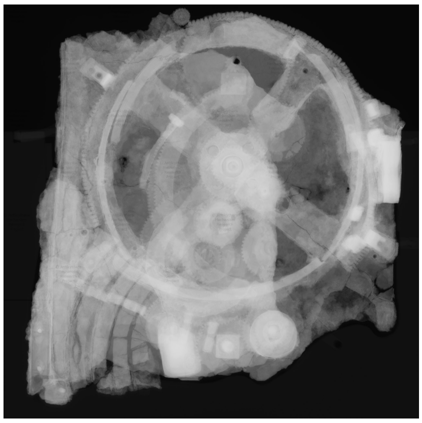
Freie Universität Berlin

Bachelor thesis

Mathematical aspects of the Antikythera Mechanism

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Abstract

In this thesis we will examine the mathematical aspects of the ancient astronomical device known as the Atnikythera Mechanism. We will look at the archeological findings of the research so far, examine the Mechanism's structure and functions. To this end, we will introduce some elements of gear design, and an algorithm for approximating irrational numbers, using continued fractions. Lastly, we will try to expand the Mechanisms function, by adding some parts the existence of which is only hypothesized.

The cover images were taken from the Antikythera Mechanism Research Project. On the left, a PTM of fragment A from the HP labs download page and on the right, its Digital Radiograph from the webpage of Shaw Inspection Systems. For more pictures and information, see: http://www.antikythera-mechanism.gr/data

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Introduction

0.1 The Antikythera Mechanism

The device that is nowadays known as the Antikythera Mechanism is a geared tooth astronomical device, discovered in the beginning of the 20th century by sponge divers in an ancient shipwreck off the coast of the uninhabited Greek island of Antikythera. The Mechanism, made of bronze, was heavily damaged from the sea water, and it was broken with some parts of it being stuck together in larger fragments, many of its gears embedded in a corroded mass, and some gears having partly survived, with only pieces of them having been found. Research on the Mechanism has been ongoing from the early 1900's to present day, and the shipwreck's site, which after the initial discovery was only reexamined by Jacques-Yves Cousteau in the 1970's, was revisited in September and October 2014 by a collaboration between Woods Hole Oceanographic Institution and the Hellenic Ephorate of Underwater Antiquities called Return to Antikythera, aiming to bring more information about the shipwreck and its cargo to the surface. The Mechanism's nature posed a question that puzzled scientists and enthusiasts alike for more than a hundred years, and every round of research shed a bit more light on previous work, disputing or proving hypotheses on the function and use of the Mechanism, models being proposed and replaced with newer ones with the passing of time. The collaborative Antikythera Mechanism Research Project, using technology developed exclusively for its purposes managed to answer the questions about the Mechanism's structure and function. Having two displays, one in the front and one in the back, the Mechanism would display the position of the sun and the moon in a geocentric system, and it included an eclilpse predictor and an Athenian calendar system used at the time in many Greek cities. The Mechanism would be inscribed with a text, a part of which was a user's manual, explaining how to operate the Mechanism and what its output would be. Part of this inscription mentions the planets and the Cosmos, which raises the question if there was more information on display on the Mechanism, as all the planets known at the time of its conception are mentioned in a partly readable inscription on the back.

0.2 Thesis Overview

This thesis is divided in five sections. We begin by presenting the history of the Mechanism's research and the findings of the AMRP, which the basis for us to see the full potential of the Mechanism. Since the gears of the Mechanism are flawed by design, we proceed to chapter 2, where we examine how gears work, and introduce the involute gear, discovered by Euler, so that the ingenious design of the Mechanism can be fully realized, and we examine how the involute curve connects to ancient greek mathematics, and especially Archimedes. The next chapter concerns continued fractions and an algorithm based on Euclidean division that will allow us to use rational approximations of irratioanl numbers, and we will compare ancient astronomical information based on observation with the best rational approximations of several irrational astronomical ratios. Chapters 4 presents an analysis of the 2005 model, using involute gears instead of the original ones. This allows for accurate calculations between meshing gears, and we can examine the accuracy of the Mechanim's intended output. We also examine a lunar anomaly device that was used to display the theory of Hipparchos for the motion of the moon. The last chapter examines the question of the Cosmos appearing on the Mechanism, and we present devices that could display the five planets mentioned in the back inscription of the Mechanism. We only aim to have the mean periods on display, as previous research has shown that the anomaly of these planets' orbit could not be realized with the technology of the Mechanism's time.

0.3 Acknowledgements

I am grateful to professor Bernold Fiedler for giving me the chance to examine the Antikythera Mechanism, and for turning my attention towards gears and gear teeth, which gave me a better understanding of the Mechanism's nature and limits. I am also very thankful to Nikos Lengas, who helped me understand the geometry and meshing of gears. Lastly, I am thankful to Anna Karnauhova for all the discussions we had during the writing of my thesis.

1 The Antikythera Mechanism

In the beginning of the 20th century an ancient shipwreck was discovered off the coast of the barely inhabited Greek island *Antikythera*. In the shipwreck, among the many treasures found, there was a mass of bronze that puzzled, and challenged, the scientific community for years to come. Further analysis and examination determined that the *Antikythera Mechanism* was a toothed gear mechanism that presented a calendar of its time, could very accurately predict eclipses based on astronomical information available at the time, and depicted the sun's and the moon's position according to the geocentric theory at the time. In this section the information known about the Mechanism's parts is presented.

1.1 Research conducted before 2005

Before analysing the results of the Antikythera Mechanism Research Project, it is interesting to know the history of the Mechanism's discovery, and the most groundbreaking projects that came along the way.

1.1.1 Discovery of the Antikythera shipwreck and early examination of its cargo

In his work "Gears from the Greeks: The Antikythera Mechanism – a Calendar Computer from ca. 80 B.C." Derek de Solla Price [19] gives a very detailed account of the research on the Mechanism starting with its discovery and ending at 1973. After Price's article was published, research continued with the most influential reconstructions being the model created by Michael Wright [26, 27] and the universally accepted reconstruction model of the

Mechanism proposed by the international Antikythera Mechanism Research Program [1].
The Antikythera shipwreck was discovered by chance shortly before Easter of 1900, when sponge drivers from the island of Symi took shelter from a storm east of port Pinakakia, near the uninhabited island of Antikythera, and dropped anchor in a depth of about 40 meters. Diving in the unknown waters below, they came upon a shipwreck 50 meters in length with marble and bronze statues scattered around it. After their return to Symi, they decided to report the findings to the authorities, and on the 6th of November 1900, professor of archaeology at the University of Athens A. Oikonomou took two of the divers and a retrieved bronze arm to Spyridon Stais, minister of education and renowned archaeologist [19]. The first operation to retrieve the treasures of the Antikythera Shipwreck started in the last days of the same month and lasted until the 30th of September of the next year. During this operation the divers recovered some objects that are considered of high artistic significance, e.g. the bronze statue named the Antikythera Youth and a bronze male head named the Philosopher of Antikythera [22]. This came at a heavy price, as one diver was killed and two were left disabled [19]. From the findings the Shipwreck was dated somewhere between 80-50 B.C., limits that have been confirmed from later research [6]. The first account of the Antikythera Mechanism fragments comes from May 1902, from minister Stais himself [2]. This started the research on the Mechanism.

1.1.2 What is this machine? Who made it, what was its use?

The first guess was that the corroded mass of bronze with the inscription was an astrolabe [23, 22]. Due to the complexity of the mechanism, this was quickly challenged (later research showed that the inscriptions and construction of the Mechanism does not correspond to an astrolabe [19, 26, 1]) and the debate drew the attention of Albert Rehm [20], who proposed that the mechanism could be some sort of planetarium, maybe a *sphere* (a sort of planetarium) like the ones designed by Archimedes. Unfortunately a large portion of his research remains unpublished[12]. Rehm was halfway right in his assumption [19, 10]. Price describes him as one of the first to read the inscriptions on the Mechanism and discovering the egyptian month name PACHON written in Greek, which in his opinion ruled out any possibility of the Mechanism being an astrolabe. Although research has shown that the Mechanism is not a sphere, his assumption that the *Cosmos* was displayed on the Mechanism was right.

The first model of the Mechanism was constructed in 1928 by Rear Admiral Ioannis Theofanidis. In his paper "The Antikythera finding" [23] he credits Hipparchos with as its creator and includes the eccentricity device for the moon's anomaly in the orbit. Although Hipparchos is no longer considered the designer of the Mechanism, his lunar theory is present in every reconstruction and Theofanidis was right to include the eccentricity device [14]. He accepted that the Mechanism is too complex to be an astrolabe and his reconstruction is a multi-purpose device which would incorporate an astrolabe in order to navigate the ship using the position of the sun, the moon and the stars [23].

1.1.3 Research conducted by Price and Karakalos

New ground was broken in the seventies when Price (who was interested in the Mechanism since the early fifties) and Ch. Karakalos used gamma-radiographs and fine x-radiographs to examine the fragments. During the years 1971-1973 they managed to get a rather accurate tooth count on the gears and to measure their dimensions with amazing precision [19]. The model they constructed based on their conclusions answered the question of the Mechanism's general function: It was a lunisolar calendar machine with representation of the sun and moon system from a geocentric perspective in the front and a calendar display and an eclipse predictor on the back. In their model Price and Karakalos included a differential turntable, that took the complexity of the Mechanism to new levels. The differential of Price and Karakalos has been replaced by a pin-and-slot device that was observed from the AMRP.

1.2 The Antikythera Mechanism Research Project

The Antikythera Mechanism Reaseach Project is a collaboration of several academic institutions, where in addition to high-definition photography, three-dimensional microfocus computed tomography developed by X-Tek Systems Ltd. and digital optical imaging using polynomial texture mapping developed by Hewlett-Packard Inc. were used, and revealed great details on the surfaces of the Mechanims, making inscripitons on layers of bronze beneath the exposed surfaces readable for the first time [10]. The members of the project managed to reveal many of the Mechanism's secrets, as they got to examine parts of the Mechanism that were unreachable before.

1.2.1 Archeological findings of the AMRP

The technology that was made available to the AMRP allowed careful examination of the fragments, the results of which left little room for speculation concerning the nature of the Mechanism [10, 11]. The inscripitons on the front and the back on the Mechanism, were revealed to contain astronomical information, with a planetarium-like display on the front dial, and an eclipse predicting display and a calendar used at the time on the back. Beyond these details, a user's manual was on the front, providing instructions and explanations of the various displays. Not all parts have been fully read and interpreted, and some parts of the back inscription are of particular interest, as they may contain crucial information about the extent of the front display [10, 11, 12]. It was from the information on the eclipse predictor that allowed a better estimate of the age of the Mechanism's design [4], which shows that the Mechanism was designed around 205 BC, even though the ship carrying it sunk around 80 BC [6], strongly indicating that Archimedes could be the designer of the Mechanism.

1.2.2 Gear parameters determined by the AMRP

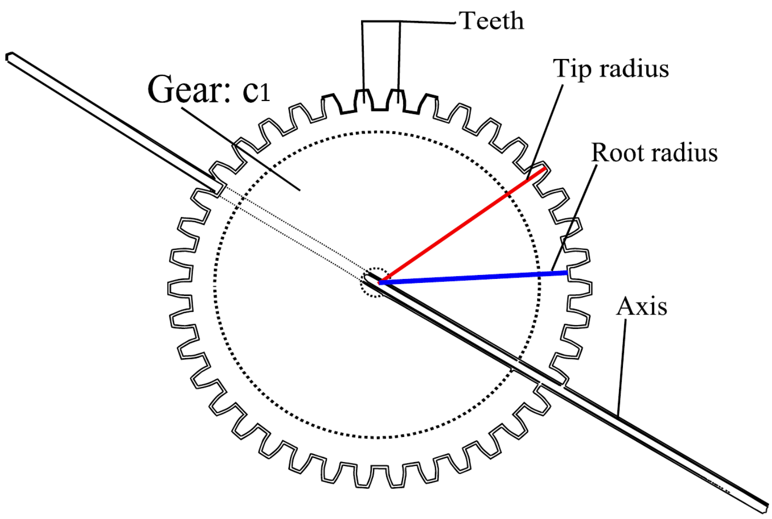

Figure 1: The terms used in this subsection, shown on a scetch of gear c1 by Anna Karnauhova

The scanning methods used did not only enable the AMRP members to read the inscriptions on the Mechanism, but also allowed researchers to better examine its gears, most of which were extensively damaged and deeply embedded in the fragments, giving them the opportunity to obtain a better image of their geometrical parameters [7]. The gears were named using the following scheme: To each gear, a unique combination of a letter and a number is given. Each letter corresponds to an axis, whereas gears on the same axis are numbered with 1 corresponding to the gear that is closest to the front of the Mechanism, and a larger number showing that the gear is further to the back. We start naming from the letter b, as the axis a would be the crank that put the Mechanism in motion.

For some of the gears, the scans allowed the researchers to obtain a definite tooth count:

Gear name	Efstathiou et al. tooth count
b3	32
d1	24
d2	127
f2	30
g2	20
l1	38
l2	53

Table 1: The tooth count for the definite gears

This was not the case for most of the teeth however. Corrosion made it difficult to simply count teeth and measure tip and root radii on the gears, as for some gears, examination of different sections would return different values. However, in an attempt to find the right parameters so that the Mechanism would be functional, the following values were accepted[7]:

Gear name	Average tip radius in mm	Average root radius in mm	Efstathiou tooth count
b1	65	63.8	223
b2	15.7	14.9	64
c1	10.3	9.4	38
c2	11	10.5	48
e1	9.7	8.6	32
e2	7.8	7.1	32
e3	52.4	51.5	223
e4	49.9	49.1	188
e5	13.1	12.2	50
e6	13.9	12.9	50
f1	14.6	13.6	53
g1	14.4	13.4	54
h1	13.7	13	60
h2	3.8	3	15
i1	13.2	12.6	60
k1	13.3	12.6	50
k2	14	13.1	50
m1	24.7	23.6	96
m2	4	3.7	15
o1	12.8	12.2	60

Table 2: The accepted tooth count for the indefinite gears

In conclusion, these are the geometrical parameters accepted of all the 27 gears found:

Gear name	Average tip radius in mm	Average root radius in mm	Efstathiou tooth count
b1	65	63.8	223
b2	15.7	14.9	64
b3	8.6	8.2	32
c1	10.3	9.4	38
c2	11	10.5	48
d1	5.6	5.1	24
d2	31.6	30.6	127
e1	9.7	8.6	32

e2	7.8	7.1	32
e3	52.4	51.5	223
e4	49.9	49.1	188
e5	13.1	12.2	50
e6	13.9	12.9	50
f1	14.6	13.6	53
f2	8.3	7.4	30
g1	14.4	13.4	54
g2	4.9	4.1	20
h1	13.7	13	60
h2	3.8	3	15
i1	13.2	12.6	60
k1	13.3	12.6	50
k2	14	13.1	50
l1	9.1	8.3	38
l2	13.1	12.5	53
m1	24.7	23.6	96
m2	4	3.7	15
o1	12.8	12.2	60

Table 3: The accepted tooth count for all the found gears

To obtain a complete model of the Mechanism, another five gears are needed. These gears have not been found, but their existence is hypothesized[10, 11, 7].

These are the geometrical parameters for these gears[7]:

Missing gear name	Tip radius	Root radius	Tooth count
m3	7.1	5.9	27
n1	13.9	12.8	53
n2	3.4	2.4	15
p1	13.8	12.8	60
p2	3.5	2.2	12

Table 4: The tooth count for the hypothesized gears

With the gear parameters listed above, a functional reconstruction of the

Antikythera Mechanism can be achieved. The shape of the teeth on the Mechanism's gears is flawed, as triangularly shaped teeth like the ones used in the Mechanism do not transmit rotation smoothly. To correct that, we need to use a more modern gear[13, 6].

2 Gears

2.1 A short history of gears

A gear can be described as a cylinder of small height with teeth placed on its perimeter. The shape the teeth have on the plane parallel to the base of the cylinder, is called tooth profile. Throughout the centuries, evolution of gears was a very slow process, as complex mechanical devices of high precision were not widely used until the construction of the first mechanical clocks[13, 6]. No original manuscripts from classical antiquity survive, and technical knowledge of the time is only known to us from medieval copies, which may not be faithful reproductions, either due to influence of later advancements or lack of understanding of the source material[13]. The earliest known device more complicated than the Mechanism is Giovanni de Dondi's Planetarium (in some texts Astrarium), constructed in 1364. De Dondi's construction breaks new ground in mechanical complexity, with 107 gears and seven displays, and what it lacked in portability with its height reaching one meter, it made up in efficiency[6]. Up to de Dondi, meshing metallic gears, used triangular teeth, sometimes with rounded tips (the tips of the Mechanism's gears were not rounded, as far as we know). De Dondi used a new profile to improve his planetarium; some gears have oval teeth, and tooth spacing variations were used. This was the time that traditional means of timekeeping such as clepsydrae were abandoned, and demand for complex and impressive mechanical clocks was booming (the first astronomical clock of the Strasbourg Cathedral was constructed between 1352 and 1354) and inaccurate gears presented a problem[6]. The first gear profile to solve this problem was the cycloid gear introduced around the middle of the 17th century and the involute gear tooth profile designed by Leonhard Euler[13].

These profiles are still in use today. Here we will examine the involute gear, as it is the more widespread of the two[13].

2.2 Basic definitions and the fundamental Law of Gears

Before examining the gear itself, we need to define rotation around a center.

2.2.1 Rotational motion on the plane

We define a real vector space with the orthonormal basis vectors

$e_x := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e_y := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In this vector space we take a vector r of fixed norm $\|r\| := \rho \in \mathbb{R}_{>0}$ and define the angle between r and the positive x-Axis as $\phi := \arctan(\frac{y_r}{x_r})$.

So r can now be written as $r = \rho(\cos(\phi)e_y + \sin(\phi)e_x)$. The unit vector in the direction of r is defined as the vector $e_r := \cos(\phi)e_y + \sin(\phi)e_x$ and the unit vector in the direction of the angle ϕ is defined as $e_\phi := \cos(\phi)e_y - \sin(\phi)e_x$. If the vector r is considered as a function of a single variable t, and differentiated we get the following:

$$\frac{dr}{dt} = \frac{d(\rho e_r)}{dt} = \rho\frac{de_r}{dt} = \rho\left(-\sin(\phi)e_y\frac{d\phi}{dt} + \cos(\phi)e_x\frac{d\phi}{dt}\right) = \rho\frac{d\phi}{dt}e_\phi$$

We will define $v_t := \rho\frac{d\phi}{dt}e_\phi$ as the tangential velocity of r, and $\omega := \frac{d\phi}{dt}$ as the angular velocity of r. Angular and tangential velocity are connected by the following relations:

$$v_t = \rho\omega\begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix},$$

$$\|v_t\| = \rho\omega$$

The term velocity is used to describe the differentials of v and ϕ, because if t represents time, these equations describe a rotating motion of constant radius ρ around a center $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Frequency. *The frequency of the motion described above is defined as* $f := \frac{\omega}{2\pi}$*. It is equivalent to* $\left[\frac{Rotations}{Timeunit}\right]$*. The period of the motion is defined as* $T := \frac{2\pi}{\omega} = \frac{1}{f}$

2.2.2 Formulating and proving the fundamental law of gears

A simple case: Two discs in contact Before introducing tooth profiles, let us examine a simple case of two discs in contact with eachother:

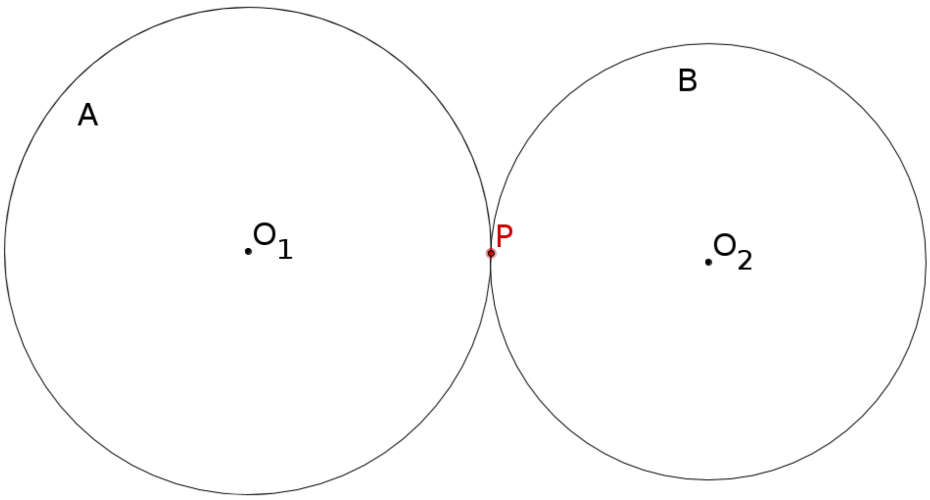

Figure 2: O_1 and O_2 are the centers of the discs 1 and 2 respectively, with P being their point of contact, called the *pitch point*

We suppose that the discs make contact in P in such a way that no fricton arises. If disc 1 starts rotating clockwise with angular velocity ω_1, then disc

2 starts rotating counterclockwise with an angular speed ω_2. Then, for the pitch point P the following holds:

$$\left.\begin{aligned} v_P &= \overline{O_1P}\omega_1 \\ v_P &= \overline{O_2P}\omega_2 \end{aligned}\right\} \qquad \Longrightarrow \frac{\omega_1}{\omega_2} = \frac{\overline{O_1P}}{\overline{O_2P}}$$

This is a very simple relation for constant frequency transmission, however there is a very important parameter that we have intentionally ignored in this scenario: that gears have teeth on their periphery and are not flat. The fundamental law of gears will provide us with the necessary geometric condition that gear teeth must satisfy if we are to have a constant frequency transmission ratio.

The fundamental law of gears: In another idealized case, we have toothed gears where all friction disappears, our gears come in contact at a single point on their teeth, named point of action and denoted with Q. The path of Q will be called line of action. The line of action can be found with relative ease, as it is on the perpendicular to tangent of the gear surfaces at Q. Setting the gears in motion will change the point of action, changing the direction of the tangent and its perpendicular. However, the pitch point and the centers of gears are not influenced. This allows to formulate the *fundamental law of gears*:

Fundamental law of gears. *For constant angular velocity ratio, the normal (i.e. the line of action) to the tooth profiles at the point of action Q must always pass through the pitch point P.*

The mathematical description of the fundamental law of gears goes as follows with the names of the points are to be taken from the following picture:

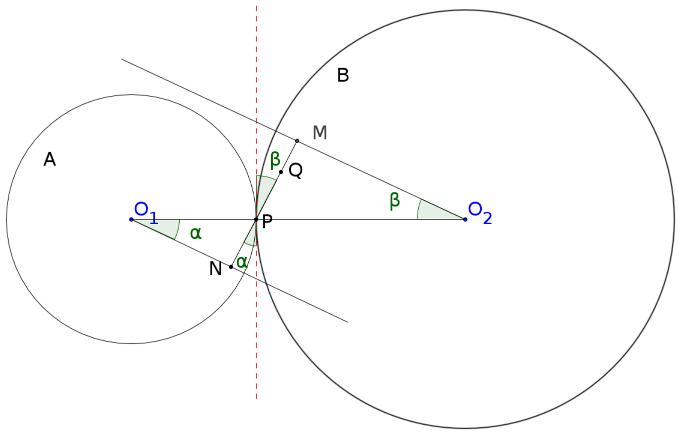

Figure 3: The general situation that will allow us to formulate the fundamental law of gears.

For the point of action Q, the angular velocities of gears A and B are ω_A and ω_B respectively. If the gears are to touch and not dig into each other, then the linear velocities on the line of action must be equal:

$$v_A \cos \angle QO_AM = v_B cos \angle QO_BN$$
$$\Leftrightarrow \omega_A \overline{O_AQ} \cos \angle QO_AM = \omega_B \overline{O_BQ} \cos \angle QO_BN$$
$$\Leftrightarrow \frac{\omega_A}{\omega_B} = \frac{\overline{O_BQ} \cos \angle QO_BN}{\overline{O_AQ} \cos \angle QO_AM} = \frac{O_BN}{O_AM}$$

Triangles $\triangle PO_AM$ and $\triangle PO_BN$ are similar, as $\angle PO_AM = \alpha = \beta =$

$\angle PO_BN$ and they have a right angle and points N and M, thus:

$$\frac{O_BN}{O_AM} = \frac{O_AP}{O_BP} \Rightarrow \frac{\omega_A}{\omega_B} = \frac{O_AP}{O_BP} \tag{1}$$

This means that the ratio of angular velocities remains constant if and only if the perpendicular to the tangent goes through the pitch point on the line connecting the centers of the gears.

2.3 Euler's idea: the involute gear

Leonhard Euler was the first to approach the search for an efficient gear profile using analysis and not geometry of his time[13]. He based the tooth profile on the involute of a circle. The circle is called the base circle, and the involute of a curve is best described as the endpoint of a string unwinding from the circle.

The involute of a circle We consider a circle around a center O, with radius ρ. We choose an arbirtary point M_0 on its surface. From there we start unwinding. This looks like this:

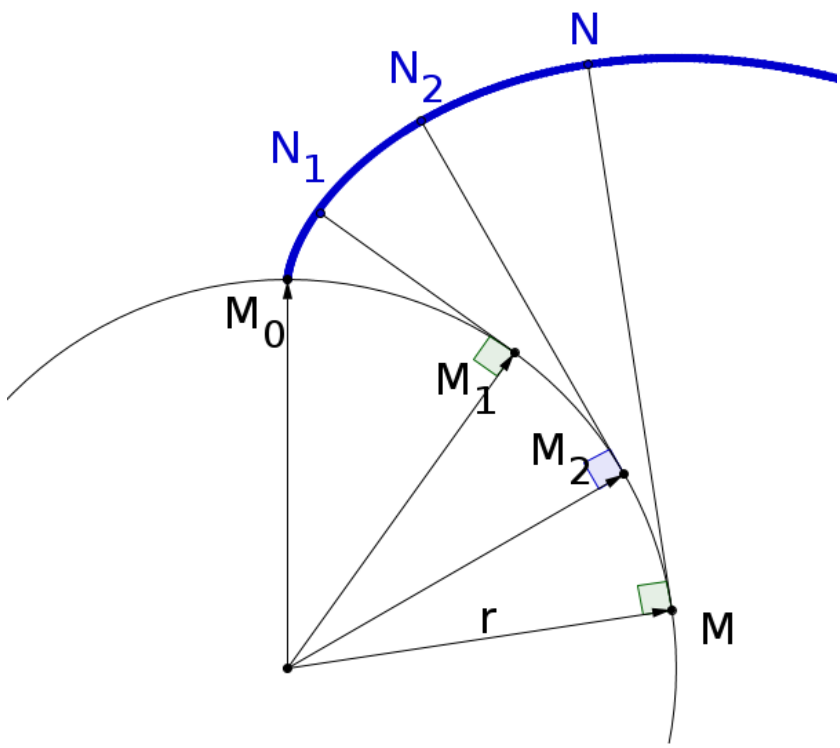

Figure 4: The involute unwinding from the circle

This is a very intuitive way to describe the involute, it is however easy to formulate in vector terms: The points M' of the involute are given by:

$OM' = r + MM'$, where r is a vectror of length ρ at an angle θ with the line OM_0 and MN' is a vector orthogonal to r with length equal to the circle segment M_0M[8]. Thus:

$$r = \rho \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$
$$MN = \int_0^\theta \rho \mathrm{d}\phi \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} = \rho\theta \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}$$

This gives the following equation for the points of the involute:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho(\sin\theta - \theta\cos\theta) \\ \rho(\cos\theta + \theta\sin\theta) \end{pmatrix}$$

Now it is time to examine the tangent and the normal of the involute at a given point. To this end, we differentiate the above equations for θ:

$$\begin{pmatrix} T_x \\ T_y \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}\theta} \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}(\rho(\sin\theta - \theta\cos\theta))}{\mathrm{d}\theta} \\ \frac{(\mathrm{d}\rho(\cos\theta + \theta\sin\theta))}{\mathrm{d}\theta} \end{pmatrix}$$
$$= \begin{pmatrix} \rho(\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta} - \frac{\mathrm{d}(\theta\cos\theta)}{\mathrm{d}\theta}) \\ \rho(\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta} + \frac{\mathrm{d}(\theta\sin\theta)}{\mathrm{d}\theta}) \end{pmatrix} = \begin{pmatrix} \rho\theta\sin\theta \\ \rho\theta\cos\theta \end{pmatrix}$$

The normal is orthogonal to the tangent and can be calculated as the vector product between the involute and a unit vector along a third axis z orthogonal to the plane of the circle,

$N = T \times e_z$ with $T_z = 0$ and $e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ [8].

$$N = \begin{pmatrix} T_x \\ T_y \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} T_y \\ -T_x \\ 0 \end{pmatrix}$$

$$\Rightarrow N = \begin{pmatrix} \rho\theta\cos\theta \\ -\rho\theta\sin\theta \\ 0 \end{pmatrix}$$

We are only interested in the x and y plane, so from now on we will consider the normal as $N = \begin{pmatrix} \rho\theta\cos\theta \\ -\rho\theta\sin\theta \end{pmatrix}$. The teeth involute gear are shaped by involutes: each side of the tooth is an involute, one is clockwise unwinding, the other counterclockwise unwinding, and the tip of the tooth is cut and rounded, instead of leaving it at the common point of its sides[8, 16].

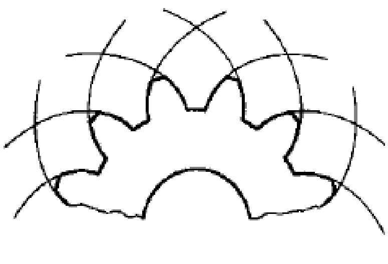

Figure 5: The shape of gear teeth is given by involutes. In modern machinery, the tips are rounded for mechanical reasons. Image source: KONSTRUKTION 2, Zahnrad- und Kegelradgetriebe Skript, TU Berlin, AG Konstruktion

For two gears to mesh properly, they must have the same ratio of tooth number to the distance between their centers to the pitch point. When two involute gears are meshing, they have a common tangent at their point of contact. This means that the normal to the involutes that shaped the sides of their teeth is also common, and it is a tangent to the base circle[13, 16], therefore, the involute gear satisfies the fundamental law of gears.

Gear trains Now that we have estabilished that involute gears transmit frequency at a constant rate, we will examine the following scenario: We are given four gears, called g1, g2, g3 and g4, with tooth counts z_1, z_2, z_3 and z_4 respectively. The gears are ordered so that g1 meshes with g2 and g3 meshes with g4, while g2 and g3 are fixed on the same axis, meaning that they rotate with the same frequency. We try to find out the relation between the frequency of g1 to the frequency of g4.

$$\begin{aligned} \frac{f_1}{f_2} &= \frac{z_2}{z_1} \\ \frac{f_3}{f_4} &= \frac{z_4}{z_3} \\ f_2 &= f_3 \\ \Rightarrow \frac{f_1}{f_4} &= \frac{f_1}{f_2}\frac{f_2}{f_4} = \frac{f_1}{f_2}\frac{f_3}{f_4} \\ \Rightarrow \frac{f_1}{f_4} &= \frac{z_2}{z_1}\frac{z_4}{z_3} \end{aligned}$$

This formula can be used for an arbirtary number of gears in the train, allowing us to calculate frequency ratios between first and last gear, only through the tooth count of the gears in the train. Gear trains are a crucial part of the Mechanism, as we will later see.

2.4 The extendened involute and Archimedes

If the involute is the trace of a string unwinding from a circle, then the extended involute is the trace of a string unwinding from a circle, the length of the string given by the arc length of a bigger circle. We will formulate this in mathematical terms, and examine a very interesting case, discovered long before the involute gear.

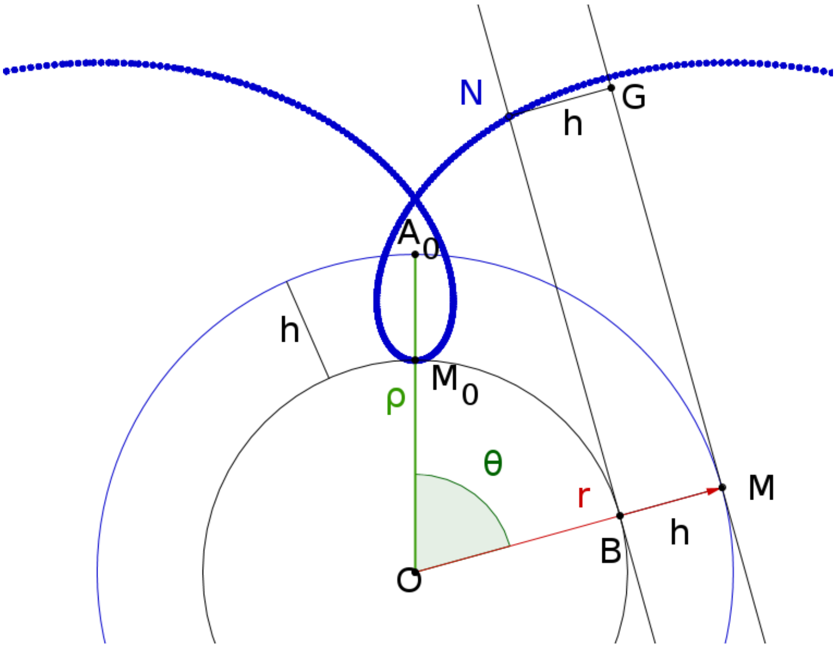

Figure 6: The construction of the extended involute

We will formulate this curve in vector terms as its every point N is given from:
$ON = OB + BN$ with OB having the direction of r but length $\rho - h$ and BN is tangential to the inner circle, with its length being equal to the arc length of MA_0 i.e. $BN = \rho\theta$.
We therefore, following the steps we did for the involute can write:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\rho - h)\sin\theta - \rho\theta\cos\theta \\ (\rho - h)\cos\theta + \rho\theta\sin\theta \end{pmatrix}$$

Obviously, if $h = 0$ we have the involute as described in the paragraph above.

The other special case is $h = \rho$ and this is what the extended involute looks like:

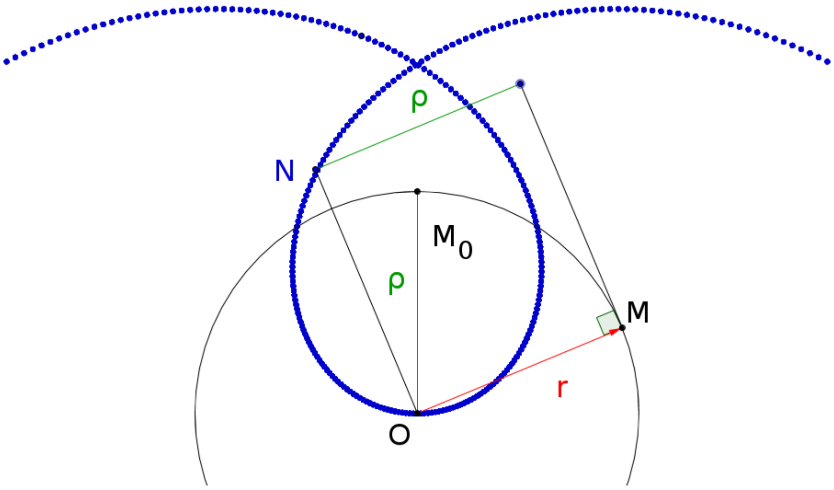

Figure 7: The extended involute for $h = \rho$

This curve is called the spiral of Archimedes [8]. It was discovered, as the name suggests by Archimedes, and it is as close as ancient mathematicians got to the involute. Unfortunately, mathematics and astronomy in Europe would decline soon after Archimedes' time [6] and the involute gear was not discovered until Euler.

2.5 Conclusion

There is no evidence that the involute was known in antiquity, nor is there any evidence to support the claim that any gear design other than triangular was used in antiquity [13, 6]. The Antikythera Mechanism's accuracy can be fully appreciated only when its gears are substituted by modern ones, which transmit frequencies accurately, and from now on, we will treat the Mechanism as if its gears' profile was shaped by involutes, instead of the original triangles. It is unfourtunate, that the knowledge that lead to the creation of the Mechanism was lost in Roman times [6] and one can only speculate as to what could have been if the developement of geared astronomical mechanisms had not come to such a sudden stop. Seeing how the Archimedean spiral is a special case of the involute, and keeping in mind that Archimedes is a very strong candidate as the Mechanism's designer, it might have been likely that at some further point a connection would have been possible, and the involute gear could be developed more than a thousand years before Euler. This assumption can not be verified due to the following obstales: The lack of evidence that anyone tried to improve on gears until the Middle Ages, and the fact that geared mechanisms in antiquity were made of bronze, having a rather short lifespan; it is estimated that friction would render the Antikythera Mechanism unusable after about twenty years [6]. This does not mean that the Antikythera Mechanism is to be thought less of, to the contrary, it is a machine that was not unsurpassed for more than a thousand years, and it is the only one machine of its kind that survived from ancient Greece to present day, even if it had to suffer extensive damage at the bottom of the sea for two thousand years [6].

The death of Archimedes by the hands of a Roman soldier is symbolical of a world-change of the first magnitude: the Greeks, with their love of abstract science, were superseded in the leadership of the European world by the practical Romans.

Alfred Whitehead [24]

3 Continued fractions and best rational approximations

In this section a rather simple yet very efficient method of approximating irrational numbers with fractions is introduced. We will examine continued fractions and how they can be used to provide us rational approximations of irrational numbers.

3.1 Definition and notation

Definition 1 [17]. *An expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_{n-1} + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}}}$$

$a_0 \in \mathbb{N} \cup \{0\}; a_i \in \mathbb{N}, i \geq 1$ *is called a simple continued fraction.*

For the moment, we will concern ourselves with finite simple continued fractions. These are simple continued fractions of the following form:

$$\xi_n := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_{n-1} + \cfrac{1}{a_n}}}}$$

For practical reasons, the notation is introduced:

$\xi_n = [a_0, a_1, \dots, a_n]$.

Example 3.1. The continued fraction $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}} = \frac{225}{157}$

can be written as: $\dfrac{225}{157} = [1, 2, 3, 4, 5]$

To bring a continued fraction to the form $\frac{p}{q}$, $p, q \in \mathbb{Z}, q \neq 0$ is quite straightforward, we only need to start calculating "from the bottom up": Invert a_n, add $\dfrac{1}{a_n}$ to a_{n-1} and keep going upwards. The generation of continued fractions from rational numbers, however is very interesting.

3.2 Expansion of rational numbers

Here we will see how to expand a rational number into a continued fraction, setting the base for the expansion of irrational numbers.

Example 3.1. Consider the fraction $\dfrac{254}{19}$. We are going to calculate its continued fraction expansion.

We begin by diving the numerator by the denominator and isolate the integer part from the rest:

$$\frac{254}{19} = 13 + \frac{7}{19}$$

We now apply the same procedure on the inverted rest:

$$\frac{19}{7} = 2 + \frac{5}{7}$$

Once more on $\frac{7}{5}$ gives: $\frac{7}{5} = 1 + \frac{2}{5}$
We stop when the enumerator of the rest is 1: $\frac{5}{2} = 2 + \frac{1}{2}$
This means, that the inverse of the rest is an integer and we stop calculating.

Substituting our results in $\frac{254}{19} = 13 + \frac{7}{19}$ yields:

$$\frac{254}{19} = 13 + \frac{7}{19} = 13 + \frac{1}{\frac{19}{7}} = 13 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = [13, 2, 1, 2, 2]$$

This example illustrated the way the algorithm works. In a more formal language the algorithm applied to a number $\xi \in \mathbb{Q}\backslash\mathbb{Z}$ is formulated as:

$$\xi = \lfloor \xi \rfloor + (\xi - \lfloor \xi \rfloor),$$

where $\lfloor \xi \rfloor$ is the floor function. Since $0 < \xi - \lfloor \xi \rfloor < 1$ a $\xi_1 := \frac{1}{\xi - \lfloor \xi \rfloor} > 1$ is defined, and the same procedure applied to ξ_1 produces a ξ_2 as follows:

$$\xi = \lfloor \xi \rfloor + (\xi - \lfloor \xi \rfloor) = \lfloor \xi \rfloor + \frac{1}{\xi_1} = \lfloor \xi \rfloor + \frac{1}{\lfloor \xi_1 \rfloor + (\xi_1 - \lfloor \xi_1 \rfloor)} = \lfloor \xi \rfloor + \cfrac{1}{\lfloor \xi_1 \rfloor + \cfrac{1}{\xi_2}}.$$

The algorithm stops at the first i for which ξ_i is an integer.

This algorithm, which is based on the euclidean division algorithm, produces an "almost unique" finite simple continued fraction for every rational number: After obtaining the continued fraction we can modify it by manipulating the last term a_n as follows:

If $a_n > 1$ we can write $a_n = a_n - 1 + \frac{1}{1}$ turning $[a_0, a_1, \ldots, a_n]$ into $[a_0, a_1, \ldots, a_n - 1, 1]$ and if $a_n = 1$ then we can write $a_{n-1} + \frac{1}{a_n} = a_{n-1} + 1$ and $[a_0, a_1, \ldots, a_{n-1}, a_n] = [a_0, a_1, \ldots, a_{n-1} + 1]$. This does not prevent us from saying that our algorithm produces a unique expansion, due to the fact that it stops for the first rest the inverse of which is an integer. This prevents the case that $a_n = 1$, since $a_{n-1}+1$ is the first such term, and the case $a_n > 1$ requires us to make an extra step, after the algorithm would have stopped.

3.3 Expansion of irrational numbers

We saw above how to obtain a continued fraction expression from a given rational number. What happens, however, when we apply our algorithm on an irrational number? Note that nowhere in the algorithm's steps (except the starting definition) it is important that ξ is a rational number.

Example 3.1. Let us try to apply the above algorithm to an irrational number: Let $r \in \mathbb{R}\backslash\mathbb{Q}$, and observe:

$$r = \lfloor r \rfloor + (r - \lfloor r \rfloor)$$

For the sake of convenience, we define a sequence $(r_i)_{i\in\mathbb{N}\cup 0}$ with $r_0 := \lfloor r \rfloor$ and r_i defined recursively as $r_i = \dfrac{1}{r_{i-1} - \lfloor r_{i-1} \rfloor}$ Our algorithm applied to r gives us:

$$
\begin{aligned}
r &= \lfloor r \rfloor + (r - \lfloor r \rfloor) = \lfloor r \rfloor + \frac{1}{r_1} \\
r_1 &= \lfloor r_1 \rfloor + (r_1 - \lfloor r_1 \rfloor) = \lfloor r_1 \rfloor + \frac{1}{r_2} \\
r_2 &= \lfloor r_2 \rfloor + (r_2 - \lfloor r_2 \rfloor) = \lfloor r_2 \rfloor + \frac{1}{r_3} \\
&\vdots \\
r_n &= \lfloor r_n \rfloor + (r_n - \lfloor r_n \rfloor) = \lfloor r_n \rfloor + \frac{1}{r_{n+1}} \\
&\vdots
\end{aligned}
$$

By substitution it follows:

$$
r = r_0 + \cfrac{1}{\lfloor r_1 \rfloor + \cfrac{1}{\lfloor r_2 \rfloor + \cfrac{1}{\ddots + \cfrac{\ddots}{\lfloor r_{n-1} \rfloor + \cfrac{1}{\lfloor r_n \rfloor + \cfrac{1}{\ddots}}}}}}
$$

This procedure is infinite. If there was a final r_i, then r would be rational which, due to its definition is not the case [17].

In example 3.1 we constructed an *infinite simple contiued fraction.* From now on, for $r \in \mathbb{R} \backslash \mathbb{Q}$ we will write $r = [r_0, \lfloor r_1 \rfloor, \lfloor r_2 \rfloor, \ldots, \lfloor r_n \rfloor, \ldots]$.

We now proceed to examine what happens when we stop this algorithm at an arbitrary $i \in \mathbb{N}$.

3.4 Best rational approximations

Once again, consider $r \in \mathbb{R}\backslash\mathbb{Q}$ such that $r = [r_0, \lfloor r_1 \rfloor, \lfloor r_2 \rfloor, \ldots, \lfloor r_n \rfloor, \ldots]$, and for tha sake of convenience write $a_i = \lfloor r_i \rfloor$ In this case we are going to stop the algorithm at a random $i \in \mathbb{N}$, getting the first $i + 1$ continuants. This term will be denoted as $\xi_{0,i} := [a_0, a_1, \ldots, a_i]$. Then the following is true:

Theorem [5]. There exist $A_i, B_i \in \mathbb{N}$, such that the fracton $\xi_{0,i} := \frac{A_i}{B_i}, i \geq 1$ is irreducible. A_i and B_i are precisely defined by the following recursion:

$$A_i = a_i A_{i-1} + A_{i-2} \tag{2}$$

$$B_i = a_i B_{i-1} + B_{i-2} \tag{3}$$

with $A_{-1} := 1, A_0 := a_0$ **;** $B_{-1} := 0, B_0 := 1$.

Proof. The proof of the theorem splits into two parts: We begin by proving the recursion, and then prove the irreducibility of the fraction. The recursion is proved by induction on i:

Induction basis:

$$\frac{A_i}{B_i} = \frac{a_i A_{i-1} + A_{i-2}}{a_i B_{i-1} + B_{i-2}}$$

and try to construct $\dfrac{A_{i+1}}{B_{i+1}}$. In order to do that, we raise the index from i to $i+1$ (induction step). This makes a_i into $a_i + \dfrac{1}{a_{i+1}}$ and the finite continued fraction

$$\xi_{0,i} := a_0 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_{i-1} + \cfrac{1}{a_i}}}}$$

into $$\xi_{0,i+1} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_{i-1} + \cfrac{1}{a_i + \cfrac{1}{a_{i+1}}}}}}$$

making $\dfrac{A_i}{B_i}$ into $\dfrac{A_{i+1}}{B_{i+1}}$.

Since the terms A_{i-1}, A_{i-2}, B_{i-1}, B_{i-2} are not affected by the change of i, $\dfrac{A_{i+1}}{B_{i+1}}$ can be written as:

$$\begin{aligned}
\frac{A_{i+1}}{B_{i+1}} &= \frac{(a_i + \frac{1}{a_{i+1}})A_{i-1} + A_{i-2}}{(a_i + \frac{1}{a_{i+1}})B_{i-1} + B_{i-2}} \\
&= \frac{(a_{i+1}a_i + 1)A_{i-1} + a_{i+1}A_{i-2}}{(a_{i+1}a_i + 1)B_{i-1} + a_{i+1}B_{i-2}} \\
&= \frac{a_{i+1}(a_iA_{i-1} + A_{i-2}) + A_{i-1}}{a_{i+1}(a_iB_{i-1} + B_{i-2}) + B_{i-1}} \\
&= \frac{a_{i+1}A_i + A_{i-1}}{a_{i+1}B_i + B_{i-1}},
\end{aligned}$$

confirming the recursive formula.

It remains to show that $\dfrac{A_i}{B_i}$, $i \in \mathbb{N}$ is irreducible.

The fraction $\dfrac{A_i}{B_i}$ is irreducible if, and only if, A_i and B_i are relatively prime, i.e. their greatest common divisor is 1, and we will prove that by using the following theorem from number theory[25]:

Any two integers $z_1, z_2 \in \mathbb{Z}$ are relatively prime if and only if there exist $x, y \in \mathbb{Z}$ such that $xz_1 + yz_2 = 1$ i.e. $gcd(z_1, z_2) = 1 \Leftrightarrow \exists x, y \in \mathbb{Z} : xz_1 + yz_2 = 1$.

This is again shown by induction on i:

Induction basis for $i = 1$:

$$A_1 B_0 - A_0 B_1 = a_0 a_1 + 1 - a_0 a_1 = 1 = (-1)^{i-1}$$

Induction step from $i = k$ to $i = k + 1$:

Let for $i = k$

$$A_k B_{k-1} - A_{k-1} B_k = (-1)^{k-1}$$

Then for $i = k + 1$

$$\begin{aligned} A_{k+1} B_k - A_k B_{k+1} &= (a_{k+1} A_k + A_{k+1}) B_k - A_k (a_{k+1} B_k + B_{k-1}) \\ &= a_{k+1} A_k B_k + A_{k-1} B_k - a_{k+1} B_k A_k - A_k B_{k-1} \\ &= A_{k-1} B_k - A_k B_{k-1} = -(-1)^{k-1} = (-1)^{k-1+1} = (-1)^k \end{aligned}$$

If $A_{k+1} B_k - A_k B_{k+1} = -1$, then for $x' := -B_k$ and $y' := -A_k$ the equation $A_{k+1} x' - y' B_{k+1} = 1$ is true.

Summing up, for A_i and B_i; $i \in \mathbb{N}$ either $A_i B_{i-1} - A_{i-1} B_i = 1$ or $A_i(-B_{i-1}) - (-A_{i-1}) B_i = 1$. Hence, $gcd(A_i, B_i) = 1 \; \forall \; i \in \mathbb{N}$, making $\frac{A_i}{B_i}$ irreducible. □

These fractions $\frac{A_i}{B_i}$ are called *best rational approximations* of r. We define a best rational approximation in the following way:

Definition 2[18]. A fraction $\frac{a}{b}$ is called a *best rational approximation* of $r \in \mathbb{R}_{>0} \backslash \mathbb{Q}$ if $|r - \frac{a}{b}| < |r - \frac{a'}{b'}|$, $a, b, a', b' \in \mathbb{N}$; $b' \leq b$[18]

And now we prove our statement in two steps: We start by finding an upper bound for the approximation error of these fractions, and then we show that these fractions minimize the error for all denominators less or equal to B_i.

Step 1 is a lemma on the bound of the error:

Lemma 1. For a $r \in \mathbb{R}\backslash\mathbb{Q}$, $r = [a_0, a_1, \dots]$ the approximation error of $\xi_{0,n} = [a_0, a_1, \dots, a_n] =: \dfrac{A_n}{B_n}$ is bound as follows:

$$\left| r - \frac{A_n}{B_n} \right| < \frac{1}{B_n B_{n+1}}$$

Proof. [5]

$$\begin{aligned}
\frac{A_n}{B_n} &= \frac{A_n}{B_n} + \sum_{i=0}^{n-1} \frac{A_i}{B_i} - \sum_{i=0}^{n-1} \frac{A_i}{B_i} \\
&= (\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}) + (\frac{A_{n-1}}{B_{n-1}} - \frac{A_{n-2}}{B_{n-2}}) + \cdots + (\frac{A_1}{B_1} - \frac{A_0}{B_0}) + \frac{A_0}{B_0} \\
&= \frac{(-1)^{n-1}}{B_n B_{n-1}} + \frac{(-1)^{n-2}}{B_{n-1} B_{n-2}} + \cdots + \frac{1}{B_1 B_0} + a_0
\end{aligned}$$

Define $c_i := \dfrac{(-1)^{i-1}}{B_i B_{i-1}}, i \in \mathbb{N}$ and , $x = a_0 + \sum_{i=1}^{\infty} c_i$, and
$\xi_{0,n} = \sum_{i=1}^{\infty} c_i$. Since $B_{i+1} > B_i$, $\lim_{i\to\infty} |c_i| = 0$, $|r - \xi_{0,n}| < |c_{n+1}|$ is true, thus proving the inequality. □

And step 2:

Theorem. For a given $r \in \mathbb{R}\backslash\mathbb{Q}$ with $r = [a_0, a_1, \dots]$ the continued fraction $\xi_{0,n} = [a_0, a_1, \dots, a_n] =: \dfrac{A_n}{B_n}$ is a best rational approximation.

Proof. Suppose that a fraction $\dfrac{p}{q}$; $p, q \in \mathbb{N}, q \leq B_n$ is a better approximation of r, i.e. $\left| r - \dfrac{p}{q} \right| < \left| r - \dfrac{A_n}{B_n} \right|$. Then,

$$\left|\frac{p}{q}-\frac{A_n}{B_n}\right| \leq \left|r-\frac{p}{q}\right| + \left|r-\frac{A_n}{B_n}\right| \leq 2\left|r-\frac{A_n}{B_n}\right| < 2\frac{1}{2B_n^2} = \frac{1}{B_n^2}$$

Whereas $\left|\frac{p}{q}-\frac{A_n}{B_n}\right| = \left|\frac{pB_n - qA_n}{qB_n}\right| > \frac{1}{B_n q} > \frac{1}{B_n^2}$, which contradicts our assumption that $\left|\frac{p}{q}-\frac{A_n}{B_n}\right| < \frac{1}{B_n^2}$

□

Let us now calculate the first four best rational approximations of $\pi = 3.14159\ldots$ [17].

$$\pi = 3 + 0.14159\cdots = 3 + \frac{1}{7.062513306}$$
$$7.062513306 = 7 + 0.062513306 = 7 + \frac{1}{15.99659441}$$
$$15.99659441 = 15 + 0.99659441 = 15 + \frac{1}{1.003417228}$$
$$1.003417228 = 1 + 0.003417228 = 1 + \frac{1}{292.6348337}$$
$$292.6348337 = 292 + 0.6348337 = 292 + \frac{1}{1.575215807}$$
$$1.575215807 = 1 + 0.575215807 = 1 + \frac{1}{1.738477957}$$

Here we break the algorithm. We have now that $\pi = [3, 7, 15, 1, 292, 1, \dots]$. This gives us the following rational approximations:

i	1	2	3	4	5
$\frac{A_i}{B_i}$	3	$\frac{22}{7}$	$\frac{333}{106}$	$\frac{355}{113}$	$\frac{103993}{33102}$

Table 5: The five best rational approximations of π

3.4.1 Ancient astronomical observations

Now we will examine some observations of ancient astronomy and compare them to the best rational approximations of the actual values. We will begin with observations concerning the sun, the earth and the moon, and proceed with the approximations of the planets known in antiquity. The periods that will be examined in this thesis[1] are given in the following table, with their continued fraction expansions[2] up to the sixth continuant.

Period	Value accepted the actual one	Continued fraction expansion
Year length measured in days	365.2421897	[365; 4, 7, 1, 3, 27, 2, ...]
Sidereal months in a year	13.368267	[13; 2, 1, 2, 1, 1, 17, ...]
Period of Mercury in the night sky	0.2408404	[0; 4,6,1,1,2,1, ...]
Period of Venus in the night sky	0.6151854	[0; 1,1,1,1,2,29, ...]
Period of Mars in the night sky	1.8808148	[1; 1,7,2,1,1,3, ...]
Period of Jupiter in the night sky	11.8617555	[11; 1,6,4,3,1,1, ...]
Period of Saturn in the night sky	29.4565217	[29;2,5,4,4,2,16, ...]

Table 6: Astronomical values with their continued fraction expansions

[1] The periods related to the sun, earth and moon system are discussed in chapter 4, the ones concerning the planets' synodic periods are discussed in chapter 5.

[2] As these values are irrational, we accept the first seven decimals for all periods, except the Sarros where we stop at four.

The actual numbers were not known in antiquity. Ancient astronomy was based on observation, and the universe was believed to be geocentric, i.e. the celestial bodies moved around the earth [15]. Some[3] observations for the above periods are given here [10, 12]:

Period	Value observed by ancient astronomers	Continued fraction expansion
Year length measured in days	$\frac{1461}{4}$	[365; 4]
Sidereal months in a year	$\frac{254}{19}$	[13; 2, 1, 2, 1, 1]
Period of Mercury in the night sky	$\frac{46}{191}$	[0;4,6,1,1,2,1]
Period of Venus in the night sky	$\frac{8}{13}$	[0;,1,1,1,1,2]
Period of Mars in the night sky	$\frac{79}{42}$	[1;1,7,2,1,1]
Period of Jupiter in the night sky	$\frac{83}{7}$	[11;1,6]
Period of Saturn in the night sky	$\frac{59}{2}$	[29;2]

Table 7: Ancient observed values with their continued fraction expansions

Comparing the two tables, we see that, unknowingly, ancient astronomers had used best rational approximations for all the observations examined here.

The Saros The Saros cycle is an eclipse prediction cycle, that is defined as 223 synodic months, or 239 sidereal months, or 242 draconic months . These duration are not equal to each other, which at first glance makes the definition of the Saros rather vague. However the greatest difference between these definitions is less than five and a half hours (between the 223 synodic and the 239 sidereal)[4], and it was accepted that the Saros was 6585 days and 8 hours long. The way the Saros appears in the Mechanism is obtained by setting the different periods equal to each other [27], as explained in the next

[3]there were several estimates for some of these periods, e.g. Mars, here we see only the most accurate

[4]A more detailed explanation of these terms is found in Appendix A

chapter.

4 AMRP 2005 model of the Mechanism

In this section the 2005 reconstruction proposed by the Antikythera Mechanism Research Project[1] will be presented. Using the information explained earlier, and expanding on it, the AMRP came up with a model of the Antikythera Mechanism which is now considered the most accurate so far. It consists of three parts, working simultaneously: the sun and moon dial on the front, and on the back, the lunisolar calendars of Meton and Kallippos, and the Saros and Exeligmos eclptic cycles. Note that the calculations in this section hold as long as the involute gear is used instead of the original triangularly shaped teeth of the Mechanism.[13]

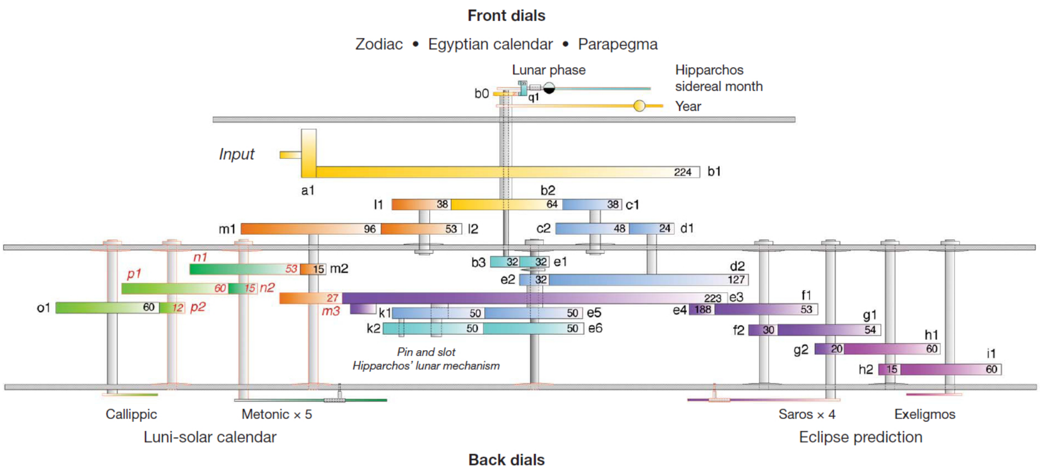

Figure 8: This is a scheme of the Mechanism's gears, viewed from the side. Every rectangle is a gear, with every letter corresponding to an axis, and coaxial gears being enumerated front-to-back, i.e. gear b1 is closer to the front than gear b2. [10]

4.1 The front dial

4.1.1 The sidereal month assembly

The geocentric system used at the time of the Mechanism's was illustrated on the front dial of the Mechanism, on which two spherical markers were representing the sun's and the moon's position in relation to the earth [10, 19]. The pointers would be rolling on circular orbits around the earth positioned in the center, with the display resembling that of a wristwatch. It is furthermore possible that the moon pointer was two-colored, consisting of a black and a white (or silver) hemisphere, showing the differrent phases of the moon, but this can only be hypothesized as the pointers have not been found [6]. The frequency with which these indices turned was determined as such:

Using the knowledge from the previous chapters, the frequency ratio between a full rotation of the sun dial and the number of rotations of the moon dial can be calculated, ignoring for the time the pin-and-slot device which does not influence the mean angular velocity, only the momentary in order to express the lunar anomaly. The ratio between the frequency of gear b1 to e2 is given by the gear train b1-b2-c1-c2-d1-d2-e2 [10] which calculates:

$$\frac{\omega_{b1}}{\omega_{e2}} = \frac{d_{b2}}{d_{c1}}\frac{d_{c2}}{d_{d1}}\frac{d_{d2}}{d_{e2}} = \frac{64}{38}\frac{48}{24}\frac{127}{32} = \frac{254}{19} \tag{4}$$

This means that 254 full *sidereal* lunar rotations correspond to 19 rotations of the earth around the sun (or of the sun around the earth from an ancient Greek astronomer's point of view). The actual number is the irrational number with the first six decimals $r = 13.368267\ldots$ The continued fraction expansion of this number is $r = [13; 2, 1, 2, 1, 1, 17\ldots]$ and it gives the following best rational approximations: $\frac{27}{2}, \frac{40}{3}, \frac{107}{8}, \frac{147}{11}, \frac{254}{19}$ and finally $\frac{4465}{334}$. The ratio in the mechanism is a best rational approximation of the irrational astronomical ratio, and it was known since the ancient Babylonians [12]. The margin of error of this approximation is bounded as follows:

$$\left| r - \frac{254}{19} \right| < \frac{1}{19 \times 334} = \frac{1}{6346} = 0.00015758$$

This error is small, and the way the approximation was implemented is also interesting: in order for the dial to work (remember, this was not only about gears rotating, but also for a useful display) a pair of gears with 249 and 19 teeth respectively would not work very well [16]. Therefore the designer was forced to use at least one extra meshing pair. This would present a problem, as the last gear of the train and the first would be rotating in opposite directions, which would contradict the geocentric point of view. This called for an odd number of meshing pairs in the train, resulting in three pairs. Having dealt with this problem, the number 254 was broken into its prime factors 127 and 2. 127 had to appear as the tooth count of a gear, and this was chosen as d2, the 2 however is unusable as a tooth count[16]. Thus, it had to be expressed as a ratio. Another problem in this train of gears was the number 19. By modern standards it is marginally low for a tooth count[16], and it is best if it could be avoided. The easiest way to bypass this, is to introduce its double, 38 in gear c1. So far the tooth count of two out of six given gears is fixed. The condition for the remaining four, is that the ratio produced by $\frac{d_{b2}}{38}\frac{d_{c2}}{d_{d1}}\frac{127}{d_{e2}}$ must be 4. The numbers chosen are relatively practical: b2, the only gear common in all parts of the Mechanism is given 64 teeth, a power of 2, which would be easy to construct by bisecting the circle and then the resulting sections, whereas c2 and d1 had 48 and 24 teeth respectively.

The above part of the Mechanism provided a fairly accurate model for the ratio between months and years, yet it could not account for the lunar anomaly that was known to the ancient Greeks. This called for a more complex gearing system, in this case a pin-and-slot device was used.

4.1.2 Hipparchos' lunar theory in the pin-and-slot device

The pin-and-slot device consists of two eccentric gears, with the driven gear, Gear 2 having its axis mounted on the driving gear, Gear 1. Gear 1 has a little pin protruding from it, whereas Gear 2 has a slot on its surface, so that the pin of Gear 1 fits into the slot. As Gear 1 is turned, the pin moves in the slot and if we consider the angular velocity of Gear 1 constant, the angular velocity of Gear 2 changes periodically as we will see.

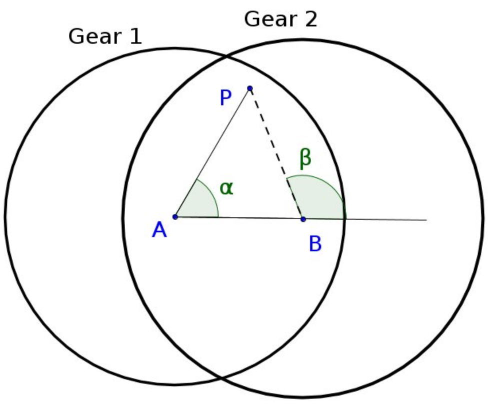

Figure 9: The general idea of the pin-and-slot device. The driving gear is Gear 1, the driven gear is Gear 2. P marks the position of the pin, fixed on Gear 1, whereas the dotted line on Gear 2 is the slot wherein the pin slides

It is interesting to note that this device took the sidereal month from gear e2 as input and adjusted it using Hipparchos's theory [14]. The ratio of the gear tooth count in it is $\frac{d_{e5}}{d_{k1}}\frac{d_{k2}}{d_{e6}}\frac{d_{e1}}{d_{b3}} = \frac{50}{50}\frac{50}{50}\frac{32}{32} = 1$ [10]. From the scans of the mechanism, the eccentricity ε is $\frac{d}{a} = \frac{1.1}{9.6} \approx 0.114583$ $\varepsilon = \frac{AB}{AP} = \frac{1.1}{9.6} \approx 0.1146$ [14]. In Eq(7) the square of ε can be ignored, giving the following approximate value for ω_2:

$$\omega_2 = \omega_1 \frac{1 - \varepsilon \cos \omega_1 t}{1 - 2\varepsilon \cos \omega_1 t} = \omega_1 (1 - \varepsilon \cos \omega_1 t)(1 - 2\varepsilon \cos \omega_1 t)^{-1}.$$

Applying the binomial theorem on $(1 - 2\varepsilon \cos \omega_1 t)^{-1}$ gives:

$$\begin{aligned} \omega_2 &\approx \omega_1(1 - \varepsilon \cos \omega_1 t)(1 + 2\varepsilon \cos \omega_1 t) \\ &= \omega_1(1 - \varepsilon \cos \omega_1 t + 2\varepsilon \cos \omega_1 t + \mathcal{O}(\varepsilon^2)) \\ &= \omega_1(1 + \varepsilon \cos \omega_1 t) = \omega_1 + \omega_1 \varepsilon \cos \omega_1 t \end{aligned}$$

Integration by dt produces the following relation between ϕ_2 and ω_1:
As evidenced by Elias Gourtsoyannis in [14] the angular speed ω_2 of the driven gear, Gear 2 is given in relation to the angular speed ω_1 of the driving gear, Gear 1 by:

$$\omega_2 = \omega_1 + \varepsilon \omega_1 \cos \omega_1 t \tag{5}$$

$$\phi_2 = \omega_1 t + \varepsilon \sin \omega_1 t$$

As evidenced by E.Gourtsoyannis et al.[14] the input of astronomical data in this formula will produce a value of 0.1142 for ε which is within the limits given by the scans of the Mechanism[14].
The moon's angular velocity can be approximated using 5 with the following astronomical data as input:

$$\text{Mean sidereal month: } T_s = 27.555 \text{ days}$$
$$\text{Mean lunar angular velocity: } \omega_1 = \frac{360°}{27.555} = \frac{13.065°}{day}$$

This input would make 5:

$$\omega_2 = 13.065 + 1.4917 \cos 13.065t$$

This gives us that $\varepsilon\omega_1 = \dfrac{1.4917^\circ}{day}$, thus $\varepsilon = \dfrac{1.4917}{13.065} =\approx 0.1142$ [14]. This value comes close to the one observed in the scans of the Mechanism. The pin-and-slot device was hypothesized to be part of a lost part of the Mechanism showing other planets on the front dial [14], this assumption has been challenged in [3] and in [12], and it is rather unlikely that such an eccentricity device could be used in a geocentric representation of the solar system for any of the planets[3, 12].

4.2 The back dials

4.2.1 The Saros and Exeligmos cycles

The fact that lunisolar eclipses occur on a regular period was known to the ancient Babylonians and from them it was passed on to ancient Greece [15]. The hellenisation of the Babylonian term for the repetition cycle of the eclipses is Saros. The Saros is roughly equal to 223 synodic months, with a duration of 6585.3223 days ($\approx$ 6585 days 7 hours 43 minutes) long. This was accepted in ancient astronomy as $6585\frac{1}{3}$ days, and it was observed that eclipses were repeated nearly identically after three Saros cycles had passed, since the $\frac{1}{3}$ of a day corresponds to a westward shift of the eclipse's path by 120°. Thus, a cycle three times the duration of the Saros was defined by the Greeks, called Exeligmos[15, 10]. The Saros and the Exeligmos were illustrated in two of the back dials of the Mechanism. The gear sequence describing them is b2-l1-l2-m1-m3-e3-e4-f1-f2-g1-g2-h1-h2-i1. The Saros dial was g1 and the Exeligmos dial was i1[10].

In order to make the Mechanism easier to use and understand, its creator made the Saros index move on a spiral, of 4 coils, while the Exeligmos circular diagram was divided in three sectors, so that each sector was one

Saros[10, 21]. This display method was used again in the calendar display. Examining the gear sequence b2-l1-l2-m1-m3-e3-e4-f1-f2-g1 gives the following approximation for the length of the Saros cycle:

$$\frac{z_{b2}}{z_{l1}}\frac{z_{l2}}{z_{m1}}\frac{z_{m3}}{z_{e3}}\frac{z_{e4}}{z_{f1}}\frac{z_{f2}}{z_{g1}} = \frac{64}{38}\frac{53}{96}\frac{27}{223}\frac{188}{53}\frac{30}{54} = \frac{516533760}{2328248448} = \frac{940}{4237} = \frac{4}{223}\frac{235}{19}$$

This ratio is the most complicated appearing in the Mechanism, and it must be understood as such:

The number 4 in the numerator is there so that one Saros is 4 full circles of the pointer, and the ratio $\frac{223}{235 \times 19}$ is obtained by the following procedure [27]:

$$\begin{aligned} 242 \text{ draconic months} &= 223 \text{ synodic months} \\ 235 \text{ synodic months} &= 254 \text{ sidereal months} \\ 254 \text{ sidereal months} &= 19 \text{ years} \end{aligned}$$

Since the Saros dial must display the eclipses, it needs to present the 242 draconic months. Thus:

$$\begin{aligned} 242 \text{ draconic } \times 235 \text{ synodic } &= 223 \text{ synodic } \times 254 \text{ sidereal} \\ 242 \text{ draconic } &= \frac{223 \text{ synodic } \times 254 \text{ sidereal}}{235 \text{ synodic}} \\ &= \frac{223 \text{ synodic } \times 19 \text{ years}}{235 \text{ synodic}} \end{aligned}$$

This lets us interpret $\frac{4}{223}\frac{235}{19}$ as $\frac{4 \text{ turns of g1}}{242 \text{ draconic months}}$ i.e. the 242 draconic months are spread across four turns of the Saros pointer, and they are equal to 19 years/turns of gear b2.

The rotation of the gear g1 is transmitted to i1 through the sequence g2-h1-h2-i1, giving us the following ratio:

$$\frac{T_{g1}}{T_{i1}} = \frac{z_{g2}}{z_{h1}}\frac{z_{h2}}{z_{i1}} = \frac{20}{60}\frac{15}{60} = \frac{1}{12}$$

This means that for every 4 rotations of g1, i1 makes one third of the circle, so that after 12 full rotations (3 times the spiral), i1 completes its rotation and an Exeligmos circle has passed. The Saros pointer would then be reset by hand to the starting position [12].

4.2.2 The Metonic and Kallippic calendars

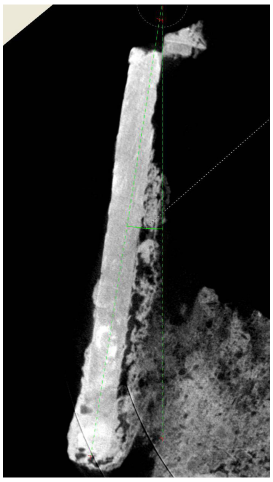

Figure 10: Scan of the pointer used for the Metonic calendar. ©2005 Antikythera Mechanism Research Project

In ancient Greece there was no universal calendar system[11]. Examination of the Mechanism's upper back dial has estabilished that it was a Metonic calendar[19, 10]. The Metonic calendar had a length of 19 years divided in 235 months, with 110 of them lasting 29 days and the remaining 125 months lasting 30 days[9, 11]. This gives a mean year length of $\frac{125 \times 30 + 110 \times 29}{19} = 365\frac{5}{19} \approx 365.2631579\ldots$ days. Inscriptions on the back of the machanisms would determine how these months would be regulated, and the names of the months used in these inscriptions indicate that the Mechanism was created in Corinth or a Corinthian colony, in northwestern Greece or quite possibly, Syracuse in Sicily[11]. The Metonic calendar was illustrated on a five turn spiral on the back of the Mechanism.

Allthough the months of the Metonic calendar were used in most of the Greek cities, albeit with different names in different regions, it was the most widely accepted calendar[11]. Next to its five turn spiral two other displays survive: a Kallippic calendar display and an Olympiad cycle display. The Olympiad dial would indicate when the most important of the Panhellenic Games (Isthmia, Olympia, Nemea, Pythia) would take place, the Olympiads being a commonplace framework to measure time in ancient greece[11]. The Kallippic calendar is a modification of the Metonic calendar, which the astronomer Kallippos came up with, based on observations of natural phaenomena. Kallippos accepted that the year had a duration of $365\frac{1}{4}$ days, and noticed that the Metonic year exceeded this by $\frac{1}{76}$. In order to correct the error of Meton, he proposed that every four Metonic cycles a day should be ommitted. The Kallippic callendar did not influence the duration of the months in the Metonic cycle, nor their arrangement. A circle divided in four quadrants was used, one full turn of the index corresponding to four Metonic cycles [10].

The tropical year has an approximate duration of $365.2421897\ldots$ days. The continued fraction expansion of this number is $[365; 4, 7, 1, 3, 27, 2, \ldots$. This continued fraction produces the following best rational approximations: $365\frac{1}{4}$, $365\frac{7}{29}$, $356\frac{8}{33}$, $356\frac{31}{128}$, $356\frac{845}{3489}$,$356\frac{1721}{7106}$, $\ldots$. It is obvious that the Metonic cycle heavily overestimated the length of the year. Kallippos came on the first best rational approximation of the year, and the mean year length in his calendar is identical to mean year length in the julian calendar. For comparison, the gregorian calendar used today, gives a mean year length of 365.2425 days.

These calendars had periods of 19 and 76 years, making their illustration hard, given that the solar year was displayed on the front dial. The designer of the Mechanism used the same trick as with the eclipse cycle display to make the output of the mechanism useful: instead of the pointer moving on the dial making circles, one for each metonic cycle, the pointer moved on a spiral, like the needle of a gramophone on a vinyl plate. The spiral had five coils, and after the Meton index had covered the spiral, the pointer would then be manually returned to its original position and the Mechanism would keep working. Meanwhile a second index would indicate the Kallippic cycle moving on a circle with clearly marked quadrants, so that the user would keep track of how many metonic cycles had passed.

The Metonic pointer was driven by the gear n1. The train leading to it, is b1-b2-l1-l2-m1-m2-n1, and the ratio of frequencies is:

$$\frac{\omega_{b1}}{\omega_{n1}} = \frac{d_{b2}}{d_{l1}} \frac{d_{l2}}{d_{m1}} \frac{d_{m2}}{d_{n1}} = \frac{64}{38} \frac{53}{96} \frac{15}{53} = \frac{960}{3648} = \frac{5}{19} \tag{6}$$

This means that 5 rotations of the index correspond to the 19 years long Metonic cycle.

The pointer for the Kallippic cycle is driven by the metonic one, the train being n1-n2-p1-p2-o1, giving a ratio of

$$\frac{\omega_{n1}}{\omega_{o1}} = \frac{d_{n2}}{d_{p1}} \frac{d_{p2}}{d_{o1}} = \frac{15}{60} \frac{20}{60} = \frac{180}{3600} = \frac{1}{20} \tag{7}$$

As expected, four times the five coil spiral corresponds to a full Kallippic cycle.

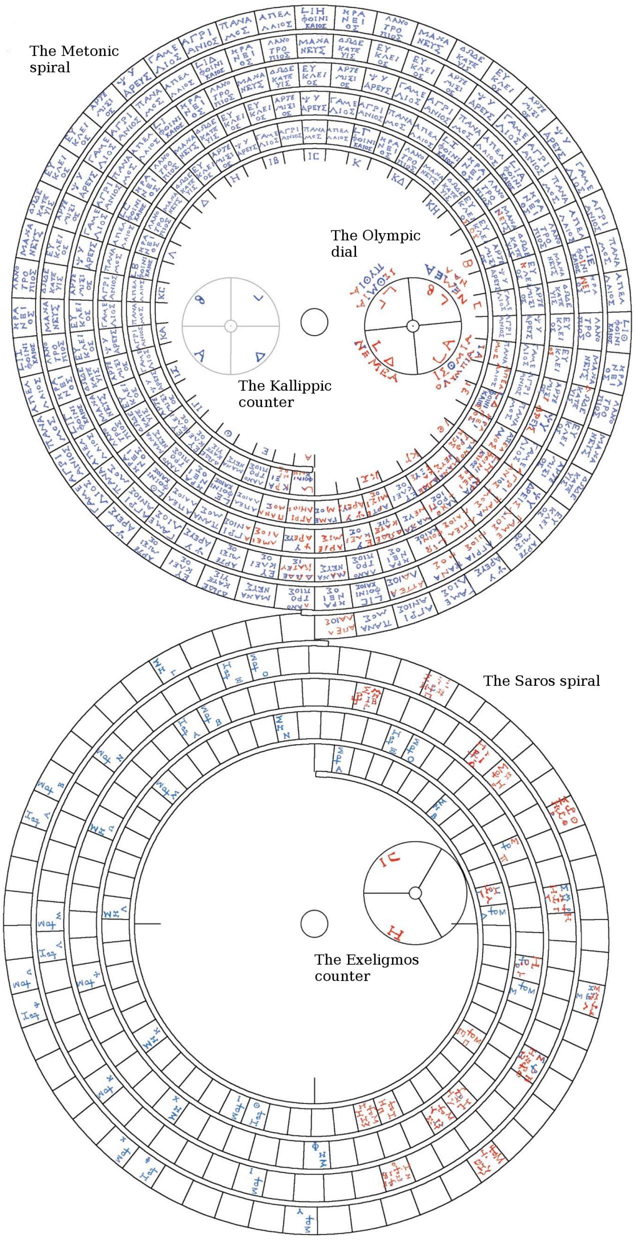

Figure 11: The back dials. The characters in blue are known to us from the scans of the Mechanism, the ones in red are hypothesized. ©2008 Macmillan publishers limited

5 The Mechanism as a Planetarium

Beyond the findings: what could have been

The Greeks knew of seven moving bodies in the sky: sun, moon, Mercury, Venus, Mars, Jupiter and Saturn. They had named them, and it is safe to assume they knew of their movements. One cannot help but ask: Would it have been possible to incorporate all seven in the Mechanism or a similar device, given the technical knowledge of the time? In this section, we try to expand on the Mechanism by adding pointers for Mercury, Venus, Mars, Jupiter and Saturn to the front dial. In this expanded model of the Mechanism, we limit ourselves by using technology already existing in the machanism and examining the inscriptions on the Mechanism and parts of the crown gear b1, which point to the existence of such parts. In this section we will examine the mathematical aspects of such an expansion, without going into the process of constructing and fitting the additional parts into the Mechanism. This was done by Freeth and Jones in [12]. Our pointers will not aim to be eccentricity devices, but will aim to show the mean frequencies of these planets in the sky, as seen by ancient astronomers.

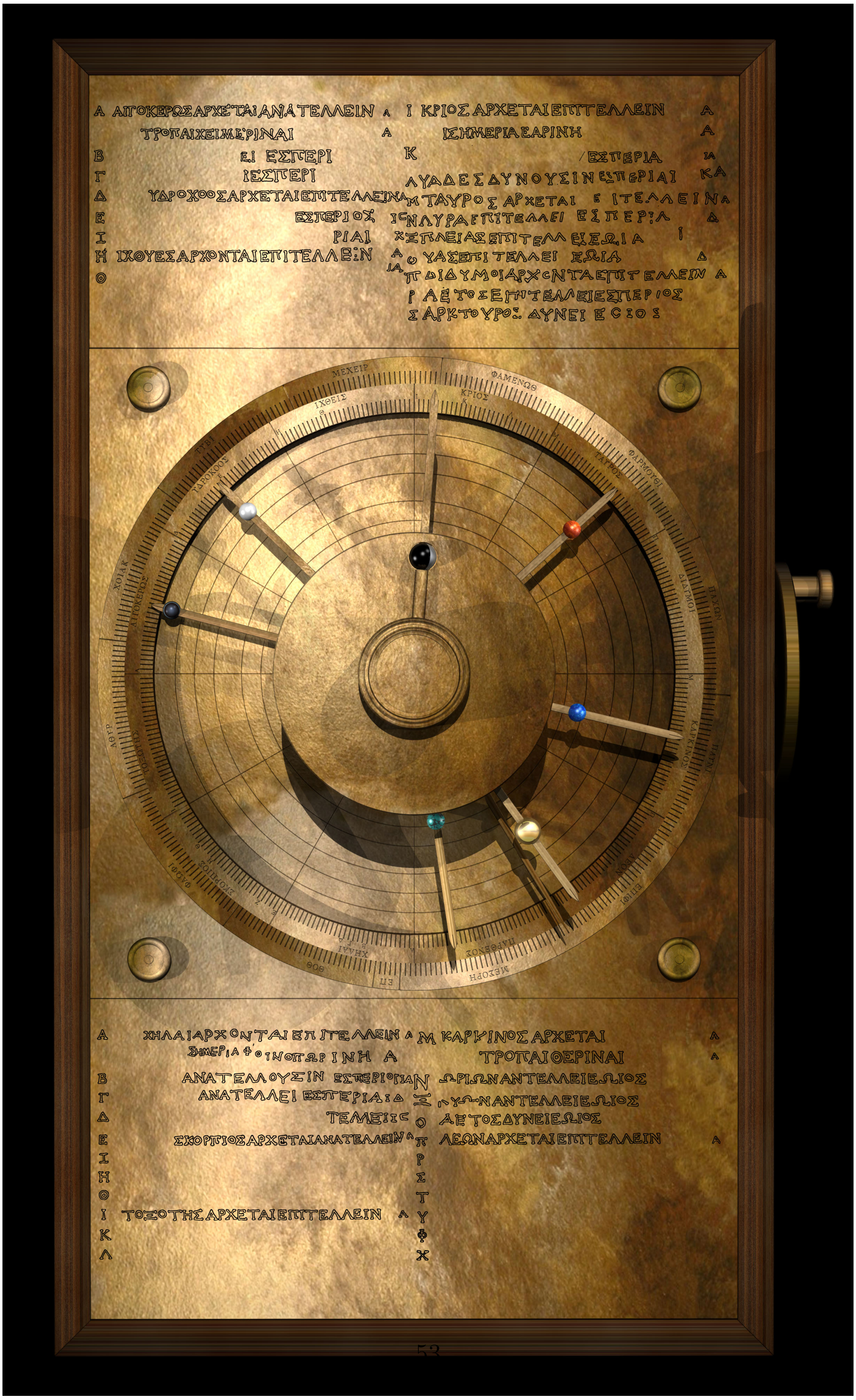

Figure 12: The Cosmos as it would appear in the front dial of the Mechanism. ©2011 Tony Freeth, Images First Ltd.

5.1 Implications of the back cover inscription and the crown gear spokes

5.1.1 The inscription on the back cover of the Antikythera Mechanism

On the back cover of the Mechanism lies an inscription that mentions all of the five planets closest to the sun, mentioned with both theophoric names associating a god of Greek mythology with each planet [5] and names describing their appearance in the sky. These five planets together with the sun and the moon are the only moving heavenly bodies known in antiquity.[6] According to greek astronomy, beyond these seven bodies lie the fixed stars of the night sky. In the inscription the termsζώνη"belt",σφαίριον"sphere",κύκλος "circle" andγνωμόνιον"pointer" can be read, and there is mention of the spheres "traveling trough"διαπορευόμενον and "revolving"περιφέρειαν. The word carrying the most weight, however, is the wordκόσμου, the genetive case of the wordκόσμος. The translation ofκόσμος in modern Greek is "world", and there are many meanings it could have in antiquity. However, the astronomical context restricts the translation to the world outside the Earth, not from a philosophical, but from a scientific point of view. This is an indication that the wholeκόσμος could have been on display on the front dial of the Mechanism [12].

[5] The latin names of the gods used by the Greeks are the ones used in modern astronomy.
[6] The word planet is derived from the greek wordπλανήτης which translates to wanderer.

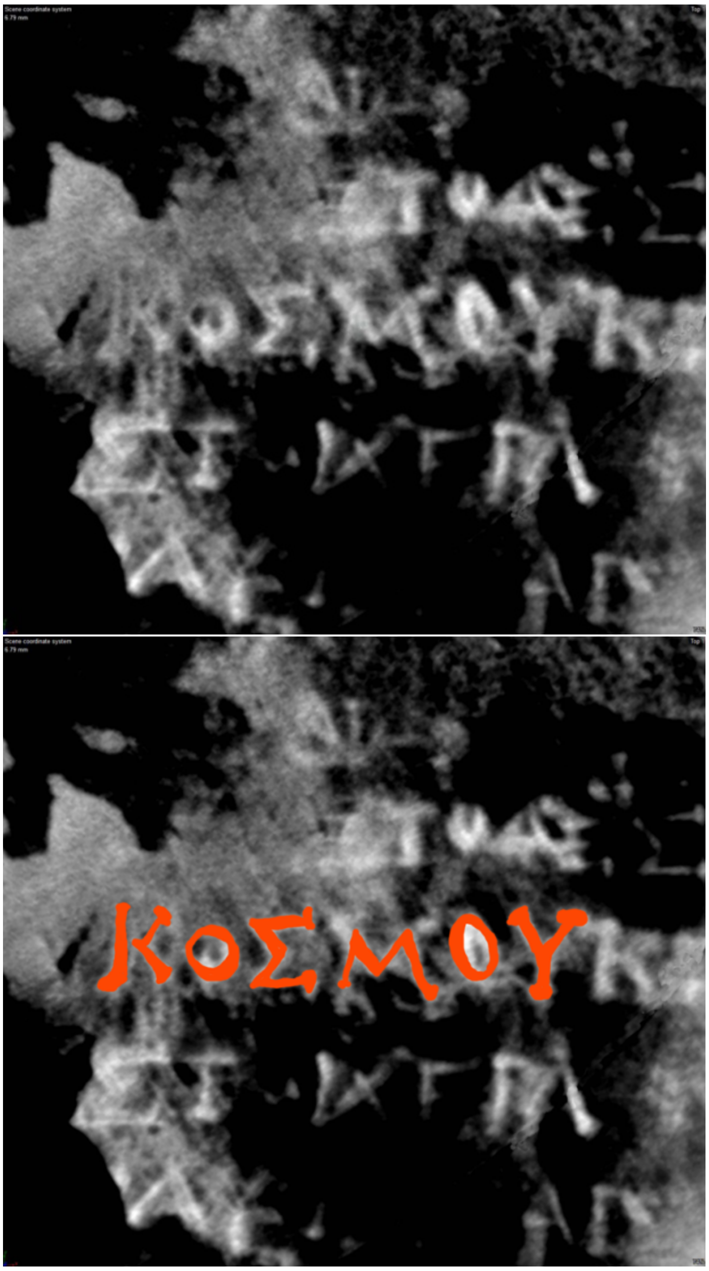

Figure 13: The word κόσμου in the inscription, is highlightened in red in the lower image [12]. ©2005 Antikythera Mechanism Research Project.

5.1.2 The spokes and pillars on b1

One the gear named b1 there can be seen traces of four spokes the purpose of which have not yet been fully explained, allthough the existence of some structure mounted on b1 has been hypothesized [19]. Besides the spokes, three pillars have been found the periphery of b1. One of the pillars is significantly longer than the other two, measuring somewhere between 28.7 and 33.7 mm, and there are indications on the gear that another three identical pillars could have been been mounted on b1. The great inaccuracy of the measurements is due to extensive corrosion and a part of the pillars tip being broken off [12].
The two shorter pillars measure 21.9 mm and 20.5 ± 1 mm respectively, with the later being broken, leading to a lack of precision concerning its length. If devices for the planets were to be mounted on b1, the pointers used should be similar to the ones already in the Mechanism (e.g. the Metonic pointer). The devices presented here are simplified versions of the ones constructed by Freeth and Jones in [12].

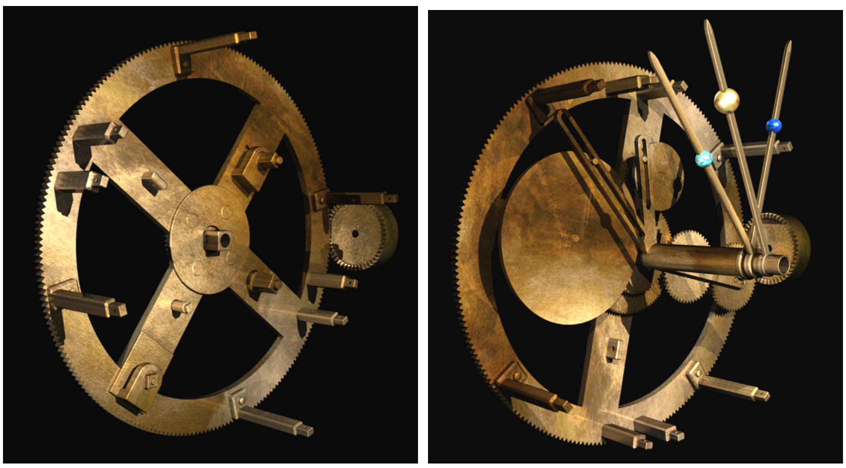

Figure 14: On the left gear b1 with the pillars and spokes, on the right some of the devices examined have been mounted [12] ©Tony Freeth, Images First Ltd

5.2 Construction of the additional parts

The new and expanded model of the Mechanism can be broken into four parts:

1. The 2005 reconstruction as examined earlier

2. The inferior planets' devices

3. The superior planets' devices

All new devices are mounted on the gear b1. The mechanics of fitting the devices present some limitations [12], but it is still possible to have a working model.[3, 12]

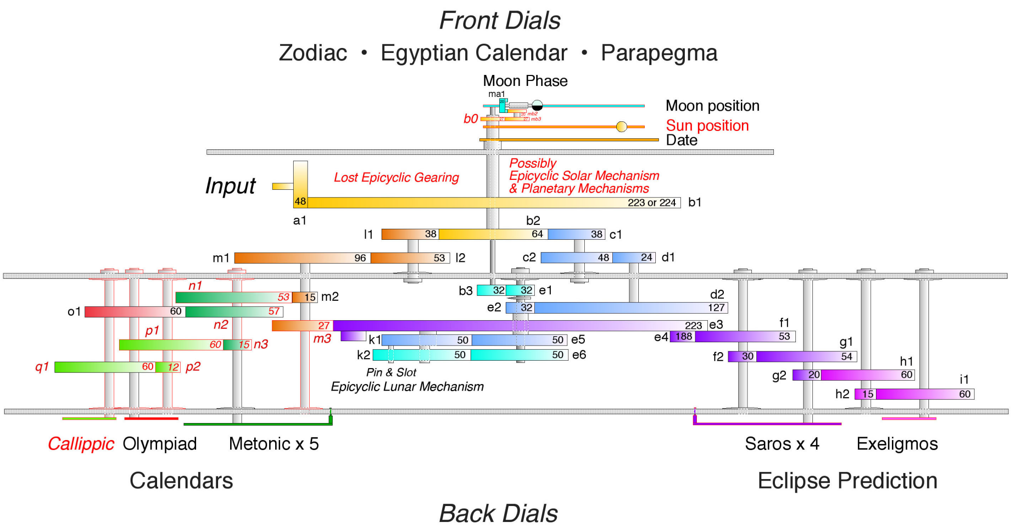

Figure 15: In this scheme, the hypothesized gears are marked red, and the position of the new mechanisms is labeled 'Lost epicyclic gearing' in front of b1. [12]

5.2.1 Mathematical background

This new scheme vastly expands the information displayed on the front dial of the Mechanism. The positions of the five planets closest to the sun (earth excluded, as the system is geocentric) are displayed, and this could be done in the way these planets were thought at the time to be positioned around the earth. The mean period of the planets, is taken from Babylonian observations [3], the accuracy of which we have examined in chapter 3. For the mean period of the planets the following procedure is followed: The number of synodic cycles is added to the number of revolutions of the planet across the sky. The resulting number is period of the planet in question measured in years. The pointers have to complete this number of rotations for every rotation of b1 [12].

Inferior planets Due to the eccentricity device being mounted on the gear b1, this number must be approximated in relation to the synodic month, thus, every approximation must be of the form $\frac{y}{x+y}$ where x is number of synodic months that have passed, and y the number of years that correspond to this period[7]. We define the error of an approximation as $\varepsilon = 360\left|\frac{1}{r'} - \frac{1}{r}\right|$. For Mercury, we start by assumimng that one full orbit around the sun is $r_{Mercury} = 0.2408404$[12]. The best approximation used by the Babylonians is $x_{Mercury} = 145$ and $y_{Mercury} = 46$, giving an approximate $r'_{Mercury} = \frac{46}{191}$. There is a mechanical issue that restricts the number of teeth we can use, as the space in the Mechanism is limited and more teeth need a larger gear[12]. Due to these mechanical restrictions, it is better to discard $x_{Mercury} = 145$ and use another, worse approximation that is not found in ancient texts, with $x_{Mercury} = 104$ and $y_{Mercury} = 33$ giving the approximate mean period of Mercury as $r'_{Mercury} = \frac{33}{137}$, with an error of 0.221 degrees per year.

The mean period of Venus will be taken from the Babylonians, with

[7]Recall from earlier that due to the moon orbiting the earth, one synodic month must be added with every year to account to the simultaneous movement of the earth around the sun and the moon around the earth

$x_{Venus} = 5$ and $y_{Venus} = 8$, making the approximate period $r'_{Venus} = \frac{8}{13}$, the actual number accepted as $r_{Venus} = 0.6151854$, with an error of 0.189 degrees per year.

Superior planets There are several approximations mentioned in ancient texts for the mean period of Mars (the actual number taken here as $r_{Mars} = 1.8808148$) here we will use the best one, with $x_{Mars} = 37$ and $y_{Mars} = -79$. The approximate mean period is therefore $r'_{Mars} = \frac{79}{42}$, with an error of 0.014 degrees per year. As we get to the gas giants Jupiter and Saturn their periods get very long in comparison to that of the Earth, and this would make the display rather inefficient. Nonetheless we will examine them, as they were known at the time, and it is possible that their position was displayed on the front dial, using Babylonian data. Jupiter's orbit lasts $r_{Jupiter} = 11.8617555$ earth years and the designer of the Mechanism would use the pair $x_{Jupiter} = 76$ and $y_{Jupiter} = -83$ given in Babylonian sources [12]. The same will be done for Satrun, whose orbit corresponds to $r_{Saturn} = 29.4565217$ earth years, and the Babylonian observation $x_{Saturn} = 57$ and $y_{Saturn} = -59$ is used. The errors of these calculations are respectively 0.012 and 0.018 degrees per year.

Approximations using continued frations It is important to test the chosen approximations for accuracy by comparing them to the best rational approximations of the actual periods. These are given in the following table:

Mean period	Continued fraction expansion	Best rational approximations						
$r_{Mercury} = 0.2408404$	[0; 4,6,1,1,2,1]	0	$\frac{1}{4}$	$\frac{6}{25}$	$\frac{7}{29}$	$\frac{13}{54}$	$\frac{33}{137}$ F	$\frac{46}{191}$ B
$r_{Venus} = 0.6151854$	[0; 1,1,1,1,2,29]	0	1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{13}$ B	$\frac{235}{382}$
$r_{Mars} = 1.8808148$	[1; 1,7,2,1,1,3]	1	2	$\frac{15}{8}$	$\frac{32}{17}$ B	$\frac{47}{25}$ B	$\frac{79}{42}$ B	$\frac{284}{151}$
$r_{Jupiter} = 11.8617555$	[11; 1,6,4,3,1,1]	11	12	$\frac{83}{7}$ B	$\frac{344}{29}$	$\frac{1115}{94}$	$\frac{1459}{123}$	$\frac{2574}{217}$
$r_{Saturn} = 29.4565217$	[29;2,5,4,4,2,16]	29	$\frac{59}{2}$ B	$\frac{324}{11}$	$\frac{1355}{46}$	$\frac{5744}{195}$	$\frac{12843}{436}$	$\frac{211232}{7171}$

Table 8: The periods of the planets known in ancient Greece and their best rational approximations

In the above table, the index B means that the best rational approximation can be found in Babylonian texts [12], and blue marks the approximations presented here. The index F in the best rational approximation for Mercury indicates that this is used by Freeth and Jones in [12] which was used due to the space inside the Mechanism not permiting the use of the Babylonian observation. Here we will take a look at both approximations.

5.2.2 The gears used for the devices

The following table contains the details of the devices used to recreate the approximate mean synodic periods of the planets known in ancient Greece. The devices for Mercury and Venus consist of two gears, whereas the devices for the superior planets consist of four gears [12]. Here, we examine the tooth count for the gears in these devices. The ratios are easy to construct, and we can stay within the numbers already existing in the Mechanism. If these devices were to illustrate the anomaly in the planets' orbit, they would be pin-and-slot devices like the one used for the moon anomaly.

Planet	Gear 1 tooth count(y)	Gear 2 tooth count (x)
Mercury(Freeth)	33	104
Mercury(Babylonians)	46	145
Venus	40	64

Table 9: The tooth count for the gears used for the inferior planets

For the more complex devices used to present the superior planets' position, the following gear parameters are used:

Planet	Gear 1 tooth count	Gear 2 tooth count	Gear 3 tooth count	Gear 4 tooth count
Mars	37	79	70	70
Jupiter	76	83	85	85
Saturn	57	59	60	60

Table 10: The tooth count for the gears used for the superior planets

Here we see what the gear b1 would look like with all the devices mounted, and a metallic plate in front of them to be used as part of the front cover of the Mechanism:

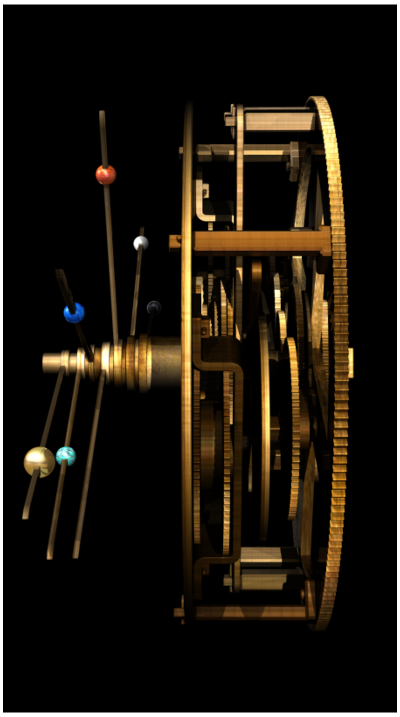

Figure 16: This is a side view of the end result of the construction [12]

The most challenging part of this construction from a mechanical point of view, is not in the devices discussed, as those are based on preexisting parts of the Mechanism; it is the hollow axis for the pointers. There have to be six layers around the axis used for the moon pointer. Whether or not that would be possible at the time of the Mechanism's construction, is open to speculation [12, 3]. The possibility that these devices showed not only the mean periods but also the anomalies in the orbital velocities of the planets has been shown to be highly unlikely, and any attempt at it has resulted in an inaccurate display [12, 3]. This comes to strong contrast to the accurate parts of the 2005 model. As the site of the wreck is visited again, the possibilty that new parts of the Mechanism will be discovered remains, and additional parts could confirm or dispute the existence of these additional devices.

5.2.3 Comparison between the approximations and the actual values

The observations of the Babylonians and the numbers used by Freeth and Jones for Mercury are all best rational approximations. We now can compare the mean periods of the planets with the approximations we used here. We will define the error as $\varepsilon = 360\left|\frac{1}{r'} - \frac{1}{r}\right|$.

Planet	Mean period r	Approximate period r'	Error ε in degrees per year
Mercury	0.240804	$\frac{33}{137}F$	0.446323
		$\frac{46}{191}B$	0.2091688
Venus	0.6151854	$\frac{8}{13}$	0.1894404
Mars	1.8808148	$\frac{79}{42}$	0.0140003
Jupiter	11.8617555	$\frac{83}{7}$	0.0118065
Saturn	29.4565217	$\frac{59}{2}$	0.0180124

Table 11: The error for the several approximations used for the mean period of the planets

These approximations are very accurate, with the ones used for the superior planets having an error of less than 0.05 degrees. This is on par with the rest of the Mechanism's accuracy as demonstrated in chapter 4. However, there has been no successful pin-and-slot devices that could illustrate the anomaly in the planets' orbits [12, 3]. Therefore it would be safer to assume that such a display would be inconsistent with the rest of the Mechanism and the planets were not on display on the front dial.

Glossary

best rational approximation A fraction $\frac{a}{b}$ is called a best rational approximation of $r \in \mathbb{R}_{>0} \backslash \mathbb{Q}$ if $|r - \frac{a}{b}| < |r - \frac{a'}{b'}|$, $a, b, a', b' \in \mathbb{N}$; $b' \leq b$. 33, 40, 46

draconic month For an explanation of the draconic month, see Appendix A. 37

Exeligmos The Exeligmos cycle is an eclipse cycle lasting three times the Saros, making the repetion of the eclipses in the Exeligmos almost identical. 43

fundamental law of gears The fundamental law of gears states that for constant angular velocity ratio, the line of action must always pass through the pitch point P . 17

gear A cylinder of small height with teeth placed on its perimeter. 14

involute gear tooth profile A gear tooth profile constructed using the involute of the base of the gear cylinder. 14

involute of a curve The curve traced by the end point of a string unwinding from another curve *See also: evolute*. 19

Kallippic calendar The calendar proposed by Kallippos, who observed an error in the Metonic calendar, and modified the length of the year by omitting one day every four Metonic cycles. 45

line of action The path the point of action follows, normal to the tangent of the tooth profiles. 17

Metonic calendar Calendar of Athenian astronomer Meton, who overestimated the duration of the year by approximately 0.04 days. *See also: Kallippic calendar*. 45

planetarium a mechanical device illustrating part of or the entire solar system, *See also sphere*. 7

point of action The point of contact between two meshing gears. 17

root radius the distance from the axis of a gear to the bottom of its teeth. 11

Saros The Saros is an eclipse repetition cycle that lasts 223 synodic months, or more accurately 6585 days, 07 hours and 43 minutes. This was observed in antiquity as 6585 days and 8 hours. These eight hours correspond to a 120° shift of the eclipses' path to the west. *See also: Exeligmos*. 37, 43

sidereal month For an explanation of the sidereal month, see Appendix A. 37

simple continued fraction An term of the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$ $a_0 \in \mathbb{N} \cup \{0\}; a_i \in \mathbb{N}, i \geq 1$. 26

sphere a type of *planetarium* constructed by Archimedes, made of bronze and illustrating the Sun, Earth and Moon configuration, predicting solar and lunar eclipses. Sadly no sphere has survived, and we know of their existence only from historical texts. 7

spiral of Archimedes The curve described by the equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\rho\theta\cos\theta \\ \rho\theta\sin\theta \end{pmatrix}$$

See also: involute . 24

synodic month For an explanation of the synodic month, see Appendix A. 37

tip radius the distance from the axis of a gear to the outermost point (tip) of its teeth. 11

tooth profile The shape of a gear's teeth, on the plane parallel to the cylinder base. 14

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A The moon's movement around the Earth

The earth rotates around the sun in an elliptical orbit, and the moon rotates around the earth, in an elliptical orbit that is not coplanar with earth's rotation around the sun. The plane of the moon's orbit is called the ecliptic. There are two periods that describe the period of the moon's orbit. One is called the *sidereal month*, and one is called the *synodic month*. Their definitions are the following: The sidereal month is the time it takes the moon to orbit the earth in relation to the stars, which can be considered stationary. The synodic month is the time it takes for the moon to return to the same position in relation to the earth and the sun. They differ in duration, because the moon has to catch up to earth and cover a greater distance in order to complete its path. The rotation of the moon around the earth is explained here:

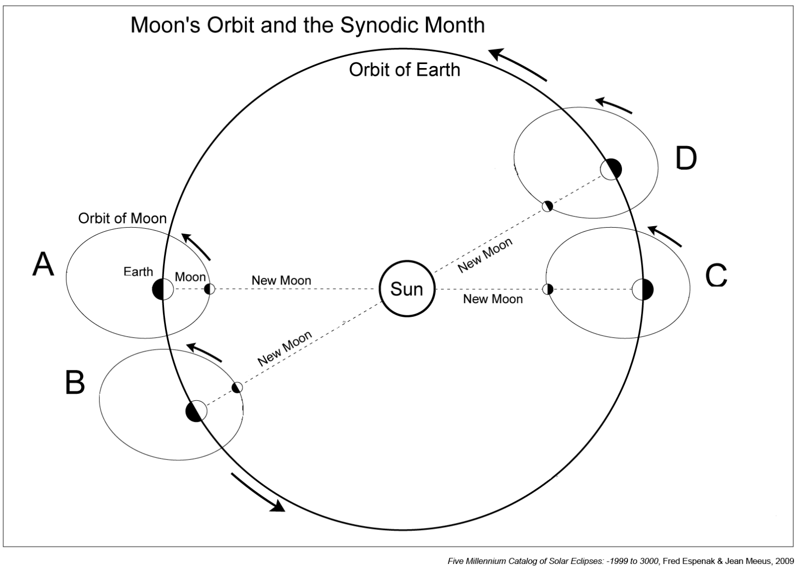

Figure 17: The rotation of the moon around the Earth. Source: Eclipse Predictions by Fred Espenak (NASA's GSFC) and Jean Meeus

The moon's orbit is at an angle with the plane of the sun and the earth, which means that the moon is on the plane of the earth's orbit twice in every lunar orbit. These two points are called nodes, one is the ascending, and the other the descending. The draconic month is the period between two consecutive ascending nodes. An eclipse occurs every time the sun, the earth and the moon are colinear. If the moon is between the earth and the sun, it is a solar exlipse, if the earth is between the sun and the moon it is a lunar eclipse. The eclipses are repeated with a certain period, a cycle that lasts approximately 223 synodic or 239 sidereal or 242 draconic months. This cycle was known to Babylonian astronomers and was called the Saros. The eclipses repeat themselves nearly identically every three Saros cycles, and this lead the Greeks to define the Exeligmos cycle. The information in this Appendix is taken from the NASA Eclipse Web Site: eclipse.gsfc.nasa.gov/eclipse.html

B Statement of authorship

I hereby testify that the presented thesis, is my own work, and that I have used no materials or sources, except the ones listed.

Anastasios Tsigkros

Date Signature