Asymptotic Behavior of Solutions of the Heat Equation and the Gagliardo-Nirenberg Inequality

熱方程式の解の漸近挙動と Gagliardo-Nirenberg 不等式

付和文抄訳

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Chapter 1

Introduction

This is a senior thesis presented to the faculty of International Christian University for the Baccalaureate degree; majoring in mathematics. The purpose of this thesis is to learn some basics of partial differential equations in order to prepare for further studies in the future. Topics of the thesis are asymptotic behavior of solutions of the heat equation and calculus inequalities derived from estimates of the solutions.

In chapter 2, solutions of the heat equation are introduced, followed by estimates of these solutions. We establish an asymptotic formula of solutions of the heat equation at the end of the chapter, which will be discussed again in the following chapters. In chapter 3, we mention a transformation that preserves structures of partial differential equations and introduce a notion of self-similar solutions. In chapter 4, we derive an Ascoli-Arzelà-type compactness theorem in order to discuss the asymptotic formula in another way.

In chapter 5, we cover inequalities that are important in analyzing partial differential equations in general; we exploit estimates of solutions of the heat equation discussed earlier. Our primary goal in this chapter is to prove the Gagliardo-Nirenberg inequality; to that end, we introduce the Marcinkiewicz interpolation theorem, the Hardy-Littlewood-Sobolev inequality, and the Sobolev inequality.

Chapter 2

Asymptotic Behavior of Solutions of the Heat Equation

2.1 The Heat Equation

We consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \qquad x \in \mathbb{R}^n, \quad t > 0. \tag{2.1}$$

Physically, the heat equation models the temperature distribution $u(x,t)$ of thermal conduction in a homogeneous medium in $\mathbb{R}^n$, with constants such as the density, the specific heat, and the thermal conductivity of the medium normalized for simplicity.

The solution $u(x,t)$ of the heat equation with the initial temperature distribution $u(x,0)$:

$$u(x,0) = f(x), \qquad x \in \mathbb{R}^n,$$

for a given real-valued function f on $\mathbb{R}^n$, is represented by

$$u(x,t) = \int_{\mathbb{R}^n} G_t(x-y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{2.2}$$

using the Gauss kernel $G_t(x) = g(x,t)$ defined by

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \qquad x \in \mathbb{R}^n, \quad t > 0,$$

under the condition that the modulus $|f(x)|$ of the function $f(x)$ does not grow too much at space infinity.

2.2 Decay Estimates of Solutions

Assume that initial data f is continuous and vanishes outside of a large ball in $\mathbb{R}^n$. We expect that the solution $u(x,t)$ also vanishes as time evolves since there is no other heat source.

Definition 2.2.1 The set $C_0(\mathbb{R}^n)$ is defined by

$$C_0(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n);\ \text{supp} f \text{ is compact}\}.$$

Proposition 2.2.1 *(Decay Estimate).*

Let $u(x,t)$ be the solution (2.2) of the heat equation (2.1) with initial data $f \in C_0(\mathbb{R}^n)$. Then

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |f(y)|\, dy, \qquad t > 0. \tag{2.3}$$

In particular, the solution $u(x,t)$ converges uniformly to 0 on $\mathbb{R}^n$ as $t \to \infty$.

Proof. For $t > 0$, we have

$$\begin{aligned} |u(x,t)| = \left| \int_{\mathbb{R}^n} G_t(x-y) f(y)\, dy \right| &\le \int_{\mathbb{R}^n} G_t(x-y)\, |f(y)|\ dy \\ &\le \int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} \{G_t(x-y)\}\, |f(y)|\ dy \le \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |f(y)|\ dy. \end{aligned}$$

■

Remark. The solution $u(x,t)$ of the heat equation converges uniformly to 0 with order at least $t^{-n/2}$ as $t \to \infty$.

Definition 2.2.2 The L^p-norm and the L^∞-norm of a measurable function f on $\mathbb{R}^n$ are defined respectively by

$$\begin{aligned} ||f||_p &:= \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx \right)^{1/p}, \qquad 1 \le p < \infty, \\ ||f||_\infty &:= \inf \{M \in \mathbb{R};\ |f(x)| \le M \text{ for almost all } x \in \mathbb{R}^n\}. \end{aligned}$$

Lemma 2.2.2 *(The Hölder Inequality for real numbers).*

Assume that $1/p + 1/p' = 1$ *for* $1 < p, p' < \infty$*. Then, for positive numbers* a *and* b*,*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Proof. Since the logarithm function $\ln x$: $(0, \infty) \to \mathbb{R}$ is concave, for $\lambda \in [0, 1]$, we have that $\lambda \ln x + (1 - \lambda) \ln y \leq \ln\big(\lambda x + (1 - \lambda)y\big)$. Substituting $\lambda = 1/p$, $x = a^p$, and $y = b^{p'}$, we get

$$\frac{1}{p} \ln a^p + \left(1 - \frac{1}{p}\right) \ln b^{p'} \leq \ln\left(\frac{1}{p} a^p + \left(1 - \frac{1}{p}\right) b^{p'}\right).$$

Hence

$$\ln a + \ln b = \ln ab \leq \ln\left(\frac{a^p}{p} + \frac{b^{p'}}{p'}\right).$$

■

Proposition 2.2.3 *(The Hölder Inequality).*

Assume that $1/p + 1/p' = 1$ *for* $1 \leq p, p' \leq \infty$*. Then, for* $f_1 \in L^p(\mathbb{R}^n)$ *and* $f_0 \in L^{p'}(\mathbb{R}^n)$*,*

$$\left|\int_{\mathbb{R}^n} f_1(x) f_0(x)\, dx\right| \leq ||f_1||_p ||f_0||_{p'}. \tag{2.4}$$

Proof. If $p = 1$ or $p = \infty$, the Hölder inequality is obvious, so suppose that $1 < p, p' < \infty$. If $||f_1||_p = 0$ or $||f_0||_{p'} = 0$, then $f_1 f_0 = 0$ and the Hölder inequality holds. Thus, we may assume that $||f_1||_p \neq 0$ and $||f_0||_{p'} \neq 0$. Integrating the Hölder inequality for real numbers (Lemma 2.2.2), with $a = |f_1(x)|/||f_1||_p$ and $b = |f_0(x)|/||f_0||_{p'}$, we obtain

$$\frac{\int_{\mathbb{R}^n} |f_1(x) f_0(x)|\, dx}{||f_1||_p ||f_0||_{p'}} \leq \frac{1}{p} \frac{1}{||f_1||_p^p} \int_{\mathbb{R}^n} |f_1(x)|^p\, dx + \frac{1}{p'} \frac{1}{||f_0||_{p'}^{p'}} \int_{\mathbb{R}^n} |f_0(x)|^{p'}\, dx = \frac{1}{p} + \frac{1}{p'} = 1.$$

■

Definition 2.2.3 Let h and f be functions on $\mathbb{R}^n$. The convolution of h and f is defined by

$$(h * f)(x) := \int_{\mathbb{R}^n} h(x - y) f(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Remark. If both functions h, $f \in C_0(\mathbb{R}^n)$, the convolution $h * f$ is well defined for each $x \in \mathbb{R}^n$, and continuous on $\mathbb{R}^n$.

Proposition 2.2.4 *(The Young Inequality).*
Let $h \in L^p(\mathbb{R}^n)$ and $f \in L^q(\mathbb{R}^n)$. Assume that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \tag{2.5}$$

for $1 \leq p, q, r \leq \infty$. Then $h * f \in L^r(\mathbb{R}^n)$ and

$$||h * f||_r \leq ||h||_p ||f||_q. \tag{2.6}$$

Proof. If $r = \infty$, the balancing relation (2.5) becomes $1/p + 1/q = 1$ and the inequality yields

$$\left| \int_{\mathbb{R}^n} h(x-y) f(y) \, dy \right| \leq ||h||_p ||f||_q$$

which follows from the Hölder inequality (Proposition 2.2.3). If $p = \infty$, since $1/r = 1/q - 1$ and $1 \leq q, r \leq \infty$, we get $q = 1$ and $r = \infty$. Similarly, if $q = \infty$, we end up having $p = 1$ and $r = \infty$. Therefore, we may assume that $1 \leq p, q, r < \infty$.

Let $0 \leq \theta < 1$ and write $|f| = |f|^{1-\theta}|f|^{\theta}$. The Hölder inequality gives us, for $x \in \mathbb{R}^n$ and $1/p + 1/p' = 1$,

$$\begin{aligned} |(h * f)(x)| &\leq \int_{\mathbb{R}^n} |h(x-y)||f(y)|^{1-\theta}|f(y)|^{\theta} \, dy \\ &\leq \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^{(1-\theta)p} \, dy \right)^{1/p} \left(\int_{\mathbb{R}^n} |f(y)|^{\theta p'} \, dy \right)^{1/p'}. \end{aligned}$$

If $p = 1$, we set $\theta = 0$. Then, $1/p' = 0$, and the inequality follows from the Hölder inequality. If $p > 1$, we have $p' \neq \infty$, so set $p'\theta = q$. Now, $(1-\theta)p = (1 - q/p')p = p - pq/p' = p + pq(1/p - 1) = p + q - pq = pq(p + q - pq)/pq = pq(1/p + 1/q - 1) = pq/r$. The inequality then yields

$$\begin{aligned} |(h * f)(x)| &\leq \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^{pq/r} \, dy \right)^{1/p} \left(\int_{\mathbb{R}^n} |f(y)|^q \, dx \right)^{q/qp'} \\ &= \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^{pq/r} \, dy \right)^{1/p} ||f||_q^{q/p'}. \end{aligned}$$

Let q' be the conjugate exponent of q, i.e. $1/q+1/q' = 1$. By the balancing relation (2.5), $1/r = 1/p + (1 - 1/q') - 1 = 1/p - 1/q'$. Hence, $1/\frac{r}{p} + 1/\frac{q'}{p} = p/r + p/q' = p(1/r + 1/q') = p(1/p - 1/q' + 1/q') = 1$, which means that the conjugate exponent of r/p is q'/p.

Now, split $|h| = |h|^{p/r}|h|^{p/q'}$ and apply the Hölder inequality.

$$\begin{aligned}
&\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^{pq/r}\, dy = \int_{\mathbb{R}^n} |h(x-y)|^{p(p/r+p/q')} |f(y)|^{pq/r}\, dy \\
&= \int_{\mathbb{R}^n} (|h(x-y)|^p |f(y)|^q)^{p/r}\, |h(x-y)|^{p(p/q')}\, dy \\
&\le \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^q\, dy \right)^{p/r} \left(\int_{\mathbb{R}^n} |h(x-y)|^p\, dy \right)^{p/q'} \\
&= \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^q\, dy \right)^{p/r} ||h||_p^{p^2/q'}.
\end{aligned}$$

Hence

$$|(h * f)(x)| \le ||h||_p^{p/q'} ||f||_q^{q/p'} \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^q\, dy \right)^{1/r}.$$

Taking the r-th power, integrating over x, and interchanging the order of integration (by Fubini's theorem),

$$\begin{aligned}
\int_{\mathbb{R}^n} |(h * f)(x)|^r\, dx &\le \int_{\mathbb{R}^n} ||h||_p^{rp/q'} ||f||_q^{rq/p'} \left(\int_{\mathbb{R}^n} |h(x-y)|^p |f(y)|^q\, dy \right) dx \\
&= ||h||_p^{rp/q'} \left(\int_{\mathbb{R}^n} |h(x-y)|^p\, dx \right) ||f||_q^{rq/p'} \left(\int_{\mathbb{R}^n} |f(y)|^q\, dy \right) \\
&= ||h||_p^{rp/q'} ||h||_p^p ||f||_q^{rq/p'} ||f||_q^q = ||h||_p^{rp/q'+p} ||f||_q^{rq/p'+q}.
\end{aligned}$$

Since $rp/q' + p = rp(1/q' + 1/r) = rp(1 - 1/q + 1/r) = r$ and $rq/p' + q = rq(1/p' + 1/r) = r$, we obtain (2.6). ■

Theorem 2.2.5 *(L^p-L^q Estimates).*
Let $u(x,t)$ be as in (2.2) with initial data $f \in C_0(\mathbb{R}^n)$ and let $1 \le q \le p \le \infty$. Then

$$||u||_p(t) \le \frac{1}{(4\pi t)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}} ||f||_q, \qquad t > 0. \tag{2.7}$$

Proof. The solution $u(x,t)$ can be written as $u = G_t * f$. By the Young Inequality (Proposition 2.2.4), for $t > 0$ and $1 \le r \le \infty$ satisfying $1/p = 1/r + 1/q - 1$,

$$||u||_p(t) = ||G_t * f||_p \le ||G_t||_r ||f||_q.$$

If $r = \infty$, we get $1/q - 1/p = 1$. Since $1 \le q \le p \le \infty$, we end up having $p = \infty$ and $q = 1$. Thus, the inequality (2.7) becomes

$$||u||_\infty(t) \le \frac{1}{(4\pi t)^{\frac{n}{2}}} ||f||_1, \qquad t > 0$$

which is nothing but (2.3). So we may assume that $1 \le r < \infty$. Then

$$\begin{aligned} ||G_t||_r^r &= \int_{\mathbb{R}^n} |G_t(x)|^r dx = \int_{\mathbb{R}^n} \left| \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)\right|^r dx \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{r|x|^2}{4t}\right) dx. \end{aligned}$$

By setting $z = \left(\frac{r}{4t}\right)^{1/2} x$, we have $dx = \left(\frac{4t}{r}\right)^{n/2} dz$ and $-|z|^2 = -\frac{r|x|^2}{4t}$. Hence

$$\begin{aligned} ||G_t||_r^r &= \frac{1}{(4\pi t)^{nr/2}} \int_{\mathbb{R}^n} \exp(-|z|^2) \left(\frac{4t}{r}\right)^{n/2} dz \\ &= \frac{1}{(4\pi t)^{nr/2}} \left(\frac{4t}{r}\right)^{n/2} \int_{\mathbb{R}^n} \exp(-|z|^2)\, dz. \end{aligned}$$

Since $\int_{\mathbb{R}^n} \exp(-|z|^2)\, dz = \pi^{n/2}$, we obtain

$$||G_t||_r^r = (4\pi t)^{\frac{n}{2}(1-r)} r^{-\frac{n}{2}}.$$

Considering the balancing relation $1/p = 1/r + 1/q - 1$, i.e. $1/r - 1 = 1/p - 1/q$,

$$\begin{aligned} ||G_t||_r &= (4\pi t)^{\frac{n}{2}\frac{1-r}{r}} r^{-\frac{n}{2r}} = (4\pi t)^{\frac{n}{2}\left(\frac{1}{r}-1\right)} r^{-\frac{n}{2r}} = (4\pi t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} r^{-\frac{n}{2r}} \\ &\le (4\pi t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} = \frac{1}{(4\pi t)^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}. \end{aligned}$$

■

Theorem 2.2.6 *(Derivative L^p-L^q Estimates).*

Let $u(x,t)$ be as in (2.2) with initial data $f \in C_0(\mathbb{R}^n)$. Then, there exists a constant $C = C(n)$ depending only on n such that

$$\left|\left| \frac{\partial u}{\partial x_j} \right|\right|_\infty (t) \le \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} ||f||_1, \qquad j = 1, \ldots, n, \quad t > 0. \tag{2.8}$$

Proof. Differentiating $u(x,t)$ under the integral, we have

$$\begin{aligned}\partial_{x_j}\big(u(x,t)\big) &= \partial_{x_j}\big((G_t * f)(x)\big) = \big((\partial_{x_j} G_t) * f\big)(x) \\ &= \int_{\mathbb{R}^n} (\partial_{x_j} G_t)(x-y) f(y)\, dy.\end{aligned}$$

Furthermore

$$\begin{aligned}\partial_{x_j}\big(G_t(x)\big) &= \frac{1}{(4\pi t)^{n/2}} \partial_{x_j}\left(\exp\left(-\frac{|x|^2}{4t}\right)\right) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \partial_{x_j}\left(-\frac{|x|^2}{4t}\right) \\ &= -\frac{2x_j}{4t}\frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).\end{aligned}$$

Now, we set $z = |x|/(2t^{1/2})$, i.e. $-z^2 = -|x|^2/(4t)$, then

$$\begin{aligned}\frac{x_j}{2t}\exp\left(-\frac{|x|^2}{4t}\right) &= \frac{x_j}{2t}\exp(-z^2) = \frac{1}{t^{1/2}}\frac{x_j}{2t^{1/2}}\exp(-z^2) \\ &\le \frac{1}{t^{1/2}}\frac{|x|}{2t^{1/2}}\exp(-z^2) = \frac{1}{t^{1/2}} z\exp(-z^2).\end{aligned}$$

Here, if $x_j \ge 0$,

$$\left|\frac{x_j}{2t}\exp\left(-\frac{|x|^2}{4t}\right)\right| \le \frac{1}{t^{1/2}} z\exp(-z^2),$$

and if $x_j < 0$,

$$\begin{aligned}\left|\frac{x_j}{2t}\exp\left(-\frac{|x|^2}{4t}\right)\right| &= \frac{|x_j|}{2t}\exp\left(-\frac{|x|^2}{4t}\right) \\ &\le \frac{1}{t^{1/2}}\frac{|x|}{2t^{1/2}}\exp(-z^2) = \frac{1}{t^{1/2}} z\exp(-z^2).\end{aligned}$$

Thus

$$\left|\frac{x_j}{2t}\exp\left(-\frac{|x|^2}{4t}\right)\right| \le \frac{1}{t^{1/2}} z\exp(-z^2).$$

Since $z\exp(-z^2)$ attains its maximum $C_1 = 1/\sqrt{2e}$ at $z = 1/\sqrt{2}$, we obtain

$$\begin{aligned}||\partial_{x_j} G_t||_\infty &= \sup_{x\in\mathbb{R}^n} |\partial_{x_j} G_t| = \sup_{x\in\mathbb{R}^n}\left|-\frac{x_j}{2t}\frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right)\right| \\ &= \sup_{x\in\mathbb{R}^n}\left|\frac{x_j}{2t}\frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right)\right| = \frac{1}{(4\pi t)^{n/2}}\sup_{x\in\mathbb{R}^n}\left|\frac{x_j}{2t}\exp\left(-\frac{|x|^2}{4t}\right)\right| \\ &\le \frac{1}{(4\pi t)^{n/2}}\frac{1}{t^{1/2}} z\exp(-z^2) \le \frac{1}{(4\pi t)^{n/2}}\frac{1}{t^{1/2}} C_1.\end{aligned}$$

Hence

$$\begin{aligned}
\left|\left|\frac{\partial u}{\partial x_j}\right|\right|_\infty &= \left|\left|\int_{\mathbb{R}^n} (\partial x_j G_t)(x-y) f(y)\, dy\right|\right|_\infty = \sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n} (\partial x_j G_t)(x-y) f(y)\, dy\right| \\
&\le \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n} |(\partial x_j G_t)(x-y) f(y)|\, dy = \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n} (\partial x_j G_t)(x-y)|f(y)|\, dy \\
&\le \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n}\{(\partial x_j G_t)(x-y)\}|f(y)|\, dy \\
&= \sup_{x\in\mathbb{R}^n}\left(\sup_{y\in\mathbb{R}^n}\{(\partial x_j G_t)(x-y)\}\int_{\mathbb{R}^n}|f(y)|\, dy\right) = \sup_{x\in\mathbb{R}^n}\{\partial x_j G_t(x)\}\int_{\mathbb{R}^n}|f(y)|\, dy \\
&\le \sup_{x\in\mathbb{R}^n}|\partial x_j G_t(x)|\int_{\mathbb{R}^n}|f(y)|\, dy = ||\partial x_j G_t||_\infty \int_{\mathbb{R}^n}|f(y)|\, dy \\
&\le \frac{1}{(4\pi t)^{n/2}}\frac{1}{t^{1/2}}C_1\int_{\mathbb{R}^n}|f(y)|\, dy = \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}}||f||_1, \qquad \text{where} \quad C := \frac{C_1}{(4\pi)^{n/2}}.
\end{aligned}$$

■

Remark. Similarly, the following holds:

$$\left|\left|\frac{\partial u}{\partial t}\right|\right|_\infty(t) \le \frac{C}{t^{\frac{n}{2}+1}}||f||_1, \qquad t > 0. \tag{2.9}$$

2.3 Asymptotic Behavior Near Time Infinity

We want to know more about how $u(x,t)$ converges to 0 as $t \to \infty$. We already know that $u(x,t)$ decays as $||u||_\infty \le (4\pi t)^{-n/2}||f||_1$ by (2.3). In the following, we will find a "nice" function $v(x,t)$ such that the L^∞-norm of $u-v$ approaches 0, i.e. $||u-v||_\infty(t) \to 0$, faster than $t^{-n/2}$ approaches 0 as $t\to\infty$. Such a function v is called a "leading term" of the decay of u, and we will see that we may choose v as a multiple of the Gauss kernel G_t.

Theorem 2.3.1 *(Asymptotic Formula).*

Let $u(x,t)$ be as in (2.2) with initial data $f \in C_0(\mathbb{R}^n)$, and $m = \int_{\mathbb{R}^n} f(y)\, dy$. Then

$$\lim_{t\to\infty} t^{n/2}||u-mg||_\infty(t) = 0. \tag{2.10}$$

Remark. We will prove the asymptotic formula (2.10) in two distinct ways: One based on the representation formula of solutions of the heat equation (2.2), and the other based on the structure of the heat equation.

Lemma 2.3.2 *(Integral Form of the Mean Value Theorem).*
Let $h \in C^1(\mathbb{R}^n)$. Then

$$h(x) - h(x-y) = \int_0^1 \langle (\nabla h)\big(x-(1-\tau)y\big), y\rangle \, d\tau, \qquad x, y \in \mathbb{R}^n. \tag{2.11}$$

Furthermore

$$|h(x-y) - h(x)| \le |y| \int_0^1 |(\nabla h)\big(x-(1-\tau)y\big)| \, d\tau, \qquad x, y \in \mathbb{R}^n. \tag{2.12}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

Proof. Set $F(s) = h(x-y+sy)$ where $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. By the fundamental theorem of calculus,

$$h(x) - h(x-y) = F(1) - F(0) = \int_0^1 F'(\tau)\, d\tau.$$

By the chain rule, we get

$$F'(\tau) = \sum_{j=1}^n \frac{\partial h}{\partial x_j}\big(x-(1-\tau)y\big) y_j = \langle (\nabla h)\big(x-(1-\tau)y\big), y\rangle,$$

hence (2.11).

Furthermore, by the Schwarz inequality, we get

$$\begin{aligned} |h(x) - h(x-y)| &\le \int_0^1 \big|\langle (\nabla h)\big(x-(1-\tau)y\big), y\rangle\big| \, d\tau \\ &\le \int_0^1 \big|(\nabla h)\big(x-(1-\tau)y\big)\big| \, |y| \, d\tau = |y| \int_0^1 \big|(\nabla h)\big(x-(1-\tau)y\big)\big| \, d\tau. \end{aligned}$$

■

Proof. [Theorem (2.3.1)]. Let $h_\eta(x) = \exp(-\eta|x|^2)$ where $\eta = 1/(4t) > 0$. Then, for $x \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}
&(4\pi t)^{n/2}\big(u(x,t) - mg(x,t)\big) \\
&= \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)\,dy - \exp\left(-\frac{|x|^2}{4t}\right)\int_{\mathbb{R}^n} f(y)\,dy \\
&= \int_{\mathbb{R}^n} h_\eta(x-y) f(y)\,dy - h_\eta(x)\int_{\mathbb{R}^n} f(y)\,dy \\
&= \int_{\mathbb{R}^n} h_\eta(x-y) f(y)\,dy - \int_{\mathbb{R}^n} h_\eta(x) f(y)\,dy \\
&= \int_{\mathbb{R}^n} \big(h_\eta(x-y) f(y) - h_\eta(x) f(y)\big)dy \\
&= \int_{\mathbb{R}^n} \big(h_\eta(x-y) - h_\eta(x)\big) f(y)\,dy.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big|(4\pi t)^{n/2}\big(u(x,t) - mg(x,t)\big)\big| = \left|\int_{\mathbb{R}^n} \big(h_\eta(x-y) - h_\eta(x)\big) f(y)\,dy\right| \\
&\le \int_{\mathbb{R}^n} \big|\big(h_\eta(x-y) - h_\eta(x)\big) f(y)\big|\,dy \le \int_{\mathbb{R}^n} |h_\eta(x-y) - h_\eta(x)||f(y)|\,dy.
\end{aligned}$$

By the integral form of the mean value theorem (Lemma 2.3.2), we have that

$$|h_\eta(x-y) - h_\eta(x)| \le |y| \int_0^1 |(\nabla h_\eta)\big(x - (1-\tau)y\big)|\,d\tau, \qquad x, y \in \mathbb{R}^n.$$

Now, setting $z = \eta^{1/2}|x| > 0$, we get

$$\begin{aligned}
&|\nabla h_\eta(x)| = |-2\eta x h_\eta(x)| \le |-2\eta x||h_\eta(x)| = |-2\eta||x||h_\eta(x)| \\
&= 2\eta^{1/2}\left(\eta^{1/2}|x|\right)\exp(-|z|^2) = 2\eta^{1/2} z \exp(-|z|^2) \le 2\eta^{1/2}/\sqrt{2e} =: 2\eta^{1/2} C_1.
\end{aligned}$$

Thus

$$|h_\eta(x-y) - h_\eta(x)| \le |y| \int_0^1 2\eta^{1/2} C_1\,d\tau \le |y| 2\eta^{1/2} C_1 = 2|y|\eta^{1/2} C_1.$$

Therefore

$$(4\pi t)^{n/2}|u(x,t) - mg(x,t)| \le \int_{\mathbb{R}^n} \left(2|y|\eta^{1/2} C_1\right)|f(y)|\,dy = \frac{C_1}{t^{1/2}} \int_{\mathbb{R}^n} |y||f(y)|\,dy,$$

which yields

$$t^{n/2}|u(x,t) - mg(x,t)| \leq \frac{C_1}{(4\pi)^{n/2}t^{1/2}} \int_{\mathbb{R}^n} |y||f(y)|\, dy.$$

Taking the supremum of the both sides, we get

$$t^{n/2}||u - mg||_\infty(t) = \frac{C_1}{(4\pi)^{n/2}t^{1/2}} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |y||f(y)|\, dy,$$

and since $f \in C_0(\mathbb{R}^n)$,

$$\lim_{t\to\infty} t^{n/2}||u - mg||_\infty(t) = 0.$$

■

Chapter 3

Structures of Equations and Self-Similar Solutions

3.1 Invariance Under Scaling

First, we refer to a transformation that reflects structures of partial differential equations; through this transformation, we consider special solutions preserving the structure of the heat equation.

Definition 3.1.1 (Rescaled Function).
For nonzero real number λ, the function u^λ is defined by

$$u^\lambda(x,t) = u(\lambda x, \lambda^2 t).$$

Proposition 3.1.1 *Assume that $u(x,t)$ satisfies the heat equation in an open set $Q \subset \mathbb{R}^n \times \mathbb{R}^n$, i.e.*

$$\partial_t u(x,t) - \Delta u(x,t) = 0, \qquad (x,t) \in Q.$$

Then, the following properties hold:

(1) For a real number μ, the function μu satisfies the heat equation in Q.

(2) The function u^λ satisfies the heat equation in

$$Q_\lambda = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}; (\lambda x, \lambda^2 t) \in Q\}.$$

Proof. The former part follows from linearity of the heat equation while the latter part follows from direct calculation: In fact, we have

$$\begin{aligned}
\partial_t u^\lambda(x,t) &= \partial_t\big(u(\lambda x, \lambda^2 t)\big) = \lambda^2(\partial_t u)(\lambda x, \lambda^2 t),\\
\partial_{x_j} u^\lambda(x,t) &= \partial_{x_j}\big(u(\lambda x, \lambda^2 t)\big) = \lambda(\partial_{x_j} u)(\lambda x, \lambda^2 t),\ (1 \le j \le n),\\
\Delta u^\lambda(x,t) &= (\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)\big(u(\lambda x, \lambda^2 t)\big) = \lambda^2(\Delta u)(\lambda x, \lambda^2 t).
\end{aligned}$$

Therefore, $\partial_t u^\lambda(x,t) - \Delta u^\lambda(x,t) = 0$. ■

Proposition 3.1.2 *Let $u(x,t)$ be the solution (2.2) of the heat equation (2.1) with initial data $f \in C_0(\mathbb{R}^n)$. Then, for any $t > 0$,*

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$

Proof. Since $\int_{\mathbb{R}^n} G_t(x)\,dx = 1$, by Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} (G_t * f)(x)\,dx = \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} G_t(x-y)f(y)\,dy\right\}dx\\
&= \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} G_t(x-y)f(y)\,dx\right\}dy = \int_{\mathbb{R}^n}\left\{f(y)\int_{\mathbb{R}^n} G_t(x-y)\,dx\right\}dy\\
&= \int_{\mathbb{R}^n} f(y)\,dy.
\end{aligned}$$

■

Proposition 3.1.3 *Let $u(x,t)$ be a continuous function on $\mathbb{R}^n \times (0,\infty)$ satisfying $0 \neq \int_{\mathbb{R}^n} u(x,t)\,dx < \infty$ and $\partial_t \int_{\mathbb{R}^n} u(x,t)\,dx = 0$ for $t > 0$. Suppose that $\mu = \mu(\lambda)$ is a positive function of $\lambda > 0$. Then the following are equivalent:*

(1) $\partial_\lambda \int_{\mathbb{R}^n} \mu u^\lambda(x,t)\,dx = 0$ for $\lambda > 0$.

(2) μ is a positive constant multiple of λ^n.

Proof. For $t > 0$, setting $z = \lambda x$,

$$\int_{\mathbb{R}^n} \mu u^\lambda(x,t)\,dx = \int_{\mathbb{R}^n} \mu u(\lambda x, \lambda^2 t)\,dx = \int_{\mathbb{R}^n} \mu u(z, \lambda^2 t)\frac{1}{\lambda^n}\,dz = \frac{\mu}{\lambda^n}\int_{\mathbb{R}^n} u(z, \lambda^2 t)\,dz.$$

By assumption, $\int_{\mathbb{R}^n} u(x,t)\,dz$ is independent of t, hence $\int_{\mathbb{R}^n} u(z,\lambda^2 t)\,dz$ is independent of $\lambda^2 t$. Therefore, $\partial_\lambda \int_{\mathbb{R}^n} \mu u^\lambda(x,t)\,dx = 0$ if and only if μ/λ^n is a positive constant. ■

Definition 3.1.2 (Scaling Transformation)

For a real-valued function $u(x,t)$ defined on $\mathbb{R}^n \times (0,\infty)$, we set

$$u_k(x,t) = k^n u(kx, k^2 t), \qquad k > 0. \tag{3.1}$$

A mapping $u \to u^\lambda$ (Definition 3.1.1) or $u \to u_k$ is called a scaling transformation.

Remark. The scaling transformation preserves its total heat (Proposition 3.1.2) and the solvability property of the heat equation (Proposition 3.1.3).

3.2 Self-Similar Solutions

Definition 3.2.1 (Self-Similar Solutions).

A solution u of the heat equation defined on $\mathbb{R}^n \times (0,\infty)$ is said to be a (forward) self-similar solution if $u = u_k$ on $\mathbb{R}^n \times (0,\infty)$ for all $k > 0$.

Lemma 3.2.1 *The Gauss kernel $G_t = g(x,t)$ is a self-similar solution of the heat equation.*

Proof. For $k > 0$, $x \in \mathbb{R}^n$, and $t > 0$,

$$\begin{aligned} g_k(x,t) &= k^k g(kx, k^2 x) \\ &= k^n \frac{1}{(4\pi k^2 t)^{n/2}} \exp\left(-\frac{|kx|^2}{4k^2 t}\right) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) = g(x,t). \end{aligned}$$

■

Proposition 3.2.2 *Let $u(x,t)$ be as in (2.2) with initial data $f \in C_0(\mathbb{R}^n)$, and $m = \int_{\mathbb{R}^n} f(y)\,dy$. Then the asymptotic formula (2.10) is equivalent to*

$$\lim_{k\to\infty} ||u_k - mg||_\infty(1) = 0. \tag{3.2}$$

Proof. It suffices to prove equivalence for the case $k^2 = t$. By Lemma 3.2.1, we know that $g = g_k$ for any $k > 0$. Letting $v = u - mg$, we get $v_k = u_k - mg$.

Furthermore, setting $z = kx$,

$$||v_k||_\infty(1) = \sup_{x\in\mathbb{R}^n} |v_k(x,1)| = \sup_{x\in\mathbb{R}^n} |k^n v(kx,k^2)|$$
$$= \sup_{x\in\mathbb{R}^n} k^n |v(kx,k^2)| = k^n \sup_{z\in\mathbb{R}^n} |v(z,k^2=t)| = t^{n/2}||v||_\infty(t).$$

■

Chapter 4

Proof of the Asymptotic Formula Based on the Structure

In this chapter, we will prove the asymptotic formula by, first, showing that any subsequence of $\{u_k\}$ has a convergent subsequence, and then, showing that all the limit functions of those subsequences are mg independent of the choice.

4.1 Ascoli-Arzelà-Type Compactness Theorem

To show that any subsequence of $\{u_k\}$ has a convergent subsequence, we introduce an Ascoli-Arzelà-type compactness theorem. To that end, we prove the Ascoli-Arzelà theorem and then modify it to obtain the compactness theorem that we need.

Definition 4.1.1 (Exhausting Sequence).
Assume that a sequence $\{M_j\}_{j=1}^{\infty}$ of compact subsets of a metric space M satisfies the following conditions:

(1) $M_j \subset M_{j+1}, \quad j \in \mathbb{N}$;

(2) $\bigcup_{j=1}^{\infty} M_j = M$; and

(3) for any compact subset $M_0 \subset M$, there exists $j_0 \in \mathbb{N}$ such that $M_0 \subset M_{j_0}$.

Then, $\{M_j\}_{j=1}^{\infty}$ is called an exhausting sequence of compact sets of M.

Definition 4.1.2 Let M be a metric space that has an exhausting sequence of compact sets. The set $C_\infty(M)$ is defined by

$$C_\infty(M) = \left\{ h \in C(M);\ \lim_{j\to\infty} \sup_{z\in M\setminus M_j} |h(z)| = 0 \right\}.$$

Definition 4.1.3 For $h \in C_\infty(M)$, we define the norm by

$$||h||_{\infty,M} = \sup_{z\in M} |h(z)|.$$

Remark. With this norm, $C_\infty(M)$ becomes a Banach space; hence $C_\infty(M)$ becomes a complete metric space with the metric $d(h_1, h_2) = ||h_1 - h_2||_{\infty,M}$ for $h_1, h_2 \in C_\infty(M)$.

Lemma 4.1.1 *Let M be a metric space that has an exhausting sequence of compact sets. Then, the set $C_\infty(M)$ is independent of the choice of a subsequence.*

Proof. Let $\{M_j\}_{j=1}^\infty$ and $\{N_j\}_{j=1}^\infty$ be two exhausting sequences of M. Setting $a_j = \sup_{x\in M}\{|f(x)|; x \in M\setminus M_j\}$ and $b_j = \sup_{x\in M}\{|f(x)|; x \in M\setminus N_j\}$ for $f \in C(M)$, we want to prove that

$$\lim_{j\to\infty} a_j = 0 \Leftrightarrow \lim_{j\to\infty} b_j = 0.$$

By definition of an exhausting sequence of compact sets, for each M_j, there exists $i = i(j) \in \mathbb{N}$ such that $M_j \subset N_{i(j)}$; hence $b_{i(j)} \leq a_j$. Since $i(j) \to \infty$ as $j \to \infty$ and b_j is non-increasing, we get $\lim_{j\to\infty} a_j = 0 \Rightarrow \lim_{j\to\infty} b_{i(j)} = 0 \Rightarrow \lim_{j\to\infty} b_j = 0$. Interchanging the roles of M_j and N_j, the converse also holds. ■

Remark. In particular, $C_\infty(M) = C(M)$ if M is compact.

Definition 4.1.4 A subset K of $C_\infty(M)$ is said to be bounded if

$$\sup_{h\in K} ||h||_{\infty,M} < \infty.$$

Remark. If K is a set of functions defined on M, boundedness of K is called uniform boundedness.

Definition 4.1.5 A subset K of $C_\infty(M)$ is said to be equicontinuous if

$$\lim_{y\to z}\sup_{h\in K}|h(z)-h(y)|=0, \qquad \text{for all } z\in M.$$

Proposition 4.1.2 *(The Ascoli-Arzelà Theorem).*

Let M be a compact set and let $K\subset C(M)$ ($=C_\infty(M)$ by Lemma 4.1.1). Then the following are equivalent:

(1) K is bounded and equicontinuous in $C(M)$.

(2) K is relatively compact in $C(M)$.

Proof. If M is empty, so is $K\subset C(M)$; hence we may assume that M is nonempty.

[(1) ⇒ (2)]. (Step 1) Let $A=\{x_l\}_{l=1}^\infty$ be an at most countable subset of M (If A is finite, set $x_l:=x_m$ for $l\geq m$.), and let $\{f_j\}_{j=1}^\infty\subset K$. Since K is bounded, i.e. $\sup_{h\in K}||h||_{\infty,M}=\sup_{h\in K}\sup_{z\in M}|h(z)|<\infty$, we know that $\{f_j(x_1)\}_{j=1}^\infty$ is bounded in $\mathbb{R}$. Hence, there exists a subsequence $\{f_j^1\}_{j=1}^\infty\subset\{f_j\}_{j=1}^\infty$ such that $f_j^1(x_1)$ converges to a limit $\tilde{f}(x_1)$. Since $\{f_j^1\}_{j=1}^\infty$ is convergent, $\{f_j^1\}_{j=1}^\infty$ is bounded. Again, for $k=2,3,\dots$, there is a subsequence $\{f_j^k\}\subset\{f_j^{k-1}\}$ such that $f_j^k(x_k)$ converges to a limit $\tilde{f}(x_k)$. Setting $g_j=f_j^j$, we get

$$\lim_{j\to\infty}f_j^j(x_k)=\lim_{j\to\infty}f_j^k(x_k)=\tilde{f}(x_k),$$

for $k=1,2,\dots$, since $\{f_j^j(x_k)\}_{j=1}^\infty\subset\{f_j^k(x_k)\}_{j=1}^\infty$ for $j>k$. Thus, at each point $x\in A=\{x_l\}_{l=1}^\infty$, there exists a convergent subsequence $\{g_j\}_{j=1}^\infty\subset\{f_j\}_{j=1}^\infty$.

(Step 2) By assumption, K is equicontinuous in $C(M)$, i.e. $\lim_{y\to x}\sup_{f\in K}|f(x)-f(y)|=0$ for all $x\in M$. Hence, for all $x\in M$, there is an open neighborhood V_{x^k} of x such that

$$\sup_{y\in V_{x^k}}\sup_{f\in K}|f(x)-f(y)|\leq\frac{1}{k}.$$

The set $\{V_{x^k}\}_{x\in M}$ is an open covering of M, and since M is compact, M is already covered by a finite subcovering $\{V_{x_i^k}\}_{i=1}^{N(k)}$ where $x_i^k\in M$ with $1\leq i\leq N(k)$. Thus, for every $k \in \mathbb{N}$, there are $N(k)$ points $x_i^k \in M$ and open neighborhoods $V_{x_i^k}$ of x_i^k satisfying

$$\sup_{f\in K}\sup_{y\in V_{x_i^k}} |f(x_i^k) - f(y)| \leq \frac{1}{k}$$

and

$$\bigcup_{i=1}^{N(k)} V_{x_i^k} = M.$$

(Step 3) Let $A^k = \{x_i^k \ ; \ 1 \leq i \leq N(k), \ k \in \mathbb{N}\}$ for $x_i^k \in M$. Assume that $\{h_j\}_{j=1}^\infty \subset K$ converges pointwise on A^k. Then, $\{h_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Hence

$$\lim_{l\to\infty}\sup_{j,m\geq l} |h_j(x) - h_m(x)| = 0. \tag{4.1}$$

For $z \in M$,

$$\begin{aligned}|h_j(z) - h_m(z)| &= |h_j(z) - h_j(x_i^k) + h_j(x_i^k) - h_m(x_i^k) + h_m(x_i^k) - h_m(z)| \\ &\leq |h_j(z) - h_j(x_j^k)| + |h_j(x_j^k) - h_m(x_i^k)| + |h_m(x_i^k) - h_m(z)| \\ &\leq 2\sup_{f\in K}|f(z) - f(x_i^k)| + |h_j(x_i^k) - h_m(x_i^k)|.\end{aligned}$$

Next, fixing k, we pick x_i^k and $V_{x_i^k} \ni z$ given in Step 2. Then

$$\begin{aligned}|h_j(z) - h_m(z)| &\leq \frac{2}{k} + |h_j(x_i^k) - h_m(x_i^k)| \\ &\leq \frac{2}{k} + \sup_{1\leq i\leq N(k)} |h_j(x_i^k) - h_m(x_i^k)|.\end{aligned}$$

Hence

$$\sup_{z\in M}|h_j(z) - h_m(z)| \leq \frac{2}{k} + \sup_{1\leq i\leq N(k)} |h_j(x_i^k) - h_m(x_i^k)|.$$

By (4.1), it follows that

$$\lim_{l\to\infty}\sup_{j,m\geq l}\sup_{z\in M}|h_j(z) - h_m(z)| \leq \frac{2}{k}.$$

That is

$$\overline{\lim_{j,m\to\infty}}\sup_{z\in M}|h_j(z) - h_m(z)| \leq \frac{2}{k}.$$

Here, $k \in \mathbb{N}$ was arbitrary, so we obtain

$$\overline{\lim_{j,m\to\infty}} \sup_{z\in M} |h_j(z) - h_m(z)| = \overline{\lim_{j,m\to\infty}} ||h_j - h_m||_{\infty,M} = 0.$$

Since $C(M)$, where M is compact, is complete with the L^∞-norm, $\{h_j\}_{j=1}^\infty$ converges uniformly to some $h \in M$.

(Step 4) For $\{f_j\}_{j=1}^\infty \subset K$, we have that $\{g_j\}_{j=1}^\infty \subset K$ converges uniformly to some $g \in C(M)$ by Step 3. Thus, $\{f_j\}_{j=1}^\infty$ contains a convergent subsequence $\{g_j\}_{j=1}^\infty$ in $C(M)$; hence K is relatively compact in $C(M)$.

[(2) $\Rightarrow$ (1)]. (Step 1) For $f \in C(M)$, we define an open ball $B(f,\epsilon)$ centered at f with radius ϵ by $B(f,\epsilon) := \{h \in C(M);\ ||f-h||_{\infty,M} < \epsilon\}$. Since K is relatively compact, we know that $\overline{K}$ has a finite subcovering of $\cup_{f\in K} B(f,\epsilon)$, i.e. $\overline{K} \subset \cup_{i=1}^{N(\epsilon)} B(f_i,\epsilon)$ for suitable $f_i \in C(M)$; $i \in \{1,\dots,N(\epsilon)\}$. Here, $B(f_i,\epsilon)$ are bounded, so is K.

(Step 2) Pick $\epsilon > 0$ and let $\{f_i\}_{i=1}^{N(\epsilon)}$ as constructed above. Since $f_i \in C(M)$, we have that, for $z \in M$, there exists a neighborhood V_z^i of z such that $|f_i(z) - f_i(y)| \le \epsilon$ for $y \in V_z^i$. Now, $V_z := \cap_{i=1}^{N(\epsilon)} V_z^i$ is also a neighborhood of z, so we obtain $|f_i(z) - f_i(y)| \le \epsilon$ for every $i \in \{1,\dots,N(\epsilon)\}$ and $y \in V_z$. By the choice of f_i, $||f - f_i||_{\infty,M} < \epsilon$ for any $f \in K$. Hence, for $y \in V_z$,

$$\begin{aligned}
|f(z) - f(y)| &= |f(z) - f_i(z) + f_i(z) - f_i(y) + f_i(y) - f(y)| \\
&\le |f(z) - f_i(z)| + |f_i(z) - f_i(y)| + |f_i(y) - f(y)| \\
&\le 2||f - f_i||_{\infty,M} + |f_i(z) - f_i(y)| \le 2\epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

That is, V_z is independent of $f \in K$, which means that, for any $\epsilon > 0$, there exists a neighborhood V_z satisfying $\sup_{f\in K} |f(z) - f(y)| \le 3\epsilon$ for all $y \in V_z$. Hence $\lim_{y\to z} \sup_{f\in K} |f(z) - f(y)| = 0$. ■

Definition 4.1.6 A subset K of $C_\infty(M)$ is said to have the equidecay property if

$$\lim_{j\to\infty} \sup_{h\in K} \sup_{z\in M\setminus M_j} |h(z)| = 0.$$

Remark. This notion is independent of the choice of an exhausting sequence of compact sets $\{M_j\}_{j=1}^{\infty}$ as is the space $C_\infty(M)$.

Lemma 4.1.3 *(Lower Semicontinuity of the Supremum).*
For $m \in \mathbb{N}$, let h_m be real-valued functions defined on a set X. Then the supremum is lower semicontinuos, i.e.

$$\sup_{x\in X} \underline{\lim}_{m\to\infty} h_m(x) \leq \underline{\lim}_{m\to\infty} \sup_{x\in X} h_m(x).$$

Proof. Since $h_m(x) \leq \sup_{x\in X} h_m(x)$, we have that, for each $x \in X$,

$$\underline{\lim}_{m\to\infty} h_m(x) \leq \underline{\lim}_{m\to\infty} \sup_{x\in X} h_m(x).$$

Hence

$$\sup_{x\in X} \underline{\lim}_{m\to\infty} h_m(x) \leq \underline{\lim}_{m\to\infty} \sup_{x\in X} h_m(x).$$

■

Proposition 4.1.4 *Let M be a topological space that has an exhausting sequence of compact sets $\{M_j\}_{j=1}^{\infty}$. Suppose that, on each M_j, the sequence $\{h_m\}_{m=1}^{\infty} \subset C_\infty(M)$ converges to a real-valued function h defined on M. Then, $h \in C_\infty(M)$. Moreover, if $H := \{h_m\}_{m=1}^{\infty}$ has the equidecay property, then $\{h_m\}_{m=1}^{\infty}$ converges uniformly to h on M.*

Proof. Since $M = \cup_{j=1}^{\infty} M_j$ and h_m converges uniformly to h on M_j, we have $h \in C(M)$. Now we want to show that

$$\lim_{j\to\infty} \sup_{x\in M\setminus M_j} |h(x)| = 0.$$

By assumption, we know that h_m converges pointwise to h on M. Using the lower semicontinuity of the supremum (Lemma 4.1.3),

$$\sup_{x\in M\setminus M_j} |h(x)| = \sup_{x\in M\setminus M_j} |\lim_{m\to\infty} h_m(x)| \leq \underline{\lim}_{m\to\infty} \sup_{x\in M\setminus M_j} |h_m(x)| \leq \sup_{\tilde{h}\in H} \sup_{x\in M\setminus M_j} |\tilde{h}(x)|.$$

By the equidecay property of H,

$$\lim_{j\to\infty} \sup_{\tilde{h}\in H} \sup_{x\in M\setminus M_j} |\tilde{h}(x)| = 0,$$

which shows that $h \in C_\infty(M)$.

Moreover,

$$\begin{aligned}
\sup_{x\in M} |h_m(x) - h(x)| &\le \sup_{x\in M_j} |h_m(x) - h(x)| + \sup_{x\in M\setminus M_j} |h_m(x) - h(x)| \\
&\le \sup_{x\in M_j} |h_m(x) - h(x)| + \sup_{x\in M\setminus M_j} (|h_m(x)| + |h(x)|) \\
&\le \sup_{x\in M_j} |h_m(x) - h(x)| + 2\sup_{\tilde{h}\in H} \sup_{x\in M\setminus M_j} |\tilde{h}(x)|.
\end{aligned}$$

Since each h_m converges uniformly to h on M_j and H has the equidecay property,

$$\overline{\lim_{m\to\infty}} ||h_m - h||_{\infty,M} \le 2\sup_{\tilde{h}\in H} \sup_{x\in M\setminus M_j} |\tilde{h}(x)| \to 0, \quad (j\to\infty).$$

Hence, by the squeeze theorem,

$$\lim_{m\to\infty} ||h_m - h||_{\infty,M} = 0.$$

■

Theorem 4.1.5 *(Ascoli-Arzelà-Type Compactness Theorem).*
Let M be a topological space that has an exhausting sequence of compact sets $\{M_j\}_{j=1}^\infty$, and let $K \subset C_\infty(M)$. Then the following are equivalent:

(1) K is bounded, equicontinuous, and having the equidecay property.

(2) K is relatively compact in $C_\infty(M)$.

Proof. [(1) ⇒ (2)]. (Step 1) Let $\{f_m\}_{m=1}^\infty \subset K$. Since M_1 is compact, by the Ascoli-Arzelà theorem (Proposition 4.1.2), we may choose a subsequence $\{f_m^1\}_{m=1}^\infty \subset \{f_m\}_{m=1}^\infty$ that converges uniformly to a function f^1 on M_1. Similarly, if $\{f_m^{k-1}\}_{m=1}^\infty$ converges uniformly to f^{k-1} on M_{k-1}, then we may choose a subsequence $\{f_m^k\}_{m=1}^\infty \subset \{f_m^{k-1}\}_{m=1}^\infty$ that converges uniformly to f^k on M_k. Since $f^j = f^k$ on M_k if $j \ge k$, there exists a function f such that $f = f^k$ on M_k. Setting $g_m = f_m^m$, $\{g_m\}_{m=1}^{\infty}$ converges uniformly to f on each M_k.

(Step 2) By Proposition 4.1.4, $\{g_m\}_{m=1}^{\infty}$ converges uniformly to f on $\cup_{k=1}^{\infty} M_k = M$.

[(2) ⇒ (1)]. (Step 1) Analogously to Step 2 of the latter part of the proof of the Ascoli-Arzelà theorem (Proposition 4.1.2), there exists a finite set $\{f_i\}_{i=1}^{N(\epsilon)}$ such that $\overline{K} \subset \cup_{i=1}^{N(\epsilon)} B(f_i, \epsilon)$. Therefore, boundedness and equicontinuity of K follows exactly the same way as in the previous proof.

(Step 2) For $f \in K$, choose f_i so that $f \in B(f_i, \epsilon)$. Here, observe that we may assume that

$$\sup_{1 \leq i \leq N(\epsilon)} \sup_{x \in M \setminus M_i} |f_i(x)| \leq \epsilon$$

by taking sufficiently large j. It then follows that

$$\begin{aligned} \sup_{M \setminus M_i} |f(x)| &\leq \sup_{M \setminus M_i} |f(x) - f_i(x)| + \sup_{M \setminus M_i} |f_i(x)| \\ &\leq ||f - f_i||_{\infty, M} + \epsilon \leq 2\epsilon. \end{aligned}$$

Thus, for each $\epsilon > 0$, we have that $\sup_{f \in K} \sup_{x \in M \setminus M_j} |f(x)| \leq 2\epsilon$ for suitable M_j. Here, $\{M_j\}_{j=1}^{\infty}$ is an increasing sequence, so we arrive at

$$\lim_{j \to \infty} \sup_{f \in K} \sup_{x \in M \setminus M_j} |f(x)| = 0,$$

hence K has the equidecay property. ■

4.2 Relative Compactness of the Family of Rescaled Functions

In this section, we shall prove relative compactness of the family of rescaled functions $\{u_k\}$. To that end, it suffices to consider behavior for large k, so we set $k \geq 1$. Here, as a shortcut, we use estimates obtained from the representation formula; however, these estimates could be obtained without the formula.

Lemma 4.2.1 *Let $u(x,t)$ be the solution (2.2) of the heat equation (2.1) with initial data $f \in C_0(\mathbb{R}^n)$. Assume that there is an open ball B_{j_0} centered at the origin with radius $j_0 > 0$ such that* $\operatorname{supp} f \subset B_{j_0}$. *Then, for $\eta \in (0,1)$,*

$$|u(x,t)| \leq \frac{||f||_1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}|x|^2 + \eta\frac{j_0}{2}|x|\right), \quad x \in \mathbb{R}^n, \ \eta \leq t \leq 1/\eta. \tag{4.2}$$

Proof. Since

$$u(x,t) = \int_{\mathbb{R}^n} g(x-y,t)f(y)\,dy = \int_{B_{j_0}} g(x-y,t)f(y)\,dy = \int_{|y|\leq j_0} g(x-y,t)f(y)\,dy,$$

we have

$$\begin{aligned}
|u(x,t)| &= \left|\int_{|y|\leq j_0} g(x-y,t)f(y)\,dy\right| \leq \int_{|y|\leq j_0} |g(x-y,t)f(y)|\,dy \\
&= \int_{|y|\leq j_0} g(x-y,t)|f(y)|\,dy \leq \sup_{|y|\leq j_0} g(x-y,t) \int_{|y|\leq j_0} |f(y)|\,dy \\
&\leq \sup_{|y|\leq j_0} g(x-y,t) \int_{\mathbb{R}^n} |f(y)|\,dy \leq \sup_{|y|\leq j_0} g(x-y,t)||f||_1.
\end{aligned}$$

Furthermore, $|x-y|^2 \geq |x|^2 - 2|x||y| + |y|^2 \geq |x|^2 - 2|x|j_0$ for $|y| \leq j_0$, so, if $\eta \leq t \leq 1/\eta$, we obtain

$$\begin{aligned}
&\sup_{|y|\leq j_0} g(x-y,t) = \sup_{|y|\leq j_0} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \\
&\leq \sup_{|y|\leq j_0} \frac{1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{|x|^2 - 2|x|j_0}{4/\eta}\right) = \frac{1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}(|x|^2 - 2|x|j_0)\right).
\end{aligned}$$

Hence

$$|u(x,t)| \leq \frac{||f||_1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}(|x|^2 - 2|x|j_0)\right) = \frac{||f||_1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}|x|^2 + \eta\frac{j_0}{2}|x|\right).$$

■

Proposition 4.2.2 *Let $M = \mathbb{R}^n \times [\eta, 1/\eta]$ for $\eta \in (0,1)$, let $u(x,t)$ be the solution (2.2) of the heat equation (2.1) with initial data $f \in C_0(\mathbb{R}^n)$, and let u_k be as in (3.1). Then, $K = \{u_k \ ; \ k \geq 1\}$ is relatively compact in $C_\infty(M)$.*

Proof. By the Ascoli-Arzelà-type compactness theorem (Theorem 4.1.5), it suffices to show that K is bounded, equicontinuous, and having the equidecay property. Note that $u_k \in C(M)$; hence $K \subset C(M)$. We show that $K \subset C_\infty(M)$ in Step 3.

(Step 1) By the decay estimate (Proposition 2.3),

$$\begin{aligned}
&||u_k||_\infty(t) = \sup_{x\in\mathbb{R}^n} |k^n u(kx, k^2 t)| = k^n \sup_{z\in\mathbb{R}^n} |u(z, k^2 t)|, \quad \text{where } z := kx \\
&\leq k^n \frac{1}{[4\pi(k^2 t)]^{n/2}} ||f||_1 = \frac{1}{(4\pi t)^{n/2}} ||f||_1.
\end{aligned}$$

If $\eta \leq t$, we get

$$||u_k||_{\infty,M} = \sup_{(x,t)\in M} |u_k(x,t)| \leq \frac{1}{(4\pi\eta)^{n/2}} ||f||_1.$$

Here, the right hand side is independent of $k > 0$, so K is bounded.

(Step 2) By derivative L^p-L^q estimates (2.8) and (2.9), we have

$$\begin{aligned}
||\partial_{x_j} u||_\infty(t) &\leq \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} ||f||_1, \quad j = 1, \dots, n, \\
||\partial_t u||_\infty(t) &\leq \frac{C}{t^{\frac{n}{2}+1}} ||f||_1.
\end{aligned}$$

Similarly to Step 1, we arrive at

$$\begin{aligned}
||\partial_{x_j} u_k||_\infty(t) &\leq \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} ||f||_1 \\
||\partial_t u_k||_\infty(t) &\leq \frac{C}{t^{\frac{n}{2}+1}} ||f||_1.
\end{aligned}$$

Hence, we can take a constant L, independent of k, such that

$$\begin{aligned}
||\partial_{x_j} u_k||_{\infty,M} &= \sup_{(x,t)\in M} |\partial_{x_j} u_k(x,t)| \leq L \\
||\partial_t u_k||_{\infty,M} &= \sup_{(x,t)\in M} |\partial_t u_k(x,t)| \leq L.
\end{aligned}$$

By the integral form of the mean value theorem (2.3.2), for $(y,s),(x,t)\in M$,

$$
\begin{aligned}
&|u_k(y,s)-u_k(x,t)| \\
&\le \sqrt{|x-y|^2+|t-s|^2}\int_0^1 |(\nabla u_k)\big((x,t)+(1-\tau)(y-x,s-t)\big)|\,d\tau \\
&= \sqrt{|x-y|^2+|t-s|^2}\int_0^1 \underbrace{|(\partial_{x_1}u_k,\ldots,\partial_{x_n}u_k,\partial_t u_k)\big(y+(x-y)\tau,s+(t-s)\tau\big)|}_{\cdots(*)}\,d\tau \\
&\le \sqrt{|x-y|^2+|t-s|^2}\int_0^1 \left(\sum_{i=1}^{n+1}L^2\right)^{1/2} d\tau = \sqrt{|x-y|^2+|t-s|^2}\int_0^1 L(n+1)^{1/2}d\tau \\
&= L(n+1)^{1/2}(|x-y|^2+|t-s|^2)^{1/2}.
\end{aligned}
$$

Note that the second inequality results from the fact that each element of the norm $(*)$ is less than or equal to L as estimated above. It then follows that, for all $(x,t)\in M$,

$$\lim_{(y,s)\to(x,t)}\sup_{u_k\in K}|u_k(y,s)-u_k(x,t)|=0,$$

i.e. K is equicontinuous.

(Step 3) Since $f\in C_0(\mathbb{R}^n)$, we can take an open ball B_{j_0} centered at the origin such that $\mathrm{supp} f\subset B_{j_0}$. Moreover, a scaled function u_k is also a solution of the heat equation with initial data $f_k(x)=k^n f(kx)$, $x\in\mathbb{R}^n$, as well as having $||f_k||_1=||f||_1$, we may assume that $\mathrm{supp} f_k\subset B_{j_0}$ for $k\ge 1$. By Lemma 4.2.1, for $j\ge j_0$ and $k\ge 1$,

$$\sup_{|x|\ge j}|u_k(x,t)|\le \frac{||f_k||_1}{(4\pi\eta)^{n/2}}\sup_{|x|\ge j}\exp\left(-\frac{\eta}{4}|x|^2+\eta\frac{j_0}{2}|x|\right),\quad \eta\le t\le 1/\eta.$$

Again, since $||f_k||_1=||f||_1$, and that $|x|^2-2j_0|x|\ge |x|^2/3$ where $|x|\ge j\ge 3j_0$, we obtain

$$\sup_{|x|\ge j}|u_k(x,t)|\le \frac{||f||_1}{(4\pi\eta)^{n/2}}\sup_{|x|\ge j}\exp\left(-\frac{\eta}{4}\frac{|x|^2}{3}\right)\to 0,\qquad (j\to\infty).$$

Hence

$$\lim_{j\to\infty}\sup_{k\ge 1}\sup_{|x|\ge j}\sup_{\eta\le t\le 1/\eta}|u_k(x,t)|=0,$$

i.e. K has the equidecay property.

Furthermore, $\sup_{|x|\ge j}\sup_{\eta\le t\le 1/\eta}|u_k(x,t)|$ is bounded by a sequence of positive numbers that is independent of k and that converges to 0 as $j\to\infty$. In particular, we single out $u_k\in C_\infty(M)$. ■

4.3 Characterization of Limit Functions

Next, we will examine the limit of $f_k(x) = k^n f(kx)$ as $k \to \infty$.

Proposition 4.3.1 *For $f \in C_0(\mathbb{R}^n$, we set $f_k(x) = k^n f(kx)$, $k \geq 1$. For any $\psi \in C(\mathbb{R}^n)$, we have*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)\psi(x)\,dx = m\psi(0), \quad \textit{where} \quad m = \int_{\mathbb{R}^n} f(x)\,dx.$$

Proof. Let $y = kx$, i.e. $dy = k^n dx$, then

$$\int_{\mathbb{R}^n} f_k(x)\psi(x)\,dx = \int_{\mathbb{R}^n} k^n f(kx)\psi(x)\,dx = \int_{\mathbb{R}^n} f(y)\psi(y/k)\,dy, \quad (k \geq 1).$$

Since

$$|\psi(y/k)| \leq \sup\{|\psi(y)|;\ y \in \mathrm{supp} f\} =: c_0 < 0,$$

we have

$$|f(y)\psi(y/k)| \leq c_0|f(y)|$$

as $y \in \mathrm{supp} f$. Thus, the right hand side is an integrable function independent of k. By the dominated convergence theorem, we arrive at

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)\psi(x)\,dx = \int_{\mathbb{R}^n} f(y) \lim_{k\to\infty} \psi(y/k)\,dy = \psi(0)\int_{\mathbb{R}^n} f(y)\,dy = m\psi(0).$$

■

Remark. Initial data f_k converges to m multiple of the Dirac δ distribution; hence the limit of initial data is not a function. Nonetheless, we can still formulate the problem defining weak solutions of the heat equation.

Definition 4.3.1 (Weak Solutions of the Heat Equation).
Let u be locally integrable in $\mathbb{R}^n \times [0, \infty)$, i.e.

$$\int_0^T \int_{|x|\leq R} |u(x,t)|\,dxdt < \infty, \quad \text{for all} \quad R, T > 0.$$

(1) For an integrable function f on $\mathbb{R}^n$, u is said to be a weak solution of the heat equation with initial data f if

$$0 = \int_{\mathbb{R}^n} \phi(x,0) f(x)\, dx + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \phi + \Delta \phi) u \, dxdt, \tag{4.3}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$.

(2) For an integrable function f on $\mathbb{R}^n$, u is said to be a weak solution of the heat equation with initial data $m\delta$ if

$$0 = m\phi(0,0) + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \phi + \Delta \phi) u \, dxdt, \tag{4.4}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$.

Before we move on to characterize the limit of the family of rescaled functions, we establish a test that gives conditions for weak solutions of the first type (1) to be indeed of the second type (2).

Theorem 4.3.2 *Let $v_i \in C(\mathbb{R}^n \times [0,\infty))$ be a weak solution of the heat equation with initial data $v_{i0} \in C(\mathbb{R}^n)$, $i \in \mathbb{N}$, and let $m \in \mathbb{R}$. Then, $v \in C(\mathbb{R}^n \times (0,\infty))$ is a weak solution of the heat equation with initial data $m\delta$ if the following hold:*

(1) (Limit of initial data) For any $\psi \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} v_{i0} \psi(x)\, dx = m\psi(0);$$

(2) (Uniform estimate)

$$\sup_{i\geq 1} \sup_{t>0} ||v_i||_1(t) < \infty; \text{ and}$$

(3) (Convergence) In any compact subset of $\mathbb{R}^n \times (0,\infty)$, the function v_i converges uniformly to v as $i \to \infty$.

Proof. Since v_i is a weak solution with initial data v_{i0}, we have that

$$0 = \int_{\mathbb{R}^n} \phi(x,0) v_{i0}(x)\, dx + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \phi + \Delta \phi) v_i \, dxdt,$$

for any $\phi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$. By (1), the first term converges to $m\phi(0,0)$ as $i \to \infty$, so we want to show that the second term converges to

$$\int_0^\infty \int_{\mathbb{R}^n} (\partial_t \phi + \Delta \phi) v \, dxdt,$$

as $i \to \infty$, and that v is locally integrable in $\mathbb{R}^n \times [0,\infty)$.

(Step 1) Set F_i, $(i \in \mathbb{N})$, and F respectively by

$$F_i(t) := \int_{\mathbb{R}^n} \big(\partial_t \phi(x,t) + \Delta \phi(x,t)\big) v_i(x,t) \, dx,$$
$$F(t) := \int_{\mathbb{R}^n} \big(\partial_t \phi(x,t) + \Delta \phi(x,t)\big) v(x,t) \, dx.$$

Then, we can take an interval $[0,T)$, independent of i, that contains supports of F_i and F. By (3), the function v_i converges uniformly to v in any compact subset of $\mathbb{R}^n \times (0,\infty)$, so we have that, for any $t > 0$,

$$\lim_{i \to \infty} F_i(t) = F(t).$$

Furthermore, we have that $F_i, F \in C(0,T)$, and

$$|F_i(t)| \leq \sup_{(x,t) \in \mathbb{R}^n \times (0,\infty)} |\partial_t \phi + \Delta \phi| \int_{\mathbb{R}^n} |v_i(x,t)| \, dx$$

whose right hand side is bounded in $t > 0$ and $i \in \mathbb{N}$ by (2). Hence, by the dominated convergence theorem,

$$\lim_{i \to \infty} \int_0^T F_i(t) \, dt = \int_0^T F(t) \, dt.$$

Since F_i and F are compactly supported, it then follows that

$$\lim_{i \to \infty} \int_0^\infty F_i(t) \, dt = \int_0^\infty F(t) \, dt.$$

(Step 2) Similarly, we obtain

$$\lim_{i \to \infty} \int_0^T \int_{B_R} |v_i(x,t)| \, dxdt = \int_0^T \int_{B_R} |v(x,t)| \, dxdt < \infty,$$

i.e. v is locally integrable in $\mathbb{R}^n \times [0,\infty)$. ∎

Proposition 4.3.3 *Let $u(x,t)$ be the solution (2.2) of the heat equation (2.1) with initial data $f \in C_0(\mathbb{R}^n)$, and let u_k be as in (3.1). Assume that a subsequence $\{u_{k''}\} \subset \{u_k;\ k \geq 1\}$ converges uniformly to a continuous function U in any compact subset of $\mathbb{R}^n \times (0,\infty)$ as $k \to \infty$. Then, U is a weak solution of the heat equation with initial data $m\delta$, where $m = \int_{\mathbb{R}^n} f\,dx$. Moreover,*

$$\sup_{t>0} ||U||_1(t) \leq ||f||_1. \tag{4.5}$$

Proof. We only need to check three conditions of Theorem 4.3.2.

(Step 1) Condition (1). follows from Proposition 4.3.1.

(Step 2) Condition (2). is satisfied as

$$||u_k||_1(t) \leq ||f_k||_1 = ||f||_1, \quad t > 0;$$

hence

$$\sup_{i\geq 1} \sup_{t>0} ||u_k||_1(t) < \infty.$$

(Step 3) Condition (3). is assumed in the proposition.

To obtain the estimate (4.5), by Fatou's Lemma,

$$||U||_1(t) = \int_{\mathbb{R}^n} \lim_{k''\to\infty} |u_{k''}(x,t)|\,dx \leq \varliminf_{k''\to\infty} ||u_{k''}||_1(t) \leq ||f||_1.$$

■

Furthermore, the following uniqueness theorem for the heat equation holds, provided that the solution u does not grow rapidly at space infinity; however, we do not carry this out in this article.

Theorem 4.3.4 *Assume that the function $v \in C(\mathbb{R}^n \times (0,\infty))$ satisfies*

$$\sup_{t>0} ||v||_1(t) < \infty,$$

and that v is a weak solution of the heat equation with initial data $m\delta$ where m is a real number. Then, v is unique and, for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, we have $v = mg$.

4.4 Completion of the Proof

Finally, we are ready to prove the asymptotic formula (3.2) which is equivalent to (2.10) shown in Proposition 2.3.1.

Proof. By Proposition 4.3.3 and Proposition 4.3.4, $\lim_{k\to\infty} u_{k''} = U$. Furthermore, similarly to Step 1 of Proposition 4.1.2, a diagonal argument yields that, for any subsequence $\{u_{k'}\} \subset \{u_k;\ k \geq 1\}$, we can take a subsequence $\{u_{k''}\} \subset \{u_{k'}\}$ that converges uniformly to a continuous function U in $\mathbb{R}^n \times [\eta, 1/\eta]$, i.e. the limit U is independent of the choice of a subsequence. ■

Chapter 5

The Gagliardo-Nirenberg Inequality

In this chapter, we cover calculus inequalities derived from estimates of solutions of the heat equation. Our primary goal is to prove the Gagliardo-Nirenberg inequality.

5.1 The Marcinkiewicz Interpolation Theorem

Definition 5.1.1 (Lorentz Space).

For a positive number q the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying

$$|f|_{q,\infty} := \sup_{\lambda>0} \lambda m_f(\lambda)^{1/q} < \infty$$

is denoted by $L^{q,\infty}(\mathbb{R}^n)$.

Remark. A Lorentz space $L^{q,\infty}(\mathbb{R}^n)$ is not a norm space with the norm $|\cdot|_{q,\infty}$; however, for $q > 1$, the norm $|\cdot|_{q,\infty}$ is equivalent to a norm. Moreover, the Lorentz space $L^{q,\infty}(\mathbb{R}^n)$ become a complete metric space with this norm.

Definition 5.1.2 (Distribution Function).

For a Lebesgue measurable function f on $\mathbb{R}^n$, its distribution function is defined by

$$m_f(\lambda) = |\{x \in \mathbb{R}^n;\ |f(x)| > \lambda\}|.$$

Here, $|A|$ denotes the Lebesgue measure of a subset $A \subset \mathbb{R}^n$.

Lemma 5.1.1 *For $p > 0$ and $|f|^p \in L^1(\mathbb{R}^n)$, we have*

(1) $m_f(\lambda) \le \lambda^{-p} \int_{\mathbb{R}^n} |f(x)|^p \, dx, \quad \lambda > 0,$ *and*

(2) $\int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} m_f(t) \, dt.$

For $p = 2$, the inequality (1) is called the Chebyshev inequality.

Proof. (1). For $\lambda > 0$, we set $F_\lambda = \{x \in \mathbb{R}^n;\ |f(x)| > \lambda\}$. Then

$$\lambda^p m_f(\lambda) = \lambda^p \int_{F_\lambda} dx = \int_{F_\lambda} \lambda^p \, dx \le \int_{F_\lambda} |f(x)|^p \, dx \le \int_{\mathbb{R}^n} |f(x)|^p \, dx.$$

(2). For a nonnegative function $h \in L^1(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n} h(x) \, dx = \int_{\mathbb{R}^n} \left(\int_0^{h(x)} 1 \, dy \right) dx.$$

By Fubini's theorem, letting $W = \{(x, s) \in \mathbb{R}^{n+1};\ 0 \le s < h(x)\}$,

$$\int_{\mathbb{R}^n} h(x) \, dx = |W|.$$

If we set $H_s = \{x \in \mathbb{R}^n;\ h(x) > s\}$, we have that $W = \{(x, s) \in \mathbb{R}^{n+1};\ x \in H_s, s \ge 0\}$, and

$$\int_{\mathbb{R}^n} h(x) \, dx = |W| = \int_0^\infty |H_s| \, ds.$$

Now, set $h = |f|^p$. Then, since

$$H_s = \{x \in \mathbb{R}^n;\ |f(x)|^p > s\} = \{x \in \mathbb{R}^n;\ |f(x)| > s^{1/p}\} = F_{s^{1/p}},$$

we obtain

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty m_f(s^{1/p}) \, ds.$$

Substituting $s = t^p$, i.e. $ds = pt^{p-1} \, dt$, we arrive at

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = p \int_0^\infty t^{p-1} m_f(t) \, dt.$$

■

Remark. It still remains valid if we replace any open set $\Omega \subset \mathbb{R}^n$ with $\mathbb{R}^n$.

Lemma 5.1.2 *For measurable functions g, g_1, g_2 satisfying $|g| \leq |g_1| + |g_2|$, we have that*

$$|g(x)| > t \quad \textit{implies} \quad |g_1(x)| > t/2 \textit{ or } |g_2(x)| > t/2, \tag{5.1}$$

for any $t > 0$ and $x \in \mathbb{R}^n$.

Proof. Suppose that (5.1) does not hold; then $|g_1(x)| + |g_2(x)| \leq t$ whereas $|g(x)| = |g_1(x)| + |g_2(x)| > t$. Contradiction. ■

Remark. By (5.1), we know that

$$\chi_{\{|g|>t\}}(x) \leq \chi_{\{|g_1|>t/2\}}(x) + \chi_{\{|g_2|>t/2\}}(x),$$

for $x \in \mathbb{R}^n$ and $t > 0$. Here, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^n$. Thus, we obtain

$$m_g(t) \leq m_{g_1}(t/2) + m_{g_2}(t/2), \qquad t > 0. \tag{5.2}$$

Theorem 5.1.3 *(The Marcinkiewicz Interpolation Theorem).*
Let $1 \leq p_i \leq q_i < \infty$ for $i = 1, 2$ where $q_1 \neq q_2$. Assume for p and q that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \tag{5.3}$$

for some $\theta \in (0, 1)$. Furthermore, suppose that T is a linear operator from $L^{p_i}(\mathbb{R}^n)$ to $L^{q_i,\infty}(\mathbb{R}^d)$ and that there exists a constant $C > 0$ satisfying

$$|Tf|_{q_i,\infty} \leq M_i||f||_p, \qquad f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n), \tag{5.4}$$

for $i = 1, 2$. Then, T extends to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. In particular, there exists a constant $C > 0$ satisfying

$$||Tf||_q \leq CM_1^\theta M_2^{1-\theta}||f||_p, \qquad f \in L^p(\mathbb{R}^n). \tag{5.5}$$

Here, C depends only on p_i, q_i, $i = 1, 2$, and p, q.

Proof. Without loss of generality, we may assume that $p_1 < p < p_2$. For $s > 0$ and $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$, we set

$$f^s(x) = \begin{cases} f(x), & |f(x)| > s, \\ 0, & |f(x)| \le s, \end{cases}$$

$$f_s(x) = \begin{cases} 0, & |f(x)| > s, \\ f(x), & |f(x)| \le s. \end{cases}$$

Then, we have $f(x) = f^s(x) + f_s(x)$ for $x \in \mathbb{R}^n$. Since T is linear, we get $Tf = Tf^s + Tf_s$, and

$$|Tf(y)| = |Tf^s(y) + Tf_s(y)| \le |Tf^s(y)| + |Tf_s(y)|,$$

for $y \in \mathbb{R}^d$. By (5.2), we obtain

$$m_{Tf}(t) \le m_{Tf^s}(t/2) + m_{Tf_s}(t/2), \quad t > 0. \tag{5.6}$$

By assumption (5.4), we have $|Tf^s|_{q_1,\infty} \le M_1 ||f^s||_{p_1}$ and $|Tf_s|_{q_2,\infty} \le M_2 ||f_s||_{p_2}$; hence, for $t, s > 0$, $\sup_{t/2>0} (t/2) m_{Tf^s}(t/2)^{1/q_1} \le M_1 ||f^s||_{p_1}$, i.e. $m_{Tf^s}(t/2) \le \left(\frac{2M_1}{t}\right)^{q_2} ||f^s||_{p_1}^{q_1}$. Similarly, we also have $m_{Tf_s}(t/2) \le \left(\frac{2M_2}{t}\right)^{q_2} ||f_s||_{p_2}^{q_2}$. Therefore,

$$m_{Tf}(t) \le \left(\frac{2M_1}{t}\right)^{q_1} ||f^s||_{p_1}^{q_1} + \left(\frac{2M_2}{t}\right)^{q_2} ||f_s||_{q_2}^{p_2}. \tag{5.7}$$

Now, observe that we may assume that s depends on t, say $s = g(t)$, with g a suitable function. Then, the task is reduced to an estimate of $||Tf||_q$.

By Lemma 5.1.1. (2),

$$\frac{1}{q} \int_{\mathbb{R}^n} |(Tf)(y)|^q \, dy = \int_0^\infty t^{q-1} m_{Tf}(t) \, dt.$$

By (5.6) and (5.7), we get

$$\frac{1}{q}\int_{\mathbb{R}^n}|(Tf)(y)|^q\,dy \le \int_0^\infty t^{q-1}\left\{\left(\frac{2M_1}{t}\right)^{q_1}||f^s||_{p_1}^{q_1}+\left(\frac{2M_2}{t}\right)^{q_2}||f_s||_{p_2}^{q_2}\right\}dt$$
$$=\int_0^\infty \frac{(2M_1)^{q_1}}{t^{q_1}}t^{q-1}||f^s||_{p_1}^{q_1}\,dt+\int_0^\infty \frac{(2M_2)^{q_2}}{t^{q_2}}t^{q-1}||f_s||_{p_2}^{q_2}\,dt$$
$$=(2M_1)^{q_1}\int_0^\infty t^{q-1-q_1}||f^s||_{p_1}^{q_1}\,dt+(2M_2)^{q_2}\int_0^\infty t^{q-1-q_2}||f_s||_{p_2}^{q_2}\,dt$$
$$=(2M_1)^{q_1}\int_0^\infty t^{q-1-q_1}\left(\int_{|f(x)|>g(t)=s}|f(x)|^{p_1}\,dx\right)^{q_1/p_1}dt$$
$$+(2M_2)^{q_2}\int_0^\infty t^{q-1-q_2}\left(\int_{|f(x)|\le g(t)=s}|f(x)|^{p_2}\,dx\right)^{q_2/p_2}dt.$$

Next, we estimate the right hand side of this inequality by choosing suitable g and t.

(The case $p_i=q_i$) If $p_i=q_i$, then $p=q$ by (5.3). In this case, we set $s=g(t)=t/A$ with a positive constant A that we will define later. Then, we have $t=As$ and

$$\frac{1}{q}\int_{\mathbb{R}^d}|Tf|^q\,dy \le (2M_1)^{q_1}\int_0^\infty (As)^{q-1-q_1}\left(\int_{|f|>s}|f|^{p_1}\,dx\right)dt$$
$$+(2M_2)^{q_2}\int_0^\infty (As)^{q-1-q_2}\left(\int_{|f|\le s}|f|^{p_2}\,dx\right)dt$$
$$=(2M_1)^{q_1}A^{q-q_1}\int_0^\infty s^{q-1-q_1}\left(\int_{|f|>s}|f|^{p_1}\,dx\right)ds$$
$$+(2M_2)^{q_2}A^{q-q_2}\int_0^\infty s^{q-1-q_2}\left(\int_{|f|\le s}|f|^{p_2}\,dx\right)ds.$$

Setting $W=\{(x,t)\in\mathbb{R}^{n+1};\ 0\le t<|f(x)|\}$ and $H_s=\{x\in\mathbb{R}^n;\ |f(x)|>s\}$, and interchanging the order of integration, then implies

$$\int_0^\infty s^{p-1-p_1}\left(\int_{|f|>s}|f|^{p_1}\,dx\right)ds=\iint_W s^{p-1-p_1}|f(x)|^{p_1}\,dsdx$$
$$=\int_{\mathbb{R}^n}|f(x)|^{p_1}\left(\int_0^{|f(x)|}s^{p-1-p_1}\,ds\right)dx=\int_{\mathbb{R}^n}|f(x)|^{p_1}\frac{|f(x)|^{p-p_1}}{p-p_1}\,dx$$
$$=\frac{1}{p-p_1}\int_{\mathbb{R}^n}|f|^p\,dx.$$

Similarly,

$$\int_0^\infty s^{p-1-p_2}\left(\int_{|f|\le s}|f|^{p_2}\,dx\right)ds=\frac{1}{p_2-p}\int_{\mathbb{R}^n}|f|^p\,dx.$$

Therefore, since $p_i = q_i$ and $p = q$,

$$\frac{1}{q}\int_{\mathbb{R}^d} |Tf|^p \, dy \leq \left(\frac{(2M_1)^{p_1} A^{p-p_1}}{p-p_1} + \frac{(2M_2)^{p_2} A^{p-p_2}}{p_2 - p} \right) \int_{\mathbb{R}^n} |f(x)|^p \, ds$$
$$=: X(A) \int_{\mathbb{R}^n} |f(x)|^p \, ds.$$

The term $X(A)$ is valid for all $A > 0$, so we will find its minimum value by differentiation with respect to A:

$$\frac{d}{dA} X(A) = (2M_1)^{p_1} A^{p-1-p_1} - (2M_2)^{p_2} A^{p-1-p_2} = 0,$$

which implies that

$$A_{\min} = 2M_1^{\frac{-p_1}{p_2-p_1}} M_2^{\frac{p_2}{p_2-p_1}}.$$

Hence

$$X(A)\Big|_{A=A_{\min}} = \frac{(2M_1)^{p_1}}{p-p_1}\left(2M_1^{\frac{-p_1}{p_2-p_1}} M_2^{\frac{p_2}{p_2-p_1}}\right)^{p-p_1} + \frac{(2M_2)^{p_2}}{p_2-p}\left(2M_1^{\frac{-p_1}{p_2-p_1}} M_2^{\frac{p_2}{p_2-p_1}}\right)^{p-p_2}$$
$$= \left(\frac{1}{p-p_1} + \frac{1}{p_2-p}\right) 2^p M_1^{\frac{p_1(p_2-p)}{p_2-p_1}} M_2^{\frac{p_2(p-p_1)}{p_2-p_1}}.$$

Here, $1/p = \theta/p_1 + (1-\theta)/p_2$, so

$$\frac{1}{p} = \frac{\theta p_2 + (1-\theta)p_1}{p_1 p_2} = \frac{\theta(p_2-p_1) + p_1}{p_1 p_2},$$

i.e.

$$\theta p = \frac{p_1(p_2-p)}{p_2-p_1}.$$

Also,

$$p(1-\theta) = p - p\theta = p - \frac{p_1(p_2-p)}{p_2-p_1} = \frac{p(p_2-p_1) - p_1(p_2-p)}{p_2-p_1} = \frac{p_2(p-p_1)}{p_2-p_1}.$$

Hence

$$2^p M_1^{\frac{p_1(p_2-p)}{p_2-p_1}} M_2^{\frac{p_2(p-p_1)}{p_2-p_1}} = 2^p M_1^{\theta p} M_2^{(1-\theta)p}.$$

Therefore

$$||Tf||_p^p \leq C^p M_1^{\theta p} M_2^{(1-\theta)p} ||f||_p^p,$$

where $C = 2\left(\frac{p}{p-p_1}+\frac{p}{p_2-p}\right)^{1/p}$. Since $L^{p_1}(\mathbb{R}^n)\cap L^{p_2}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the operator T extends uniquely to a bounded operator on $L^p(\mathbb{R}^n)$; i.e. (5.5) is valid for all $f \in L^p(\mathbb{R}^n)$.

(The case of general exponents) It suffices to consider $f \in L^{p_1}(\mathbb{R}^n)\cap L^{p_2}(\mathbb{R}^n)$. First, suppose that $q_1 < q_2$. In this case, set $s = g(t) = (t/A)^{1/\mu}$ with $A, \mu > 0$. Similarly to the case $p_i = q_i$, we will estimate the right hand side of the inequality:

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^d}|(Tf)(y)|^q\,dy \le{}& (2M_1)^{q_1}\int_0^\infty t^{q-1-q_1}\left(\int_{|f(x)|>g(t)}|f(x)|^{p_1}\,dx\right)^{q_1/p_1}dt\\ &+(2M_2)^{q_2}\int_0^\infty t^{q-1-q_2}\left(\int_{|f(x)|\le g(t)}|f(x)|^{p_2}\,dx\right)^{q_2/p_2}dt\\ &=: (2M_1)^{q_1}J_1+(2M_2)^{q_2}J_2.\end{aligned}$$

Calculation shows that

$$J_1 \le A^{q-q_1}\frac{1}{q-q_1}||f||_p^{pq_1/p_1},$$

and that

$$J_2 \le A^{q-q_2}\frac{1}{q_2-q}\left(\int_{\mathbb{R}^n}|f|^{(q-q_2)\mu\frac{p_2}{q_2}}\,dx\right)^{q_2/p_2}.$$

Furthermore, (5.3) gives us

$$\frac{\theta}{1-\theta}=\frac{1/q-1/q_2}{1/q_1-1/q}=\frac{1/p-1/p_2}{1/p_1-1/p};$$

hence

$$\mu=\frac{q_1}{p_1}\frac{p-p_1}{q-q_1}=\frac{q_2}{p_2}\frac{p-p_2}{q-q_2}.$$

Therefore

$$J_2 \le A^{q-q_2}\frac{1}{q_2-q}||f||_p^{pq_2/p_2}.$$

Combining J_1 and J_2, finding the minimum value, and organizing the right hand side, then implies that

$$||Tf||_q \le C' M_1^\theta M_2^{1-\theta}||f||_p,$$

where $C' = 2\left(\frac{q}{q-q_1}+\frac{q}{q_2-q}\right)^{1/q}$, i.e. (5.5) holds for $q_1 < q_2$.

For the case $q_1 > q_2$, we only need to interchange roles of q_1 and q_2. ■

5.2 The Hardy-Littlewood-Sobolev Inequality

Definition 5.2.1 (Inverse of the Laplacian).

For $0 < \alpha < n$, the operator $(-\Delta)^{-\alpha/2}$ is defined by

$$(-\Delta)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta} f)\, dt, \qquad f \in C_0(\mathbb{R}^n).$$

Here, we define $e^{t\Delta} f := G_t * f$.

Definition 5.2.2 (The Riesz Potential).

Let $0 < \alpha < n$. The Riesz potential $I_\alpha(f)$ of a function f is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy = \frac{1}{|x|^{n-\alpha}} * f, \qquad x \in \mathbb{R}^n. \tag{5.8}$$

Lemma 5.2.1 *We have that*

$$\begin{aligned} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt &= \int_0^\infty t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y) f(y)\, dy \right) dt \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x-y)\, dt \right) f(y)\, dy. \end{aligned} \tag{5.9}$$

for $f \in C_0(\mathbb{R}^n)$.

Proof. We need to check integrability: By L^p-L^q estimates (Theorem 2.2.5), we have

$$||e^{t\Delta} f||_\infty \leq \frac{1}{(4\pi t)^{\frac{n}{2}(0-0)}} ||f||_\infty,$$

and

$$||e^{t\Delta} f||_\infty \leq \frac{1}{(4\pi t)^{\frac{n}{2}(1-0)}} ||f||_1.$$

Since $f \in C_0(\mathbb{R}^n)$ and $0 < \alpha < n$, we get

$$\int_0^\infty t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y) f(y)\, dy \right) dt \leq ||f||_\infty \int_0^1 t^{\frac{\alpha}{2}-1}\, dt < \infty,$$

and

$$\int_1^\infty t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y) f(y)\, dy \right) dt \leq \frac{||f||_1}{(4\pi)^{n/2}} \int_1^\infty t^{\frac{\alpha}{2}-\frac{n}{2}-1}\, dt < \infty.$$

■

Lemma 5.2.2 *We have that*

$$(-\Delta)^{-\alpha/2} f = C(n,\alpha) I_\alpha(f)$$

where

$$C(n,\alpha) = \frac{\Gamma\left(n/2-\alpha/2\right)}{\Gamma\left(\alpha/2\right) 2^\alpha \pi^{n/2}}.$$

Proof. By (5.9),

$$\int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x)\, dt = \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} t^{\frac{\alpha}{2}-1}\, dt$$
$$= \frac{1}{\pi^{n/2}4^{\alpha/2}} \int_0^\infty \tau^{\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\tau} \frac{d\tau}{|x|^{n-\alpha}} = \frac{\Gamma(n/2-\alpha/2)}{2^\alpha \pi^{n/2}|x|^{n-\alpha}}.$$

Hence

$$(-\Delta)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} (e^{t\Delta} f)\, dt = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} (G_t * f)\, dt$$
$$= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x)\, dt * f = \frac{1}{\Gamma(\alpha/2)} \frac{\Gamma(n/2-\alpha/2)}{2^\alpha \pi^{n/2}|x|^{n-\alpha}} * f$$
$$= \frac{\Gamma\left(n/2-\alpha/2\right)}{\Gamma\left(\alpha/2\right) 2^\alpha \pi^{n/2}} \frac{1}{|x|^{n-\alpha}} * f = C(n,\alpha) I_\alpha(f).$$

■

Lemma 5.2.3 *We define $T_\alpha(f)$ by*

$$T_\alpha(f) = \int_0^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt.$$

Assume for α, p, r that $0 < \alpha < n$, $1 \le p < \infty$, $1 < r < \infty$, and

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}. \tag{5.10}$$

Then, there exists a constant $C = C(p,\alpha,n)$ such that

$$|T_\alpha(f)|_{r,\infty} \le C||f||_p, \qquad f \in L^p(\mathbb{R}^n). \tag{5.11}$$

Proof. For $S > 0$, set

$$F^S = \int_0^S t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt, \quad \text{and} \quad F_S = \int_S^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt.$$

Then,

$$|\big(T_\alpha(f)\big)(x)| \leq |F^S(x)| + |F_S(x)|,$$

for $x \in \mathbb{R}^n$. By (5.2), we have $m_{T_\alpha(f)}(t) \leq m_{F^S}(t/2) + m_{F_S}(t/2)$ for $t > 0$, and since $||e^{t\Delta} f||_\infty \leq ||f||_p$,

$$||F_S||_\infty \leq \int_S^\infty t^{\frac{\alpha}{2}-1} ||e^{t\Delta} f||_\infty \, dt \leq C \int_S^\infty t^{\frac{\alpha}{2}-1} t^{-\frac{n}{2p}} \, dt ||f||_p = C' S^{-\frac{n}{2p}} ||f||_p,$$

where $C' = 2rC/n$. Hence, if we set S to be $t/4 = C' S^{-n/(2p)}$, i.e. $S = (4C'||f||_p/t)^{2r/n}$, we get $m_{F_S}(t/2) = 0$ for $t > 0$. By Lemma 5.1.1, we have $m_{F^S}(t/2) \leq (t/2)^{-p} ||F^S||_p^p$, and since $||e^{t\Delta} f||_p \leq ||f||_p$,

$$||F^S||_p \leq ||f||_p \int_0^S t^{\frac{\alpha}{2}-1} \, dt = \frac{2}{\alpha} ||f||_p S^{\alpha/2}.$$

Since $S^{\alpha p/2} = (4C'||f||_p/t)^{(2r/n)(\alpha p/2)}$, we obtain

$$m_{F^S}(t/2) \leq C'' t^{-p} t^{-\alpha pr/n} ||f||_p^{p+\alpha pr/n},$$

where $C'' = 2^p (2/\alpha)^p (4C')^{\alpha pr/n}$. By assumption (5.10), $1 + \alpha r/n = r/p$, so

$$m_{T_\alpha(f)}(t) \leq m_{F^S(t/2)} + 0 \leq C'' t^{-r} ||f||_p^r,$$

for $t > 0$. ■

Theorem 5.2.4 *(The Hardy-Littlewood-Sobolev Inequality).*
Let $0 < \alpha < n$. Assume for $1 < p, r < \infty$ that

$$\frac{1}{r} = \frac{n-\alpha}{n} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{\alpha}{n}.$$

Then, there exists a constant $C = C(\alpha, p)$ such that, for all $f \in C_0(\mathbb{R}^n)$,

$$||I_\alpha(f)||_r \leq C ||f||_p, \qquad f \in C_0(\mathbb{R}^n);$$

hence, the operator I_α extends uniquely to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

Proof. For p and r, we can take $1 < p_i, r_i < \infty$ $(i = 1, 2)$ and $\theta \in (0, 1)$ so that $r_1 \neq r_2$ and

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \quad \frac{1}{r_i} = \frac{1}{p_i} - \frac{\alpha}{n}.$$

By Lemma 5.2.3, T_α is a bounded linear operator from $L^{p_i}(\mathbb{R}^n)$ to $L^{r_i,\infty}(\mathbb{R}^n)$, so by Theorem 5.1.3, T_α extends to a bounded linear operator form $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. Since

$$I_\alpha(f) = \frac{(-\Delta)^{-\alpha/2} f}{C(n/\alpha)} = \frac{1}{C(n,\alpha)\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta} f)\, dt = \frac{1}{C(n,\alpha)\Gamma(\alpha/2)} T_\alpha(f)$$

by Lemma 5.2.2, the inequality holds. ■

5.3 The Sobolev Inequality

Proposition 5.3.1 *Let $n \geq 3$. For any $f \in C_0^\infty(\mathbb{R}^n)$,*

(1) $(-\Delta)^{-1}\Delta f = -f$, and

(2) $(-\Delta)^{-1} f = E * f$,

where $E(x) = 1/\big((n-2)|S^{n-1}||x|^{n-2}\big)$ for $x \in \mathbb{R}^n$. Here, $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of the $(n-1)$-dimensional unit sphere.

Proof. (1). By definition,

$$(-\Delta)^{-1}\Delta f = \frac{1}{\Gamma(1)} \int_0^\infty e^{t\Delta}\Delta f\, dt = \int_0^\infty e^{t\Delta}\Delta f\, dt = \lim_{m\to\infty} \int_0^m e^{t\Delta}\Delta f\, dt.$$

Since

$$||e^{t\Delta}\Delta f||_\infty \leq \frac{1}{(4\pi t)^{\frac{n}{2}\left(\frac{1}{1}-\frac{1}{\infty}\right)}} ||\Delta f||_1 = \frac{1}{(4\pi t)^{\frac{n}{2}}} ||\Delta f||_1 =: \frac{C}{t^{n/2}} ||\Delta f||_1,$$

the convergence above is uniform in $x \in \mathbb{R}^n$. Moreover, $e^{t\Delta}\Delta f = \Delta e^{t\Delta} f$, so

$$\int_0^m e^{t\Delta}\Delta f\, dt = \int_0^m \Delta e^{t\Delta} f\, dt = \int_0^m \frac{d}{dt} e^{t\Delta} f\, dt = e^{m\Delta} f - f.$$

Since $||e^{m\Delta} f||_\infty \leq \frac{C}{m^{n/2}} ||f||_1 \to 0$ as $m \to \infty$, we have that $(-\Delta)^{-1}\Delta f = -f$.

(2). Choosing $\alpha = 2$, the inverse of the Laplacian becomes

$$(-\Delta)^{-1}f = C(n,2)I_2(f) = \frac{1}{(n-2)|S^{n-1}|}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}\,dy$$
$$= \int_{\mathbb{R}^n}\frac{1}{(n-2)|S^{n-1}|}\frac{f(y)}{|x-y|^{n-2}}\,dy = \frac{1}{(n-2)|S^{n-1}|}\frac{1}{|x|^{n-2}} * f = E * f,$$

where $C(n,2) = \Gamma(n/2-1)/\big(\Gamma(1)2^2\pi^{n/2}\big)$. Here, we used $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. ■

Proposition 5.3.2 *For $x \in \mathbb{R}^2 \setminus \{0\}$, set $E(x) = -\ln|x|/2\pi$. Then*

(1) $\lim_{\alpha\uparrow 2}||(-\Delta)^{-\alpha/2}\Delta f + f||_\infty = 0$*, for $f \in C_0^\infty(\mathbb{R}^2)$, and*

(2) $\lim_{\alpha\uparrow 2}\big((-\Delta)^{-\alpha/2}h\big)(x) = (E * h)(x)$*, for $x \in \mathbb{R}^2$ and $h \in C_0^\infty(\mathbb{R}^2)$ satisfying* $\int_{\mathbb{R}^2} h(x)\,dx = 0$.

Proof. (1). Fix $\epsilon > 0$, and split the interval of integration:

$$(-\Delta)^{-\alpha/2}\Delta f = \frac{1}{\Gamma(\alpha/2)}\left(\int_\epsilon^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)\,dt + \int_0^\epsilon t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)\,dt\right)$$
$$=: \frac{1}{\Gamma(\alpha/2)}(J_1 + J_2).$$

Regarding J_1, we estimate

$$J_1 = \int_\epsilon^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)\,dt = \int_\epsilon^\infty t^{\frac{\alpha}{2}-1}\left(\frac{d}{dt}e^{t\Delta}f\right)\,dt$$
$$= t^{\frac{\alpha}{2}-1}(e^{t\Delta}f)\Big|_\epsilon^\infty - \int_\epsilon^\infty \left(\frac{\alpha}{2}-1\right)t^{\frac{\alpha}{2}-2}(e^{t\Delta}f)\,dt$$
$$= -\epsilon^{\frac{\alpha}{2}-1}(e^{t\Delta}f) - \left(\frac{\alpha}{2}-1\right)\int_\epsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta}f)\,dt.$$

Since $||e^{t\Delta}f||_\infty \le (4\pi t)^{-n/2}||f||_1 =: C_1 t^{-n/2}||f||_1$, we get

$$\left|\left|\int_\epsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta}f)\,dt\right|\right|_\infty \le C_1||f||_1\int_\epsilon^\infty t^{\frac{\alpha}{2}-2-\frac{n}{2}}\,dt = C_1||f||_1\frac{\epsilon^{\frac{\alpha}{2}-1-\frac{n}{2}}}{\frac{\alpha}{2}-1-\frac{n}{2}}.$$

Furthermore

$$||J_1 + e^{\epsilon\Delta}f||_\infty = \left|\left|-\epsilon^{\frac{\alpha}{2}-1}(e^{\epsilon\Delta}f) - \left(\frac{\alpha}{2}-1\right)\int_\epsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta}f)\,dt + e^{\epsilon\Delta}f\right|\right|_\infty$$
$$\le -\left(\frac{\alpha}{2}-1\right)\left|\left|\int_\epsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta}f)\,dt\right|\right|_\infty + \left|\left|-\epsilon^{\frac{\alpha}{2}-1}(e^{\epsilon\Delta}f) + e^{\epsilon\Delta}f\right|\right|_\infty$$
$$\le -\left(\frac{\alpha}{2}-1\right)C_1||f||_1\frac{\epsilon^{\frac{\alpha}{2}-1-\frac{n}{2}}}{\frac{\alpha}{2}-1-\frac{n}{2}} + \left|\left|-\epsilon^{\frac{\alpha}{2}-1}(e^{\epsilon\Delta}f) + e^{\epsilon\Delta}f\right|\right|_\infty \to 0.$$

as $\alpha \uparrow 2$. Regarding J_2, we estimate

$$||J_2||_\infty = \left|\left|\int_0^\epsilon t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)\,dt\right|\right|_\infty \le \int_0^\epsilon \left|\left|t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)\right|\right|_\infty dt$$

$$\le \int_0^\epsilon \left|t^{\frac{\alpha}{2}-1}\right| ||e^{t\Delta}\Delta f||_\infty\,dt \le \int_0^\epsilon t^{\frac{\alpha}{2}-1}\,dt\,||\Delta f||_\infty = \frac{2}{\alpha}\epsilon^{\frac{\alpha}{2}}||\Delta f||_\infty \to \epsilon||\Delta f||_\infty$$

as $\alpha \uparrow 2$. Therefore

$$\overline{\lim_{\alpha\uparrow 2}}\,||(-\Delta)^{-\alpha/2}\Delta f + f||_\infty \le ||f - e^{\epsilon\Delta}f||_\infty + \epsilon||\Delta f||_\infty \to 0$$

as $\epsilon \to 0$.

(2). For $0 < \alpha < 2$, we have

$$(-\Delta)^{-\alpha/2}h = C(2,\alpha)I_\alpha(h) = \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\left(\frac{1}{|x|^{2-\alpha}} * h\right).$$

Note that, since $\int_{\mathbb{R}^2} h(y)\,dy = 0$,

$$\frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\left(\frac{1}{|x|^{2-\alpha}} - 1\right) * h = \int_{\mathbb{R}^2}\frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\left(\frac{1}{|x-y|^{2-\alpha}} - 1\right)h(y)\,dy$$
$$= \int_{\mathbb{R}^2}\frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\frac{1}{|x-y|^{2-\alpha}}h(y)\,dy + \int_{\mathbb{R}^2}\frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}h(y)\,dy$$
$$= \int_{\mathbb{R}^2}\frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\frac{1}{|x-y|^{2-\alpha}}h(y)\,dy + 0 = \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\left(\frac{1}{|x|^{2-\alpha}} * h\right).$$

Hence

$$C(2,\alpha)I_\alpha(h) = E_\alpha * h, \quad \text{where} \quad E_\alpha(x) := \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)2^\alpha\pi}\left(\frac{1}{|x|^{2-\alpha}} - 1\right).$$

Now, $\Gamma(1-\alpha/2) = \big(2/(2-\alpha)\big)\Gamma(2-\alpha/2)$ implies that $\Gamma(1-\alpha/2) \to \infty$ with the principle term $2/(2-\alpha)$. Set $\delta = 2-\alpha$. If $x \neq 0$, we have

$$\frac{|x|^{-\delta} - 1}{\delta} = \frac{\exp(-\delta\ln|x|) - \exp(-0\ln|x|)}{\delta} \to -\ln|x|$$

as $\delta \to 0$, i.e.

$$\lim_{\alpha\uparrow 2} E_\alpha(x) = -\frac{1}{2\pi}\ln|x|$$

for any $x \in \mathbb{R}^2 \backslash \{0\}$. To interchange limit and integration, we appeal to the dominated convergence theorem. Indeed, we have

$$\big((-\Delta)^{-\alpha/2} h\big)(x) = \int_{\mathbb{R}^2} E_\alpha(x-y)h(y)\,dy = \int_{\mathbb{R}^2} E_\alpha(y)h(x-y)\,dy$$

as h is compactly supported, and by the mean value theorem of integral form (2.11), for $y \neq 0$,

$$\begin{aligned} |y|^{-\delta} - 1 = |y|^{-\delta} - |y|^{-(\delta-\delta)} &= \int_0^1 \big(-\ln|y| e^{-\{\delta-(1-\tau)\delta\}\ln|y|}\big)\,\delta\,d\tau \\ = \int_0^1 \big(-\ln|y| e^{-\delta\tau \ln|y|}\big)\,\delta\,d\tau &= -\delta \ln|y| \int_0^1 e^{-\delta\tau\ln|y|}\,d\tau, \end{aligned}$$

which implies

$$\left|\frac{|y|^{-\delta}-1}{\delta}\right| \leq \begin{cases} |\ln|y||\,|y|^{-1}, & 0<|y|\leq 1, \\ \ln|y|, & |y|\geq 1, \end{cases}$$

for $0<\delta<1$. Hence

$$\sup_{1<\alpha<2} |E_\alpha(y)| \leq \begin{cases} C_2|\ln|y||\,|y|^{-1}, & 0<|y|\leq 1, \\ C_2\ln|y|, & |y|\geq 1, \end{cases}$$

where

$$C_2 = \sup_{1<\alpha<2} \frac{\Gamma(2-\alpha/2)}{2^{\alpha-1}\pi\Gamma(\alpha/2)}.$$

Thus, $|E_\alpha(y)|$ is bounded from above by a locally integrable function independent of $\alpha \in (1,2)$. Therefore, by the dominated convergence theorem,

$$\lim_{\alpha\uparrow 2} \big((-\Delta)^{-\alpha/2}h\big)(x) = (E * h)(x).$$

■

Remark. We have that $E * \Delta f = -f$ for $f \in C_0^\infty(\mathbb{R}^2)$.

Definition 5.3.1 For $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, we define the scaled function h_λ by $h_\lambda(x) = h(\lambda x)$ for $\lambda > 0$. The function h is said to be positively homogeneous of degree d if there exists d such that $h_\lambda(x) = \lambda^d h(x)$ for $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$.

Lemma 5.3.3 *We have the following:*

(1) Let $n \geq 2$. Assume, if $n = 2$, that $|\alpha| \geq 1$. Then

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} |\partial_x^\alpha E(x)|\, |x|^{n-2+|\alpha|} < \infty.$$

(2) For $1 \leq j \leq n$, $\partial_{x_j} E$ is locally integrable on $\mathbb{R}^n$.

Proof. (1). If h is positively homogeneous of degree d, then $\partial_{x_j} E$ is positively homogeneous of degree $d - |\alpha|$. In fact, if there is a d such that $h_\lambda(x) = \lambda^d h(x)$ for $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$,

$$(\partial_x^\alpha h)_\lambda(x) = (\partial_x^\alpha h)(\lambda x) = \lambda^{-|\alpha|} \partial_x^\alpha h_\lambda(x) = \lambda^{-|\alpha|} \partial_x^\alpha \big(\lambda^d h(x)\big) = \lambda^{d-|\alpha|} (\partial_x^\alpha h)(x).$$

For $n \geq 3$, i.e. $E(x) = 1/\big((n-2)|S^{n-1}||x|^{n-2}\big)$ where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$,

$$E_\lambda(x) = E(\lambda x) = |\lambda|^{2-n} E(x) = \lambda^{2-n} E(x);$$

hence E is positively homogeneous of degree $2 - n$. Therefore, $\partial_x^\alpha E$ is positively homogeneous of degree $2 - n - |\alpha|$.

For $n = 2$, i.e. $E(x) = -\ln|x|/(2\pi)$, we use direct calculation:

$$E_\lambda(x) = E(\lambda x) = -\frac{1}{2\pi} \ln|\lambda x| = -\frac{1}{2\pi} \ln \lambda - \frac{1}{2\pi} \ln|x| = -\frac{1}{2\pi} \ln \lambda E(x),$$

and

$$(\partial_x^\alpha E)_\lambda(x) = (\partial_x^\alpha E)(\lambda x) = \lambda^{-|\alpha|} \partial_x^\alpha \big(E(x) - \frac{1}{2\pi} \ln \lambda\big) = \lambda^{-|\alpha|} \partial_x^\alpha E(x);$$

hence $\partial_x^\alpha E$ is positively homogeneous of degree $-|\alpha| = 2 - n - |\alpha|$.

Now, consider $x \in \mathbb{R}^n$ on a sphere centered at the origin, i.e. $|x| = r > 0$. Since $\partial_x^\alpha E$ is positively homogeneous of degree $2 - n - |\alpha|$, we get

$$\partial_x^\alpha E(x) = \partial_x^\alpha E\left(|x| \frac{x}{|x|}\right) = |x|^{2-n-|\alpha|} \partial_x^\alpha E\left(\frac{x}{|x|}\right).$$

Here, $\partial_x^\alpha E$ is continuous on the sphere, and the sphere is compact; hence

$$C := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \left|(\partial_x^\alpha E)\left(\frac{x}{|x|}\right)\right| < \infty.$$

(2). By (1), there exists C independent of x such that

$$|\partial_{x_j} E(x)| \leq \frac{C}{|x|^{n-1}}$$

Hence

$$\int_{B_R} |\partial_{x_j} E(x)|\, dx \leq C \int_{B_R} \frac{dx}{|x|^{n-1}} = C|S^{n-1}| \int_0^R r^{n-1} r^{1-n}\, dt = CR|S^{n-1}| < \infty.$$

Thus, $\partial_{x_j} E$ is locally integrable on $\mathbb{R}$. ■

Theorem 5.3.4 *(The Sobolev Inequality).*
Let $n \geq 2$. Assume for r and r_* that $1 \leq r < \infty$, $1 < r_* < \infty$, and $1/r_* = 1/r - 1/n$. Then

$$||u||_{r_*} \leq C||\nabla u||_r, \tag{5.12}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Proof. We use direct calculation for the case $r = 1$, and use the Hardy-Littlewood-Sobolev inequality for the case $1 < r < \infty$.

(The case $r = 1$; $r_* = n/(n-1)$) By the fundamental theorem of calculus, for $u \in C_0^1(\mathbb{R}^n)$,

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, s_n)\, ds$$

for $1 \leq i \leq n$. Hence

$$|u(x)| \leq \int_{-\infty}^{x_i} |\partial_{x_i} u(x_1, \ldots, s, \ldots, x_n)|\, ds \leq \int_{-\infty}^{\infty} |\partial_{x_i} u(x_1, \ldots, s, \ldots, x_n)|\, ds,$$

which implies

$$|u(x)|^n \leq \prod_{i=1}^{n} \int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i;$$

hence

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i \right)^{\frac{1}{n-1}}.$$

Integrating this inequality with respect to x_1, by the Hölder inequality,

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 \leq \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i \right)^{\frac{1}{n-1}} dx_1 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i \right)^{\frac{1}{n-1}} dx_1$$
$$= \left(\int_{-\infty}^{\infty} |\partial_{x_1} u|\, dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i \right)^{\frac{1}{n-1}} dx_1$$
$$\leq \left(\int_{-\infty}^{\infty} |\partial_{x_1} u|\, dx_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_{x_i} u|\, dx_i dx_1 \right)^{\frac{1}{n-1}}.$$

Iterating this procedure with respect to other variables,

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, dx \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_{x_i} u|\, dx \right)^{\frac{1}{n-1}}.$$

Since the geometric mean precedes the arithmetic mean, i.e.

$$\left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

where $a_i \geq 0$, we get

$$||u||_{r_*} = ||u||_{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_{x_i} u|\, dx \right)^{\frac{1}{n}} \leq \frac{1}{n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} |\partial_{x_i} u|\, dx \leq \frac{\sqrt{n}}{n} ||\nabla u||_1.$$

(The case $1 < r < \infty$) By Lemma 5.3.3, we have

$$|\nabla E(x)| = \sqrt{\left(\partial_{x_1} E(x)\right)^2 + \cdots + \left(\partial_{x_n} E(x)\right)^2} \leq \sqrt{\left(\frac{C_1}{|x|^{n-2+1}} \right)^2 + \cdots + \left(\frac{C_n}{|x|^{n-2+1}} \right)^2}$$
$$= \sqrt{\frac{C_1^2 + \cdots + C_n^2}{|x|^{2(n-1)}}} =: \frac{C''}{|x|^{n-1}}.$$

By Proposition 5.3.1 and Proposition 5.3.2,

$$u(x) = -\int_{\mathbb{R}^n} E(x-y) \Delta u(y)\, dy;$$

hence

$$|u(x)| \leq \int_{\mathbb{R}^n} |\langle (\nabla E)(x-y), \nabla u(y) \rangle|\, dy \leq \int_{\mathbb{R}^n} |\nabla E(x-y)|\, |\nabla u(y)|\, dy$$
$$\leq \int_{\mathbb{R}^n} \frac{C''}{|x-y|^{n-1}} |\nabla u(y)|\, dy = C'' I_1(|\nabla u|)(x).$$

By the Hardy-Littlewood-Sobolev inequality (Theorem 5.2.4),

$$||u||_{r_*} \leq C'' ||I_1(|\nabla u|)||_{r_*} \leq C' ||\, |\nabla u|\, ||_r =: C ||\nabla u||_r.$$

■

5.4 The Gagliardo-Nirenberg Inequality

Theorem 5.4.1 *(The Gagliardo-Nirenberg Inequality).*

Assume for $1 \le p, q, r \le \infty$ *and* $\sigma \in [0,1]$ *that*

$$\frac{1}{p} = (1-\sigma)\frac{1}{q} + \sigma\left(\frac{1}{r} - \frac{1}{n}\right). \tag{5.13}$$

We also assume, if $n \ge 2$*, that*

$$p \ne \infty, \quad \text{or} \quad r \ne n. \tag{5.14}$$

Then, there exists a constant $C = C(p,q,r,n) > 0$ *such that*

$$||u||_p \le C||u||_q^{1-\sigma}||\nabla u||_r^{\sigma}, \tag{5.15}$$

for all $u \in C_0^1(\mathbb{R}^n)$*. If* $n = 1$*,* $p = \infty$*, and* $r = 1$*, the inequality (5.15) holds with* $C = 1$*.*

Proof. If $\sigma = 1$, i.e. $1/p = 1/r - 1/n$, then this is nothing but the Sobolev inequality (5.12). Suppose $\sigma = 0$. Then, we have $1/p = 1/q$ and the inequality becomes $||u||_p \le C||u||_q$ which is trivial. Hence, we may assume that $0 < \sigma < 1$. To prove the inequality, we use the duality characterization of the L^p-norm of u:

$$||u||_p = \sup\left\{\int_{\mathbb{R}^n} u(x)\phi(x)\,dx;\ ||\phi||_{p'} \le 1, \phi \in C_0^\infty(\mathbb{R}^n)\right\},$$

where p' is the conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

(The case $p > q$ and $p \ge r$) Since

$$e^{t\Delta}u - u = \int_0^t \frac{d}{d\tau}e^{\tau\Delta}u\,d\tau = \int_0^t \Delta e^{\tau\Delta}u\,d\tau,$$

we obtain, for $t > 0$,

$$\begin{aligned}
||u||_p &= \sup\left\{\int_{\mathbb{R}^n} u\left(e^{t\Delta}\phi - \int_0^t \Delta e^{\tau\Delta}\phi\,d\tau\right)dx;\ ||\phi||_{p'} \le 1, \phi \in C_0^\infty(\mathbb{R}^n)\right\} \\
&\le \sup\left\{\left|\int_{\mathbb{R}^n} ue^{t\Delta}\phi\,dx\right| + \left|\int_0^t\int_{\mathbb{R}^n} u\Delta e^{\tau\Delta}\phi\,dxd\tau\right|;\ ||\phi||_{p'} \le 1, \phi \in C_0^\infty(\mathbb{R}^n)\right\} \\
&=: \sup\{I_1(\phi) + I_2(\phi);\ ||\phi||_{p'} \le 1, \phi \in C_0^\infty(\mathbb{R}^n)\}.
\end{aligned}$$

Regarding I_1, by the Hölder inequality and $L^{q'}$-$L^{p'}$ estimate, for $t > 0$,

$$I_1(\phi) \le ||u||_q ||e^{t\Delta}\phi||_{q'} \le ||u||_q \, C_1 t^{-\frac{n}{2}(\frac{1}{p'}-\frac{1}{q'})} ||\phi||_{p'} \le C_1 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} ||u||_q ||\phi||_{p'}.$$

Regarding I_2, by integration by parts, for $t > 0$,

$$I_2(\phi) = \left| \int_0^t \int_{\mathbb{R}^n} \langle \nabla u, \nabla e^{\tau\Delta}\phi \rangle \, dx d\tau \right|.$$

By the Hölder inequality and $L^{r'}$-$L^{p'}$ estimate,

$$I_2(\phi) = \int_0^t ||\nabla u||_r ||\nabla e^{\tau\Delta}\phi||_{r'} \, d\tau \le C_2 \tau^{-(\frac{1}{2}+\alpha)} ||\phi||_{p'}$$

where

$$\alpha := \frac{n}{2}\left(\frac{1}{p'} - \frac{1}{r'}\right) = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right).$$

Now, I_2 is integrable with respect to τ on the interval $(0, t)$ if and only if $1/2 + \alpha < 1$, i.e. $1/p > 1/r - 1/n$. Here, by assumptions $p > q$ and $0 < \sigma < 1$, we have that $1/p > 1/r - 1/n$; hence I_2 is integrable. Therefore, for $t > 0$,

$$I_2(\phi) \le C_2 ||\nabla u||_r ||\phi||_{p'} \int_0^t \tau^{-(\frac{1}{2}+\alpha)} \, d\tau = \frac{C_2}{\frac{1}{2}-\alpha} ||\nabla u||_r ||\phi||_{p'} t^{\frac{1}{2}-\alpha} =: C_3 ||\nabla u||_r ||\phi||_{p'} t^{\frac{1}{2}-\alpha}.$$

Thus

$$||u||_p \le C_1 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} ||u||_q + C_3 t^{\frac{1}{2}-\alpha} ||\nabla u||_r.$$

Choosing $t > 0$ so that the two terms on the right-hand side equal, we arrive at

$$||u||_p \le C ||u||_q^{1-\sigma} ||\nabla u||_r^{\sigma}.$$

(Cases excluded above) The negation of the case $p > q$ and $p \ge r$ is $p \le q$ or $p < r$; however, the case $p \le q$ and $p < r$ does not occur. In fact, suppose that $p \le q$ and $p < r$, i.e. $1/q \le 1/p$ and $1/r < 1/p$. If $1/q \le 1/p$, the assumption (5.13) implies that $1/q < 1/p < 1/r - 1/n$; hence $1/p \le 1/r - 1/n$, i.e. $1/n \le 1/r - 1/p$. Contradiction. Thus, we only need to check two cases $p \le q$ and $p < r$.

(The case $r \leq p \leq q$) By assumption (5.13), $r \leq p \leq q$ implies that $r \leq p \leq q$. Here, we appeal to the Sobolev inequality (5.12). First, suppose that $p < q$. Observe that $r_* \leq p < q$ for $1/r_* = 1/r - 1/n$ and replace r with q, q with r_*, and p with σ. Then, we get

$$||u||_p \leq ||u||_{r_*}^{\sigma} ||u||_q^{1-\sigma}$$

where

$$\frac{1}{p} = \frac{\sigma}{r_*} + \frac{1-\sigma}{q}$$

Therefore

$$||u||_p \leq (C_| |\nabla u||_r)^{\sigma} ||u||_q^{1-\sigma} =: C' ||u||_q^{1-\sigma} ||\nabla u||_r^{\sigma}.$$

Next, suppose that $p = q$. By assumption (5.13), $1/q = 1/p \leq 1/r_* = (1/r - 1/n) < 1/r$; hence $1/q = 1/p = 1/r_* < 1/r$. We break this case into even more cases $r_* < \infty$ and $r_* = \infty$:

If $r_* < \infty$, then $0 < 1/r_* < 1/r$, i.e. $1 \leq r < r_* < \infty$. By the Sobolev inequality (5.12), we arrive at (5.15).

If $r_* = \infty$, then $1/q = 1/p = 1/r_* = 0$, i.e. $p = q = r_* = \infty$. By (5.14), we get $n = 1$; hence $r = 1$. Thus, $||u||_\infty \leq C ||u||_\infty^{1-\sigma} ||\frac{du}{dx}||_1^{\sigma}$, which is trivial.

(The case $p < r$) By assumption (5.13), $p < r$ implies that $q < p < r$. Moreover, $(1/r - 1/n) < 1/r < 1/q$, which is equivalent to existence of $\sigma_1 \in (0, 1)$ satisfying

$$\frac{1}{p} = \sigma_1 \left(\frac{1}{r} - \frac{1}{n} \right) + (1 - \sigma_1) \frac{1}{q}.$$

By the case $p > q$ and $p \geq r$,

$$||u||_r \leq C ||u||_q^{1-\sigma_1} ||\nabla u||_r^{\sigma_1}.$$

Now, observe that $||u||_p \leq ||u||_q^{\rho} ||u||_r^{1-\rho}$. Then

$$\begin{aligned} ||u||_p &\leq ||u||_q^{\rho} ||u||_r^{1-\rho} \leq ||u||_q^{\rho} \left(C ||u||_q^{1-\sigma_1} ||\nabla u||_r^{\sigma_1} \right)^{1-\rho} \\ &= ||u||_q^{\rho} C^{1-\rho} ||u||_q^{(1-\sigma_1)(1-\rho)} ||\nabla u||_r^{\sigma_1(1-\rho)} =: C' ||u||_q^{\rho} ||u||_q^{(1-\sigma_1)(1-\rho)} ||\nabla u||_r^{\sigma_1(1-\rho)}. \end{aligned}$$

Setting $\rho = (1/p - 1/r)/(1/q - 1/r)$ and $\sigma = \sigma_1(1-\rho)$, we arrive at (5.15). ∎

Remark. For the case $p = 2$, $\sigma = 1 - 2/(n+2)$, $q = 1$, and $r = 2$ in (5.15) is called the Nash inequality:

$$||u||_2^2 \le C||u||_1^{\frac{4}{n+2}}||\nabla u||_2^{2-\frac{4}{n+2}}.$$

Moreover, the inequality (5.15) holds for even larger sets of functions; namely, for any $u \in L^q(\mathbb{R}^n)$ satisfying $\partial_{x_l} u \in L^r(\mathbb{R}^n)$, $l = 1, 2, \ldots, n$, with condition $p \ge q$, $r < \infty$, and $q < \infty$. In particular, (5.15) implies $u \in L^p(\mathbb{R}^n)$. Here, $\partial_{x_l} u$ is the derivative in the sense of distributions.

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和文抄訳

本論文の主題は，熱方程式の解の漸近挙動と Gagliardo-Nirenberg 不等式である．第二章で，熱方程式の解及びその評価を導入し、解の漸近挙動について考察する．第三章において，解の漸近挙動を方程式の構造から論じるために、自己相似解の概念を導入し，スケール変換の性質をみる．第四章で，Ascoli-Arzelà 型コンパクト性定理を用いて，解の漸近挙動を方程式の構造から論じる．第五章では，熱方程式の解の評価を用いて，一般の偏微分方程式を解析するために有用だと思われる不等式を扱う．特に，Gagliardo-Nirenberg 不等式を証明することを主眼とし，そのために Marcinkiewicz の補間定理，Hardy-Littlewood-Sobolev 不等式，そして Sobolev 不等式を示す.

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