differential equation of mixed type, a deeper understanding of the intersection of stable and unstable manifold of the steady state in mixed type equations turns out to be crucial. As one of the main results we prove the existence of stable and unstable manifolds for general functional differential equations.

We apply our results to the one-dimensional equation of elasticity with nonlocal energy. In particular we prove that a travelling wave is structural stable if and only if the underlying shock wave is compressive.

1 Introduction

In this article we are interested in the equation

$$\partial_t u(t,x) = (K * u(t,\cdot))(x) - u(t,x) - f(u(t,x)), \tag{1}$$

where the kernel K of the convolution $(K * u(\cdot))(x) = \int_{\mathbb{R}} K(x-y)u(y)dy$ is even, nonnegative, with unit integral. We want to assume in this paper that K has compact support in the interval $[-a,a]$ for some $a > 0$. The function $f : \mathbb{R} \to \mathbb{R}$ is bistable in most application, though we do not need to make that restriction. One can view equation (1) as a nonlocal analogue of the usual reaction reaction-diffusion equation

$$\partial_t u(t,x) = u_{xx}(t,x) - f(u(t,x)).$$

As such, (1) may model a variety of physical and biological phenomena involving media with properties varying in space. If we think of f as an bistable function, a motivation for studying (1) lies in the fact that it is a gradient flow

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for a natural generalization of the usual Ginzburg-Landau functional for an order parameter describing the state of a solid material.
The equation (1) has been subject to a lot of recent research, see for example [1, 2, 3, 17] and the references therein. The authors mostly focused on bistable nonlinearities $f(u) = u(u-1)(u+1)$ and the existence and stability of travelling waves connecting the end states ± 1. Travelling wave solutions of (1) have the form

$$u(t, x) = h(x - c_* t),$$

for some profile $h(\cdot)$ and a certain travelling wave speed $c_* \neq 0$. Inserting this ansatz into (1) and letting $\xi = x - c_* t$ we obtain the travelling wave equation

$$-c_* v'(\xi) = L v_\xi - v(\xi) - f(v(\xi)), \tag{2}$$

where $v_\xi \in C^0([-a, a], \mathbb{R})$ denotes the function $v_\xi(\theta) := v(\xi + \theta)$. Here, the linear map L is defined by

$$\begin{aligned} L : C^0([-a, a], \mathbb{R}) &\rightarrow \mathbb{R} \\ L(\phi(\cdot)) &= \int_{-a}^{a} K(y)\phi(y)dy. \end{aligned}$$

Note that the travelling wave solution $u(t, x)$ induces a heteroclinic solution $h(\xi)$ of (2). On account of the term $L v_\xi$ the resulting equation (2) is a *functional differential equation of mixed type*. It is known that equations of this type are typically not well posed, see [21, 20]. Furthermore, not much is known about the general theory of mixed type equations; however, there has been made recent progress [5, 6, 7, 9, 13, 14, 15, 16, 21].
In this paper we are interested in the question under what conditions travelling wave solutions of (1) are *structurally stable*. This means that we are concerned whether a small perturbation of the linear operator L or the nonlinearity f still result in the existence of a travelling wave for the perturbed system near the original travelling wave. Let us remark that this question is of importance since only structurally stable travelling waves are likely to be observed in physical experiments which (1) may model.
We now explain our results in more detail and assume that there exists a travelling wave $u(t, x) = h(x - c_* t)$, $c_* \neq 0$, connecting two simple zeros $x_\pm$ of the continuous differentiable nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$. That is, we assume

a) $\lim_{\xi \rightarrow \pm\infty} h(\xi) = x_\pm$

b) $f(x_\pm) = 0$ and $f'(x_\pm) \neq 0$

and state the following hypothesis:

Hypothesis 1
The function $h'(\cdot)$ is not in the range of the operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R}, \mathbb{R}) &\rightarrow L^2(\mathbb{R}, \mathbb{R}) \\ (\mathcal{L}v(\cdot))(t) &= \dot{v}(t) + \frac{1}{c_*}\left(L v_t - v(t) - f'(h(t))v(t)\right) \end{aligned}$$

The following theorem is the main result of this paper. The proof will be postponed to section 5.

Theorem 1
Fix two steady states $x_+, x_- \in \mathbb{R}$ with $x_+ \neq x_-$. Assume that the above assumptions on f, K hold and that there exists a unique (up to translations) travelling wave solution $u(t,x) = h(x - c_* t)$, such that a) and b) are satisfied. Then the following holds:

i) If $f'(x_-) > 0 > f'(x_+)$ the travelling wave $u(t,x)$ is structurally stable: If $\tilde{L}, \tilde{f}$ are small perturbations of L and f (where we refer to theorem 8 for the class of admissible perturbations), then the perturbed system

$$\partial_t u(t,x) = \tilde{L}u(t, \cdot + x) - u(t,x) - \tilde{f}(u(t,x)), \tag{3}$$

possesses a travelling wave solution $\tilde{u}(t,x) = \tilde{h}(t - cx)$ for every wave speed $c \approx c_*$. Furthermore, the function $\tilde{h}$ is unique (up to translations) for fixed $c \approx c_*$ with the property that it is a heteroclinic solution of the travelling wave equation of (35) connecting some states $\tilde{x}_\pm$ near $x_\pm$.

ii) Assume that sign$(f'(x_+))$ = sign$(f'(x_-))$. Then the existence of a travelling wave of the perturbed equation is a codimension-one-phenomenon (and we refer to the statement of theorem 8 for a definition of codimension in this context). However, if in this case hypothesis 1 is true then there exists a travelling wave solution $u(t,x) = \tilde{h}(x - ct)$ for the perturbed equation (1) and some unique wave speed $c \approx c_*$.

iii) If $f'(x_+) > 0 > f'(x_-)$ then the travelling wave $u(t,x)$ is structurally unstable: The existence of a travelling wave solution is a codimension-two-phenomenon (and we refer again to theorem 8 for the precise definition of codimension).

If case *iii)* is satisfied we call the travelling wave u *structurally unstable* and *structurally stable* in case of *i)*.
At a formal level we can and should interpret this theorem in the following way. Let us choose a perturbed linear operator $\tilde{L}$, a nonlinear function $\tilde{f}$ and consider the equation:

$$\begin{aligned} -c \cdot v'(\xi) &= (1-\alpha)\left(Lv_\xi - v(\xi) - f(v(\xi))\right) \\ &+ \alpha(\tilde{L}v_\xi - v(\xi) - \alpha\tilde{f}(v(\xi))) \end{aligned} \tag{4}$$

Then the right hand side of (4) depends differentiably on the parameter α. For $\alpha = 0$ we identify the original travelling wave equation (2) and for $\alpha = 1$ the perturbed equation. We will show that under the assumptions of theorem 1 the heteroclinic solution $h(\xi)$ of (4) for $\alpha = 0, c = c_*$ lies in the intersection of the stable and unstable manifold of x_+ and x_-, respectively. More precisely, equation (4) induces an abstract equation and h induces a heteroclinic solution of that equation, such that the induced heteroclinic solution lies in the intersection of stable and unstable manifold associated to this abstract equation (see the next section for the definition of the abstract setting). Both of these manifolds depend on the parameters α and c. Theorem 1 now asserts that under the condition $\text{sign}(f'(x_+)) = -\text{sign}(f'(x_-))$ the intersection of stable and unstable manifold is transverse. Thus small changes in the parameter α or the wave c still result in an intersection of stable and unstable manifold and therefore provide the existence of a one-parameter family of travelling wave solutions of the perturbed equation.

If on the other hand $\text{sign}(f'(x_+)) = \text{sign}(f'(x_-))$, we will show that the intersection of stable and unstable manifold is *non*transverse: The sum of the tangent spaces is of codimension one in the ambient space and we cannot expect an intersection of stable and unstable manifold after small changes of α. It is in this sense that we will consider the heteroclinic orbit $h(\xi)$ as a *codimension-one-phenomenon*. However, enlarging the space by the "parameter" c yields extended stable and unstable manifolds which generically intersect transversely in the extended phase space (namely, if hypothesis 1 is true) and we refer to the proof of theorem 8 for a verification of these facts. In this way we still conclude the existence of a travelling wave solution for the perturbed equation for some speed c, but now $c \approx c_*$ is uniquely determined by the perturbation parameter α. In case *iii*) of the theorem the (nonextended) manifolds have codimension two in the ambient space. Thus, the parameter c cannot "produce" a transversal intersection of the corresponding extended manifolds.

Following these arguments we observe that is quite important to know, when hypothesis 1 is satisfied. Restricting our attention to the special class of bistable nonlinearities, as has been done in [1, 2, 3, 17], the results of [1] actually imply that hypothesis 1 is always met. Thus if we perturb equation (1) slightly, we get the existence of a travelling wave solution for a unique wave speed $c \approx c_*$. For precise statements we refer the reader to section 6.

We caution the reader that the above interpretations of theorem 1 in terms of intersection of stable and unstable manifolds are only formal so far. The existence of stable and unstable manifolds for general functional differential equations of mixed type has not been proved up to now. The proof of existence of their existence in section 4 is one of the main achievements of this work. In particular, we can include the case of pure delay- and advance-delay differential equations. Theorem 1 can also be shown by an application of the Lyapunov-Schmidt reduction, see [15] for a similar problem. Indeed, our analysis relies on the calculation of some Fredholmindex, see theorem 5 and the key lemma 4. However, we have decided to work with invariant manifolds instead. We think that this approach is not only more geometricly but also provides a useful machinery for tackling quite different problems occuring in general functional differential equations of mixed type, see also [9]. Moreover, we will discuss a global bifurcation scenario in a subsequent paper [8], where a Lyapunov-Schmidt reduction fails and invariant manifolds prove very helpful.

It is also interesting to note that the structural stability of the travelling wave u in case i) of theorem 1 depends on the hyperbolicity assumption b) and a sign condition of the stationary states $x_\pm$ *only*. Both conditions are purely finite dimensional even though the travelling wave equation (2) has an infinite

dimensional character and no restrictions on the travelling wave u have been made. We will use this fact in section 7 to make an interesting observation for the one-dimensional equation of elasticity with nonlocal energy. Very roughly we will show the following: Assuming the existence of a travelling wave solution u for the regularized equation of elasticity we will prove that u is structurally stable if and only if the underlying shock wave is compressive.
The paper is organized as follows. In the second chapter we will review some basic facts about functional differential equations of mixed type and introduce the functional analytic framework in which we will work. In chapter 4 we will prove the existence of stable and unstable manifolds for general functional differential equations of mixed type. We will use these manifolds to prove theorem 1 in chapter 5. Finally we will apply our results to the one-dimensional equation of elasticity with nonlocal energy in chapter 7.

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2 The framework

We will shortly introduce the functional analytic framework with which we will work. Our aim is to study the persistence of a heteroclinic solution $h = h(t)$ of the equation

$$-c\dot{x}(t) = Lx_t - x(t) - f(x(t)) \tag{5}$$

for $c = c_*$ by slightly perturbing the linear operator L, the nonlinearity f or the speed c_*. Throughout the next sections we to make the following assumptions on f and K.

Hypothesis 2
Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2-function and there exist two simple zeros $x_\pm$, that is

$$f(x_\pm) = 0 \quad \text{and} \quad f'(x_\pm) \neq 0.$$

The continuous function $K : \mathbb{R} \to \mathbb{R}$ has compact support in the interval $[-a, a]$ for some suitable $a > 0$. Moreover, K is even, $K \geq 0$ and $\int_{\mathbb{R}} K(x)dx = 1$.

Instead of working equation (5) directly we will consider the abtsract equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} \frac{-1}{c_*}(L\phi(t, \cdot) - \phi(t, 0) - f(\phi(t, 0))) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix} =: F((\xi(t), \phi(t, \cdot))). \tag{6}$$

Here $F : X \to Y$, where

$$\begin{aligned} Y &:= \mathbb{C}^N \times L^2([-a, b], \mathbb{C}^N), \\ X &:= \{(\xi, \varphi) \in Y \mid \varphi \in H^1([-a, b], \mathbb{C}^N) \text{ and } \varphi(0) = \xi\} \end{aligned}$$

and $b = a$. We should point out that we define the spaces X, Y in this more general setting only for later reference.

In general, one could also work with equation (5) directly. However, the advantage of working with the abstract equation (6) lies in the fact that the domain of the linear operator (which is induced by linearizing at a steady state) does *not* depend on the equation. In fact, this was one of the main reasons to introduce the "sun-star-setting" (which is similar to our abstract setting) for purely delay differential equations, [12].
The following next lemma clarifies the connection between solutions of (6) and our original equation (1). We first specify the notion of a solution of (6):

Definition 1
We call a continuous function $U(t) : [t_1, t_2) \to Y$ a solution of (6) on (t_1, t_2) and $-\infty < t_1 < t_2 \leq \infty$, if $t \to U(t)$ is continuous regarded as a map on (t_1, t_2) with values in X, if $t \to U(t)$ is differentiable regarded as a map on (t_1, t_2) with values in Y and satisfies (6) on (t_1, t_2).
We call a continuous function $U(t) : [t_1, t_2) \to Y$ a solution of (6) on $(-\infty, t_2)$ and $t_2 \in \mathbb{R}$, if $t \to U(t)$ is continuous regarded as a map on $(-\infty, t_2)$ with values in X, if $t \to U(t)$ is differentiable regarded as a map on $(-\infty, t_2)$ with values in Y and (6) is satisfied on $(-\infty, t_2)$.

We can now state the following lemma

Lemma 1
Let

$$U(t) = \begin{pmatrix} \xi(t) \\ \varphi(t)(\cdot) \end{pmatrix}$$

be a solution of (6) on $(t_1 - b, t_2 + c)$. Then $\varphi(t)(\theta) = \xi(t + \theta)$ for all $t \in (t_1 - a, b + t_2)$ and $\theta \in [-a, b]$ with $t + \theta \in (t_1 - a, t_2 + b)$. Furthermore $\xi(t)$ solves (5) on the interval (t_1, t_2).

Proof
In order to prove the lemma it suffices to show

$$\varphi(t + \theta)(0) = \varphi(t)(\theta)$$

for all $t \in (t_1 - a, t_2 + b)$ and $\theta \in [-a, b]$ with $t + \theta \in (t_1 - a, t_2 + b)$, since $\varphi(t)(0) = \xi(t)$ for all t. For $t \in (t_1 - a, t_2 + b)$ we introduce the coordinates $(\tau, \theta) = (t + \theta, \theta)$ and consider

$$[\tilde{\varphi}(\tau)](\theta) := [\varphi(\tau - \theta)](\theta).$$

Let now $t \in (t_1 - a, t_2 + b)$ and $t + \theta \in (t_1 - a, t_2 + b)$ then we have $\tau \in (t_1 - a, t_2 + b)$ and $\tau - \theta \in (t_1 - a, t_2 + b)$. Since by assumption $\partial_t \varphi = \partial_\theta \varphi$ holds on the interval $(t_1 - a, t_2 + b)$ with respect to the coordinates (t, θ), we can deduce the identity $\tilde{\varphi}(\tau, \theta) = \tilde{\varphi}(\tau, 0)$ with respect to (τ, θ) for almost every τ. Since $\tilde{\varphi}(\tau, 0) = [\varphi(\tau)](0)$ and $[\varphi(\tau)](0) = \xi(\tau)$ depends continuously on τ, we have $\tilde{\varphi}(\tau, \zeta) = \tilde{\varphi}(\tau, 0)$ for every τ. This shows $\varphi(\tau - \theta)(0) = \varphi(\tau)(0)$ for all τ and θ and we have $\varphi(t + \theta)(0) = \varphi(t)(\theta)$. $\square$

We will work with the abstract equation (6) in the sequel. Note that the heteroclinic solution $h(t)$ of (5) induces via $H(t) := (h(t), h_t)$ a solution of the abstract equation (6). Since we are interested in the existence of stable and unstable manifolds near $H(t)$, we have to deal with linear equations, which come up naturally after linearizing the abstract equation (6) along $H(t)$. This will be done in the following section.

3 Linear mixed type equations

In this chapter we want to review some well-known facts about linear functional differential equations of mixed type, see also [14, 21].

3.1 Definitions and basic facts

Let us start by considering a nonautonomous linear equation

$$\dot{x}(t) = L(t)x_t. \tag{7}$$

We want to assume that $L(t)$ has the following form

Hypothesis 3
$L(t) \in L(C^0([-a,b],\mathbb{C}^N),\mathbb{C}^N)$ can be represented as

$$L(t)\varphi(\cdot) = \int_{-a}^{b} p(t,\theta)\varphi(\theta)d\theta + \sum_{k=1}^{m} A_k(t)\varphi(r_k),$$

where $t \mapsto p(t,\cdot) \in BC^0(\mathbb{R}, C^0([-a,b],\mathbb{C}^{N\times N}))$ and $A_k(\cdot)$ are elements of $BC^0(\mathbb{R},\mathbb{C}^{N\times N})$. We want to assume further that the functions $A_1(\cdot)$ and $A_m(\cdot)$ do not vanish identically and $-a = r_1 < \ldots < r_m = b$.

Definition 2
We call a function $x \in L^2([-a,\tau),\mathbb{C}^N)$ a solution of (7) for some $a < \tau \leq \infty$ and some initial condition $\phi \in L^2([-a,b],\mathbb{C}^N)$, if $x \in H^1_{loc}([0,\tau),\mathbb{C})$, $x_0 = \phi$ and (7) is satisfied for almost every $t \in [0,\tau)$.

We define the closed and densely defined linear operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R},\mathbb{C}^N) \subset L^2(\mathbb{R},\mathbb{C}^N) &\to L^2(\mathbb{R},\mathbb{C}^N) \\ (\mathcal{L}v)(t) &= \partial_t v(t) - L(t)v_t \end{aligned} \tag{8}$$

and recall the definition of a Fredholmoperator, see also [11].

Definition 3
Let Z be a Banach space. We call a closed and densely defined operator $\Gamma : \mathcal{D}(\Gamma) \subset Z \to Z$ a Fredholmoperator, if $Rg(\Gamma)$ is a closed subspace of finite codimension and $dim(ker(\Gamma)) < \infty$. Furthermore we can define the Fredholmindex i by

$$i = dim(ker(\Gamma)) - codim(Rg(\Gamma))$$

The next definition will be important in order to obtain a sufficient condition for $\mathcal{L}$ in terms of the asymptotic equations $\dot{x}(t) = L_\pm x_t$ to be a Fredholmoperator.

Definition 4
The equation (7) is called **asymptotically constant**, *if the limits* $p_\pm(\cdot) := \lim_{t\to\pm\infty} p(t,\cdot)$ *and* $A_k^\pm := \lim_{t\to\pm\infty} A_k(t)$ *exist in* $L^2([-a,b],\mathbb{C}^{N\times N})$ *and* $\mathbb{C}^{N\times N}$ *respectively. Equation (7) is called* **asymptotically hyperbolic**, *if the characteristic equations*

$$det \triangle_\pm (\lambda) := det \left(\lambda id - \left(\int_{-a}^{b} p_\pm(\theta) e^{\lambda\theta} d\theta + \sum_{k=1}^{m} A_k^\pm e^{\lambda r_k} \right) \right) = 0 \qquad (9)$$

associated with the limiting equations at $t = \pm\infty$ *have no solutions on the imaginary axis, that is* $det \triangle_\pm (is) \neq 0$ *for all* $s \in \mathbb{R}$. *If the coefficients do not depend on* t *and if the characteristic equation* $det\triangle(\lambda)$ *has no imaginary zeros* λ, *we call the operator* L **hyperbolic**.

The following result is due to Mallet-Paret [14].

Theorem 2
If $\mathcal{L}$ *is asymptotically hyperbolic, then* $\mathcal{L}$ *is a Fredholmoperator.*

3.2 Exponential dichotomies

We can relate equation (7) to the following abstract equation

$$\partial_t V(t) = \mathcal{A}(t)V(t), \qquad (10)$$

where the linear operator $\mathcal{A}(t) : X \subset Y \to Y$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} \xi \\ \varphi \end{pmatrix} = \begin{pmatrix} L(t)\varphi \\ \partial_\theta \varphi \end{pmatrix}$$

for $(\xi,\varphi) \in X$. Indeed, consider a solution $x \in H^1_{loc}(\mathbb{R},\mathbb{C}^N)$ of $\dot{x} = L(t)x_t$ and define

$$V(t) := \begin{pmatrix} x(t) \\ x_t \end{pmatrix}. \qquad (11)$$

Then $V(t) \in X$ for all $t \in \mathbb{R}$ and $V(t)$ defines a solution of (10). On the other hand, if $t \mapsto V(t)$ is a continuous function with values in X and differentiable with values in Y, such that (10) is satisfied for all $t \in \mathbb{R}$, then one can show that (11) holds for some solution $x(\cdot)$ of (7), see [6, 21].
Equation (10) induces a densely defined operator $\mathcal{T} : \mathcal{D}(\mathcal{T}) \subset L^2(\mathbb{R},Y) \to L^2(\mathbb{R},Y)$, where

$$\mathcal{T} : V(\cdot) \mapsto \partial_t V(\cdot) - \mathcal{A}(\cdot)V(\cdot)$$

and more explicitly

$$\left(\mathcal{T} \begin{pmatrix} \xi(\cdot) \\ \Phi(\cdot) \end{pmatrix} \right)(t) = \begin{pmatrix} \partial_t \xi(t) - \int_{-a}^{b} p(t,\theta)[\Phi(t)(\theta)]d\theta - \sum_{k=1}^{m} A_k(t)[\Phi(t)(r_k)] \\ \partial_t \Phi(t)(\cdot) - \partial_\theta \Phi(t)(\cdot). \end{pmatrix}.$$

Here $\mathcal{D}(\mathcal{T})$ is defined by

$$\mathcal{D}(\mathcal{T}) = \{(\xi(\cdot), \Phi(\cdot)) \in L^2(\mathbb{R}, Y) : \quad (\partial_t - \partial_\theta)\Phi(\cdot)(\cdot) \in L^2(\mathbb{R} \times I, \mathbb{C}^N), \\ \xi(\cdot) \in H^1(\mathbb{R}, \mathbb{C}^N), \Phi(t)(0) = \xi(t) \ \forall t\},$$

where we often prefer to write $\Phi(t, \cdot)$ instead of $\Phi(t)(\cdot)$. It is shown in [21] that $\mathcal{T}$ is indeed a closed operator. Since we will work in the sequel with the abstract equation (10) we are interested in a relation between $\mathcal{T}$ and $\mathcal{L}$ (defined in (8)). We therefore state the following theorem, which has been proven in [6, 21].

Theorem 3
If the operator $\mathcal{L}$ is a Fredholmoperator, $\mathcal{T}$ is also a Fredholmoperator. Furthermore, the indices are the same, that is $i_{\mathcal{L}} = i_{\mathcal{T}}$.

We are now concerned about the existence of *exponential dichotomies* of equation (10). Roughly speaking, we want to know whether there exists a projection $P(t) : Y \to Y$, such that we can solve equation (10) in forward time for initial values in $\mathrm{Rg}(P(t))$ and in backward time for initial values in $\ker(P(t))$. We need the following hypothesis:

Hypothesis 4
If $x(\cdot)$ is a solution of $\mathcal{L}x = 0$ and $x_\tau = 0$ for $x_\tau \in H^1([-a, b], \mathbb{C}^N)$ and some $\tau \in \mathbb{R}$ then $x \equiv 0$.

The next theorem, which has been proved in [21] and has been extended to a larger class of equations in [6], guarantees the existence of exponential dichotomies provided that the equation $\dot{x}(t) = L(t)x_t$ is asymptotic hyperbolic.

Theorem 4 (Exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$)
Let $\mathcal{L}$ be a Fredholmoperator and hypothesis 4 be satisfied. Then equation (10) possesses an exponential dichotomy on J, where $J = \mathbb{R}_+$ and $J = \mathbb{R}_-$. That is, there exist constants $K, \alpha > 0$ and a family of strongly continuous projections $P(t) : Y \to Y$, $t \in J$, with the following properties. For $U \in Y$ and $t_0 \in J$

- *there exists a continuous function $V^s(\cdot) : [t_0, \infty) \cap J \to Y$, such that $V^s(t_0) = P(t_0)U$. Furthermore $V^s(t) \in Rg(P(t))$ and we have the estimate $|V^s(t)|_Y \leq Ke^{-\alpha|t-t_0|}|U|_Y$ for all $t \geq t_0$ with $t, t_0 \in J$.*

- *There exists a continuous function $V^u(\cdot) : (-\infty, t_0] \cap J \to Y$, such that $V^u(t_0) = (id - P(t_0))U$. Moreover $V^u(t) \in \ker(P(t))$ and $|V^u(t)|_Y \leq Ke^{-\alpha|t-t_0|}|U|_Y$ for all $t_0 \geq t$ with $t, t_0 \in J$.*

Moreover, if $U \in X$, then the functions $V^s(t)$ and $V^u(t)$ define strong solutions of (10) for $t \geq t_0$ and $t_0 \geq t$ respectively. If $U \in (Y \setminus X) \cap Rg(P(t_0))$ and $U = (\xi, \phi(\cdot))$ then $V^s(t) = (x(t), x_t)$, where $x : [-a + t_0, \infty) \to \mathbb{R}^N$ denotes the unique solution of $\dot{x}(t) = L(t)x_t$ on (t_0, ∞) with $x_{t_0} = \phi$. A similar statement holds vor $V^u(t)$.

3.3 Computing Fredholm-indices

Let us now consider the special linear equation (2), namely

$$\dot{x}(t) = \frac{1}{c_*}\left[\int_{-a}^{a} K(\theta)x(t+\theta)d\theta - x(t) - f'(h(t))x(t)\right] =: L(t)x_t, \qquad (12)$$

which is asymptotically hyperbolic on account of hypothesis 2. In particular we observe that $L(t) \to L_\pm$, where

$$L_\pm\phi = \frac{1}{c_*}\left[\int_{-a}^{a} K(\theta)\phi(\theta)d\theta - \phi(0) + f'(x_\pm)\phi(0)\right]$$

for $t \to \pm\infty$ and the equations $\dot{y}(t) = L_\pm y_t$ do not possess solutions of the form

$$y(t) = e^{i\kappa t}y^*$$

for some $\kappa \in \mathbb{R}$ and some $y^* \in \mathbb{C}$. By theorem 2 the operator

$$(\mathcal{L}x)(t) := \dot{x}(t) - L(t)x_t$$

is a Fredholmoperator. We are now interested in an effective way to compute the relevant Fredholmindex $i = i_{\mathcal{L}}$ of $\mathcal{L}$.

In order to state the next theorem, which goes back to Mallet-Paret [14], we need to define the *crossing number* $\mathrm{cross}(L^\rho)$. This number is defined for a family L^ρ of operators, which satisfy hypothesis 3 and depend continuously on a parameter $\rho \in [0,1]$. More precisely we want to assume that $L^\rho : C^0([-a,b],\mathbb{C}) \to \mathbb{C}$, L^0, L^1 are both hyperbolic and there exist only finitely many values $0 < \rho^1 < \ldots, < \rho^n < 1$, such that L^{ρ^j} is not hyperbolic. Fix any such ρ^j and let $\{\lambda_{j,k}\}_{k=1}^{K_j}$ denote the corresponding zeros of the associated characteristic equation

$$\det(\triangle_{L^{\rho^j}}(\lambda)) = 0 \qquad (13)$$

with $\mathrm{Re}(\lambda_{j,k}) = 0$. We list these eigenvalues with repetitions, according to their multiplicity as roots of the characteristic equation (13). Let M^j denote the sum of their multiplicities. For ρ near ρ^j, with $\pm(\rho-\rho^j) > 0$, this equation has exactly M^j eigenvalues near the imaginary axis, $M^j_{L_\pm}$ with $\mathrm{Re}\lambda < 0$ and $M^j_{R_\pm}$ with $\mathrm{Re}\lambda > 0$, where $M^j_{L_\pm} + M^j_{L_\pm} = M^j$. The net crossing number of eigenvalues at $\rho = \rho^j$ is given by $M^j_{R_+} - M^j_{R_-}$. Finally we define

$$\mathrm{cross}(L^\rho) = \sum_{j=1}^{n} M^j_{R_+} - M^j_{R_-}.$$

Theorem 5

Let L^ρ for $0 \leq \rho \leq 1$ be a continuously varying one-parameter family of operators of the form

$$L^\rho\phi(\cdot) = \frac{1}{c_*}\left[\int_{-a}^{a} K(\theta)\phi(\theta)d\theta - \phi(0)(1-\rho f'(x_-) - (1-\rho)f'(x_+))\right],$$

and suppose the operators $L_- = L^0, L_+ = L^1$ are hyperbolic. Then there are only finitely many values

$$\rho^1 < \ldots < \rho^n \in [0,1],$$

for which L^{ρ^j} is not hyperbolic and the Fredholmindex i of $\mathcal{L}$ satisfies

$$i = -cross(L^\rho).$$

Proof
Choose an approximation $L^m\phi$ of the integral $\int_{-a}^{a} K(\theta)\phi(\theta)d\theta$ by Riemann sums

$$L^m\phi := \sum_{j=1}^{m} K(\theta_i)\phi(\theta_i)(\theta_{i+1} - \theta_i)$$

for some partition $-a = \theta_1 \leq \ldots \leq \theta_m = a$, where we restrict ourselves to *symmetric* partitions. Now observe that the operator $\mathcal{L}$ and $\tilde{\mathcal{L}}$ have the same Fredholmindex if $m > 0$ is large enough, where

$$\begin{aligned} \tilde{\mathcal{L}} : H^1(\mathbb{R},\mathbb{R}) &\rightarrow L^2(\mathbb{R},\mathbb{R}) \\ (\tilde{\mathcal{L}}x(\cdot))(t) &= \dot{x}(t) - \frac{1}{c_*}(L^m x_t - x(t) - f'(h(t))x(t)). \end{aligned}$$

Indeed, note that $\tilde{\mathcal{L}}$ also is asymptotically hyperbolic and therefore is a Fredholm operator. Consider a "path" $\mathcal{L}^\eta$ for $\eta \in [0,1]$ from $\mathcal{L}$ to $\tilde{\mathcal{L}}$, where

$$(\mathcal{L}^\eta x(\cdot))(t) = \dot{x}(t) - \frac{1}{c_*}\left[(1-\eta)\int_{-a}^{a} K(\theta)x_t d\theta - \eta L^m x_t - x(t) - f'(h(t))x(t)\right].$$

Then $\mathcal{L}^0 = \mathcal{L}$ and $\mathcal{L}^1 = \tilde{\mathcal{L}}$. Furthermore each $\mathcal{L}^\eta$ is asymptotically constant and hyperbolic: Let us illustrate the relevant calculation for the limit $t \rightarrow \infty$, then for $\omega \in \mathbb{R}$

$$\begin{aligned} 0 &= \det\triangle(i\omega) := i\omega - \frac{1}{c_*}\left[(1-\eta)\int_{-a}^{a} K(\theta)e^{i\omega}d\theta - \eta L^m e^{i\omega} - 1 - f'(x_+)\right] \\ &\Leftrightarrow \quad \omega = 0 \quad \wedge \quad f'(x_+) - (1-\eta) + 1 - \eta\left[\sum_{j=1}^{m} K(\theta_i)(\theta_{i+1} - \theta_i)\right] = 0 \end{aligned}$$

which is satisfied if m is large enough, since $f'(x_\pm) \neq 0$ and

$$\sum_{j=1}^{m} K(\theta_i)(\theta_{i+1} - \theta_i) \rightarrow \int_{-a}^{a} K(\theta)d\theta = 1$$

for $m \rightarrow \infty$. This calculation shows that $\mathcal{L}^\eta$ is always a Fredholmoperator. Thus $\mathcal{L}^0$ and $\mathcal{L}^1$ must have the same index.
Let us now consider for $\rho \in [0,1]$ the family

$$\tilde{L}^\rho\phi := \frac{1}{c_*}\left[L^m\phi(\theta) - \phi(0)(1 - \rho f'(x_-) - (1-\rho)f'(x_+))\right].$$

Then theorem C in [14] tells us that the Fredholmindex of $\tilde{\mathcal{L}}$ (and thus also the index of $\mathcal{L}$) can be computed by the net number $i = -\text{cross}(\tilde{L}^\rho)$. It is now easy to check that the net numbers of the familys L^ρ and $\tilde{L}^\rho$ are identical and thus the theorem is proved. □

4 Stable and unstable manifolds

In this section we construct stable and unstable manifold of the steady state zero of the abstract system

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} \frac{-1}{c}(L\phi(t,\cdot) - \phi(t,0) - f(\phi(t,0))) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} \tag{14}$$

near the point $H(0) = (h(0), h_0)$ of the heteroclinic orbit. Since invariant manifolds are a powerful tool and for future reference, we decide to consider the more general equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} G(\phi(t,\cdot),\mu) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} =: \mathcal{G}((\phi(t,\cdot),\xi(t)),\mu), \tag{15}$$

where we want to impose the following hypothesis.

Hypothesis 5
Let $G : C^0([-a,b],\mathbb{R}^N) \times \mathbb{R}^p \to \mathbb{R}^N$ be a C^2-map for some $p \in \mathbb{N}$ and $a \geq 0$, $b \geq 0$, such that

a) (15) possesses a homoclinic or heteroclinic orbit $H(t) = (h(t), h_t)$ for some parameter μ_* with $\lim_{t\to\pm\infty} h(t) = x_\pm$.

b) $D_1G(h_t,\mu_*) =: \tilde{L}(t)$ satisfies the hypotheses 4,3 and $\tilde{L}(t)$ is asymptotically hyperbolic.

On account of assumption b) and theorem 4 the linearization of (15) along $H(t)$, namely

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} D_1G(h_t,\mu_*)\phi(t,\cdot) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} =: A(t) \begin{pmatrix} \xi(t) \\ \phi(t,\cdot) \end{pmatrix} \tag{16}$$

possesses exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ with solution operators $\Phi_+^{s/u}$ and $\Phi_-^{s/u}$, respectively. Here, for $V \in Y$, we have set $\Phi_+^s(t,t_0)V := V^s(t)$, where $V^s(t)$ denotes the continuous function occuring in the statement of theorem 4 with $V^s(t_0) = V$. Similiarly, the other operators Φ_+^u and $\Phi_-^{s/u}$ are defined. By theorem 4 we have

$$\begin{aligned} \|\Phi_+^s(t,s)\|_{L(Y,Y)} &\leq Me^{-\kappa|t-s|}, & \|\Phi_+^u(s,t)\|_{L(Y,Y)} &\leq Me^{-\kappa|t-s|} \\ \|\Phi_-^s(t,s)\|_{L(Y,Y)} &\leq Me^{-\kappa|t-s|}, & \|\Phi_-^u(s,t)\|_{L(Y,Y)} &\leq Me^{-\kappa|t-s|}, \end{aligned} \tag{17}$$

for some $M, \kappa > 0$ and $t \geq s \geq 0$ in the first and $s \leq t \leq 0$ in the second equation. We introduce some notation next.

Notation
Let us define the space $\tilde{X} := \{(\xi,\phi) \in \mathbb{R}^N \times C^0([-a,b],\mathbb{R}^N) : \phi(0) = \xi\}$. Moreover, let $E_+^{s/u} := \mathrm{Rg}(\Phi_+^{s/u}(0,0)\big|_Y)$ and $E_-^{s/u} := \mathrm{Rg}(\Phi_-^{s/u}(0,0)\big|_Y)$. Finally we define $\tilde{E}_+^{s/u} := E_+^{s/u} \cap \tilde{X}$ and $\tilde{E}_-^{s/u} := E_-^{s/u} \cap \tilde{X}$, where these spaces are considered as subspaces of $\tilde{X}$ with the $\tilde{X}$-norm.

Theorem 6 (The stable manifold)
Assume that hypothesis 5 is met. Then equation (15) possesses a C^2*-manifold* $W^s(x_+) = W^s_\mu(x_+) \subset \tilde{X}$ *near* $H(0)$ *with the following properties:*

a) *The tangent space of* $W^s_{\mu_*}(x_+)$ *at* $H(0) \in W^s_{\mu_*}(x_+)$ *is* $\tilde{E}^s_+$.

b) *If* $W_+ \in W^s_\mu(x_+)$ *and* $W_+ \approx H(0)$ *with respect to the* $\tilde{X}$*-norm, then there exists a continuous function* $W(t) : \mathbb{R}_+ \to \tilde{X}$*, such that* $W(\cdot) \in W^s_\mu(x_+)$ *on some time interval* $[0, t_*)$*,* $W(0) = W_+$ *and* $W(t) = (\xi(t), \xi_t)$*, where* $\xi(t) : [-a, \infty) \to \mathbb{R}^N$ *is a solution of* $\dot{x}(t) = G(x_t, \mu)$*. Moreover,* $\xi(t)$ *converges to zero with exponential rate* $\kappa > 0$ *for* $t \to \infty$*.*

c) *If there is a solution* $W(t)$ *of the abstract equation (15), such that* $W(t) \to 0$ *with exponential rate* κ *for* $t \to \infty$ *and if* $W(0)$ *is close enough to* $H(0)$ *with respect to the* $\tilde{X}$*-norm, then* $W(0) \in W^s_\mu(x_+)$*.*

d) $W^s_\mu(x_+)$ *is two times continuously differentiable with respect to* μ*. That is, we can represent* $W^s_\mu(x_+)$ *in the form* $W^s_\mu(x_+) = H(0) + graph(\Psi(\cdot, \mu))$*, where* $\Psi(\cdot, \mu) : \tilde{E}^s_+ \times (\mathbb{R}^p \cap U(\mu_*)) \to \tilde{E}^u_+$ *is two times continuously differentiable with respect to the parameter* μ*, and where* $U(\mu_*)$ *denotes a small neighborhood of* μ_**.*

Proof
Since $\tilde{L}(t)$ satisfies hypothesis 3, it can be represented in the form

$$\tilde{L}(t)\varphi(\cdot) = \int_{-a}^{b} p(t, \theta)\varphi(\theta)d\theta + \sum_{k=1}^{m} A_k(t)\varphi(r_k),$$

where $A_k(\cdot)$ and $p(\cdot, \cdot)$ are as in hypothesis 3. Now consider a solution $W(t)$ of (15) and write $W(t) = H(t) + V(t)$. Then $V(t) = (\eta(t), \psi(t, \cdot))$ solves the equation

$$\dot{V}(t) = A(t)V(t) + \mathcal{J}(t, V(t), \mu), \tag{18}$$

where $\mathcal{J} : \mathbb{R} \times \hat{X} \times \mathbb{R} \to \hat{X}$ is defined by

$$\mathcal{J}(t, (\xi, \varphi), \mu) := \begin{pmatrix} G(h_t + \varphi, \mu) - D_1 G(h_t, \mu_*)\varphi - G(h_t, \mu_*) \\ 0 \end{pmatrix} \tag{19}$$

and $\hat{X} := \mathbb{R}^N \times C^0([-a, b], \mathbb{R}^N)$. Let us now solve (18) and consider the fixed-point equation

$$\begin{aligned} V(t) &= \Phi^s_+(t, 0)V^s_0 + \int_0^t \Phi^s_+(t, s)\mathcal{J}_{mod}(s, V(s), \mu)ds \\ &+ \int_\infty^t \Phi^u_+(t, s)\mathcal{J}_{mod}(s, V(s), \mu)ds. \end{aligned} \tag{20}$$

Here, $V^s_0 \in \tilde{E}^s_+$ and $\mathcal{J}_{mod}$ denotes the modified nonlinearity

$$\mathcal{J}_{mod}(s, V, \mu) := \chi_\varepsilon(\|V\|_{\hat{X}})\mathcal{J}(s, V, \mu),$$

where $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is a bump-function with compact support in $[-\varepsilon, \varepsilon]$. Moreover, we look for fixed points in the space $BC^{-\gamma} := BC^{-\gamma}(\mathbb{R}_+, \tilde{X})$, equipped with the norm $\|V(\cdot)\|_{-\gamma} := \sup_{\xi \geq 0} e^{\gamma|\xi|}\|V(\xi)\|_{\tilde{X}}$. Let us now explain in which sense the right hand side of (20) is defined. Note first that the solution operators Φ^s_+, Φ^u_+ map $\tilde{Z} := \mathbb{R}^N \times L^\infty([-a,b], \mathbb{R}^N)$ boundedly into $\tilde{Z}$. Moreover, they satisfy (17) also with respect to the $L(\tilde{Z}, \tilde{Z})$-norm, see [7]. We now consider the integrals, which appear in (20), as weak* integrals; these are introduced in the appendix. The results of the appendix also show that each of the two integrals is actually an element of $\tilde{X}$ for each fixed t. Let us point out that we cannot consider the integrals in the usual Lebesgue sense, since even in the delay case $a > 0, b = 0$ the map $s \mapsto \Phi^s_+(t,s)U$ is not necessarily Lebesgue integrable if regarded as a map with values in the space $\hat{X}$ and $U \in \hat{X} \setminus \tilde{X}$. However, the integrals are well defined as weak* integrals and lemma 6 in the appendix shows that we can estimate these integrals by estimating the integrands.
If we now choose $0 < \gamma < \kappa$ and let ε small enough, then (20) has a fixed point $V_*(\cdot)$ for every V^s_0 and $\mu \approx \mu_*$ by the contraction mapping theorem. Furthermore, $V_*(\cdot)$ depends C^2 on μ and V^s_0. Indeed, note that the proof of lemma 5 also shows that the weak* integral commutes with the operator ∂_μ. If we now let $\Psi(V^s_0, \mu) := \Phi^u_+(0,0)V_*(0)$ and define

$$W^s_\mu(x_+) = \text{graph}(\Psi(\cdot, \mu)) + H(0) \tag{21}$$

then all assertions of the theorem except point b) are proved. Now consider $V^s_0 \in \tilde{E}^s_+(0)$ and assume that (20) is satisfied for some $V(\cdot) \in BC^{-\gamma}(\mathbb{R}_+, \tilde{X})$. Let us show that $V(t)$ has the form $V(t) = (\xi^*(t), \xi^*_t)$ for some continuous function $\xi^*(t) : (-a, \infty) \to \mathbb{R}^N$ with $\xi^* \in H^1([0,\infty), \mathbb{R}^N)$. Let us modify the nonlinearity $\mathcal{J}_{mod}$ once more and consider $\mathcal{J}^\delta(t, \cdot) : \hat{X} \to X$, where

$$\mathcal{J}^\delta(t, V) := \begin{pmatrix} g(t,V) \\ l(t,V,\delta)(\cdot) \end{pmatrix} \tag{22}$$

and

$$g(t,(\xi,\varphi)) = g(t,(\xi,\varphi),\mu) := G(h_t + \varphi, \mu) - D_1G(h_t, \mu_*)\varphi - G(h_t, \mu_*)$$

for $V = (\xi, \varphi) \in \hat{X}$. Moreover, the map $l(t,V,\delta)(\cdot) \in C^\infty([-a,b], \mathbb{C}^N)$ denotes the bump-function

$$l(t,(\xi,\varphi),\delta)(\theta) := \begin{cases} 2 \cdot g(t,(\xi,\varphi)) 2^{\frac{1}{(\theta/\delta)^2 - 1}} & \theta \in (-\delta, \delta) \\ 0 & \text{else} \end{cases} .$$

For the sake of simplicity we often suppress the parameter μ from now on. Note that the L^2-norm of l with respect to the θ-variable tends to zero for $\delta \to 0$ and $l(t,(\xi,\varphi))(0) = g(t,(\xi,\varphi)) \in \mathbb{C}^N$. This shows that $\mathcal{J}^\delta$ takes values in X for every $V \in \hat{X}$. Let us now additionally approximate $V^s_0 \in \hat{X}$ by elements $V^\delta_s \in X$ with respect to the Y-norm and consider the fixed-point equation

$$\begin{aligned} \tilde{V}(t) &= \Phi^s_+(t,0)V^\delta_s + \int_0^t \Phi^s_+(t,s)\mathcal{J}^\delta(s, \tilde{V}(s))ds \\ &+ \int_\infty^t \Phi^u_+(t,s)\mathcal{J}^\delta(s,\tilde{V}(s))ds \end{aligned} \tag{23}$$

on the space $BC^{-\gamma}(\mathbb{R}_+, \hat{X})$. Choose some $\delta > 0$. Then for every fixed $t > 0$ and $(\xi, \phi), (\eta, \psi) \in \hat{X}$

$$\|l(t, (\xi, \varphi), \delta) - l(t, (\eta, \psi), \delta)\|_{C^0([-a,b],\mathbb{R}^N)} \leq L \cdot \|(\xi, \phi) - (\eta, \psi)\|_{\hat{X}},$$

where L denotes the globally small Lipschitz-constant of the map $(\xi, \phi) \mapsto \chi(\|\phi\|_{C^0} + |\xi|_{\mathbb{R}^N})g(t, (\xi, \phi))$. If $L > 0$ is small enough, then the right hand side of (23) defines a contraction in $BC^{-\gamma}([0, \infty), \hat{X})$. Note that $L > 0$ can be chosen uniformly small with respect to $t, \delta > 0$. This shows that there exists a fixed point $\tilde{V} = \tilde{V}^\delta \in BC^{-\gamma}([0, \infty), \hat{X})$, which solves (23). Moreover, we can differentiate $t \to \tilde{V}(t)$, regarded as a map with values in Y, and obtain the equation

$$\tilde{V}(t) = A(t)\tilde{V}(t) + \mathcal{J}^\delta(t, \tilde{V}(t)), \tag{24}$$

where $\mathcal{J}^\delta$ has been defined in (22). In particular $\xi(t)$, where $(\xi(t), \phi(t, \cdot)) = \tilde{V}(t)$, is a solution of

$$\dot{\xi}(t) = \int_{-a}^{b} p(t, \theta)\phi(t, \theta)d\theta + \sum_{k=1}^{m} A_k(t)\phi(t, r_k) + g(t, (\xi(t), \phi(t, \cdot))).$$

Integrating this equation gives

$$\begin{aligned} \xi(t) &= \xi(0) + \int_0^t \int_{-a}^{b} p(s, \theta)\phi(s, \theta)d\theta ds \\ &+ \int_0^t \sum_{k=1}^{m} A_k(s)\phi(s, r_k)ds + \int_0^t g(s, (\xi(s), \phi(s, \cdot))ds. \end{aligned} \tag{25}$$

We can now see from equation (23) that for $t \in [-a, M]$ and some arbitrary $M > 0$ the norm $\|\phi(t, \cdot)\|_{C^0([-a,b],\mathbb{R}^N)}$ can be bounded uniformly with respect to $\delta > 0$ by some constant $C = C(M) > 0$. Together with equation (25) we conclude the existence of a convergent subsequence $\xi(\cdot) = \xi^{\delta_n}(\cdot)$ on $[-a, M]$ by Arzela-Ascoli, which converges uniformly to some continuous function $\xi^*(\cdot)$. Let us now consider the C^0-component of equation (24), namely

$$\partial_t \phi(t, \theta) = \partial_\theta \phi(t, \theta) + l(t, (\xi, \phi(t, \cdot)), \delta)(\theta).$$

This equation can be solved by the method of characteristics, see Smoller [24], and we conclude the identity (note that $(\xi(t), \phi(t, \theta)) \in X$, which implies $\xi(t) = \phi(t, 0)$)

$$\phi(t, \theta) = \xi(t + \theta) + \int_0^\theta \tilde{l}(t - s + \theta, \delta)(s)ds \tag{26}$$

for $t + \theta \geq 0$, where we have set $\tilde{l}(t, \delta)(\theta) := l(t, (\xi(t), \phi(t, \cdot)), \delta)(\theta)$. Since $\tilde{l}(t, \delta)(\cdot)$ converges to zero with respect to the $L^2([-a, b], \mathbb{R}^N)$-norm for $\delta \to 0$ and $\xi = \xi^\delta \to \xi_*$ for $\delta \to 0$ uniformly on bounded intervals, we conclude $\phi(t, \cdot) = \phi^\delta(t, \cdot) \to \phi^*(t, \cdot) = \xi(t + \cdot)$ for $t + \theta \geq 0$ with respect to the sup-norm. For $-b \leq t+\theta \leq 0$ let us note that $\phi^\delta(t, \theta) \to \phi(0, t+\theta)$ for $\delta \to 0$, where $\phi(0, \cdot)$ denotes the H^1-component of our initial vector $V^s \in \tilde{E}^s_+$, which can be obtained by inserting $t = 0$ in equation (20). Let us define $\xi_*(t) := \phi(0,t)$ for $-a \leq t \leq 0$. We now insert the function $\phi^\delta(t,\cdot)$ in equation (25), pass to the limit $\delta \searrow 0$ and obtain

$$\begin{aligned} \xi^*(t) &= \xi(0) + \int_0^t \int_{-a}^b p(s,\theta)\xi^*(s+\theta)d\theta ds \\ &+ \int_0^t \sum_{k=1}^m A_k(s)\xi^*(s+r_k)ds + \int_0^t g(s,(\xi^*(s),\xi^*(s+\cdot))ds. \end{aligned}$$

This shows that $\xi^*(t)$ is a solution of $\dot{z}(t) = D_1G(t,h_t,\mu_*)z_t + g(t,(z(t),z_t),\mu)$ for $t \geq 0$. Since, finally, the sequence $V^\delta \in BC^{-\gamma}(\mathbb{R}_+,\hat{X})$ is uniformly bounded in $BC^{-\gamma}(\mathbb{R}_+,\hat{X})$ with respect to $\delta > 0$, we conclude that $\xi^*(\cdot) \in BC^{-\gamma}(\mathbb{R}_+,\mathbb{R}^N)$ and therefore $V(t) = (\xi^*(t),\xi^*_t)$, which shows claim b). □

Analogously we can prove the existence of an unstable manifold $W^u(x_-)$ near $H(0)$:

Theorem 7 (The unstable manifold)
Assume that hypothesis 5 is met. Then equation (15) possesses a C^2-manifold $W^u(x_-) = W^u_\mu(x_-) \subset \tilde{X}$ near $H(0)$, with the following properties:

a) *The tangent space of $W^u_{\mu_*}(x_-)$ at $H(0) \in W^u_{\mu_*}(x_-)$ is $\tilde{E}^u_-$.*

b) *If $W_- \in W^u_\mu(x_-)$ and $W_+ \approx H(0)$ with respect to the $\tilde{X}$-norm, then there exists a continuous function $W(t) : \mathbb{R}_- \to \tilde{X}$, such that $W(\cdot) \in W^u_\mu(x_-)$ on some time interval $(-t_*,0]$, $W(0) = W_-$ and $W(t) = (\xi(t),\xi_t)$, where $\xi(t) : (-\infty,b] \to \mathbb{R}^N$ is a solution of $\dot{x}(t) = G(x_t,\mu)$ which converges to zero with exponential rate $\kappa > 0$ as $t \to -\infty$.*

c) *If there is a solution $W(t)$ of the abstract equation (15), such that $W(t) \to 0$ with exponential rate κ for $t \to -\infty$ and if $W(0)$ is close enough to $H(0)$ with respect to the $\tilde{X}$-norm, then $W(0) \in W^u_\mu(x_-)$.*

d) *$W^u_\mu(x_-)$ is two times continuously differentiable with respect to μ: We can represent $W^u_\mu(x_-)$ in the form $W^u_\mu(x_-) = H(0) + graph(\Psi(\cdot,\mu))$ and $\Psi(\cdot,\mu) : \tilde{E}^u_- \times (\mathbb{R}^p \cap U(\mu_*)) \to \tilde{E}^s_-$ is two times continuously differentiable with respect to the parameter μ, if $U(\mu_*)$ denotes a small neighborhood of μ_*.*

Remark
Note that we introduced the space $\tilde{X}$, since $G : C^0 \times \mathbb{R}^p \to \mathbb{R}^N$ may not be defined for functions in L^2. But if we consider the special case of a well defined map $G : L^2 \times \mathbb{R}^p \to \mathbb{R}^N$, we can work in the space Y directly and stable, unstable manifolds are submanifolds of Y.

4.1 Stable and unstable manifolds for the original abstract equation

Let us now check that equation (14) possesses stable and unstable manifolds near $H(0)$. Note that on account of the last remark in the previous section we can consider the stable and unstable manifolds as submanifolds of Y. In order to check, whether hypothesis 5 is satisfied, we compute the linearization of (14) along the homoclinic solution $H(t)$, which is

$$\begin{aligned} \begin{pmatrix} \partial_t \eta(t) \\ \partial_t \psi(t,\cdot) \end{pmatrix} &= \begin{pmatrix} \frac{-1}{c_*}(L\psi(t,\cdot) - \psi(t,0) - f'(h(t))\psi(t,0)) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} \\ &= A(t) \begin{pmatrix} \eta(t) \\ \psi(t,\cdot) \end{pmatrix}. \end{aligned} \tag{27}$$

Lemma 2
Equation (14) possesses a stable manifold $W_c^s(x_+)$ and an unstable manifold $W_c^u(x_-)$ near $H(0)$, which are submanifolds of the space Y.

Proof
We want to apply theorem 4 and have to check, if the hypothesis 4 is satisfied and if the operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R},\mathbb{C}) &\rightarrow L^2(\mathbb{R},\mathbb{C}) \\ (\mathcal{L}v)(t) &= \partial_t v(t) + \frac{1}{c_*}(Lv_t - v(t) - f'(h(t))v(t)). \end{aligned}$$

is a Fredholmoperator. Note that $\mathcal{L}$ is asymptotically constant and induces operators

$$L_\pm \phi = \frac{-1}{c_*}(L\phi - \phi(0) - f'(x_\pm)\phi(0)).$$

Let us now consider the characteristic equations, which are

$$\det(\triangle)_\pm(\lambda) := \lambda + \frac{1}{c_*}(Le^{\lambda\cdot} - 1 - f'(x_\pm)).$$

Looking for purely imaginary zeros $\lambda = i\sigma$ we observe that

$$\det(\triangle)_\pm(i\sigma) = 0 \quad \Leftrightarrow \quad \lambda = 0 \text{ and } f'(x_\pm) = 0$$

on account of $L(\sin(\sigma\cdot)) = 0$ (by eveness of the kernel K) and $L(1) = 1$. Since we assumed $f'(x_\pm) \neq 0$, we conclude that $\mathcal{L}$ is asymptotically hyperbolic and therefore a Fredholmoperator by theorem 2.
Let us now consider a kernel-element $v \in H^1(\mathbb{R},\mathbb{R})$ with $v_\tau = 0$ for some $\tau \in \mathbb{R}$. Define $\zeta := \sup\{t \geq \tau : v(t) = 0\}$. If $\zeta < \infty$, consider some $s < \zeta$, such that either $v_s \geq 0$ or $v_s \leq 0$ but $v_s \neq 0$. Then we have $\dot{v}(s) = \frac{-1}{c_*}(Lv_s - v(s) - f'(h(s))v(s))$ and we conclude

$$0 = Lv_s$$

and therefore $v_s \equiv 0$, since the kernel of L is a positive positive function and v_s does not change sign. This contradicts $v_s \neq 0$ and therefore $\zeta = \infty$. Similarly one can show, that $v(s) = 0$ for all $s \leq \tau$. □

5 Persistence of the the travelling wave solution

In this section we want to study under what conditions the travelling wave solution $u(t,x) = h(x - c_* t)$ of (1) is structurally stable. Since we know that the heteroclinic solution $H(t) = (h(t), h_t)$ lies in the intersection $W^s_{c_*}(x_+) \cap W^u_{c_*}(x_-)$, we are interested in the transversality of the two manifolds $W^s_{c_*}(x_+)$, $W^u_{c_*}(x_-)$ at the point $H(0)$. We therefore want to calculate the codimension of Ω in the ambient space Y, where

$$\Omega := T_{H(0)} W^s_{c_*}(x_+) + T_{H(0)} W^u_{c_*}(x_-). \tag{28}$$

On account of our assumption that the heteroclinic solution $H(t)$ is unique up to translations, we observe

$$T_{H(0)} W^s_{c_*}(x_+) \cap T_{H(0)} W^u_{c_*}(x_-) = \text{ span} \langle H'(0) \rangle .$$

Our next lemma helps us to relate the codimension of Ω to a Fredholmindex of the linearization (27) along the heteroclinic orbit.

Lemma 3
Consider the operator

$$\begin{array}{rcl} \mathcal{T} : \mathcal{D}(\mathcal{T}) \subset L^2(\mathbb{R}, Y) & \rightarrow & L^2(\mathbb{R}, Y) \\ (\mathcal{T} V(\cdot))(t) & = & \dot{V}(t) - A(t) V(t), \end{array}$$

where $A(t)$ is given in (27) and $\mathcal{D}(\mathcal{T})$ has been defined in section 3.2. Then $\mathcal{T}$ is a Fredholmoperator with index $i_{\mathcal{T}}$ and we have the identity

$$codim_Y(\Omega) = 1 - i_{\mathcal{T}}.$$

Proof
Let us consider the map $\iota : Y \times Y \rightarrow Y$, which is defined by

$$\iota : (\Gamma, \Psi) \mapsto \Phi^s_+(0,0)\Gamma - \Phi^u_-(0,0)\Psi.$$

A short calculation of the adjoint operator $\iota^* : Y \rightarrow Y \times Y$ gives the representation

$$\iota^* : z \mapsto ((\Phi^s_+(0,0))^* z, -(\Phi^u_-(0,0))^* z).$$

If $\Psi^0 \in \ker(\iota^*)$ then:

$$\begin{array}{rcl} \Psi^0 \in \mathrm{Rg}(id - \Phi^s_+(0,0)^*) & \Longleftrightarrow & \Psi^0 \in \mathrm{Rg}((id - \Phi^s_+(0,0))^*) \\ & \Longleftrightarrow & \Psi^0 \in \mathrm{Rg}(\Phi^u_+(0,0)^*) \end{array}$$

and similiarly $\Psi^0 \in \mathrm{Rg}(\Phi^s_-(0,0)^*)$. This shows that $\Psi^0 \in \mathrm{Rg}(\Phi^s_-(0,0)^*) \cap \mathrm{Rg}(\Phi^u_+(0,0)^*)$. More precisely, we have proved $\Psi^0 = \Phi^s(0,0)^*\Psi^0 = \Phi^u_+(0,0)^*\Psi^0$. The adjoint operators $\Phi^u_+(\zeta,t)^*, \Phi^s_-(\zeta,t)^*$ are defined for $t \geq \zeta \geq 0$ and $t \leq \zeta \leq 0$ respectively. Now Ψ^0 defines via

$$\tilde{W}(t) = \left\{ \begin{array}{lcl} (\Phi^u_+(0,t))^*\Psi^0 & : & t \geq 0 \\ (\Phi^s_-(0,t))^*\Psi^0 & : & 0 \geq t \end{array} \right. \tag{29}$$

a solution of the adjoint equation $\dot{W}(t) = -\mathcal{A}(t)^* W(t)$. Thus every element of $\ker(\iota^*)$ defines an element of $\ker(\mathcal{T}^*)$.
On the other hand any element $W(\cdot)$ in $\ker(\mathcal{T}^*)$ is a solution of the adjoint equation $\dot{W}(t) = -\mathcal{A}(t)^* W(t)$ and can be represented in the form (29) for some appropriate $\Psi^0 \in Y$, which defines a kernel element of ι^* and therefore lies in a complement of $\Omega = \mathrm{Rg}(\iota)$. The claim now follows from the identity

$$i_{\mathcal{T}} = \dim(\mathrm{kern}(\mathcal{T})) - \mathrm{codim}(\mathrm{Rg}(\mathcal{T})) = 1 - \dim(\mathrm{kern}(\mathcal{T}^*)).$$

□

This lemma together with theorem 5 gives us the possibility of calculating the codimension of the space Ω, defined in (28). We can now prove the next lemma, which is the key step in proving theorem 1.

Lemma 4
The manifolds $W^s_{c_*}(x_+)$ and $W^u_{c_*}(x_-)$ are transversal at $H(0)$ if and only if the states x_+ and x_- satisfy

$$sign(f'(x_-)) > 0 > sign(f'(x_+)). \tag{30}$$

More precisely, the space Ω satisfies $\Omega = Y$ if (30) is true and has codimension one in Y if $sign(f'(x_-)) = sign(f'(x_+))$. Finally, $codim_Y \Omega = 2$, if $f'(x_+) > 0 > f'(x_-)$.

Proof
Let us first note that the sum Ω of the tangent spaces is always a *closed* subspace of Y, see lemma 3.19 in [6] for a proof. For the proof of this theorem it is therefore sufficient to calculate the Fredholmindex $i_{\mathcal{T}}$ of $\mathcal{T}$ by the previous lemma 3. We will do this with the help of theorem 5 and consider the operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R}, \mathbb{C}) &\rightarrow L^2(\mathbb{R}, \mathbb{C}) \\ (\mathcal{L}v)(t) &= \partial_t v(t) - L(t)v(t) \\ &= \partial_t v(t) + \frac{1}{c_*}\left(Lv_t - v(t) - f'(h(t))v(t)\right). \end{aligned}$$

Let us now choose for $\rho \in [0,1]$ the homotopy

$$L^\rho \phi(\cdot) := \frac{1}{c_*}(L(\phi(\cdot)) - \phi(0)) - \frac{1}{c_*}(\rho Df(x_+)\phi(0) + (1-\rho)Df(x_-)\phi(0)).$$

Then we observe $L^1\phi = L(\infty)\phi$ and $L^0\phi = L(-\infty)\phi$, where we have set $L(\pm\infty) := \lim_{t\to\pm\infty} L(t)$. Let us now check that the assumptions of theorem 5 are satisfied, that is, there are only finitely many values ρ where the characteristic equation

$$\triangle^\rho(\lambda) := \lambda - \left(\frac{1}{c_*}(L(e^{\lambda \cdot}) - 1) - \frac{1}{c_*}(\rho Df(x_+) + (1-\rho)Df(x_-))\right)$$

possesses purely imaginary zeros $\lambda = i\sigma$. Indeed, making the ansatz $\lambda = i\sigma$ leads to

$$\triangle^\rho(i\sigma) = i\sigma - \left(\frac{1}{c_*}(L(\cos(\sigma \cdot)) - 1) - \frac{1}{c_*}(\rho Df(x_+) + (1-\rho)Df(x_-))\right).$$

We note that only $i\sigma = 0$ is a possible root, which reduces the equation to

$$0 = \rho Df(x_+) + (1-\rho)Df(x_-). \tag{31}$$

Thus, the condition (31) can never be satisfied if $\text{sign}(f'(x_+)) = \text{sign}(f'(x_-))$. Under this assumption we conclude with the help of theorem 5 and theorem 3 that $i_\mathcal{T} = i_\mathcal{L} = -\text{cross}(L^\rho) = 0$ which proves the lemma in view of lemma 3. Now let us assume that $\text{sign}(f'(x_+)) \neq \text{sign}(f'(x_-))$ and consider the case $f'(x_-) > 0 > f'(x_+)$. Then there is exactly one value $\rho_* \in (0,1)$, such that (31) is satisfied for $\rho = \rho_*$. We easily check that $i\sigma = 0$ is a simple zero of $\triangle^{\rho_*}(\cdot)$, since

$$\partial_\lambda \triangle^{\rho_*}(\lambda)\big|_{\lambda=0} = 1 - \frac{1}{c_*}L(\theta) = 1 \neq 0.$$

In particular there exists a curve $\lambda = \lambda(\rho)$ with $\lambda(\rho_*) = 0$ and $\triangle^\rho(\lambda(\rho)) = 0$ by the implicit function theorem. We can calculate the derivative of this function and obtain

$$\begin{aligned}\partial_\rho \eta(\rho)\big|_{\rho=\rho_*} &= -\left[\partial_\lambda \triangle(0)^{\rho_*}\right]^{-1} \cdot \partial_\rho \triangle^{\rho_*}(0) \\ &= \frac{1}{c_*}(f'(x_+) - f'(x_-)) < 0\end{aligned}$$

and therefore $\text{cross}(L^{\rho_*}) = -1 < 0$. Finally this proves

$$i_\mathcal{T} = i_\mathcal{L} = -\text{cross}(L^{\rho_*}) = 1,$$

which shows that the codimension of Ω in the ambient space Y is zero. This together with the fact that Ω is closed proves $\Omega = Y$. Finally, if $f'(x_+) > 0 > f'(x_-)$ then the calculation shows that $\text{codim}_Y \Omega = 2$. □

We can now prove our main result.

Theorem 8
Fix two states $x_+, x_- \in \mathbb{R}$ with $x_+ \neq x_-$. Assume that f and K satisfy hypothesis 2 hold and that there exists a unique (up to translations) travelling wave solution $u(t,x) = h(x - c_* t)$, which satisfies the conditions a) and b) of the introduction. Then the following holds:

i) If $\text{sign}(f'(x_-)) > 0 > \text{sign}(f'(x_+))$ then the travelling wave $u(t,x)$ is structurally stable: Choose a linear operator $\tilde{L} : C^0 \to \mathbb{R}$ and a nonlinearity $\tilde{f} : \mathbb{R} \to \mathbb{R}$, such that the differences $\|\tilde{L} - L\|_{L(C^0([-a,a],\mathbb{R}),\mathbb{R})}$ and $\|f - \tilde{f}\|_{BC^1(\mathbb{R},\mathbb{R})}$ are small enough, then the perturbed system

$$\partial_t u(t,x) = \tilde{L}u(t, \cdot + x) - u(t,x) - \tilde{f}(u(t,x)) \tag{32}$$

possesses a travelling wave solution $\tilde{u}(t,x) = \tilde{h}(t - \tilde{c}x)$ for every $c \approx c_*$ and the states $\tilde{x}_\pm := \lim_{\xi \to \pm\infty} \tilde{h}(\xi)$ are near the states $x_\pm$.

ii) Assume that $\text{sign}(f'(x_+)) = \text{sign}(f'(x_-))$. Supply system (1) with a parameter α, such that $\alpha \mapsto f(\cdot, \alpha) \in BC^1(\mathbb{R},\mathbb{R})$ and $\alpha \mapsto L^\alpha$ are *differentiable and* $L^0 = L, f(\cdot,0) = f$ *holds. Assume hypothesis 1 is not satisfied. Then there exists a* C^2 *function* $\gamma : (\mathbb{R} \cap B_\varepsilon(0)) \times (-\alpha_*, \alpha_*) \times (-\varepsilon + c_*, c_* + \varepsilon) \to \mathbb{R}$ *such that the existence of a travelling wave of the perturbed system with wave speeed* $c \approx c_*$ *for the parameter* α *corresponds to a zero of* $\gamma(\cdot,\alpha,c) = 0$*. Furthermore,* $\gamma(0,0,c_*) = 0$ *and* $\partial_1\gamma(0,0,c_*) = 0$*.*

On the other hand if hypothesis 1 is true then there exists a travelling wave solution $u(t,x) = \tilde{h}(x - ct)$ *for the perturbed equation (1) with wave speed* $c = c(\alpha) \approx c_*$ *for small enough* $\alpha \neq 0$*. Moreover, the function* $c = c(\alpha)$ *is* C^2 *and* $c(0) = c_*$*.*

iii) If $f'(x_+) > 0 > f'(x_-)$ *then the travelling wave* $u(t,x)$ *is structurally unstable: The existence of a travelling wave corresponds to a zero of a* C^2*-function* $\eta(\cdot,\alpha,c)$*, where* $\eta : (\mathbb{R} \cap B_\varepsilon(0)) \times (-\alpha_*, \alpha_*) \times (-\varepsilon + c_*, c_* + \varepsilon) \to \mathbb{R}^2$ *and* $\eta(0,0,c_*) = 0$*,* $\partial_1\eta(0,0,c_*) = 0$*.*

Proof of Theorem 8
Supply system (1) with a parameter α, such that $\alpha \mapsto f(\cdot,\alpha) \in BC^1(\mathbb{R},\mathbb{R})$, $\alpha \mapsto L^\alpha$ is two times continuously differentiable and $L^0 = L, f(\cdot,0) = f$ holds. Without loss of generality we assume that $f(x_\pm,\alpha) = 0$ for all α. We then observe that the manifolds $W^s(x_+) = W^s_{c,\alpha}(x_+)$ and $W^u(x_-) = W^u_{c,\alpha}(x_-)$ also depend two times differentiably on α and c.
We have already proved that $W^u_{c,\alpha}(x_-)$ and $W^s_{c,\alpha}(x_+)$ can be represented in the form

$$W^u_{c,\alpha}(x_-) = H(0) + \mathrm{graph}(\Psi^u(\cdot,c,\alpha)) \qquad W^s_{c,\alpha}(x_+) = H(0) + \mathrm{graph}(\Psi^s(\cdot,c,\alpha))$$

locally near $H(0)$, where $\Psi^u(\cdot,c,\alpha) : \mathrm{Rg}(\Phi^u_-(0,0)) \to \mathrm{Rg}(\Phi^s_-(0,0))$ and $\Psi^s(\cdot,c,\alpha)$: $\mathrm{Rg}(\Phi^s_+(0,0)) \to \mathrm{Rg}(\Phi^u_+(0,0))$, see (21). We therefore can define (for $|\alpha_*|$ small enough) the function

$$\begin{aligned} \Xi &: \mathrm{Rg}(\Phi^u_-(0,0)) \times \mathrm{Rg}(\Phi^u_-(0,0)) \times (-\alpha_*,\alpha_*) \times (c_* - \varepsilon, c_* + \varepsilon) \to Y \\ \Xi &: (V^u, V^s, \alpha, c) \mapsto \Psi^u(V^u,c,\alpha) - \Psi^s(V^s,c,\alpha) + V^u - V^s. \end{aligned}$$

and note that $\Xi(0,0,0,c_*) = 0$ and $D_{(V^u,V^s)}\Xi(0,0,0,c_*)$ has a one-dimensional kernel. Furthermore, any zero of Ξ defines an intersection point of the stable and unstable manifold and therefore induces a heteroclinic orbit. Let us now consider the case $f'(x_-) > 0 > f'(x_+)$. Then $D_{(V^u,V^s)}\Xi(0,0,0,c_*)$ is surjective with one-dimensional kernel. The claim of theorem 1 now follows directly from the implicit function theorem.
Now we restrict our attention to the case $\mathrm{sign} f'(x_+) = \mathrm{sign} f'(x_+)$ and assume that hypothesis 1 is satisfied. In this case

$$\mathrm{Rg}\left\{D_{(V^u,V^s)}\Xi(0,0,0,c_*)\right\} = \Omega,$$

possesses a one-dimensional complement in Y, where Ω has been defined in (28). Hence, we can write $Y = \Omega + \mathrm{span}\langle\Psi_0\rangle$, where Ψ_0 is defined in the proof of lemma 3. After Lyapunov-Schmidt reduction, zeros of Ξ correspond to zeros of a reduced map

$$\gamma(\cdot,\alpha,c) : \ker\left\{D_{(V^u,V^s)}\Xi(0,0,0,c_*)\right\} \to \mathrm{span}\langle\Psi_0\rangle\,,$$

where $\partial_1\gamma(0,0,c_*)=0$. We now show that $\partial_c\gamma(0,0,c)\big|_{c=c_*}\neq 0$ if hypothesis 1 is satisfied. Indeed, it suffices to show that

$$\left\langle \partial_c\Psi^u(0,c,0)\big|_{c=c_*} - \partial_c\Psi^s(0,c,0)\big|_{c=c_*}, \Psi^0\right\rangle_Y \neq 0.$$

More explicitly, this reads

$$\left\langle \int_\infty^0 \Phi_+^u(0,\xi)\partial_c G(\xi,0,0,c_*)d\xi - \int_{-\infty}^0 \Phi_-^s(0,\xi)\partial_c G(\xi,0,0,c_*)d\xi, \Psi^0\right\rangle_Y \neq 0$$

(where G has been defined in (19)) and this is equivalent to

$$\int_\infty^\infty \left\langle \partial_c G(\xi,0,0,c_*), \tilde{W}(\xi)\right\rangle_Y d\xi \neq 0, \tag{33}$$

where $\tilde{W}$, defined in (29), is the unique (up to scalar multiples) solution of the adjoint $\mathcal{T}^*\tilde{W}=0$ and we remind that $i_{\mathcal{T}}=0$, $\dim(\ker(\mathcal{T}^*))=1$ in the case of $\operatorname{sign}(f'(x_+))=\operatorname{sign}(f'(x_-))$. Now a closer look at the definition of G in (19) shows that $\partial_c G(t,0,0,c_*)=h'(t)$ and therefore (33) is equivalent to hypothesis 1. We conclude that if hypothesis 1 is satisfied, we get a zero of Ξ for $\alpha\approx\alpha_*$ which induces an intersection point $(H^\alpha,c^\alpha)\approx(H(0),c_*)$ of stable and unstable manifold. This point induces a travelling wave $u^\alpha(t,x)=\tilde{h}^\alpha(x-c^\alpha t)$ for the perturbed equation for some C^2-function $\tilde{h}^\alpha:\mathbb{R}\to\mathbb{R}$ with $(\tilde{h}^\alpha(0),\tilde{h}_0^\alpha)=H^\alpha\in Y$.
The other cases of theorem 8 now follow analogously: Note that the range of $D_{(V^s,V^u)}\Xi(0,0,0,c_*)$ has codimension two if $f'(x_+)>0>f'(x_-)$. □

6 A special case: Bistable nonlinearities

In this section we consider the special case of bistable nonlinearities, which has been considered in [1, 2, 3, 4, 17]. Let us specify these class of functions by the following definition.

Definition 5
Let $\mathcal{C}$ denote the set of all functions $f\in C^2(\mathbb{R},\mathbb{R})$, such that

- $f(\pm a)=0$ *and* $0<f'(\pm a)$
- *f has only one zero γ in $(-a,a)$ and no zeros outside $[-a,a]$.*

Then the following results have been proved by Bates et al (see Theorem 4.1 in [1]):

Theorem 9 (Bates et al)
Assume that $g(u):=u+f(u)$ has at most three intervals of monotonicity, that is

$$g'>0 \text{ on } [-a,\beta)\cup(\zeta,a], \qquad g'<0 \text{ on } (\beta,\zeta) \tag{34}$$

for some $\beta\le\zeta$. Then there exists a unique travelling wave solution $u(t,x)=h(x-c_*t)$ of (1) connecting the states $x_\pm$ for a unique travelling wave speed c_* in the class of all continuous functions h.

Together with lemma 2.2 of [1], which states that hypothesis 1 is always satisfied, we have the following corollary.

Corollary 1
Consider equation (1) for some nonlinearity $f \in \mathcal{C}$ and assume that (34) is satisfied. Choose a linear operator $\tilde{L} : C^0([-a,a],\mathbb{R}) \to \mathbb{R}$ and a differentiable nonlinearity $\tilde{f} : \mathbb{R} \to \mathbb{R}$. If the differences $\|\tilde{L} - L\|_{L(C^0([-a,a],\mathbb{R}),\mathbb{R})}$ and $\|f - \tilde{f}\|_{BC^1(\mathbb{R},\mathbb{R})}$ are small enough, the perturbed system

$$\partial_t u(t,x) = \tilde{L}u(t,\cdot + x) - u(t,x) - \tilde{f}(u(t,x)) \tag{35}$$

possesses a travelling wave solution $\tilde{u}(t,x) = \tilde{h}(t - \tilde{c}x)$ for a unique $c \approx c_*$. The states $\tilde{x}_\pm := \lim_{\xi \to \pm\infty} \tilde{h}(\xi)$ are near the states $x_\pm$. Furthermore the travelling wave profile $\tilde{h}$ is unique (up to translations); that is, $\tilde{h}$ is unique with the property to be a heteroclinic solution of the perturbed travelling wave equation (2) connecting the states $\tilde{x}_\pm$ near $x_\pm$.

7 Application: The one-dimensional equation of elasticity

One of our main motivations in studying the structural stability of travelling waves of integrodifferential equations of the form (1) is the one-dimensional system of elasticity, which is given by

$$\begin{pmatrix} \partial_t w - \partial_x v \\ \partial_t v - \partial_x \sigma(w) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{36}$$

where $(x,t) \in \mathbb{R} \times (0,T)$, $T > 0$. Here, $w(x,t)$ denotes the strain, $v(x,t)$ is the velocity and $\sigma : (-1,\infty) \to \mathbb{R}$ denotes the stress strain relation. It is well known that (36) produces shock wave solutions, see [18, 19]. These are solutions of the form

$$\begin{pmatrix} w^0(x,t) \\ v^0(x,t) \end{pmatrix} = \begin{cases} (W_-,V_-)^T & : \quad x - ct < 0 \\ (W_+,V_+)^T & : \quad x - ct > 0 \end{cases} \tag{37}$$

for some appropriate elements $(W_\pm, V_\pm) \in \mathbb{R}^2$ and some speed $c \in \mathbb{R}$.
In order to single out unique physically relevant shock wave solutions one can regularize (36) by supplementing terms which model the effects of viscosity and capillarity:

$$\begin{pmatrix} \partial_t w - \partial_x v \\ \partial_t v - \partial_x \sigma(w) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu\varepsilon\partial_{xx}^2 v - \nu\partial_x \left[K^\varepsilon(\partial_x w)\right] \end{pmatrix}, \tag{38}$$

where μ and ν are nonnegative viscosity and capillarity constants and

$$K^\varepsilon[w](x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} K\left(\frac{x-y}{\varepsilon}\right)\{w(y) - w(x)\}dy.$$

The equation (38) has been studied in [18, 19]. For simplicity we restrict to functions K which have compact support in the interval $[-a,a]$ for some $a > 0$.

Let us observe that the right hand side of (38) vanishes if we let $\varepsilon = 0$. In this sense we can think of the right hand side as a regularisation of the conservation law (36).
Focusing on the the case $\nu = \varepsilon = 1$ and making a travelling wave ansatz $(w(x,t), v(x,t)) = (W(x - c_* t), V(x - c_* t))$, with limits $\lim_{\xi\to\pm\infty} W(\xi) = W_\pm$, $\lim_{\xi\to\pm\infty} V(\xi) = V_\pm$ we arrive at the travelling wave equation

$$\begin{aligned} c_*\mu W'(\tau) &= (LW_\tau - W(\tau)) - \left[-\sigma(W(\tau)) + c_*^2 W(\tau) - c_*^2 W_- + \sigma(W_-)\right] \\ &= (LW_\tau - W(\tau)) - f(W(\tau)), \end{aligned} \tag{39}$$

where

$$\begin{aligned} L &: C^0([-a,a],\mathbb{R}) \to \mathbb{R} \\ L &: \phi \mapsto \int_{-a}^{a} K(\theta)\phi(\theta)d\theta. \end{aligned}$$

Indeed, let W be any heteroclinic solution of the travelling wave equation (39) with $\lim_{\xi\to-\infty} W(\xi) = W_-$. For any given V_- we can define $V(\cdot)$ by the equation

$$c_* W(\tau) - V(\tau) = -c_* W_- - V_-, \tag{40}$$

which is the integrated first equation of (38) after travelling wave ansatz. Then

$$(w^\varepsilon(t,x), v^\varepsilon(t,x)) := \left(W\left(\frac{x - c_* t}{\varepsilon}\right), V\left(\frac{x - c_* t}{\varepsilon}\right)\right) \tag{41}$$

defines a travelling wave solution of (38); see also [18] for more details. Moreover, the solution $(w^\varepsilon, v^\varepsilon)$ converges pointwise for $\varepsilon \to 0$ to a shock wave solution (w^0, v^0), if $x - ct \neq 0$ is fixed. In this case we say that (w^0, v^0) can be *realized* by a travelling wave profile (W, V) and we can think of the shock wave (w^0, v^0) as a physically relevant solution (we refer to [18, 19] for more backround on this subject).
We are now aiming at a relation between the "nature" of the shock wave (w^0, v^0) and the structural stability of the travelling wave profile (W, V) as a solution of a functional differential equation of mixed type. Let set $U_+ = (W_+, V_+)$ and $U_- = (W_-, V_-)$ and make the following definition, see also [18, 19, 24].

Definition 6
Assume that the matrices $Df(U_+), Df(U_-)$ have real eigenvalues $\lambda_1(U_+), \lambda_2(U_+)$ and $\lambda_1(U_-), \lambda_2(U_-)$, respectively. Then we call a shock wave (w^0, v^0), defined as in (37),

i) compressive, *if $\lambda_1(U_-) > c > \lambda_1(U_+)$ or $\lambda_2(U_-) > c > \lambda_2(U_+)$*

ii) undercompressive, *if $\lambda_2(U_-) > c > \lambda_1(U_+)$ and $\lambda_2(U_+) > c > \lambda_1(U_-)$.*

A straightforward application of theorem 1 yields the following result.

Theorem 10
Assume that a shock wave solution (w^0, v^0) of (36) can be realized by a travelling wave profile (W, V), where W is a solution of (39), connecting W_+, W_-, and V is a solution of (40) for some $V_- \in \mathbb{R}$, such that $W_\pm, V_\pm, c_*$ satisfy the Rankine-Hugoniot conditions

$$-c_*(W_+ - W_-) = V_+ - V_-, \qquad -c_*(V_+ - V_-) = \sigma(W_+) - \sigma(W_-).$$

We now fix the end states W_-, V_- and obtain the following result: The solution (W, V) is structurally stable if and only if the shock wave (w^0, v^0) is compressive. Thus, small perturbations of L_1 with respect to the $L(C^0([-a, a], \mathbb{R}), \mathbb{R})$-norm or the nonlinearity σ with respect to the $BC^1(\mathbb{R}, \mathbb{R})$-norm result in the existence of a solution $(\tilde{W}, \tilde{V})$ for every speed $c \approx c_*$, such that $\lim_{\xi\to\pm\infty} \tilde{W}(\xi) = \tilde{W}_\pm$, where $\tilde{W}_- = W_-$, and $\tilde{W}$ is a solution of the perturbed travelling wave equation (39) with c_* replaced by c. We also have $\lim_{\xi\to\pm\infty} \tilde{V}(\xi) = \tilde{V}_\pm$, where $\tilde{V}_- := V_-$, and $\tilde{V}$ solves (40) with c instead of c_* and $(\tilde{W}, \tilde{V})$ defines via (41) a shock wave in the limit $\varepsilon \to 0$, where c_* has to be replaced by c.

Proof
It is enough to consider the persistence of W as a heteroclinic sultion of (39) after perturbation of L_1 or σ. Indeed, any heteroclinic solution $\tilde{W}$ of the perturbed travelling wave equation with limits $\lim_{\xi\to\pm\infty} \tilde{W}(\xi) = \tilde{W}_\pm$ induces a solution $\tilde{V}$ via

$$c\tilde{W}(\tau) - \tilde{V}(\tau) = -cW_- - V_-$$

as long as the triple $(\tilde{W}_\pm, \tilde{V}_\pm, \tilde{c})$ satisfies the (perturbed) Rankine Hugoniot conditions $-\tilde{c}(\tilde{W}_+ - \tilde{W}_-) = -\tilde{c}(\tilde{W}_+ - W_-) = \tilde{V}_+ - V_-$ and $-\tilde{c}(\tilde{V}_+ - V_-) = \tilde{\sigma}(\tilde{W}_+) - \tilde{\sigma}(W_-)$. But now we can apply theorem 8 to equation (39). This proves theorem 10. □

8 Appendix: The weak* integral

In this section we want to clarify, in which sense the integral

$$\int_0^t T(t, s)G(s)ds \tag{42}$$

is well defined, if $s \to G(s) = (g(s), 0)$ maps smoothly into the space $\hat{X} = \mathbb{C}^N \times C^0([-a, b], \mathbb{C}^N)$. Let us make the following assumption concerning $T(t, s)$.

Assumption 1
Let $L(t) \in BC^0(\mathbb{R}, L(C^0([-a, b], \mathbb{C}^N), \mathbb{C}^N))$ and let $L(t) \to L_\pm$ with respect to the operator norm as $|t| \to \infty$, where $L_\pm \in L(C^0([-a, b], \mathbb{C}^N), \mathbb{C}^N)$ are such that the equations $\dot{y}(t) = L_\pm y_t$ are hyperbolic. Then

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} \xi(t) \\ \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} L(t)\phi(t, \cdot) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix} \tag{43}$$

possesses an exponential dichotomy on $\mathbb{R}_+$ (respectively $\mathbb{R}_-$) with associated solution operators $\Phi^s_+(\tau, \sigma)$, $\Phi^u_+(\sigma, \tau)$ for $\tau \geq \sigma \geq 0$ (respectively $\Phi^s_-(\sigma, \tau)$, $\Phi^u_-(\tau, \sigma)$ and $\tau \leq \sigma \leq 0$). We now consider the case that $T(t, s)$ is one of these solution operators on $\mathbb{R}_+$.

Let us now choose some element

$$(\eta, \psi) \in \hat{Y} := \mathbb{C}^N \times L^1([-a, b], \mathbb{C}^N)$$

and note that

$$s \mapsto \langle T(t,s)G(s), (\eta, \psi)\rangle \in L^1([0,t], \mathbb{R}), \tag{44}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\hat{Y}^* = \tilde{Z} = \mathbb{C}^N \times L^\infty([-a, b], \mathbb{C}^N)$ and $\hat{Y}$; that is

$$\langle (\xi, \phi), (\eta, \psi)\rangle = \xi \cdot \eta + \int_{-a}^{b} \phi(\theta)\psi(\theta) d\theta$$

for $(\xi, \phi) \in \tilde{Z}$ and $(\eta, \psi) \in \hat{Y}$. Here, $\tilde{Z}$ can be identified with the dual space of $\hat{Y}$. Hence, there exists a unique $Q \in \tilde{Z}$, such that

$$\langle Q, (\eta, \psi)\rangle = \int_0^t \langle T(t,s)G(s), (\eta, \psi)\rangle \, ds \tag{45}$$

for every $(\eta, \psi) \in \hat{Y}$; see the appendix of [12]. Note that if $s \mapsto G(s)$ is continuous and takes values in X, then the weak* integral coincides with the usual Riemann integral.

Definition 7
We set $\int_0^t T(t,s)G(s)ds := Q$ *and call* Q *the weak** *integral.*

We therefore view the integral term in (42) as a weak* integral, which is an element of $\hat{Y}^* = \tilde{Z}$ by definition. Let us now prove that the integral is actually an element of $\tilde{X} = \{(\xi, \phi) \in \mathbb{C}^N \times C^0([-a, b], \mathbb{C}^N) : \phi(0) = \xi\}$.

Lemma 5
For each fixed $t \geq 0$ *we have* $\int_0^t T(t,s)G(s)ds \in \tilde{X}$.

Proof
Consider

$$F^\delta(t) := \int_0^t T(t,s) \begin{pmatrix} g(s) \\ g(s) \cdot l(\delta)(\cdot) \end{pmatrix} ds, \tag{46}$$

where

$$l(\delta)(\theta) := \begin{cases} 2 \cdot 2^{\frac{1}{(\theta/\delta)^2 - 1}} & \theta \in (-\delta, \delta) \\ 0 & \text{else} \end{cases}$$

Hence, for fixed $\delta > 0$, (46) defines an element in X for each fixed t. Moreover, the integral can be defined as the usual Riemann integral since the integrand is continuous when considered as a map with values in X. We can now differentiate $F^\delta(t)$ in the space Y and obtain

$$\begin{aligned} \partial_t F^\delta(t) = \partial_t \begin{pmatrix} f^\delta(t) \\ \xi^\delta(t, \cdot) \end{pmatrix} &= T(t,t) \begin{pmatrix} g(t) \\ g(t) l(\delta)(\cdot) \end{pmatrix} + \mathcal{A}(t) F^\delta(t) \\ &= \begin{pmatrix} \zeta(t) \\ h(t, \cdot) \end{pmatrix} + \begin{pmatrix} L(t)[\xi^\delta(t, \cdot)] \\ \partial_\theta \xi^\delta(t, \cdot) \end{pmatrix}. \end{aligned} \tag{47}$$

Let us take a closer look at the second component of (47). Since $F^\delta(t) \in X$ for each fixed t, δ and therefore $\xi(t,0) = f(t)$, we obtain from

$$\partial_t \xi^\delta(t,\theta) = \partial_\theta \xi^\delta(t,\theta) + h(t,\theta)$$

via the method of characteristics the identity

$$\xi^\delta(t,\theta) = \begin{cases} f^\delta(t+\theta) + \int_0^\theta h(t+\theta-\eta,\theta)d\eta & t+\theta \geqslant 0 \\ \xi^\delta(0,\theta+t) + \int_0^t h(t,\theta+t-\eta)d\eta, & -a \leqslant t+\theta < 0. \end{cases} \tag{48}$$

Note that $\xi^\delta(0,\cdot) = 0$ and $f^\delta(0) = 0$ for any $\delta > 0$. Since $T(t,t) : Y \to Y$ is a bounded projection for each t, we conclude that $h(t,\cdot) \to \sigma(t,\cdot)$ for some function $\sigma(t,\cdot) \in L^2$ as $\delta \searrow 0$ in $L^2([-a,b],\mathbb{C}^N)$ because $g(t)l(\delta)(\cdot)$ converges in L^2 as $\delta \searrow 0$. Moreover, the integral in (46) converges with respect to the Y-norm to

$$F^0(t) = \int_0^t T(t,s) \begin{pmatrix} g(s) \\ 0 \end{pmatrix} ds$$

as $\delta \to \infty$. Let us write $F^0(t) = (f(t), \xi(t,\cdot))$. Convergence of (46) in Y implies by definition that $f^\delta(t) \to f(t)$ for fixed t as $\delta \searrow 0$. Therefore, we can pass to the limit $\delta \searrow 0$ in (48) and get

$$\xi(t,\theta) = f(t+\theta) + \int_0^\theta \sigma(t+\theta-\eta,\theta)d\eta \tag{49}$$

as long as $t+\theta \geqslant 0$. Hence, $\xi(t,\cdot)$ is continuous if the spatial variable θ satisfies $t+\theta \geq 0$. In particular we conclude that

$$\xi(t,0) = f(t)$$

for all $t \geqslant 0$. Because $\xi^\delta(t,\cdot)$ also converges with respect to the sup-norm in the region $t+\theta \leqslant 0$ and fixed t (namely, it converges to zero), we conclude that $\xi(t,\cdot) \in C^0$, which proves that $(f(t),\xi(t,\cdot)) \in \tilde{X}$.
Finally, we note that $F^0(t)$ actually coincides with the weak* integral; that is $F^0(t) = Q$. Indeed, $F^0(t) \in \tilde{X}$. Moreover, for any $(\xi,\psi(\cdot)) \in \mathbb{C}^N \times L^1([-a,b],\mathbb{C}^N)$ the identity

$$\left\langle F^\delta(t), (\xi,\psi(\cdot)) \right\rangle = \int_0^t \left\langle T(t,s)G^\delta(s), (\xi,\psi(\cdot)) \right\rangle ds$$

holds. Passing to the limit $\delta \searrow 0$ we obtain

$$\left\langle F^0(t), (\xi,\psi(\cdot)) \right\rangle = \int_0^t \langle T(t,s)G(s), (\xi,\psi(\cdot)) \rangle ds.$$

By uniqueness this shows that $Q = F^0(t)$. □

Lemma 6
The function $v : t \to \int_0^t T(t,s)\mathcal{G}(s)ds$ is continuous as a function from $[0,\infty)$ to $\hat{X}$ and

$$\|v(t)\|_{\hat{X}} \leqslant \int_0^t Me^{\alpha(t-s)}ds \cdot \sup_{0\leqslant s\leqslant t} \|\mathcal{G}(s)\|_{\hat{X}},$$

if $T(t,s)$ satisfies the estimate $\|T(t,s)\|_{L(\tilde{Z},\tilde{Z})} \leqslant Me^{\alpha(t-s)}$ for $t \geq s \geq 0$ and some $\alpha \in \mathbb{R}$, where, as before, $\tilde{Z} = \mathbb{C}^N \times L^\infty$.

Proof
Note that the integral is well defined with values in $\hat{X}$ by the previous lemma. Since the map $t \rightarrow \int_0^t T(t,s)\mathcal{G}(s)ds$ is continuous when regarded with values in $\hat{Y}^* = \tilde{Z}$ (see lemma 2.1, page 54 in [12]) and the norm of L^∞ restricted to C^0 coincides with the usual norm in C^0, the claim concerning continuity follows immediately by lemma 2.3 in [12]. □

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