Multiplicity results of a Neumann Problem for p(x)-Laplacian Systems

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Abstract

In this paper, we study a Neumann problem for elliptic systems with variable exponents and we obtain the existence of at least three nontrivial solutions by using an equivalent variational approach to a recent Ricceri's three critical points theorem [12].

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1 Introduction

Let Ω be a bounded domain of $I\!R^N, N \geq 2$ with a smooth boundary $\partial\Omega$. We consider the following Neumann problem for the corresponding elliptic system

$$(S_1)\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v) & \text{in } \Omega \\ -div(|\nabla u|^{q(x)-2}\nabla u) + |v|^{q(x)-2}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $(p,q) \in C(\overline{\Omega})^2, p(x) > 1, q(x) > 1$, for every $x \in \Omega$. F is a C^1-function on $\Omega \times [0,\infty) \times [0,\infty)$ satisfying

$$(1.1) \qquad |F_t(x,t,s)| \leq C\, t^{\alpha(x)} s^{\beta(x)+1}, \quad |F_s(x,t,s)| \leq C\, t^{\alpha(x)+1} s^{\beta(x)} \quad \forall (x,t,s)$$

for some $\alpha, \beta \in C_+(\overline{\Omega})^2$ with

$$(1.2) \qquad (\alpha^+ + 1)/p^- + (\beta^+ + 1)/q^- < 1,$$

where

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x) \ \ \forall h \in C_+(\overline{\Omega}).$$

$$F(x,t,s) < 0, \text{ for all } t,s \in]0,1[, \tag{1.3}$$

$$F(x,t,s) > M, \text{ for all } t > t_0 \text{ and } s > s_0 \tag{1.4}$$

where M is a positive constant and $t_0, s_0 > 1$.
G is assumed to be a measurable function with respect to x in Ω for every (s,t) in $I\!R \times I\!R$, and is a C^1-function with respect to (s,t) in $I\!R \times I\!R$ for almost every x in Ω and satisfies

$$\sup_{|(t,s)|\leq k} (|G_t(x,t,s)| + |G_s(x,t,s)|) \leq h_k(x) \tag{1.5}$$

for all $k > 0$ and some $h_k \in L^1(\Omega)$. Here div(— $\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian, the generalization of the classical $p-$Laplacian operator, and ν is the outward unit normal to $\partial\Omega$.

Recently, elliptic equations with variable exponents have been extensively investigated and have received much attention. They have been the subject of recent developments in nonlinear elasticity theory and electrorheological fluids dynamics [14]. In that context, let us mention that there appeared a series of papers on problems which lead to spaces with variable exponent, we refer the reader to Fan al. [8], [9], Ružicka [15] and the references therein.

Let us point out that when $p(x) = p =$ constant, there is a large literature which deal with problems involving the p-Laplacian with Dirichlet boundary conditions both in the scalar case and elliptic systems in bounded or unbounded domains.

Note that many papers deal with problems related to the p-Laplacian with Neumann conditions in the scalar case. We can cite, among others, the articles [1] and [4] and the references therein for details. The case of $p(x)-$Laplacian with Neumann conditions on the scalar case has been studied by Dai [5], Mihăilescu [10] and Shi and Ding [16].

Finally, it would be interesting at this stage to refer the reader to the recent work [6] where the authors established the existence of at least three solutions for elliptic systems involving the p-Laplacian with Neumann boundary conditions.
Our objective is to study the Neumann problem for such a system of the type (S_1). Precisely, based on a recent result due to Ricceri [12], we are interested in the existence and multiplicity of weak nontrivial solutions for system (S_1) in the Sobolev space $W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$.
Along this paper we fix $p(x), q(x) > N$, for any $x \in \overline{\Omega}$ and look for the existence result to such a problem.
First recall that a weak solution of system (S_1) is any $(u,v) \in W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$ such that

$$\begin{cases} \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + a(x)|u|^{p(x)-2}u\varphi)\,dx = \int_\Omega (\lambda F_u(x,u,v) + \mu G_u(x,u,v))\varphi\,dx \\ \int_\Omega (|\nabla v|^{q(x)-2}\nabla v\nabla\psi + b(x)|v|^{q(x)-2}v\psi)\,dx = \int_\Omega (\lambda F_v(x,u,v) + \mu G_v(x,u,v))\psi\,dx \end{cases}$$

for all $(\varphi, \psi) \in W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$.

Remark 1.1 *Let us remark that* (1.2) *and* (1.5) *guarantees that integrals given in the right side are well defined.*

Initially, we let $\alpha(x) = \beta(x)$ for all $x \in \overline{\Omega}$ and we will consider the following Neumann problem

$$(S_1)' \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda(|u|^{\alpha(x)-1}u|v|^{\alpha(x)+1} - v) + \mu|u|^{\gamma_1(x)-2}u & \text{in } \Omega \\ -\Delta_{q(x)}v + |v|^{q(x)-2}v = \lambda(|u|^{\alpha(x)+1}|v|^{\alpha(x)-1}v - u) + \mu|v|^{\gamma_2(x)-2}v & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where

$$\gamma_1, \gamma_2 \in C_+(\overline{\Omega})^2,\ \gamma_1^+ < p^- \text{ and } \gamma_2^+ < q^-, \tag{1.6}$$

and α satisfies

$$(\alpha^+ + 1)(1/p^- + 1/q^-) < 1. \tag{1.7}$$

To prove the existence of at least three weak solutions for each of the given systems $(S_1)'$ and (S_1), we will use the following result proved in [12] that, on the basis of [2], can be equivalently stated as follows

Theorem 1.1 *Let X be a reflexive real Banach space; $\Phi : X \longrightarrow I\!R$ is bounded on each bounded subset of X, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $J : X \longrightarrow I\!R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that*

$$(i) \qquad \lim_{\|u\| \longrightarrow \infty} (\Phi(u) + \lambda J(u)) = +\infty \ \textit{ for all } \ \lambda \in]0, +\infty[$$

and that there are $r \in I\!R, u_0, u_1 \in X$ such that

(ii) $\Phi(u_0) < r < \Phi(u_1)$

$$(iii) \quad \inf_{u \in \Phi^{-1}(]-\infty, r])} J(u) > \frac{(\Phi(u_1) - r)J(u_0) + (r - \Phi(u_0))J(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

Then, there exist an open interval A of $]0, +\infty[$ and a positive real number t such that for every $\lambda \in A$, and every continuously Gâteaux differentiable functional $\Psi : X \longrightarrow I\!R$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\Phi^{'}(u) + \lambda J^{'}(u) + \mu \Psi^{'}(u) = 0$$

has at least three solutions in X whose norms are less than t.

2 Preliminaries

We list some well known definitions and basic properties and recall some background facts concerning generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is an open subset of $I\!R^N$ and introduce some notations used below. For more details about these spaces, we refer the reader to the book of Musielak [11] and the papers of Kováčik al. [7] and Fan al. [8], [9].

Set

$$L_+^\infty(\Omega) = \{h;\ h \in L^\infty(\Omega), \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in L_+^\infty(\Omega)$ we define

$$h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x).$$

For any $p(x) \in L_+^\infty(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \text{is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\, dx \le 1\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [7, Theorem 2.5], the Hölder inequality holds [7, Theorem 2.1], they are reflexive if and only if $1 < p^- \le p^+ < \infty$ [7, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [7, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [7, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \le r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$|\int_\Omega uv\, dx| \le (\frac{1}{p^-} + \frac{1}{p'^-})|u|_{p(x)}|v|_{p'(x)}$$

holds true. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping

$$\rho_{p(x)} : L^{p(x)}(\Omega) \to I\!R$$

defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+};$$

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-};$$
$$|u_n - u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u_n - u) \to 0.$$

We also consider the weighted variable exponent Lebesgue spaces. We define also the variable Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\; |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\| = \inf\,\{\mu > 0;\; \int_\Omega (|\frac{\nabla u(x)}{\mu}|^{p(x)} + |\frac{u(x)}{\mu}|^{p(x)})\,dx \le 1\}.$$

Set

$$I_{p(x)}(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx.$$

For all $u \in W^{1,p(x)}(\Omega)$ the following relations hold

$$\|u\| > 1 \;\Rightarrow\; \|u\|^{p^-} \le I_{p(x)}(u) \le \|u\|^{p^+}; \tag{1.8}$$

$$\|u\| < 1 \;\Rightarrow\; \|u\|^{p^+} \le I_{p(x)}(u) \le \|u\|^{p^-}. \tag{1.9}$$

Finally, we remember some embedding results regarding variable exponent Lebesgue-Sobolev spaces. For the continuous embedding between variable exponent Lebesgue-Sobolev spaces we refer to [8, Theorem 1.1]: if $p : \Omega \to I\!\!R$ is Lipschitz continuous and $p^+ < N$, then for any $q \in L^\infty_+(\Omega)$ with $p(x) \le q(x) \le \frac{Np(x)}{N-p(x)}$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. In what concerns if $N < p^- = \operatorname{ess\,inf}_{x\in\overline{\Omega}} p(x) \le p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [9] we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$. Since $N < p^-$ it follows that $W^{1,p^-}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we obtain that $W^{1,p(x)}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$.

From now on, E is the space $W^{1,p(x)}(\Omega) \times W^{1,q(x)}(\Omega)$ with the norm

$$\|(u,v)\| = \|u\|_1 + \|v\|_2,$$

where $\|u\|_1 = \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}\,dx\right)^{1/p(x)}$, $\|v\|_2 = \left(\int_\Omega |\nabla v|^{q(x)} + |v|^{q(x)}\,dx\right)^{1/q(x)}$,

which is clearly equivalent to the usual one, on the space $(C(\overline{\Omega}))^2$ we consider the norm $\|(u,v)\|_\infty = \sup_{x\in\bar{\Omega}}(|u(x)| + |v(x)|)$. When $p^-, q^- > N$, Sobolev's Theorem implies that E is compactly embedded in $(C(\overline{\Omega}))^2$, hence

$$c = \sup_{(u,v)\in E\setminus\{(0,0)\}} \frac{\|(u,v)\|_\infty}{\|(u,v)\|} < +\infty. \tag{1.10}$$

Proposition 2.1 *Let $I : E \to E^*$ be the operator defined by*

$$\begin{aligned} I(u,v)(\varphi,\psi) &= \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + |u|^{p(x)-2}u\varphi\right)dx \\ &+ \int_\Omega \left(|\nabla v|^{q(x)-2}\nabla v\nabla\psi + |v|^{q(x)-2}v\psi\right)dx \end{aligned}$$

for all $u, v, \varphi, \psi \in E$. Then I admits a continuous inverse on E^*.

Proof.
Denoting by $\langle ., . \rangle$ the usual inner product in $I\!R^N$. We have for some positive constant C_1

$$\begin{aligned} \frac{\langle I(u,v),(u,v)\rangle}{\|(u,v)\|} &\geq C_1 \frac{I_{p(x)}(u) + I_{q(x)}(v)}{\|u\|_1 + \|v\|_2} \\ &\geq C_1 \frac{\|u\|_1^{p^-} + \|v\|_2^{q^-}}{\|u\|_1 + \|v\|_2}. \end{aligned}$$

Hence,

$$\lim_{\|(u,v)\|\to+\infty} \frac{\langle I(u,v),(u,v)\rangle}{\|(u,v)\|} = +\infty$$

since $\lim\limits_{x+y\to+\infty} \dfrac{x^m + y^k}{x+y} = +\infty$, for $x, y \in I\!R_+$ and $m, k > 1$. Then I is coercive. Moreover, by simple arguments we can easily verify that I is hemicontinuous. It remains to show that I is uniformly monotone. Indeed, recall first the well known inequality

$$|x-y|^m \leq 2^m \langle |x|^{m-2}x - |y|^{m-2}y, x-y\rangle, \ \forall x, y \in I\!R^N, \forall m \geq 2.$$

Thus, it is easy to see that

$$\begin{aligned} \langle I(u_1,v_1) - I(u_2,v_2), (u_1-u_2, v_1-v_2)\rangle &\geq \frac{1}{2^{p^+}} \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} + |u_1-u_2|^{p(x)}\,dx \\ &+ \frac{1}{2^{q^+}} \int_\Omega |\nabla v_1 - \nabla v_2|^{q(x)} + |v_1-v_2|^{p(x)}\,dx, \\ &\geq c(p^+,q^+)(\|u_1-u_2\|_1^{p^+} + \|v_1-v_2\|_2^{q^+}), \end{aligned}$$

for every u_1, v_1, u_2 and v_2 belonging to E. This means that I is uniformly monotone operator in E. Therefore, the conclusion of Proposition 2.1 follows directly from the result (Theorem 26. A) of [18].

Our main results are the following

Theorem 2.1 *Suppose $(p,q) \in C(\overline{\Omega})^2$ with $p, q > N$, for every $x \in \Omega$ and let α satisfying* (1.7). *Then there exists an open interval A of $]0, +\infty[$ and a positive real number t such that, for every $\lambda \in A$ and every γ_1, γ_2 satisfying (1.6), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$ the system $(S_1)'$ has at least three weak solutions whose norms are less than t.*

Theorem 2.2 *Suppose $p, q > N$. Let α, β satisfying (1.2). Assume that F is a C^1- function satisfying (1.1), (1.3) and (1.4). Then, there exist an open interval A of $]0,+\infty[$ and a positive real number t such that, for every $\lambda \in A$ and every C^1-function $G(.,t,s)$ satisfying (1.5), there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$ the system (S_1) has at least three weak solutions whose norms are less than t.*

3 Proofs of Theorem 2.1 and Theorem 2.2

We begin by setting

$$\Phi(u,v) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)}) + \frac{1}{q(x)}(|\nabla v|^{q(x)} + |v|^{q(x)})\,dx$$

$$J(u,v) = -\int_\Omega (\frac{1}{(\alpha(x)+1)}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1} - uv)\,dx$$

and

$$\Psi(u,v) = \int_\Omega \frac{1}{\gamma_1(x)}|u|^{\gamma_1(x)} + \frac{1}{\gamma_2(x)}|v|^{\gamma_2(x)}\,dx.$$

for each $(u,v) \in E$. It is well known that Φ and J are well defined and continuously Gâteaux differentiable with

$$\begin{aligned}\Phi^{'}(u,v)(\varphi,\psi) &= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi)\,dx \\ &+ \int_\Omega (|\nabla v|^{q(x)-2}\nabla v \nabla\psi + |v|^{q(x)-2}v\psi)\,dx\end{aligned}$$

and

$$J^{'}(u,v)(\varphi,\psi) = -\int_\Omega (|u|^{\alpha(x)-1}u\varphi|v|^{\alpha(x)+1} - v\varphi + |u|^{\alpha(x)+1}|v|^{\alpha(x)-1}v\psi - u\psi)\,dx$$

for all $(u,v),(\varphi,\psi)$ in E. Note that $J^{'}$ is compact and Φ is clearly weakly lower semi-continuous and bounded on each bounded subset of E. Proposition 2.1 ensures that $\Phi^{'}$ admits a continuous inverse on E^*. Moreover, it is easy to see that

$$\lim_{\|(u,v)\| \longrightarrow +\infty} (\Phi(u,v) + \lambda J(u,v)) = +\infty$$

for all $\lambda \in]0,+\infty[$. Indeed, since $(\alpha^+ + 1)(1/p^- + 1/q^-) < 1$, there exist $p_1, p_2 \in]1, p^-[$, $q_1, q_2 \in]1, q^-[$ such that

$$(\alpha^+ + 1)(1/p_1 + 1/q_1) = 1 \text{ and } (\alpha^- + 1)(1/p_2 + 1/q_2) = 1.$$

On the other hand, since E is embedded in $(C^0(\overline{\Omega}))^2$, we have

$$J(u,v) \geq -\frac{|\Omega|}{(\alpha^- + 1)}(\|u\|_\infty^{\alpha^+ + 1}\|v\|_\infty^{\alpha^+ + 1} + \|u\|_\infty^{\alpha^- + 1}\|v\|_\infty^{\alpha^- + 1}),$$

where $|\Omega|$ denotes the measure of Ω. This implies in view of Young's inequality that

$$J(u,v) \geq -C_2\left(\frac{\alpha^+ + 1}{p_1}\|u\|_\infty^{p_1} + \frac{\alpha^+ + 1}{q_1}\|v\|_\infty^{q_1} + \frac{\alpha^- + 1}{p_2}\|u\|_\infty^{p_2} + \frac{\alpha^- + 1}{q_2}\|v\|_\infty^{q_2}\right),$$

for some positive constant C_2. Then, the coercivity of $\Phi+\lambda J$ can be easily deduced from (1.6) since $p_1, p_2 < p^-, q_1, q_2 < q^-$ and $\lambda > 0$ and from the fact that $\|u\|_\infty \leq D_1\|u\|_1$ and $\|v\|_\infty \leq D_2\|v\|_2$ for some positive constants D_1 and D_2.
Let

$$H(x,u,v) = \frac{1}{(\alpha(x)+1)}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1} - uv.$$

Since $\alpha(x) > 0$ for all $x \in \Omega$ we have

$$\lim_{|(u,v)|\to\infty} H(x,u,v) = +\infty.$$

Choose $\delta > 1$ such that

$$\frac{1}{(\alpha(x)+1)}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1} - uv > 0$$

for all $u, v > \delta$. Hence we get

$$H(x,u,v) \geq 0 = H(x,0,0) \geq H(x,\tau_1,\tau_2), \ \ \forall u,v > \delta,\ \tau_1,\tau_2 \in]0,1[. \tag{1.11}$$

Let a, b be two real numbers such that $0 < a < \min\{1, c\}$, with c given by (1.8) and $b > \delta$ such that $\min\{b^{p^-}|\Omega|, b^{q^-}|\Omega|\} > 1$. Then from (1.10) we obtain

$$\int_\Omega \sup_{0\leq u,v\leq a}\left(\frac{1}{(\alpha(x)+1)}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1} - uv\right)dx \leq 0 < \min\left\{\frac{1}{c^{p^+}}\frac{a^{p^+}}{b^{p^-}}, \frac{1}{c^{q^+}}\frac{a^{q^+}}{b^{q^-}}\right\}\int_\Omega H(x,b,b)\,dx.$$

Define the real number r by

$$r = \min\left\{\frac{1}{p^+}\left(\frac{a}{c}\right)^{p^+}, \frac{1}{q^+}\left(\frac{a}{c}\right)^{q^+}\right\},$$

By choosing

$$(u_0(x), v_0(x)) = (0,0) \text{ and } (u_1(x), v_1(x)) = (b,b) \quad \text{for every} \quad x \in \Omega.$$

we have

$$\Phi(u_0,v_0) = J(u_0,v_0) = 0, \Phi(u_1,v_1) = \int_\Omega \frac{1}{p(x)}|b|^{p(x)} + \frac{1}{q(x)}|b|^{q(x)}\,dx$$

and

$$J(u_1,v_1) = -\int_\Omega H(x,b,b)\,dx.$$

Then clearly we have

$$\Phi(u_1,v_1) \geq \left(\frac{1}{p^+}b^{p^-} + \frac{1}{q^+}b^{q^-}\right)|\Omega| > r.$$

Thus we deduce that

$$\Phi(u_0, v_0) < r < \Phi(u_1, v_1).$$

On the other hand, we have

$$-\frac{(\Phi(u_1,v_1)-r)J(u_0,v_0)+(r-\Phi(u_0,v_0))J(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)} = r\frac{\int_\Omega H(x,b,b)\,dx}{\int_\Omega \frac{1}{p(x)}|b|^{p(x)}+\frac{1}{q(x)}|b|^{q(x)}\,dx} > 0.$$

We also have

$$\frac{1}{p^+}I_{p(x)}(u)+\frac{1}{q^+}I_{q(x)}(v) \leq \Phi(u,v),$$

which implies that

$$I_{p(x)}(u) \leq p^+r < 1 \ \text{ and } \ I_{q(x)}(v) \leq q^+r < 1,$$

for all $x \in \Omega$ and for all $(u,v) \in E$ such that $\Phi(u,v) \leq r$. Using (1.7), we get

$$\|u\|_1 \leq 1 \ \text{ and } \ \|v\|_2 \leq 1.$$

This implies that

$$\frac{1}{p^+}\|u\|_1^{p^+}+\frac{1}{q^+}\|v\|_2^{q^+} \leq \Phi(u,v) \leq r.$$

Taking into account that

$$|u(x)| \leq c(p^+r)^{\frac{1}{p^+}} < a \ \text{ and } \ |v(x)| \leq c(q^+r)^{\frac{1}{q^+}} < a$$

for all $x \in \Omega$ and for all $(u,v) \in E$ such that $\Phi(u,v) \leq r$, with $c = \sup_{(u,v)\in E}\frac{\|(u,v)\|_\infty}{\|(u,v)\|}$.
It follows

$$-\inf_{(u,v)\in\Phi^{-1}(]-\infty,r])} J(u,v) = \sup_{\Phi(u,v)\leq r} -J(u,v) \leq \int_\Omega \sup_{0\leq u,v\leq a}\left(\frac{1}{\alpha(x)+1}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1}-uv\right)dx \leq 0.$$

Consequently, we obtain

$$\inf_{(u,v)\in\Phi^{-1}(]-\infty,r])} J(u,v) > \frac{(\Phi(u_1,v_1)-r)J(u_0,v_0)+(r-\Phi(u_0,v_0))J(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)}.$$

Moreover, the functional

$$\Psi(u,v) = \int_\Omega \frac{1}{\gamma_1(x)}|u|^{\gamma_1(x)}+\frac{1}{\gamma_2(x)}|v|^{\gamma_2(x)}\,dx$$

is continuously Gâteaux differentiable on E, with compact derivative. So, in view of Theorem 1.1, the proof of Theorem 2.1 is achieved.
Now for the proof of Theorem 2.2, write now

$$J(u,v) = -\int_\Omega F(x,u,v)\,dx$$

for each $(u,v)\in E$. Now, since $G:\Omega\times I\!R\times I\!R\longrightarrow I\!R$ is a Carathéodory function satisfying the condition (1.5), then the functional $\Psi(u,v)=\int_\Omega G(x,u,v)\,dx$ is well defined and continuously Gâteaux differentiable on E, with compact derivative, and one has

$$\Psi^{'}(u,v)(\varphi,\psi)=\int_\Omega G_u(x,u,v)\varphi+G_v(x,u,v)\psi\,dx$$

for all $(u,v),(\varphi,\psi)\in E$. Then, the conclusion of Theorem 2.2 follows immediately when adapting the same technique as in the proof of Theorem 2.1 and using the growth condition (1.1) and (1.3), (1.4). Indeed, let the functional Φ similarly defined as before and consider now

$$J(u,v)=-\int_\Omega F(x,u,v)\,dx$$

and

$$|F(x,u,v)|\leq C\,u^{\alpha(x)+1}v^{\beta(x)+1}.$$

Remark that J is well defined and continuously Gâteaux differentiable with

$$J^{'}(u,v)(\varphi,\psi)=-\int_\Omega F_u(x,u,v)\varphi+F_v(x,u,v)\psi\,dx$$

for all $(u,v),(\varphi,\psi)$ in E. Note also that $J^{'}$ is compact. Now, it remains to verify the conditions (i), (ii) and (iii) of Theorem 1.1.

To claim (i), we have

$$J(u,v)\geq -C\int_\Omega |u|^{\alpha(x)+1}|v|^{\beta(x)+1}\,dx. \tag{1.11}$$

Without loss of generality, we will distinguish two cases since $|(u,v)|$ is considered to tend to infinity. We will restrict our selves to the cases when $|u|\to\infty,|v|$ bounded and when $|u|,|v|\to\infty$ and prove the coercivity of $\Phi+\lambda J$. In view of (1.2), there exists $p_2<p^-$ and $q_2<q^-$ such that $\frac{\alpha^++1}{p_2}+\frac{\beta^++1}{q_2}=1$. Hence from (1.11) and Young's inequality we get

$$\begin{aligned} J(u,v) &\geq -C\int_\Omega \|u\|_\infty^{\alpha^++1}\max(\|v\|_\infty^{\beta^++1},\|v\|_\infty^{\beta^-+1})\,dx\\ &\geq -C|\Omega|(\frac{\alpha^++1}{p_2}\|u\|_\infty^{p_2}+\frac{\beta^++1}{q_2}\max(\|v\|_\infty^{q_2},1)). \end{aligned}$$

This implies the coercivity of $\Phi+\lambda J$ since E is continuously in $C(\overline{\Omega})^2$ and $p_2<p^-$. On the other hand, let $|u|,|v|\to\infty$, we have

$$\begin{aligned} J(u,v) &\geq -C\int_\Omega \|u\|_\infty^{\alpha^++1}\|v\|_\infty^{\beta^++1}\,dx\\ &\geq -C|\Omega|(\frac{\alpha^++1}{p_2}\|u\|_\infty^{p_2}+\frac{\beta^++1}{q_2}\|v\|_\infty^{q_2}). \end{aligned}$$

Similarly as in the proof of the previous theorem, this implies that

$$\lim_{|(u,v)|\to\infty}(\Phi(u,v)+\lambda J(u,v))=\infty$$

since $p_2 < p^-, q_2 < q^-$ and E is continuously in $C(\overline{\Omega})^2$.
Now, from (1.4) , we may choose $\delta > 0$ such that

$$F(x, u, v) \geq 0$$

for all $u, v > \delta$. Then using (1.3) we obtain

$$F(x, u, v) \geq 0 = F(x, 0, 0) \geq F(x, \tau_1, \tau_2), \;\; \forall u, v > \delta,\ \tau_1, \tau_2 \in]0, 1[. \tag{1.10}$$

Then adapting the same technique as in the proof of the previous theorem, we deduce (ii) and (iii). This completes the proof of Theorem 2.2.

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