Attractors of viscous balance laws

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1 Introduction

This thesis is devoted to the study of global attractors of a class of singularly perturbed scalar parabolic equations depending on a small parameter ε. These equations possess a special structure allowing for a detailed description of the global attractor. Moreover, many properties of the attractor can be deduced using mainly information on equilibria and their variational equations. This leads to the study of a class of singularly perturbed boundary value problems which in general have many solutions.

The main part of this paper describes how to find solutions of these boundary value problems and how to determine their stability considered as equilibria of the parabolic equation. As proposed by Allen and O'Malley [AO90] for problems where qualitative information is sought rather than high order approximations we use phase plane methods to describe all solutions of the boundary value problem. Some special difficulties arise thereby from our choice of Neumann boundary conditions.

As ε tends to zero one expects typically that the global attractor has either a very simple structure (e.g. consists of one stable equilibrium only) or that its dimension tends to infinity. The rather surprising main result of this dissertation consists of the proof that for a class of nonlinearities the dimension of the global attractors can be very high but nevertheless stays bounded as ε tends to zero. We also show that the dimension can stabilize at any given dimension.

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1.1 Global attractors of scalar parabolic equations

Semiflows generated by scalar semilinear parabolic equations are a class of infinite-dimensional dynamical systems whose qualitative behavior has been an object of intensive research during the last fifteen years. It has been shown that the equation

$$u_t = u_{xx} + h(x, u, u_x) \ , \ h \in C^2 \tag{1}$$

with Neumann boundary conditions

$$u_x(0) = u_x(1) = 0$$

and initial condition

$$u(0, x) = u_0(x)$$

gives rise to a (local) C^1-semiflow on the Sobolev space $X \subseteq W^{2,2}([0,1], \mathbb{R})$ of functions satisfying the boundary conditions at $x = 0$ and $x = 1$. The associated semigroup T assigns to each pair (t, u_0) the solution profile $u(t, \cdot)$ of (1) at time $t > 0$ that satisfies the initial condition (1.1) at time $t = 0$. If the nonlinearity h satisfies some growth and sign conditions, the semiflow is global and dissipative, i.e. solutions exist for all (positive) times and there exists a large ball $B \subseteq X$ such that every solution $u(t)$ will eventually stay in this ball. Due to the smoothing properties of the Laplacian, $T(t, \cdot)$ is compact for all $t > 0$. Under these conditions a global attractor $\mathcal{A}$ exists, defined as a maximal compact invariant set in B that attracts all bounded subset of X, see e.g. the monograph of Hale [Hal88] for theorems in this rather general setting of compact and dissipative semigroups. This global attractor consists of all orbits that are defined for all (positive and negative) times t and that are uniformly bounded.

There are two special features of scalar parabolic equations that allow for a more precise description of the global attractor:
(a) a gradient structure and
(b) nodal properties.

Concerning (a), already in the sixties Zelenyak [Zel68] showed that equation (1) possesses a Lyapunov functional. Except at equilibria, this Lyapunov functional decreases along trajectories. For h not depending on u_x and Neumann boundary conditions, the Lyapunov functional can be given explicitly as

$$V(u, x) := \int_0^1 \left(\frac{1}{2}u_x^2 - H(x, u)\right) dx$$

where h is the derivative of H with respect to u.
Then
$$\frac{d}{dt}\, V(u(t), x) = -\int_0^1 u_t^2 \; dx < 0.$$
If h also depends on u_x the construction of such a Lyapunov functional becomes considerably more difficult, nevertheless there is still one. Since this Lyapunov functional can be shown to be bounded on bounded sets, any orbit that stays uniformly bounded for all $t \geq 0$ will tend to the set of equilibria of (1), i.e. the set of time independent solutions. In other words, the ω-limit set of a single point $u_0 \in X$ is contained in the set E of all equilibria. Matano [Mat78] showed that it even consists of exactly one equilibrium. The same arguments hold for negative t if a trajectory is defined there. So, for every trajectory that is defined and bounded for all $t < 0$, the α-limit set is also an equilibrium. Using the above characterization of the global attractor $\mathcal{A}$ as the union of all uniformly bounded trajectories that are defined also for all negative t, we obtain the following description of the global attractor:

Proposition 1.1 *$\mathcal{A} = \bigcup_{e \in E} W^u(e)$, where E is the set of all equilibria and $W^u(e)$ is the unstable set of e. It consists of*

- *the set E of equilibria and*
- *heteroclinic orbits connecting different equilibria.*

To refine this description, consider now the eigenvalue problem associated with the linearization of (1) at an equilibrium v.

$$\left.\begin{array}{rcl} w_{xx} + \partial_u h(x, v(x), v_x(x))w + \partial_p h(x, v(x), v_x(x))w_x & = & \lambda w \\ w_x(0) = w_x(1) & = & 0 \end{array}\right\} \qquad (2)$$

Definition 1.2 *An equilibrium v is called* **hyperbolic** *if 0 is not an eigenvalue of the linearization at v, i.e. if (2) has no nontrivial solution for $\lambda = 0$.*

Definition 1.3 *The* **Morse index** *$i(v)$ of a hyperbolic equilibrium v is the number of positive eigenvalues of the linearization at v.*

If all equilibria are hyperbolic then the global attractor is the finite union of equilibria and their unstable manifolds.

It is a classical observation by Sturm that the eigenvalues are connected to the oscillation properties of the eigenfunctions. There is a sequence of simple eigenvalues

$$\lambda_0 > \lambda_1 > \ldots, \ \lambda_n \to -\infty \ \text{ as } n \to \infty$$

and the eigenfunction w_k associated with λ_k has exactly k zeroes in the open interval $(0, 1)$.

If $h = h(u, u_x)$ does not depend explicitly on x, there is an important relation between the Morse index of an equilibrium v and the number $z(v_x)$ of strict sign changes of v_x defined as

$$\begin{aligned} z(v) &:= \sup\{n \in \mathbb{N}; \exists 0 < x_1 < \ldots < x_n < 1 \text{ with } v(x_i) \cdot v(x_{i+1}) < 0 \ , \ 1 \leq i < n\} \\ z(0) &:= 0. \end{aligned}$$

Proposition 1.4 *If $h = h(u, u_x)$ does not depend on x, then the Morse index of any nonconstant hyperbolic equilibrium v is either $z(v_x)$ or $z(v_x) + 1$.*

Proof: Suppose that $i(v) = n$ such that $\lambda_{n-1} > 0 > \lambda_n$. Differentiating the equilibrium equation

$$v_{xx} + h(v, v_x) = 0$$

with respect to x shows that v_x is a solution of the linearized equation but does not satisfy the Neumann boundary conditions. Nevertheless, the Sturm comparison theorem is applicable and yields that between two consecutive zeroes of u_{n-1} there is a zero of v_x and between two consecutive zeroes of v_x there is a zero of u_n. Since for any nonconstant equilibrium v the zeroes of v_x are simple, v_x has exactly $z(v_x) - 1$ zeroes in the interior of the interval $(0, 1)$ and another two at $x = 0$ and $x = 1$ due to the Neumann boundary conditions. Therefore, v_x has at least n and not more than $n + 1$ zeroes. Translating this back into the notions of $i(v)$ and $z(v_x)$ completes the proof.

$\square$

Remark: If the nonconstant equilibrium v is not hyperbolic one can prove along exactly the same lines that its center-unstable manifold has dimension $z(v_x)$ or $z(v_x) + 1$. In this case v has exactly one zero eigenvalue and the center-unstable manifold is a manifold with boundary, see [Hen85].

The second peculiarity of scalar parabolic equations is the existence of a discrete Lyapunov functional connected to the zero number $z(u(\cdot))$ of the solution u in $(0, 1)$. Already Sturm recognized that the zero number $z(u^1(t, \cdot) - u^2(t, \cdot))$ of the difference of two solutions u^1 and u^2 is non-increasing in time. This result was rediscovered several times during the last 150 years. The most recent version is due to Angenent [Ang88] who also shows that $z(u^1(t, \cdot) - u^2(t, \cdot))$ drops strictly at time t if $u^1(t, \cdot) - u^2(t, \cdot)$ possesses a multiple zero. One consequence of those nodal properties is the Morse-Smale property of the attractor: There can only exist heteroclinic connections from equilibria with higher Morse index to such with lower Morse index.

1.2 Equilibria, permutations and connections

Since the global attractor is the union of equilibria and connecting orbits, given a specific equation, one may want to find the equilibrium solutions first and think about connecting orbits afterwards. The equilibria are simply solutions of the boundary value problem

$$\left.\begin{array}{rcl} u_{xx} - h(x, u, u_x) & = & 0 \\ u_x(0) = u_x(1) & = & 0. \end{array}\right\} \qquad (3)$$

The next step in the description of the attractor consists of finding criteria whether two given equilibria are connected by a heteroclinic orbit or not. This question was first adressed by Brunovský and Fiedler [BF88],[BF89] in the case of a nonlinearity depending on u only with hyperbolic equilibria. Later, Fiedler and Rocha [FR96a], could show that also in case $h = h(x, u, u_x)$ all information on the connections can already be derived from the equilibrium solutions. Their work uses the observation of Fusco and Rocha [FR91] that all information on the Morse indices of the equilibria as well as on zero numbers of differences of equilibria is contained in the ordering of the equilibrium solutions at $x = 0$ and $x = 1$. Their results are formulated by means of a permutation π that is induced by the two orderings of the equilibria in the following way:
While at the left boundary $x = 0$

$$v_1(0) < v_2(0) < \ldots < v_k(0),$$

the equilibria satisfy at the right boundary $x = 1$

$$v_{\pi(1)}(1) < v_{\pi(2)}(1) < \ldots < v_{\pi(k)}(1).$$

This ordering is related to a shooting approach to find the equilibria of (1). Consider the initial value problem

$$\begin{aligned} v' &= w \\ w' &= -h(x,v,w) \\ x' &= 1 \end{aligned}$$

with

$$v(0) = v_0\ ,\quad w(0) = 0\ ,\quad x(0) = 0.$$

This choice of initial conditions ensures that the left boundary conditions will be satisfied. The **shooting surface** is just the union of all these solutions and as **shooting curve** $\mathcal{S}$ we denote the intersection of the shooting surface with the plane $\{x = 1\}$. Clearly, $v(x)$ solves the boundary value problem (3) if, and only if, $w(1)$ lies on the u-axis, i.e. if $(v(1), w(1))$ is a point of intersection between $\mathcal{S}$ and the u-axis.

The following proposition from [FR91] gives a relation between the permutation π and the Morse indices and zero numbers.

Proposition 1.5 [Fusco & Rocha] *Let the semiflow generated by (1) be dissipative. Let $v_1, v_2, \ldots, v_l$ be the equilibria of (1) with associated permutation π and assume that all equilibria are hyperbolic. Then, the Morse index $i(v_m)$ of the equilibrium v_m is given by*

$$i(v_m) = \sum_{j=1}^{m-1} (-1)^{j+1}\ \mathrm{sign}\ (\pi^{-1}(j+1) - \pi^{-1}(j)). \tag{4}$$

(empty sums equal zero.)

For any $1 \le m < n \le l$, the zero number $z(v_n - v_m)$ is given by

$$\begin{aligned} z(v_n - v_m) \;=\; i(v_m) \;&+\; \tfrac{1}{2}[(-1)^n\ \mathrm{sign}\ (\pi^{-1}(n) - \pi^{-1}(m)) - 1] \\ &+\; \textstyle\sum_{j=m+1}^{n-1} (-1)^j\ \mathrm{sign}\ (\pi^{-1}(j) - \pi^{-1}(m)). \end{aligned}$$

Fiedler and Rocha later showed that this information suffices to decide whether two equilibria are connected.

Proposition 1.6 [Fiedler & Rocha] *Under the assumptions of proposition 1.5 the permutation π determines which equilibria are connected by a heteroclinic orbit and which are not. More precisely:*

(i) Two equilibria v and w with Morse indices $i(v), i(w)$ are connected if, and only if, there exists a sequence

$$w = e_0, e_1, \ldots, e_n = v$$

of equilibria with $i(e_{k+1}) = i(e_k) + 1$ such that e_{k+1} and e_k are connected by a heteroclinic orbit.

(ii) Two equilibria v and w with Morse indices $i(v)$ and $i(w) = i(v) - 1$ are not connected if $z(v - w) \neq i(w)$ ('Morse blocking').

(iii) Two equilibria v and w with Morse indices $i(v)$ and $i(w) = i(v) - 1$ are not connected if there exists a third equilibrium $\bar{w}$ such that $z(v - \bar{w}) = z(w - \bar{w}) = z(v - w)$ and the value $\bar{w}(x)$ lies in between $v(x)$ and $w(x)$, at $x = 0$ ('zero number blocking').

(iv) Two equilibria v and w with Morse indices $i(v)$ and $i(w) = i(v) - 1$ are connected if neither (ii) nor (iii) holds.

1.3 Singular perturbations

We introduce now a small positive parameter ε in front of the diffusivity term of the parabolic equation (1), which thereby becomes

$$\left.\begin{array}{l} u_t = \varepsilon u_{xx} + h(x, u, u_x) \\ u_x(0, t) = u_x(1, t) = 0. \end{array}\right\} \tag{5}$$

It is easy to check that most of the statements above do not depend on ε. It neither influences the global existence nor the dissipativeness and compactness of the (ε-dependent) semigroup. Thus, for each fixed $\varepsilon > 0$ equation (5) possesses a global attractor $\mathcal{A}_\varepsilon$. In general, this attractor $\mathcal{A}_\varepsilon$ will vary with ε. In particular, hyperbolicity of all equilibria will not hold for all ε as the following 'classical' example shows. Consider a nonlinearity $h = h(u)$ not depending on x and u_x, e.g. the cubic $h(u) = u(1 - u^2)$. Chafee and Infante [CI74] showed that the equilibrium $u \equiv 0$ undergoes a sequence of pitchfork bifurcations at values $\varepsilon = (l\pi)^{-2}$, $l = 1, 2, \ldots$. At each of these pitchfork bifurcations two new equilibria appear and the Morse index of $u \equiv 0$ is increased by one. So there are two problems in getting a description of the attractor as ε tends to 0: the number of equilibria and the dimension of the attractor both tend to infinity. In terms of the shooting curve this is visible as the fact that $\mathcal{S}$ winds around the equilibrium $u \equiv 0$ an increasing number of times as ϵ becomes smaller and smaller. It is not difficult to see that this behaviour is typical for nonlinearities depending on u only. In this case the equilibria satisfy the equation

$$\varepsilon u_{xx} + h(u) = 0,$$

which has a Hamiltonian structure. By rescaling $x = \sqrt{\varepsilon}\xi$ the equilibrium equation becomes

$$u_{\xi\xi} + h(u) = 0, \tag{6}$$

and does not depend on ε any more, only the boundary condition at $x = 1$ reads now $u_\xi(\varepsilon^{-1/2}) = 0$ and contains the parameter ε. If h has at least two zeroes then (6) admits for families of periodic orbits that accumulate onto a homoclinic orbit or a pair of heteroclinic orbits. A solution of (6) following one of these periodic orbits with period p, say $\frac{k}{2}$ turns is a solution of the boundary value problem if $\varepsilon^{-1/2} = \frac{k}{2} \cdot p$. It is now easy to see that the following alternative holds, depending on whether h has exactly one or more than one zero: Either there is only one (spatially homogenous) equilibrium that does not depend on ε, or the attractor $\mathcal{A}_\varepsilon$ blows up in the way described above.

A natural question to ask is, whether the same is true for more general nonlinearities h. There are some results indicating that for $h = h(x, u)$ the behaviour is rather more complicated. While for h not depending on x and the gradient u_x all nonconstant equilibria are unstable, Angenent, Mallet-Paret and Peletier [AMPP87] found stable solutions which develop a transition layer. Later, Hale and Sakamoto [HS88] described also unstable equilibrium solutions with transition layers.

Below, a class of nonlinearities h of the special form $h(x, u, u_x) = (f(u))_x + g(u)$ will be shown to cause a different behaviour of $\mathcal{A}_\varepsilon$: The dimension of $\mathcal{A}_\varepsilon$ remains bounded for all small ε and in some cases all the attractors are equivalent in a sense that is to be specified in the next section. This shows that convection can prevent, at least in some cases, the attractor from blowing up.

1.4 Equivalence of attractors

Our description of attractors of singularly perturbed parabolic equations is to a large extend based on the following notion of equivalence of attractors.

Definition 1.7 *The global attractors $\mathcal{A}_h$ and $\mathcal{A}_k$ corresponding to the scalar parabolic equations*

$$u_t = u_{xx} + h(x, u, u_x),\ u_x(0, t) = u_x(1, t) = 0$$

and

$$u_t = u_{xx} + k(x, u, u_x),\ u_x(0, t) = u_x(1, t) = 0$$

are called (C^0)**-equivalent** *if there is a homeomorphism between the global attractors $\mathcal{A}_h$ and $\mathcal{A}_k$ that maps orbits onto orbits preserving the direction of time.*

Note that C^0-equivalence implies connection equivalence. Here two attractors are called **connection equivalent** if

(i) the equilibria of $\mathcal{A}_h$ are in one-to-one correspondance with the equilibria of $\mathcal{A}_k$ and

(ii) two equilibria of $\mathcal{A}_h$ are connected by a heteroclinic orbit iff the corresponding equilibria of $\mathcal{A}_k$ are connected.

The main result about C^0-equivalence was obtained only recently by Fiedler and Rocha [FR96b].

Proposition 1.8 [Fiedler & Rocha] *If for two nonlinearities h and k all equilibria are hyperbolic and the two permutations π_h and π_k induced by the shooting curves are identical, then the two attractors $\mathcal{A}_h$ and $\mathcal{A}_k$ are C^0-equivalent.*

This notion of equivalence can be adapted to the singularly perturbed setting. In that case there are no different nonlinearities, but one wants to compare the global attractors of (5) with the same nonlinearity h but different values of ε. To this end, time is rescaled by a factor ε which does not alter the global attractor. Dividing the equation by this factor ε yields

$$u_t = u_{xx} + \frac{h(x, u, u_x)}{\varepsilon}$$

such that different values of ε only affect the nonlinear part and definition 1.7 applies directly.

The ultimate goal would be to show that for a certain class of nonlinearities described below and sufficiently small ε there exists a finite ε-independent number of equilibria, all hyperbolic, and the global attractors $\mathcal{A}_\varepsilon$ are all C^0-equivalent. At the moment, we are only able to prove such a result for a rather restricted class of nonlinearities. Nevertheless, one should consider the following theorems as a (first) step in this direction.

1.5 The Main Theorems

Now all ingredients are collected such that the main results of this paper can be formulated. Theorem 1.1 tells that the dimension of the global attractor $\mathcal{A}_\varepsilon$ remains bounded as $\varepsilon \searrow 0$ for a class of nonlinearities h of the form

$$h(u, u_x) = -(f(u))_x + g(u)$$

.

Theorem 1.1 *Consider the singularly perturbed parabolic equation of the special form*

$$u_t + (f(u))_x = \varepsilon u_{xx} + g(u), \quad f, g \in C^3.$$

Assume that

(H1) *g is a dissipative function*

(H2) *the critical points of f are quadratic folds, i.e. $f'(u) = 0 \Rightarrow f''(0) \neq 0$*

(H3) *the derivative of f does not vanish at zeroes of g.*

Then the dimension of $\mathcal{A}_\varepsilon$ remains bounded as $\varepsilon \to 0$.

Under stronger conditions the dimension of $\mathcal{A}_\varepsilon$ not only stays bounded but it stabilizes.

Theorem 1.2 *There exists an open (with respect to the strong Whitney topology) class of functions f and g such that the following holds: There is an ε_0 and some integer d such that for all $0 < \varepsilon \leq \varepsilon_0$ the dimension of the global attractor $\mathcal{A}_\varepsilon$ is contained in $\{d, d+1\}$.*

The precise conditions on f and g will be given as the additional hypotheses (H4)-(H7) in chapter 4. It can be checked easily that the conditions (H1)-(H3) persist under C^2-small perturbations. Dissipativeness of g implies that only a finite range of u is really important for the asymptotic behavior of solutions and hence for the global attractor $\mathcal{A}_\varepsilon$. Assuming dissipativeness, the other conditions (H2)-(H7) are open and dense with respect to the Whitney topology.
The dimension of the $\mathcal{A}_\varepsilon$ can stabilize at any integer $d \geq 1$ but there is a special case where we have achieved our goal to prove C^0-equivalence of the attractors $\mathcal{A}_\varepsilon$:

Theorem 1.3 *If the conditions of theorem 1.2 are satisfied and f is a convex function then the global attractor is (at most) two-dimensional for all $0 < \varepsilon \leq \varepsilon_0$ and for all $0 < \varepsilon' < \varepsilon \leq \varepsilon_0$ the global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_{\varepsilon'}$ are C^0-equivalent.*

The rest of the paper is organized in the following way: Chapter 2 deals with viscous balance laws. Chapter 3 contains a first investigation of equilibrium solutions to the viscous balance law and the proof of theorem 1.1. In chapter 4 the reduced system that corresponds to $\varepsilon = 0$ is examined. In chapter 5 the relation between the reduced system and the system for $\varepsilon > 0$ is investigated. To this end we use a normal form which is derived in the appendix. In chapter 6, theorem 1.2 is proved and the existence of (in general, many) equilibrium solutions is shown. After establishing uniqueness of solutions for some cases in chapter 7, the planar case of theorem 1.3 is treated in chapter 8. The paper concludes with a short discussion.

2 Balance Laws

Since in the rest of this paper viscous balance laws are considered, we will show in this section that they fit in the above setting. Viscous balance law is a term used for a scalar parabolic equation of the special form

$$u_t + (f(u))_x = g(u) + \varepsilon u_{xx}. \tag{7}$$

We will study this equation on the unit interval with Neumann boundary conditions

$$u_x(0) = u_x(1) = 0$$

and initial data

$$u(0, \cdot) = u_0(\cdot) \in W^{1,2}.$$

Here f and g are of class C^2 and g is a dissipative function, i.e.

$$u \cdot g(u) < 0 \quad \forall\, |u| > R \tag{8}$$

with some (large) constant R. The parameter ε is very small and adds some dissipativeness to the usual "balance law"

$$u_t + (f(u))_x = g(u). \tag{9}$$

Balance laws are a generalization of conservation laws

$$u_t + (f(u))_x = 0$$

where a source term $g(u)$ is present.
As with conservation laws, for balance laws there are in general no global smooth solutions even for arbitrarily smooth initial data. After a finite time, shocks are formed. Often such first order hyperbolic equations are considered with x on the whole real line. As Kruzhkov [Kru70] showed, under some admissibility condition for any bounded measurable u_0 there is a unique solution of the hyperbolic equation (9). On finite time intervals the solution of the viscous balance law (7) converges to this unique solution as ε tends to 0.
Only recently conservation as well as balance laws on an interval have attracted more attention. This is mostly due to the occurence of steep transition layers that move very slowly [RW95]. With these property, these second order scalar equations serve as crude models for phase transitions or semiconductors [BH95]. Especially, they are used as test problems for numerical analysists who are interested in the numerical treatment of more complicated (and hopefully more realistic) problems, e.g. higher dimensional equations describing phase transitions.
One difference between scalar conservation laws and balance laws is the fact that solutions of balance laws need not decay to a spatially homogenous state as time tends to infinity. In the case of $x \in \mathbb{R}$ and periodic initial data it was shown 1970 by Glimm and Lax that solutions of scalar conservation laws decay like $\mathcal{O}(t^{-1})$ to their spatial average when the time t tends to infinity. This is not true for balance laws: There may exists periodic solutions and at least in the case of f being convex, a Poincaré-Bendixson type result holds: Every solution either tends to an equilibrium or to a periodic orbit, which is a traveling wave then. This result was

shown by Lyberopoulos [Lyb94], Fan & Hale [FH95] and also by Sinestrari [Sin95b] and ressembles very much the results in the parabolic case treated in Angenent & Fiedler [AF88] and Fiedler & Mallet-Paret [FMP89] although the methods are quite different.
Local existence of weak solutions of solutions for the parabolic equation

$$u_t + (f(u))_x = g(u) + \varepsilon u_{xx} \tag{10}$$

with Neumann boundary conditions can be shown by semigroup methods as in the book of Henry [Hen81]. The (unbounded) linear operator εu_{xx} together with the Neumann boundary conditions generates an analytical semigroup on the space $W^{1,2}(0,1)$ and the smoothness assumptions on f and g are sufficient to guarantee a local solution of (10). This solution lies in the domain of the infinitesimal generator, i.e. in the space $X \subseteq W^{2,2}$ of functions which satisfy the Neumann boundary conditions. Furthermore, the time derivative $u_t(t,\cdot)$ is Hölder continuous so by elliptic Schauder regularity theory the solution is a classical solution. This allows to talk about derivatives of the solution, to use maximum principles involving u_{xx}, etc..
Although this could also be derived from general theorems by Amann [Ama85], it will be shown here in an elementary way that the local solutions of the viscous balance law (10) exist globally in time and that dissipativeness is guaranted by the sign condition (8) on g.
Global existence of solutions will be shown via some a-priori estimates on u and the derivative u_x:

Lemma 2.1 *If u satisfies equation (7) for all $t \in [0,T]$, then:*
(i) $\|u(T,\cdot)\|_{L^\infty} \leq C(u_0)$ *independent of* ε
(ii) $\|u_x(T,\cdot)\|_{L^2} \leq C(u_0,\varepsilon)$

Proof: (i) follows from a maximum principle. Due to the dissipativeness condition (8), in any positive maximum $u(x_0,t_0)$ with $0 < t_0 \leq T$ and $|u(x_0,t_0)| > R$

$$u_t = \varepsilon \underbrace{u_{xx}}_{\leq 0} - \underbrace{f'(u)u_x}_{=0} + \underbrace{g(u)}_{<0} < 0.$$

In the same way we can conclude that in any negative minimum $u(x_0,t_0)$ with $0 < t_0 \leq T$ and $u(x_0,t_0) < -R$ we have $u_t > 0$. Therefore, the L^∞-norm decreases as long as $u(t,\cdot)$ takes values outside $[-R,R]$ and hence

$$\|u\|_{L^2} \le \|u\|_{L^\infty} \le \max\{R, \|u(0,\cdot)\|_{L^\infty}\} =: C.$$

To proof claim (ii), first note that for $u \in W^{2,2}$

$$\|u_x\|_{L^2}^2 = \int_0^1 u_x^2 dx \le \int_0^1 |u u_{xx}| dx \le \|u\|_{L^2}\|u_{xx}\|_{L^2}$$

hence for any solution u

$$-\|u_{xx}\|_{L^2}^2 \le -C_0 \cdot \|u_x\|_{L^2}^4$$

where the constant

$$C_0 := \frac{1}{C^2}$$

depends on u_0. Furthermore,

$$\begin{aligned}\int_0^1 f'(u) u_x u_{xx}\, dx &\le \max_{|u|\le C} |f'(u)| \cdot \|u_x\|_{L^2}\|u_{xx}\|_{L^2} \\ &\le \frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2 + \frac{C}{2\varepsilon}\|u_x\|_{L^2}^2.\end{aligned}$$

Multiplying equation (7) by u_{xx} one obtains after integrating

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|u_x\|_{L^2}^2 &= -\varepsilon\|u_{xx}\|_{L^2}^2 - \int_0^1 f'(u) u_x u_{xx}\, dx + \int_0^1 g'(u) u_x^2\, dx \\ &\le -\frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2 - \frac{\varepsilon C_0}{2}\|u_x\|_{L^2}^4 + \frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2 + \frac{C}{2\epsilon}\|u_x\|_{L^2}^2 + C(u_0)\|u_x\|_{L^2}^2 \\ &\le 0 \qquad\qquad \text{for } \|u_x\|_{L^2} \ge C(u_0,\varepsilon) \\ &\Rightarrow \|u_x(t,\cdot)\|_{L^2} \le \max\{\|u_x(0,\cdot)\|_{L^2}, C(u_0,\varepsilon)\}\end{aligned}$$

□

This lemma implies immediately that all solutions exist globally in time and that (forward) orbits are bounded in $W^{1,2}$. Due to the variational structure of equation (7), for any $u_0 \in W^{1,2}$ the ω-limit set of u_0 is contained in the union of the equilibrium solutions of (7). To prove dissipativeness, it remains only to show that for any fixed ε the equilibrium solutions form a bounded set. Below we will focus a lot on equilibrium solutions and for this reason we postpone the proof of boundedness to lemma 3.4. Here we only note that the boundedness of the set of equilibrium solutions implies the dissipativeness of the semiflow.

3 Equilibrium solutions

3.1 A singularly perturbed boundary value problem

Since, by definition, equilibrium solutions do not depend on time t, we will write for these solutions simply $u(x)$ instead of $u(x,t)$. Equilibrium solutions of (7) are solutions of the boundary value problem

$$\left.\begin{array}{rcl} \varepsilon u_{xx} - (f(u))_x + g(u) & = & 0 \\ u_x(0) = u_x(1) & = & 0. \end{array}\right\} \tag{11}$$

This singularly perturbed boundary value problem can be written as a first order system

$$\left.\begin{array}{rcl} \varepsilon u_x & = & v + f(u) \\ v_x & = & -g(u) \\ u_x(0) & = & u_x(1) = 0, \end{array}\right\} \tag{12}$$

a choice of coordinates which is sometimes called the "Liénard plane" in contrast to the more common "phase plane" where $v = u_x$. Concerning the shooting curve $\mathcal{S}$, there is no important difference between the usual phase plane and the Liénard plane. The u-coordinate is always the same and hence the permutation π used in proposition 1.5 and proposition 1.8 can also be read off from the Liénard plane.

Note that the boundary condition in (12) could also be written in the form

$$v(x) + f(u(x)) = 0 \quad \text{at } x = 0 \text{ and } x = 1 \quad ,$$

so, geometrically speaking, we are looking for trajectories of (12) which take exactly "time" $\Delta x = 1$ to join two points on the curve $v + f(u) = 0$. To avoid too much confusion between time and space variables, we recast (12) in the form

$$\left.\begin{array}{rcl} \varepsilon u' & = & v + f(u) \\ v' & = & -g(u) \\ x' & = & 1 \end{array}\right\} \tag{13}$$

$$0 < s < 1$$
$$u'(0) = u'(1) = x(0) = 0$$

where $'$ denotes differentiation with respect to a new variable s that looks more like a time variable than x does, although obviously $x \equiv s$. Below, methods from singular perturbation theory are used that compare the system for small ε with some limiting systems for $\varepsilon = 0$. There are different possibilities to perform this limit, leading to the so called "slow" and "fast" systems which both describe a part of the limiting behaviour of system (12). The difference originates in a different scaling of the variable s.

Setting $\varepsilon = 0$ in equation (12), we arrive at the "slow system"

$$\begin{aligned} 0 &= v + f(u) \\ v' &= -g(u). \end{aligned}$$

Here the motion is confined to a curve given by the first of the two equations, while the second one describes the flow along this curve. Since this curve will play a special rôle in the next chapters it deserves a name on its own:

Definition 3.1 *The curve $\mathcal{C}$ given by the equation $v + f(u) = 0$ in the (u, v)-plane is called the* **singular curve**.

Note that $u_x = 0$ exactly where the trajectory hits or crosses the curve $\mathcal{C}$. Later, when we have to determine $z(u_x)$ to use lemma 1.4 we will use this property. We can simply count the number of intersections between the trajectory and the curve $\mathcal{C}$.

System (12) can also be scaled in another way. If the second equation is multiplied by ε and the variable s is rescaled according to $s = \varepsilon\sigma$, we arrive at

$$\begin{aligned} \dot{u} &= v + f(u) \\ \dot{v} &= -\varepsilon g(u). \end{aligned}$$

with $\dot{}$ denoting the derivative with respect to the fast variable σ. Putting $\varepsilon = 0$, the "fast system"

$$\begin{aligned} \dot{u} &= v + f(u) \\ \dot{v} &= 0 \end{aligned}$$

is obtained. Here, the singular curve consists of equilibrium points only. According to the stability of these equilibria, parts of $\mathcal{C}$ where $f' > 0$ are called unstable arcs of $\mathcal{C}$, while the parts with $f' < 0$ are called stable arcs.

Before theorem 1.2 is proved using singular perturbation theory, we will give a pedestrian proof of theorem 1.1. Although it is rather elementary, it contains some information on system (12) that will prove useful later.

3.2 Proof of theorem 1.1

From the three assumptions theorem 1.1, we have already used one: The dissipativeness (H1) of g was necessary for the existence of a global attractor. The condition (H2) that all zeroes of f' be simple can probably be weakened on the price of longer and more complicated calculations. Let us just mention in short that the assumption (H3) which states that no zeroes of g lie on the fold points of the singular curve $\mathcal{C}$ is necessary to prevent a blow-up of $\mathcal{A}_\varepsilon$ similar to the Chafee-Infante example presented in the introduction.

We begin with a short outline of the proof. Recall the characterization of $\mathcal{A}_\varepsilon$ as the unstable manifold of the set E of equilibria. If all equilibria are hyperbolic then they are isolated and the attractor is contained in the union of the unstable manifolds of all single equilibria. The hyperbolicity of all equilibria is not guaranteed under the weak assumptions of theorem 1.1, such that we have to consider the possibility of a non-hyperbolic attractor, too. In this case a bound on the dimension of the center-unstable manifolds of the equilibria will be derived. We will distinguish between spatially homogenous and non-homogenous equilibrium solutions. It will turn out, that for the homogenous solutions the linearization has at most one non-negative eigenvalue. For the non-homogenous ones in view of lemma 1.4 it is necessary to prove that for some integer N and all small ε the equilibria have at most N extrema. This will be an immediate consequence of lemma 3.6 where we will prove that three extrema of an equilibrium solution u cannot be arbitrarily close to each other. On the way to this lemma, we collect some properties of the two-dimensional system

$$\left.\begin{aligned} \varepsilon u' &= v + f(u) \\ v' &= -g(u) \end{aligned}\right\} \tag{14}$$

for small ε.

Observe first that the equilibrium points of system (14) lie on the curve $\mathcal{C}$ and have as u-coordinates exactly the zeroes of g. Due to assumption (H3), the eigenvalues

$$\mu_{1,2} = \frac{f'(u_0) \pm \sqrt{f'(u_0)^2 - 4\varepsilon g'(u_0)}}{2\varepsilon}$$

of the linearization at such an equilibrium (u_0, v_0) turn out to be real for small ε and behave asymptotically like

$$\mu_1 \sim \frac{f'(u_0)}{\varepsilon} \quad \text{and} \quad \mu_2 \sim \frac{g'(u_0)}{f'(u_0)}.$$

If $g'(u_0) \neq 0$ the corresponding equilibrium of (14) is hyperbolic and a saddle exactly if $g'(u_0) < 0$. If all zeroes of g are simple, the equilibria are alternately saddles and sources or sinks.
Each of these zeroes of g corresponds to one homogenous equilibrium solution. The next step consists of showing that (e.g. in contrast to the Chafee-Infante case) these homogenous equilibrium solutions cannot become very unstable as ε decreases.

Lemma 3.2 *If $g(\tilde{u}) = 0$ then for ε sufficiently small the first eigenvalue λ_0 of the equilibrium solution $u \equiv \tilde{u}$ is*

$$\lambda_0 = g'(\tilde{u})$$

and all other eigenvalues are strictly negative. In particular, $u \equiv \tilde{u}$ is hyperbolic iff $g'(\tilde{u}) \neq 0$ and in this case the Morse index is 0 for $g'(\tilde{u}) < 0$ resp. 1 for $g'(\tilde{u}) > 0$.

Proof: For a homogenous equilibrium solution the eigenvalue equation is just a linear second-order boundary-value problem

$$\begin{aligned} \varepsilon w_{xx} - f'(\tilde{u}) w_x + g'(\tilde{u}) w &= \lambda w \\ w_x(0) = w_x(1) &= 0. \end{aligned}$$

The solutions of this second-order equation can be found easily to be of the form

$$w(x) = c_1 e^{\mu_-(\varepsilon)x} + c_2 e^{\mu_+(\varepsilon)x}$$

where

$$\mu_\pm = \frac{f'(\tilde{u}) \pm \sqrt{f'(\tilde{u})^2 - 4\varepsilon(g'(\tilde{u}) - \lambda)}}{2\varepsilon}.$$

Thus, $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ are real if

$$f'(\tilde{u})^2 - 4\varepsilon(g'(\tilde{u}) - \lambda) > 0.$$

With real $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ the boundary value problem only has a solution if $\mu_+(\varepsilon) = 0$ and this holds exactly for $\lambda = g'(\tilde{u})$. Other eigenvalues require $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ to be complex which is only possible for

$$\lambda < -\frac{f'(\tilde{u})^2}{4\varepsilon} + g'(\tilde{u}).$$

Thus, for all sufficiently small ε, $\lambda = g'(\tilde{u})$ has to be the *first* eigenvalue.

□

The next definition keeps track of the fact that our interest is not in any trajectory of (14), but mainly in those that correspond to solutions of the Neumann boundary value problem.

Definition 3.3 *An* **admissible trajectory** *of system (14) is a trajectory that corresponds to a solution of the boundary value problem, i.e. it is a finite piece of a trajectory $u(s)$ that satisfies $u'(0) = u'(1) = 0$.*

Let us now state a simple lemma that allows us to restrict our attention to a finite range of u.

Lemma 3.4 *Let f and g be as in theorem 1.1 and denote with $u_{min} and u_{max}$ the minimal, resp. maximal zero of g. Then:*

(i) For any admissible trajectory $(u(s), v(s))$ of (14)

$$u_{min} \leq u(s) \leq u_{max} \quad \forall s \in [0, 1].$$

(ii) There are constants C, ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ along any admissible trajectory of (14)

$$u'(s) \leq \frac{C}{\varepsilon} \qquad \forall s \in [0, 1].$$

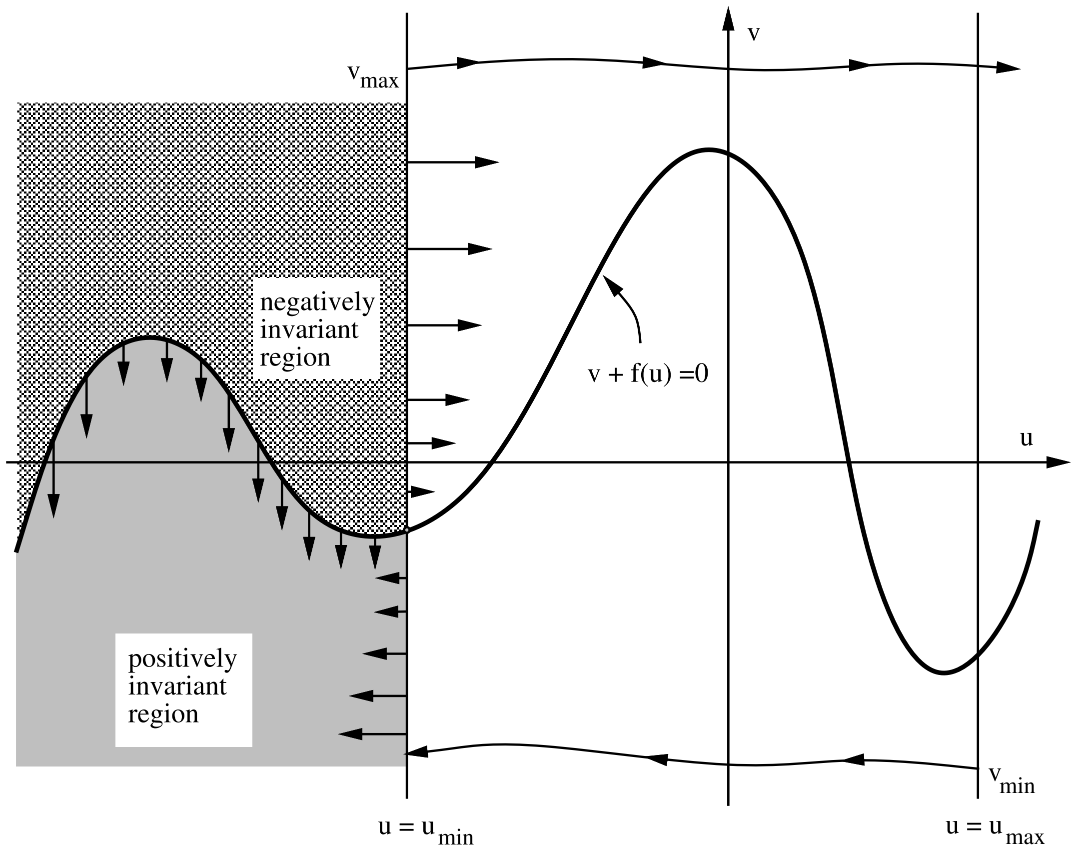


Figure 1: Admissible solutions are confined to a bounded region in the Liénard plane

Proof : To prove part (i) indirectly, suppose that for some $s \in [0,1]$ we have $u(s) < u_{min}$. The curve $v + f(u) = 0$ divides the region $\{(u,v); u < u_{min}\}$ into two parts. Since the vector field is vertical on the curve $\{v + f(u) = 0\}$ and horizontal on the line $u = u_{min}$ the region $\{(u,v);\ u < u_{min}, v < f(u)\}$ is positively invariant while $\{(u,v);\ u < u_{min}, v > f(u)\}$ is negatively invariant, see figure 1.
If $(u(s), v(s))$ lies in the positively invariant region for some s then the right boundary condition can obviously not be satisfied. If $(u(s), v(s))$ lies in the upper, negatively invariant region there is no chance of satisfying the left boundary condition. So a solution of the boundary value problem may not enter one of the two regions. It cannot stay on the curve $\{v + f(u) = 0\}$ either since by assumption u_{min} is the leftmost zero of g. Thus, there can be no solution of the boundary value problem that takes a value less than u_{min}. The argument excluding $u(s) > u_{max}$ is similar.
Claim (ii) obviously follows by showing that

$$\varepsilon u'(s) = v(s) + f(u(s)) \leq C.$$

To see this take

$$v_{max} > \sup\{-f(u)\ ;\ u_{min} \leq u \leq u_{max}\} + 1$$

and

$$v_{min} < \inf\{-f(u)\ ;\ u_{min} \leq u \leq u_{max}\} - 1.$$

For ε small enough the trajectory starting in (u_{min}, v_{max}) will be almost horizontal and intersects the line $u = u_{max}$ without having hit the singular curve before. Similarly, the trajectory starting in (u_{max}, v_{min}) stays below the singular curve until it hits the line $u = u_{min}$. Therefore, all admissible trajectories are for small ε confined to a bounded region of the (u, v)-plane.

□

The restriction of admissible trajectories to a bounded region also yields an upper bound on v':
Along any admissible trajectory we have

$$|v'(s)| \leq \max_{u_{min} \leq u \leq u_{max}} |g(u)| =: C_g.$$

There are also some restrictions on trajectories that are close to the curve $\mathcal{C}$:

Lemma 3.5 *Consider again the system (14) and assume that $f'(u) < 0$ ($f'(u) > 0$) for all $u \in [u_-, u_+]$.*
Then:

(i) For all sufficiently large k and all small ε trajectories can leave (enter) a region

$$\{(u, v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

only at $u = u_-$ or $u = u_+$.

(ii) If furthermore u_- and u_+ are two adjacent zeroes of g, then there is a positively invariant region between the curves $\mathcal{C}$ and $v + f(u) - k\varepsilon g(u) = 0$ for k large and all ε small (a negatively invariant region between $\mathcal{C}$ and a curve $v + f(u) + k\varepsilon g(u) = 0$).

(iii) In this case the two equilibria $(u_-, -f(u_-))$ and $(u_+, -f(u_+))$ on the singular curve are connected by a heteroclinic orbit.

Remark: The heteroclinic orbits of (iii) are part of the **slow manifold**, an invariant manifold that exists for $\varepsilon > 0$ near the singular curve $\mathcal{C}$ except in a neighborhood of the fold points, cf. [Fen79].

Proof:

(i) For definiteness, we suppose that $f'(u) < 0$ for $u \in [u_-, u_+]$. The case $f'(u) > 0$ can be treated in the same way.

Along a curve $v + f(u) + k\varepsilon = 0$ the trajectories written as $v = v(u)$ satisfy

$$\left|\frac{dv(u)}{du}\right| = \left|\frac{g(u)}{k}\right|.$$

Since g is bounded on $[u_-, u_+]$, by choosing

$$k > \max_{u_- \leq u \leq u_+} \left|\frac{g(u)}{f'(u)}\right|$$

one can achieve that the vector field is directed into the region

$$\{(u, v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

along the whole curve so that trajectories can leave the region only via the left or the right boundary. Especially, trajectories that enter a neighborhood of a stable arc of $\mathcal{C}$ can leave such a neighborhood only near a fold.

(ii): To establish the invariant regions, one has to check that along curves $v + f(u) - k\varepsilon g(u) = 0$ trajectories satisfy

$$\left|\frac{dv}{du}\right| = \frac{1}{k}.$$

which is for k large enough and all small ε certainly smaller than the infimum

$$\inf_{u_- \leq u \leq u_+} |f'(u) - k\varepsilon g'(u)|$$

of the slope of the curve.

Then the existence of invariant regions is easily established by distinguishing the two cases $g > 0$ and $g < 0$ on (u_-, u_+). In both cases one finds a positively invariant region either above or below $\mathcal{C}$. It is easy to check that for $f' > 0$ there are negatively invariant regions.

(iii): A simple argument proves the heteroclinic connection between the two equilibria: Since the two equilibria are adjacent, one of them is a saddle and the other a

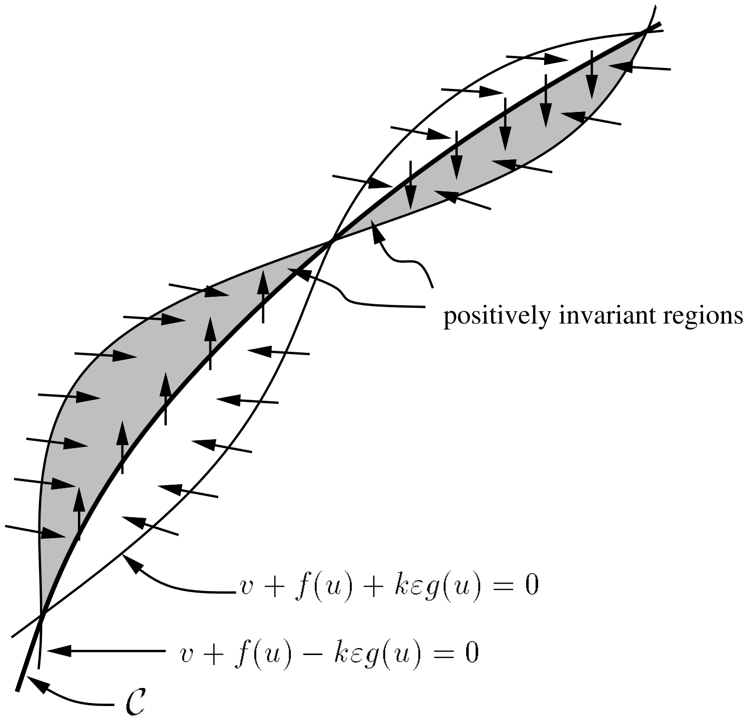


Figure 2: Invariant regions near the singular curve $\mathcal{C}$

sink. Consider the eigenvector to the unstable eigenvalue λ_2 of the saddle. A short computation shows that the (one-dimensional) invariant manifold corresponding to λ_2 is directed into the invariant region if only k is chosen large enough. Since this region contains in its interior neither equilibria nor periodic orbits (u' has a definite sign there) the invariant manifold must connect to the sink lying on the boundary of the invariant region.

□

Figure 2 shows a situation with $f' < 0$ and two positively invariant regions. There are heteroclinic orbits from the upper and lower equilibrium to the one in between which are not shown in the figure.
Note also that admissible trajectories may not enter one of the regions enclosed by $\mathcal{C}$ and such a heteroclinic orbit because once inside such a region they cannot reach the curve $\mathcal{C}$ again to satisfy the boundary condition.
Now we state the lemma which is crucial for proving theorem 1.1 since it shows that for all small ε and any solution u of the boundary value problem (11) the number

$z(u_x)$ will not exceed a certain bound.

Lemma 3.6 *Let* $(u(s), v(s))$, $0 \leq s \leq 1$ *be a nonconstant admissible trajectory. Then there exists a* $\varepsilon_0 > 0$ *and some* $\delta > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the following holds: If* $s_1 < s_2 < s_3$ *are three different zeroes of* u', *then* $s_3 - s_1 > \delta$.

Proof: The proof concentrates on s_2 and shows that $s_3 - s_2 > \delta$ or $s_2 - s_1 > \delta$. We may without restriction assume, that there are no other zeroes of u' in the intervals (s_1, s_2) and (s_2, s_3). The arguments differ depending on whether $(u(s_2), v(s_2))$ lies near a fold or on a stable or unstable arc of $\mathcal{C}$. To this end the interval $[u_{min}, u_{max}]$ is divided into several parts. It was assumed that all zeroes of f' are simple, so they cannot accumulate and there is a finite number of points $\overline{u}_1 < \overline{u}_2 < \ldots < \overline{u}_F$ in $[u_{min}, u_{max}]$ with

$$f'(\overline{u}_i) = 0.$$

The case of $g(\overline{u}_i) < 0$ and $f''(\overline{u}_i) > 0$ is treated here in detail, but all other combinations of signs for $g(\overline{u}_i)$ and $f''(\overline{u}_i)$ lead to similar results.
It is possible to find a neighborhood $[\alpha_i, \beta_i]$ of $\overline{u}_i$ such that

$$f(\alpha_i) = f(\beta_i)$$

and both

$$g(u) > c \ \text{ and } \ f''(u) > c$$

hold for all u in the whole interval $[\alpha_i, \beta_i]$ and some $c > 0$.
The condition $g(u) > c$ implies that all trajectories that cross $\mathcal{C}$ between α_i and β_i will do this from above. It is easy to see that these trajectories can leave the region enclosed by $\mathcal{C}$ and the horizontal line $v = -f(\alpha_i)$ only through just this line. Between α_i and β_i, one can find $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ with

$$\alpha_i < \tilde{\alpha}_i < \overline{u}_i < \tilde{\beta}_i < \beta_i,$$

$$f(\tilde{\alpha}_i) = f(\tilde{\beta}_i)$$

and

$$f(\tilde{\alpha}_i) - f(\alpha_i) = \frac{1}{2}(f(\overline{u}_i) - f(\alpha_i)),$$

compare figure 3.

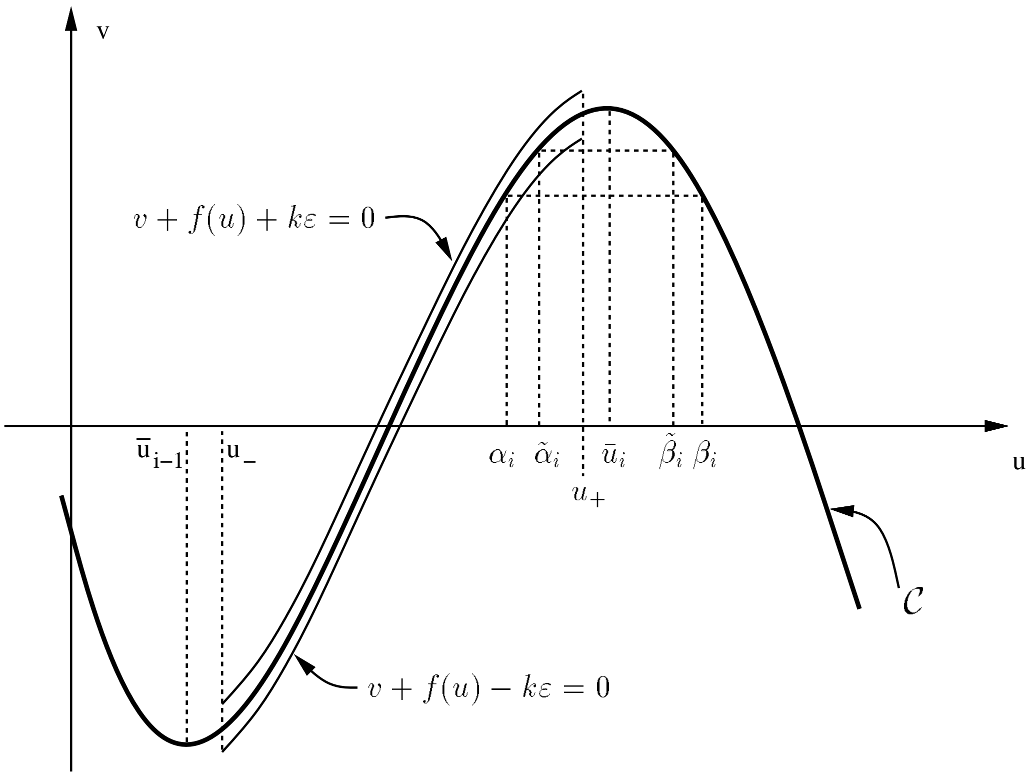


Figure 3

Consider the case that an admissible trajectory hits the curve $\mathcal{C}$ between $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ at the time $s = s_2$. Since $|v'|$ was bounded by C_g independent of ε, it will take the trajectory a time of at least

$$\Delta s_i := \frac{f(\tilde{\alpha}_i) - f(\alpha_i)}{C_g}$$

to leave the region enclosed by $\mathcal{C}$ and the line $v = -f(\alpha_i)$. During this time it cannot hit the singular curve again. Thus, if s_2 lies in the interval $[\tilde{\alpha}_i, \tilde{\beta}_i]$, we have

$$s_3 - s_2 \geq \Delta s_i.$$

The same situation appears if $g(\overline{u}_i) > 0$ and $f''(\overline{u}_i) < 0$. For the other two cases where $g(\overline{u}_i) > 0$ and $f''(\overline{u}_i) > 0$ have the same sign a similar reasoning shows that a trajectory that hits $\mathcal{C}$ near the fold cannot have hit it for a certain time before and hence

$$s_2 - s_1 \geq \Delta s_i.$$

Thus, if s_2 lies in some interval $[\tilde{\alpha}_i, \tilde{\beta}_i]$ the lemma is proved by chosing δ smaller than the infimum of the Δs_i.

If the trajectory intersects the singular curve on a stable arc in some interval $[\tilde{\beta}_{i-1}, \tilde{\alpha}_i]$ and ε is sufficiently small then the trajectory is in one of the strips

$$\{(u,v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

of lemma 3.5 that can only be left at their left or right boundary.
Two things can be shown:
1) A trajectory cannot intersect the singular curve again before leaving the strip.
2) The trajectory stays inside the strip for at least the time $\Delta s_i/2$.
To prove the first claim, remember that an admissible trajectory may not enter one of the regions enclosed by $\mathcal{C}$ and a heteroclinic orbit connecting two adjacent equilibria on the same (stable or unstable) arc of $\mathcal{C}$. Therefore, an admissible trajectory can cross the curve $\mathcal{C}$ only between a fold and the nearest equilibria on either side. By crossing $\mathcal{C}$ there, the trajectory enters a region enclosed by $\mathcal{C}$ and a curve $v + f(u) + k\varepsilon g(u) = 0$ with large $|k|$. As was shown in lemma 3.5(ii), this region can be left only at $u = u_-$ or $u = u_+$.
The second claim, concerning the time it takes a trajectory to leave the invariant strip, is proved here for the case $f'(u_+) < 0$ and $g(u_+) < 0$, i.e. of a trajectory that follows a stable arc of $\mathcal{C}$ to its right end. However, all other cases can be treated in the same way changing signs appropriately and reversing time if necessary. The idea is simple again: By choosing ε small enough, the trajectory has to cover a certain v-distance near $\mathcal{C}$ and since the velocity in v-direction is bounded by C_g this will take a certain amount of time. More precisely, if u_- is chosen close to $\overline{u}_{i-1}$ and u_+ close to $\overline{u}_i$, and furthermore ε is sufficiently small, then

$$-f(u_+) - k\varepsilon + f(\alpha_i) \geq \frac{1}{4}\left(-f(\overline{u}_i) + f(\alpha_i)\right).$$

If a trajectory intersects the singular curve on the stable arc below $v = -f(\alpha_i)$ and leaves the invariant strip at $u = u_+$ the v-coordinate has to increase by at least

$$\frac{1}{4}\left(-f(\overline{u}_i) + f(\alpha_i)\right).$$

This implies that a trajectory needs at least the time $\Delta s_i/2$ from a horizontal section $v = -f(\alpha_i)$ to the point where it leaves the invariant strip.
So, in the case that $(u(s_2), v(s_2))$ lies on a stable arc of $\mathcal{C}$ we have shown that

$$s_3 - s_2 \geq \inf_i \Delta s_i/2$$

independent of ε. The case that $(u(s_2), v(s_2))$ lies on an unstable arc is similar and leads to the result that

$$s_2 - s_1 \geq \inf_i \Delta s_i/2.$$

Choosing δ smaller than the infimum of the $\Delta s_i/2$ completes the proof of lemma 3.6.

$\square$

We are now able to finish the proof of theorem 1.1. It was shown that for any non-homogenous equilibrium solution u of the viscous balance law the zero number of u_x can be estimated by

$$z(u_x) \leq \frac{2}{\delta} + 1.$$

This ensures that the dimension of the center-unstable manifold of any non-homogenous equilibrium solution of the viscous balance law does not exceed a certain ε-independent bound $\frac{2}{\delta} + 2$. The spatially homogenous equilibria have been shown in lemma 3.2 to have an at most one-dimensional unstable resp. center manifold.

Assume first that for a given ε there is finite number of equilibria which are all hyperbolic. Then the global attractor is the union of the unstable manifolds of these equilibria. Each of these manifolds has a dimension not exceeding $\frac{2}{\delta} + 2$ so this gives an upper bound on the dimension of $\mathcal{A}_\varepsilon$ as well and the theorem is proved for this case.

In the other cases with non-hyperbolic and possibly infinitely many equilibria the arguments have to be refined.

The set of equilibria is a closed subset of $\mathcal{A}_\varepsilon$ in the space X where the semiflow is defined and hence a compact set. Consider a local center-unstable manifold $W^{cu}_{loc}(u)$ of an equilibrium u. Despite of non-uniqueness, it contains all the solutions from a neighborhood $\mathcal{N}(u)$ that are bounded backward in time, especially all the equilibria contained in $\mathcal{N}(u)$. Due to compactness of the set of equilibria a finite union of local center-unstable manifolds

$$\bigcup_{i=1}^{k} W^{cu}_{loc}(u_i)$$

suffices to cover all equilibria. We claim that the global attractor is contained in the set

$$\mathcal{W} := \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k} T(n, W^{cu}_{loc}(u_i))$$

where $T(n, \cdot)$ is the time-n-map of the semiflow. The dimension of $\mathcal{W}$ is not greater than the maximal dimension of the center-unstable manifolds since applying the semiflow to a set over a finite time does not increase the dimension. Recall that the global attractor consists of equilibria and heteroclinic orbits only, so it remains only to show that every heteroclinic orbit $h(t)$ is contained in $\mathcal{W}$. The $\alpha-$limit set of h is an equilibrium $h_{-\infty}$ and there is a i_0 such that $h_{-\infty}$ lies in $W^{cu}_{loc}(u_{i_0})$. Moreover, there exists a time $t_- < 0$ such that $h(t)$ lies in $\mathcal{N}(u_{i_0})$ for all times $t < t_-$. Therefore, $h(t) \in W^{cu}_{loc}(u_{i_0})$ for $t < t_-$. So it is clear that the whole heteroclinic orbit h is contained in

$$\bigcup_{n=1}^{\infty} T(n, W^{cu}_{loc}(u_{i_0}))$$

finishing thereby the proof of theorem 1.1.

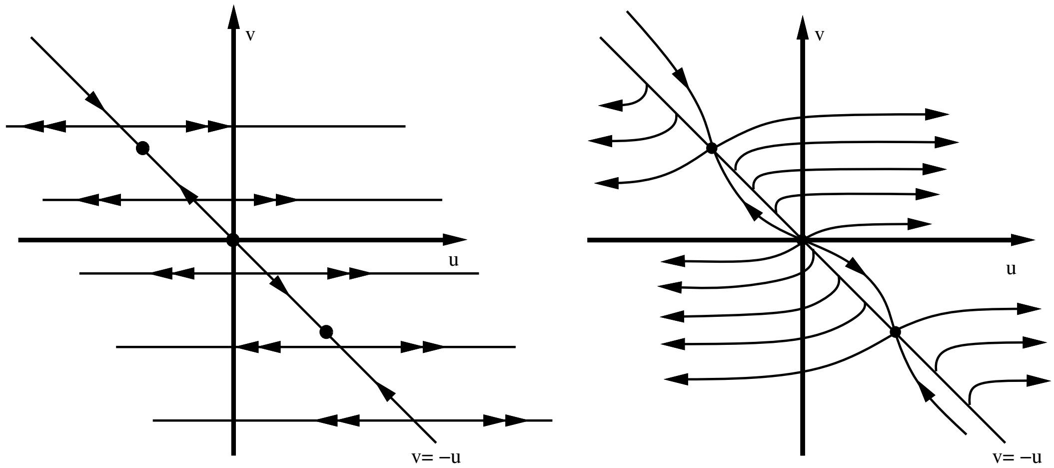
□

4 The reduced system

To get not only an upper bound but sharper estimates on the dimension of the global attractor it is necessary to find some (or better: all) equilibrium solutions of the parabolic equation and to determine their Morse indices.

In the last chapter, the "slow" and "fast" system were introduced, the slow system being defined only on $\mathcal{C}$ while for the fast system the curve $\mathcal{C}$ consists of equilibrium points only.

The combination of the "slow" system on the curve $v + f(u) = 0$ and the "fast" system off this curve indicates where in the phase plane one has to look for solutions of the boundary value problem. This combination is often refered to as the **reduced system**. Solutions of the reduced system consist alternately of slow parts where they solve the slow system and jumps according to the fast system. In the Liénard plane we will consider **trajectories of the reduced system** consisting of slow motion along the singular curve and horizontal trajectories of the fast system. Due to the different time scales, only the slow parts contribute to the time along a trajectory of the reduced system while the "velocity" along the fast parts is infinite. One might expect to find solutions for small nonzero ε near such trajectories of the reduced systems. Unfortunately, this is not always true. Unlike for the singularly



phase portrait of the reduced system ($\varepsilon = 0$)

phase portrait of the perturbed system ($\varepsilon > 0$)

Figure 4

perturbed initial value problem there need not be a solution of the singularly perturbed boundary value problem near a solution of the reduced system even if this solution of the reduced system satisfies the boundary conditions.

As a simple example consider (11) with $f(u) = u$ and $g(u) = u - u^3$. There are infinitely many solutions of the reduced system that follow the singular curve $v = -u$ but for $\varepsilon > 0$ any trajectory starting on this curve is either constant or tends to infinity. These trajectories will never come back to the singular curve and therefore (11) has no nonconstant solution.

As the example shows, some difficulties are caused by the heteroclinic orbits whose existence was established in lemma 3.5 (iii). They block the curve $\mathcal{C}$ from one side such that trajectories can leave or reach $\mathcal{C}$ only from the other side. To take account of that and also of the fact that $\mathcal{C}$ is invariant only for the reduced system at $\varepsilon = 0$ we introduce the notion of admissible solutions also for the reduced system. This will be done in such a way that the admissible solutions of the reduced system will correspond to the solutions of the boundary value problem (12). In some sense they will be the limit of admissible solutions as $\varepsilon \searrow 0$.

To avoid lots of different but similar cases we introduce a suggestive colorful notation

first:
To any point on $\mathcal{C}$ we associate two colors, one for its left side and one for its right side. In fact, we will give the same color to segments of $\mathcal{C}$ that lie either between two equilibria, between two fold points or between an equilibrium and a fold according to the following rules:

- All points between two equilibria on the same arc of $\mathcal{C}$ get the color red on the side of $\mathcal{C}$ where the heteroclinic orbit connecting the two equilibria is located. The other side is colored green.
- Between a fold and the nearest equilibrium or between two folds the side where the invariant region from lemma 3.5(ii) is gets the color yellow while the opposite side is given green.
- Equilibria are green on both sides.

or in other words: If $g(u) \geq 0$ then the left side is green while for $g(u) \leq 0$ the right side is green. The opposite side is red or yellow depending on the fact whether the point $(u, -f(u))$ lies between two equilibria on the same arc of $\mathcal{C}$ or not.
The interpretation of this coloring is easy: Red sides are the forbidden ones. No admissible trajectory of the system (14) leaves $\mathcal{C}$ to the red side or reaches it from the red side. In contrast, green sides are the sides to which trajectories leave a neighborhood of $\mathcal{C}$ or from which they enter. The yellow parts are somehow in between: The curve $\mathcal{C}$ is not quite blocked from that side but trajectories will only creep along $\mathcal{C}$ to the next fold without intersecting the singular curve. Especially, a trajectory cannot end after jumping to a yellow segment because the boundary condition cannot be satisfied there.
In view of this interpretation the following definition seems reasonable:

Definition 4.1 *A solution $(u(s), v(s))$, $0 \leq s \leq T$ of the reduced system is called an* **admissible solution** *of the reduced boundary value problem on the interval $[0, T]$ if the following properties hold:*
(i) $(u(0), (v(0))$ and $(u(T), v(T))$ both lie on $\mathcal{C}$.
(ii) The solution does neither begin with a nonconstant slow part on an unstable arc of $\mathcal{C}$, nor does it end with a nonconstant slow part on a stable arc.
(iii) The solution does neither jump to $\mathcal{C}$ from a red side nor does it leave $\mathcal{C}$ to a red side.

Remarks:

1) All nonconstant solutions of the reduced system in the preceding example are not admissible due to condition (ii).

2) We will only be interested in admissible solutions on ontervals with length near 1. If no interval length is specified, we always assume $T = 1$.

3) Admissible trajectories can start or end at an equilibrium point and spend an arbitrarily long time there. They cannot reach or leave an equilibrium via the singular curve since this would take an infinite time.

To avoid some technical complications, the following assumptions on f and g will be posed for the proof of theorem 1.2:

(H4) The zeroes of g are simple.

(H5) If $f(u_1) = f(u_2)$ and either $g(u_1) = g(u_2) = 0$ or $f'(u_1) = f'(u_2) = 0$ or $f'(u_1) = g(u_2) = 0$ then there is a u_3 between u_1 and u_2 with $f(u_3) = f(u_1)$, i.e. there is no fast connection between two equilibria, between two folds or between a fold and an equilibrium on $\mathcal{C}$.

It is well known that (H4) is a generic condition on g. Similarly, (H5) holds for an open set of f and g and moreover an arbitrarily (C^2)-small perturbation of g suffices to break a possible fast connection between two fold or equilibrium points. In fact, a local perturbation of g will be sufficient, such that there can be no problem with the dissipativeness (H1).

These assumptions simplify the reduced system considerably and enable us to give a not too long classification of all admissible solutions. The important observation is that a trajectory, once it has reached a stable arc of $\mathcal{C}$ either will terminate there (i.e. converge to an equilibrium on this stable arc) or will follow the stable arc to a fold and has to leave $\mathcal{C}$ there to either escape to infinity or to jump onto another stable arc. Similarly, a backward trajectory that has reached an unstable arc can only jump to other unstable arcs. Consequently, any trajectory can be divided into two parts one of which may be empty: First, it follows always unstable arcs, afterwards all slow parts are only along stable arcs. We will, rather loosely, speak of the **unstable part** and the **stable part of the trajectory**. Dividing trajectories in this way has an important reason: While one can easily check that trajectories do not vary continuously with respect to the initial condition (consider for example a trajectory that jumps from an unstable arc immediately to a fold point of the singular curve $\mathcal{C}$, but misses this fold point if the initial point is removed slightly), a small variation of a point (u, v) between the unstable and the stable part of the trajectory will only have small effects on the trajectory. Taking into account the possibilities of a start or end at an equilibrium point, etc., there remain six different possibilities. The number can be reduced if symmetry with respect to time reversal is considered. Type I comprises trajectories that start or end at an equilibrium, type II are the typical trajectories that consist of an unstable part and a stable part. Type III contains three different degenerate sorts of trajectories that do not persist under small perturbations of a point (u, v) on the fast part between the unstable and the stable part of the trajectory.

Type III solutions do not occur for generic choices of f and g as we will explain later. For this reason they will be excluded by assumption (H6). We list them here with the other types to give a complete classification of all possible trajectories.

Type Ia The trajectory starts at an equilibrium point on an unstable arc of $\mathcal{C}$, and jumps from there immediately to another point on $\mathcal{C}$. This point has to be on a stable arc of $\mathcal{C}$ since forward orbits cannot reach an unstable arc and reaching a fold point on a trajectory of the fast system is forbidden by assumption (H5). The trajectory may now follow the stable arc to the next fold point and jump from there to another stable arc. There may be an arbitrary number of further jumps from a fold point to a stable arc of $\mathcal{C}$.

Type Ib The same as type Ia but with time reversed: Trajectories start on an unstable arc, leave this arc somewhere and jump to a fold point where they follow the unstable arc. From this unstable arc the trajectory may again jump to a fold point. This can happen several times but after a number of such jumps the trajectory does not jump to a fold but settles down at an equilibrium on a stable arc of $\mathcal{C}$.

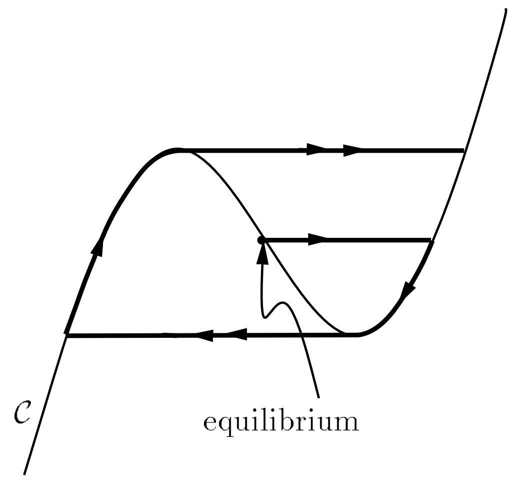


Figure 5: A type Ia trajectory of the reduced system

Type II The solution starts on an unstable arc, jumps from there immediately to a fold point and continues on the unstable arc that emanates from this fold point. From this unstable arc it may again jump to another fold point and continue again along an unstable arc of $\mathcal{C}$. After a number of jumps to fold points it jumps somewhere from an unstable directly to a stable arc. It continues along this stable arc to a fold point, jumps from there to another stable arc, and so on. An arbitrary number of jumps from a fold to other stable arcs may follow.

To avoid additional cases type II comprises also solutions that start already on a stable arc and do several jumps to other stable arcs, i.e. the unstable part of the trajectory may be empty. Similarly, the trajectory may consist of an unstable part only.

Type IIIa The trajectory starts on an unstable arc and jumps from there immediately to a fold point. From there it continues along the unstable arc and jumps again to a fold point. This can happen several times. At one such fold point, the trajectory takes the stable arc instead of the unstable one. It can continue along the stable arc to a fold point and jump from there to another stable arc of $\mathcal{C}$. Again, a number of jumps from a fold to a stable arc may follow.

Type IIIb The same as type IIIa but with time reversed.

Type IIIc The trajectory starts on an unstable arc and jumps immediately to a fold point of $\mathcal{C}$, continues along the unstable arc and jumps from there to another fold point, and so on. After one of these jumps to a fold point it continues along the fast trajectory to a stable arc of $\mathcal{C}$. The trajectory may then follow the

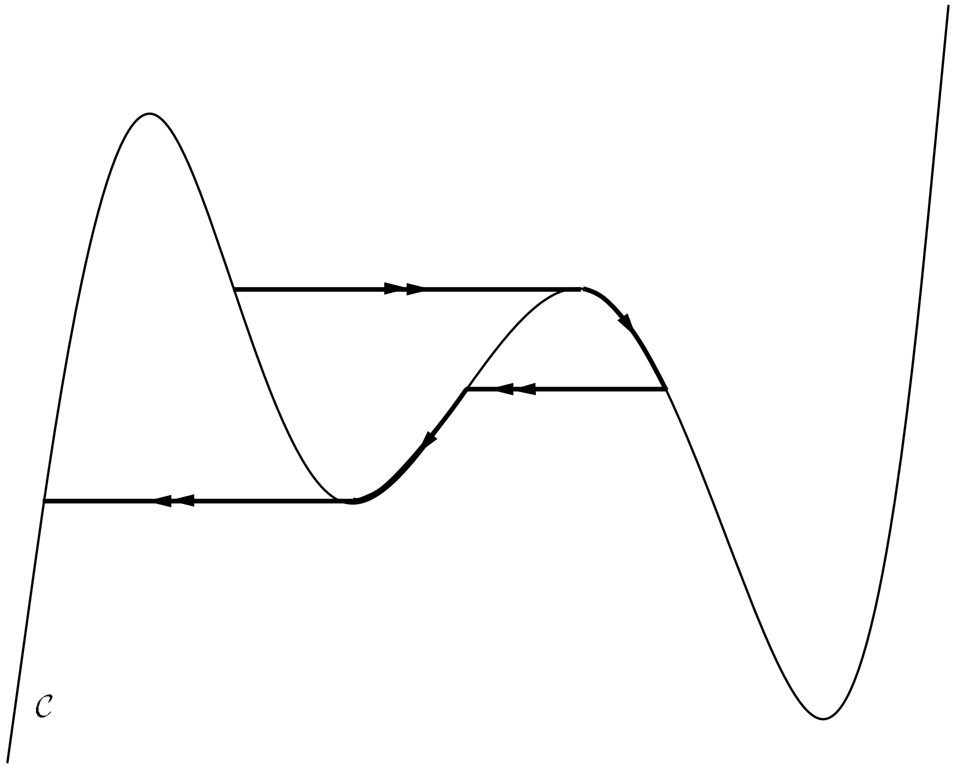


Figure 6: A type II trajectory of the reduced system

singular curve to a fold point and jump from there to another stable arc of $\mathcal{C}$. This can be followed by an arbitrary (finite) number of jumps to stable arcs.

Type III trajectories are depicted in figure 7.

The following considerations show that this is a complete classification of all possible trajectories. Starting from any point that does not lie on $\mathcal{C}$ the forward and backward trajectories are uniquely defined except possibly at points where the trajectory jumps onto a fold point. At a fold point, the trajectory could follow the stable or the unstable arc or continue on a fast part. Following the unstable arc corresponds to type II, following the stable arc to type IIIa and following the fast trajectory to a type IIIc trajectory. For backward trajectories the situation is similar but with type IIIb replacing type IIIa. We decide that forward trajectories will at a fold always take the stable arc while backward trajectories will always follow the unstable arc. By this convention we get unique forward and backward trajectories for each initial value $(u_0, v_0) \notin \mathcal{C}$ and (u_0, v_0) lies on the fast part that separates the unstable form the stable part of the trajectory. This will be very important in view of the time maps for trajectories of the reduced equation we are going to introduce in a moment. The reader may check that no trajectories are excluded by this convention: It is possible to find a point $(u_0, v_0) \notin \mathcal{C}$ on any given trajectory, such that this trajectory is exactly the (unique) trajectory through (u_0, v_0). Now let $D_j(u_0, v_0)$ denote the j-th **drop point** of the trajectory, i.e. the j-th point where a slow part of the trajectory begins. If this slow part extends to some fold point, we denote this point with $F_j(u_0, v_0)$. From there the trajectory may either escape or jump to $D_{j+1}(u_0, v_0)$, etc. . The backward trajectory gives analogously drop points $D_{-i}(u_0, v_0)$ and fold points $F_{-i}(u_0, v_0)$.

Definition 4.2 *For each point (u, v) not lying on $\mathcal{C}$ and $i = 1, 2, \ldots$ the time map $T_i(u, v)$ is defined as the i-th time at which a (forward) trajectory starting in (u, v) hits a green side of $\mathcal{C}$. Analogously, $T_i(u, v), i = -1, -2, \ldots$ are the time maps for the backward trajectory. Denote with $P_i(u, v), i \neq 0$ the corresponding points where the forward or backward trajectories intersect $\mathcal{C}$ after time T_i.*

Remarks:

1) If (u, v) lies on a trajectory of the fast system that joins an unstable and a stable arc of $\mathcal{C}$ it may happen that both $T_1(u, v) = T_{-1}(u, v) = 0$. 2) The reason why only hitting a *green* side of C gives rise to one of the time maps is the following: We will later find trajectories of the system with $\varepsilon > 0$ near the trajectory of the reduced system. These trajectories will intersect the curve $\mathcal{C}$ near the points $P_i(u, v)$ but not near points where the trajectory of the reduced system hits a red or yellow side.

Note that the time maps T_i may not be defined everywhere, but behave nicely almost everywhere: Consider a point $(u_0, v_0) \notin \mathcal{C}$ and assume that this point does not lie on a trajectory of the fast system that begins or ends neither at a fold nor at an equilibrium point. Then the same is true for all (u, v) in a neighborhood of (u_0, v_0). Moreover, the trajectories through (u, v) and (u_0, v_0) coincide almost, i.e. except for the first (forward and backward) fast parts and tiny pieces of the first slow parts. Therefore, all time maps T_i that are defined in (u_0, v_0) are defined and continuous in neighborhood of (u_0, v_0).

Let us now state in terms of the time maps under which conditions an admissible solution of the reduced system exists:

Lemma 4.3 *An admissible solution of the reduced system on the interval $[0, 1]$ through (u, v) exists if and only if one of the following conditions is satisfied:*

(ia) The complete (i.e. forward and backward) trajectory through (u,v) is of type Ia and there is an integer $j>0$ such that the (forward) time map $T_j(u,v)$ satisfies

$$T_j(u,v) \leq 1.$$

(ib) The complete (i.e. forward and backward) trajectory through (u,v) is of type Ib and there is an integer $i>0$ such that the (backward) time map $T_{-i}(u,v)$ satisfies

$$T_{-i}(u,v) \leq 1.$$

(ii) The trajectory through (u,v) is of type II and there are $i,j \geq -1$ such that

$$T_{-i}(u,v) + T_j(u,v) = 1.$$

(iii) The trajectory through (u,v) is of type III and there are $i,j>0$ such that

$$T_{-i}(u,v) + T_j(u,v) = 1.$$

Remark: $i=-1$ in case (ii) captures the possibility of a trajectory consisting only of a stable part while $j=-1$ corresponds to a trajectory without a stable part. In both cases the admissible solution does actually not pass through the point (u,v) but joins $P_1(u,v)$ and $P_j(u,v)$ or $P_{-j}(u,v)$ and $P_{-1}(u,v)$.

Proof:
(ia) Since (u,v) lies between the unstable and the stable part of a trajectory of type Ia it has to be exactly on a trajectory of the fast system that joins an equilibrium point $P_{-1}(u,v)$ on an unstable arc and a point on a stable arc of $\mathcal{C}$. If for some j the time map $T_j<1$ then an admissible solutions is easily constructed: Just stay for the time $1-T_j$ at the equilibrium $P_{-1}(u,v)$ and follow afterwards the trajectory for exactly time T_j to the point $P_j(u,v)$.
The argument for (ib) is analogous.
(ii) In this case (u,v) lies on the trajectory of the fast system that separates the unstable from the stable part of the trajectory. It is obvious that an admissible solution can only exist if the condition on some of the time maps is satisfied. Then the trajectory that joins $P_{-i}(u,v)$ and $P_j(u,v)$ is an admissible solution of the reduced system.

(iii) If (u,v) lies on a type IIIa trajectory then it is located on a trajectory of the fast system that joins an unstable arc to a fold point such that $T_{-1} = T_1 = 0$. Again it is clear that the condition is necessary for the existence of an admissible solution. In case of a type IIIc trajectory the argument is the same as in (ii).

□

In contrast to trajectories of type I or II the trajectories of type III do not occur generically. To see this, note that the drop points for a type III trajectory can only be points that are joined to a fold point by a trajectory of the fast system. The number of such points between u_{min} and u_{max} is finite since there are only finitely many extrema of f between u_{min} and u_{max} and since to every extremum there are at most two drop points, one for the forward and one for the backward trajectory of the fast system that passes through the extremum.
Consequently, the differences $|T_i - T_{i-1}|$ can only take finitely many different values corresponding to the times a trajectory takes between a drop point and a fold. So, generically, as $T_{-1} = T_1 = 0$ and the difference between T_i and T_{i-1} takes only certain values, there will be no type III solution. Also, generically the finitely many trajectories that just "touch" a fold point, i.e. trajectories like the one in the middle picture of figure 7, will not yield an admissible solution on the interval $[0,1]$.
For this reason, we will exclude all these solutions to keep things as simple as possible:

(H6) There are no admissible trajectories on $[0,1]$ of type III, see figure 7.

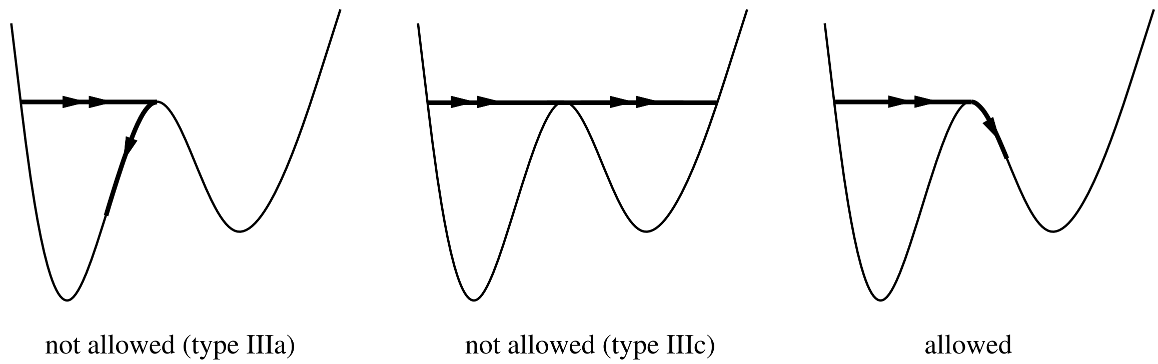


Figure 7

The classification of trajectories into different types was necessary because there is a different transversality condition that is necessary in order that solutions persist for small nonzero ε:

(H7) For any type Ia admissible solution of the reduced system with $T_j(u,v)$ as in lemma 4.3 we require $T_j(u,v) < 1$. Similarly, type Ib solutions must satisfy $T_{-i}(u,v) < 1$.

For any type II solution of the reduced system with time maps taken from a point (u_0, v_0) as in lemma 4.3, the mapping

$$v \longmapsto (T_{-i}(u,v) + T_j(u,v) - 1)(v - v_0)$$

has the same sign in a neighborhood of $v = v_0$ and there is a constant C such that $|T_{-i}(u,v) + T_j(u,v) - 1| \geq C|v - v_0|$.

For type III solutions an analogous transversality condition could also be formulated but would be more complicated. The reason is that one has to embed the type III solution correctly into a one-parameter family of trajectories. Then, for this family, transversality of the time maps has to be satisfied.
(H7) like all the other conditions is satisfied for a generic choice of f and g since it concerns only the reduced system and a small (local) change of g makes it possible to change the time maps appropriately.
We mention one important consequence of this transversality condition (H7) for the reduced system:

Lemma 4.4 *The reduced system possesses only finitely many admissible solutions on the interval* $[0,1]$.

Proof:
Admissible solutions of the reduced system share some properties with the admissible solutions for $\varepsilon > 0$ which we have proved in the last chapter: They are contained in the interval $[u_{min}, u_{max}]$ and any admissible solution of the reduced system consists of a number of fast and slow parts. The velocity in v-direction is bounded by C_g such that every slow part takes a certain minimal time. This is a consequence of the fact that on the fast parts, the v-coordinate remains unchanged while on slow parts of the trajectory v solves a differential equation which has a right hand side

bounded by C_g. Thus, there is a number K such that any admisssible trajectory consists of at most K slow and fast parts. Suppose now that there were infinitely many admissible solutions of the reduced system for the interval $[0, 1]$. Then there have to be infinitely many admissible solutions of type II since at each of the finitely many equilibrium points in $[u_{min}, u_{max}]$ at most K type I solutions can begin or end and thus the number of type I solutions is bounded. If there are infinitely many type II solutions, it is possible to choose a sequence (u_n, v_n), $n = 1, 2, \ldots$ of points with the following properties:

1) There are $i, j > -1$ independent of n such that $T_{-i}(u_n, v_n) + T_j(u_n, v_n) = 1$ for all n.

2) The points (u_n, v_n) converge to some point (u_∞, v_∞).

Three cases have to be considered now leading to different contradictions:

Case A: (u_∞, v_∞) is a fold point or lies on a trajectory of the fast system that connects to a fold point. In this case one can easily check that the trajectory through the fold point yields an admissible solution of type III in contradiction to (H6).

Case B: (u_∞, v_∞) is an equilibrium point or lies on a trajectory of the fast system that connects to an equilibrium point. Suppose for definiteness that the (forward) drop points $D_1(u_n, v_n)$ approach an equilibrium point. Then $T_2(u_n, v_n)$, if defined at all, is bigger than 1 for n sufficiently large. So j has to be 1 and $T_j(u_n, v_n) = 0$ which yields $T_{-i}(u_n, v_n) = 1$ and there is type Ib trajectory to the equilibrium point that does not satisfy the transversality condition (H7). The same reasoning is possible if the backward drop points $D_{-1}(u_n, v_n)$ approach an equilibrium point.

Case C: If neither case A nor case B holds the points (u_n, v_n) can be chosen in a way such that $(u_\infty, v_\infty) \notin \mathcal{C}$. Then the trajectory through (u_∞, v_∞) yields a type II admissible solution but since the (u_n, v_n) approach (u_∞, v_∞) the transversality condition (H7) is not satisfied.

Since each of the three cases leads to a contradiction the number of admissible solutions has to be finite.

$\square$

To find equilibrium solutions of our viscous balance law, we want to show that under the transversality assumption (H7) admissible solutions of the reduced system persist for small $\varepsilon > 0$. It will be proved that for any admissible solution of the reduced system and any small $\varepsilon > 0$ the system (12) has a solution nearby. Nearby means that the trajectory of the solution for $\varepsilon > 0$ lies in a neighborhood of the solution of the reduced system in the (u, v)-plane. In particular, the solution for $\varepsilon > 0$ follows the same arcs of $\mathcal{C}$ and has the same jumps.

To this end, it will be necessary to know something about trajectories for small nonzero ε. The next chapter provides us with the necessary information.

5 Asymptotic behavior of trajectories and time maps

Having limit cycles and their period in mind, Mishchenko and Rozov [MR80] carried out the asymptotic analysis of solutions of system (14) for small positive ε. They partitioned the trajectories into several pieces where different asymptotic expansions are valid. Four different types were distinguished by them:

(i) the slow part where the trajectory creeps along the singular curve,

(ii) the junction part where the trajectory leaves a vicinity of the singular curve near an extremum of f,

(iii) the fast part and

(iv) the drop part where after a fast transition the trajectory reenters a neighborhood of the singular curve.

For our discussion of boundary value problems we also have to consider how a trajectory starts off from the singular curve or settles down at it. It turns out that one has to distinguish the cases that a trajectory leaves the curve near an equilibrium point on $\mathcal{C}$ or somewhere else.

In figure 8 it is shown how a trajectory is decomposed into the different parts.

The trajectory starts at the point A near an equilibrium R and after leaving a neighborhood of the singular curve it moves fast from B to C. The fast part is

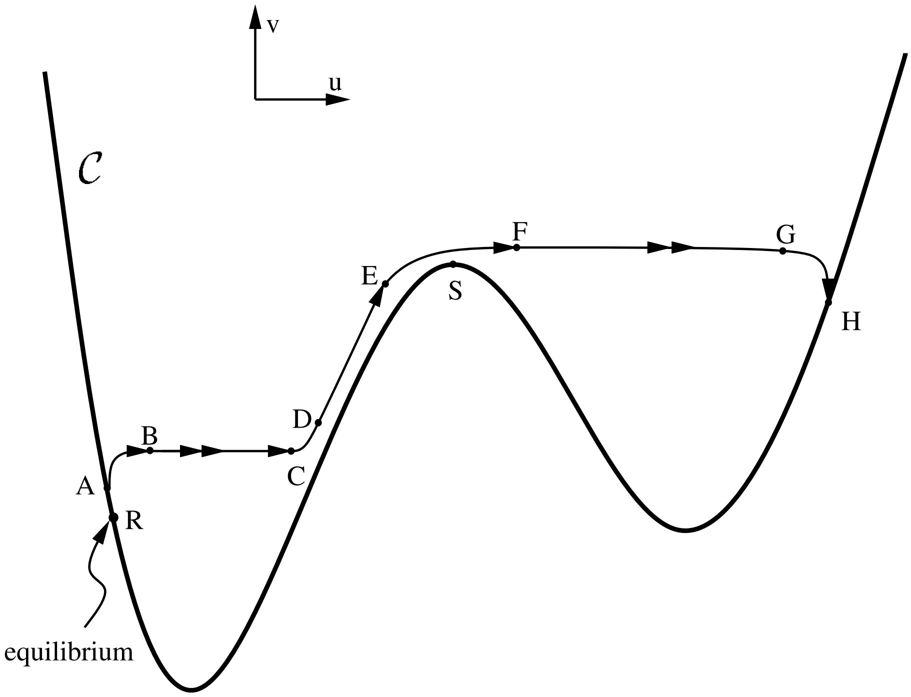


Figure 8

followed by a drop part CD and from D to E there is the slow motion along the singular curve. The curve is left via the junction part EF and after a second fast motion part FG there is the terminal part GH where the trajectory returns to the singular curve. The points A and H are chosen to lie on the singular curve while B, C, F and G are on fixed sections u = constant and D and E are on sections v = constant.

Assume for the moment that there is such a trajectory for any sufficiently small ε although this will be proved only in the next chapter.

The relation to trajectories of the reduced system is rather clear: For $\varepsilon > 0$ there are junction and drop parts between the slow and fast parts of a trajectory but these parts disappear in the limit $\varepsilon \searrow 0$.

Mishchenko and Rozov give in their monograph estimates for the time a trajectory stays in each of the different regions and also for the trajectories and their distance to $\mathcal{C}$. Fortunately, for our purposes the leading terms of their expansions suffice. Writing u_A for the u-coordinate of the point A in figure 1, v_D for the v-coordinate of D etc., their results can be summarized as follows:

Proposition 5.1 **[Mishchenko & Rozov]** *Assume that sections are chosen near the singular curve and near the fold points as described above. Then there exists* $\varepsilon_1 > 0$ *and constants* $0 < c < C$ *such that for all* $0 < \varepsilon \leq \varepsilon_1$

(i) the time T_{DE} *for the slow part* DE *satisfies*

$$\left| T_{DE} - \int_{v_D}^{v_E} \frac{dv}{g(u_{\mathcal{C}}(v))} \right| \leq C\varepsilon \qquad \textit{[equation (III.3.4) of [MR80]].}$$

(ii) the time T_{EF} *for the junction part* EF *satisfies*

$$\left| T_{EF} - \int_{v_E}^{v_S} \frac{dv}{g(u_{\mathcal{C}}(v))} \right| \leq C\varepsilon^{2/3} \qquad \textit{[equation (III.4.21)]}$$

with

$$v_F - v_S \leq C\varepsilon^{2/3} \qquad \textit{[equation (II.16.10)]}\,.$$

and

$$v + f(u) \geq c\varepsilon^{2/3} \qquad \textit{[equation (II.16.12)]}$$

along the whole junction part.

(iii) the time T_{FG} *for the fast part* FG *is*

$$T_{FG} = \varepsilon \int_{u_F}^{u_G} \frac{du}{v_F + f(u)} + \mathcal{O}(\varepsilon^{5/3}) \leq C\varepsilon \qquad \textit{[equation (III.5.2)]}$$

and

$$|v_F - v_G| = \mathcal{O}(\varepsilon).$$

(iv) the time T_{CD} *for the drop part* CD *satisfies*

$$\left| T_{CD} - \int_{v_C}^{v_D} \frac{dv}{g(u_{\mathcal{C}}(v))} \right| \leq C\varepsilon \ln \frac{1}{\varepsilon} \qquad \textit{[equation (III.6.12)]}\,.$$

Here $u_{\mathcal{C}}(v)$ and $v_{\mathcal{C}}(u)$ are local parametrizations of the singular curve $\mathcal{C}$ over u and v respectively.

The estimates for the initial part AB and the terminal part GH will be derived in the appendix by means of a normal form since this takes a lot of additional notation that is not connected to the rest of the paper. The following statements will be shown there:

Proposition 5.2 *There is an ε_1 such that for any $0 < \varepsilon \leq \varepsilon_1$ and any prescribed time $T > 0$ one can find an initial point A on the singular curve $\mathcal{C}$ such that the initial time from A to B is*

$$T_{AB} = T.$$

The distance from A to the equilibrium point R on $\mathcal{C}$ will be very small, or more precisely, exponentially small:

$$|v_R - v_A| \leq C \cdot \varepsilon^{-1} exp(-ct/\varepsilon).$$

Also, the v-coordinate does not change much between A and B:

$$|v_B - v_A| \leq C \cdot \varepsilon^{-1} exp(-ct/\varepsilon).$$

The time for the terminal part GH is

$$T_{GH} \leq C \cdot \varepsilon \ln \frac{1}{\varepsilon}$$

and the v-coordinate at G and H satisfy

$$|v_G - v_H| \leq C \cdot \varepsilon \ln \frac{1}{\varepsilon}.$$

Putting together all these estimates we obtain for the time from A to H

$$T_{AH} = T_{AB} + \int_{v_R}^{v_S} \frac{dv}{g(u_{\mathcal{C}}(v))} + \mathcal{O}(\varepsilon^{2/3})$$

where $u_{\mathcal{C}}(v)$ is a parametrization of the stable arc of $\mathcal{C}$ to the left of S. Therefore no knowledge about the true trajectory is used to compute the lowest order terms of the expansions. Only the slow system has to be integrated. This idea will be exploited in the next section to construct admissible solutions for $\varepsilon > 0$ from admissible solutions of the reduced system.
To this end we define time maps also for the system with $\varepsilon > 0$ and state the relation to the time maps of the reduced system.

Definition 5.3 *For any point $(u_0, v_0) \notin \mathcal{C}$ and any $\varepsilon > 0$ denote with $T_j^\varepsilon(u_0, v_0)$, $j = 1, 2, \ldots$, the time at which the forward trajectory through (u_0, v_0) intersects $\mathcal{C}$ for the j-th time. The according point on $\mathcal{C}$ will be called $P_j^\varepsilon(u_0, v_0)$. Time maps $T_{-i}^\varepsilon(u_0, v_0)$ with $i = 1, 2, \ldots$ are analogously defined for the backward trajectory.*

Lemma 5.4 *Consider a point $(u_0, v_0) \notin \mathcal{C}$ such that the forward trajectory of the fast system through (u_0, v_0) does neither connect to a fold point nor to an equilibrium point. Then the time maps $T_j(u_0, v_0)$ and $T_j^\varepsilon(u_0, v_0)$ satisfy*

$$|T_j(u_0, v_0) - T_j^\varepsilon(u_0, v_0)| = \mathcal{O}(\varepsilon^{2/3})$$

as $\varepsilon \searrow 0$ for any $j > 0$ for which they are defined.

Proof: Two ingredients are needed. First, the asymptotic estimates of Mishchenko and Rozov tell how close the trajectory for $\varepsilon > 0$ is to the trajectory of the reduced system. Then, recalling the definitions of time maps, green sides, red sides, etc. will finish the proof. For the first part we choose a small neighborhood $\mathcal{N}_{\varepsilon,\rho}$ of the singular curve in the following way:

$$\mathcal{N}_{\varepsilon,\rho} := \{(u, v)\ ;\ \exists(\tilde{u}, \tilde{v}) \in \mathcal{C} : |u - \tilde{u}| \leq \rho \text{ and } |v - \tilde{v}| \leq \varepsilon^{1/2}\}$$
$$\text{and } (u_0, v_0) \notin \mathcal{N}_{\varepsilon,\rho}.$$

Then we choose ε small enough such that all the invariant strips and regions we have found in chapter 3 are contained in $\mathcal{N}_{\varepsilon,\rho}$. In analogy to the drop points $D_k(u_0, v_0)$ of a trajectory of the reduced system we define for $\varepsilon > 0$ the drop point $D_k^\varepsilon(u_0, v_0)$ as the k-th point where the trajectory enters $\mathcal{N}_{\varepsilon,\rho}$. The point F_k^ε is defined as the point where the trajectory leaves $\mathcal{N}_{\varepsilon,\rho}$ for the k-th time. Note that the construction of $\mathcal{N}_{\varepsilon,\rho}$ near fold points is done in such a way that the trajectory for $\varepsilon > 0$ through (u_0, v_0) will miss $\mathcal{N}_{\varepsilon,\rho}$ iff the trajectory of the reduced system through (u_0, v_0) misses the curve $\mathcal{C}$ near the fold. So, for ε small enough there is a one-to-one correspondance between the drop points $D_k(u_0, v_0)$ and $D_k^\varepsilon(u_0, v_0)$. Hence, the following estimates are consequences of lemma 5.1 and lemma 5.2:

$$\begin{aligned}
|v_{D_1(u_0,v_0)} - v_{D_1^\varepsilon(u_0,v_0)}| &= \mathcal{O}(\varepsilon) \\
|v_{F_k(u_0,v_0)} - v_{F_k^\varepsilon(u_0,v_0)}| &= \mathcal{O}(\varepsilon^{2/3})\,, \quad k \geq 1 \\
|v_{D_k(u_0,v_0)} - v_{D_k^\varepsilon(u_0,v_0)}| &= \mathcal{O}(\varepsilon^{2/3})\,, \quad k > 1
\end{aligned}$$

where $v_{D_1(u_0,v_0)}$ denotes again the v-coordinate of the point $D_1(u_0, v_0)$, etc. . Thus the trajectory for $\varepsilon > 0$ is $\mathcal{O}(\varepsilon^{2/3})$-close to the trajectory of the reduced system. Now recall that our choice of colours for the sides of $\mathcal{C}$ was done exactly in a way such that trajectories of system (12) that enter $\mathcal{N}_{\varepsilon,\rho}$ from a green side near $D_k^\varepsilon(u_0, v_0)$ will intersect the singular curve there before they creep along the slow manifold, while trajectories approaching from a red or yellow side will not intersect the singular curve $\mathcal{C}$ there because $\mathcal{C}$ is "blocked" by the slow manifold. Thus, if the time map $T_j(u_0, v_0)$ corresponds to the drop point $D_k(u_0, v_0)$ then the time map $T_j^\varepsilon(u_0, v_0)$ corresponds to a point $P_j^\varepsilon(u_0, v_0)$ which is close to the drop point $D_k^\varepsilon(u_0, v_0)$. It is shown in the appendix that

$$|v_{D_k^\varepsilon(u_0,v_0)} - v_{P_j^\varepsilon(u_0,v_0)}| = \mathcal{O}(\varepsilon \ln(\varepsilon^{-1})) \text{ as } \varepsilon \searrow 0.$$

The time estimates given in the lemma follow now immediately by the time estimates of Mishchenko and Rozov and the appendix.

$\square$

Remarks:

1) Of course there is again an analogous statement for backward time maps if the backward trajectory of the fast system through (u_0, v_0) does not connect to a fold or an equilibrium point of the slow system.

2) The neighborhood $\mathcal{N}_{\varepsilon,\rho}$ depends on the point (u_0, v_0). In particular, we do not get a uniform estimate for a sequence of points that approaches a fold point.

6 Proof of theorem 1.2

The conditions for theorem 1.2 were given at several places as (H1)-(H7). We have indicated there why each of the hypotheses (H2)-(H7) holds for an open, dense set of f and g. Thus, proving that under these hypotheses the dimension of the global attractor stabilizes at some level proves that this stabilization holds for an open, dense set f and g. Only the dissipativeness (H1) of g is required in addition. The proof of theorem 1.2 splits into several parts. After proving persistence of type I and type II solutions separately, we show that there are no other admissible solutions for sufficiently small ε. To show that the dimension of the global attractor $\mathcal{A}_\varepsilon$ satisfies

$$\dim \mathcal{A}_\varepsilon \in \{d, d+1\}$$

for some integer d We will show below that for all small ε there is an equilibrium solution u of the viscous balance law with $z(u_x) = d$ and no equilibrium solution with $z(u_x) > d$. From lemma 1.4 we know that there is an equilibrium such that the linearization has d or $d+1$ positive eigenvalues. This gives immediately a lower bound d for the dimension of $\mathcal{A}_\varepsilon$ since the global attractor contains at least the d- or $(d+1)$-dimensional center-unstable manifold of this equilibrium. The upper bound $d+1$ follows as in the proof of theorem 1.1. There we have shown that the dimension of the global attractor is not bigger than the maximal dimension of the center-unstable manifolds of the equilibria.

Although the dimension of the global attractor $\mathcal{A}_\varepsilon$ stabilizes as $\varepsilon \searrow 0$, we cannot exclude the case that the number of equilibria nevertheless becomes infinite. This is mainly a question of uniqueness: We are not able to prove that there is exactly one admissible solution to any admissible solution of the reduced system. So, many admissible solutions for $\varepsilon > 0$ may correspond to the same admissible solution of the reduced system and this number may even tend to infinity as $\varepsilon \searrow 0$.

The number d can be found in the following way: Let

$$d_{Ia} := \max\{1+j\,; \exists \text{ type Ia trajectory with } T_j(u_0, v_0) < 1\},$$

where the maximum is taken over all type Ia trajectories of the reduced system. This definition reflects the fact that for any sufficiently small ε there is an equilibrium solution u of the viscous balance law (7) with zero number $z(u_x) = j+1$ as we will show soon. Here we use the fact that j counts how often the trajectory of the reduced system hits a *green* side of $\mathcal{C}$. The corresponding statement for $\varepsilon > 0$ is that the trajectory crosses $\mathcal{C}$ exactly j times. Thus, $j+1$ counts the number of intersections between the trajectory with $\varepsilon > 0$ and $\mathcal{C}$. (We have to take $j+1$ since an admissible trajectory starts already on $\mathcal{C}$). The number d_{Ib} is defined similarly as

$$d_{Ib} := \max\{1+i\,; \exists \text{ type Ib trajectory with } T_{-i}(u_0, v_0) < 1\}.$$

The analogous number d_{II} for type II trajectories of the reduced system is

$$d_{II} := \max\{i+j\,; \exists \text{ type II trajectory with } T_{-i}(u_0, v_0) + T_j(u_0, v_0) = 1\}$$

since in this case there will be an equilibrium solution u of the viscous balance law with $z(u_x) = i+j$. Then

$$d := \max\{d_{Ia}, d_{Ib}, d_{II}\}$$

6.1 Persistence of type I solutions

We will restrict ourselves to the case of a type Ia solution since the proof for type Ib trajectories follows by reversing "time" s in the ordinary differential equation for the equilibria of the viscous balance law. So consider a type Ia trajectory starting at an equilibrium point $(u_E, -f(u_E))$ and passing through the point (u_0, v_0) such that $T_j(u_0, v_0) < 1$ for some j. We want to find a solution for $\varepsilon > 0$ nearby that intersects the curve $\mathcal{C}$ exactly $j+1$ times. Choose T_- and T_+ with

$$T_- < 1 - T_j(u_0, v_0) < T_+$$

and a section $u = u_0$. As is shown in the appendix, for any small ε there are points $(u_-, -f(u_-))$ and $(u_+, -f(u_+))$ on $\mathcal{C}$ close to the equilibrium $(u_E, -f(u_E))$ such that the trajectories take exactly time $T_\pm$ from $(u_\pm, -f(u_\pm))$ to the section $u = u_0$. These trajectories hit this section $u = u_0$ in points $(u_0, \hat{v}_\pm)$ which are $\mathcal{O}(\varepsilon)$-close to v_0. Using continuity of the time maps and lemma 5.4 we get for the time map T_j

$$\left|T_j(u_0, v_0) - T_j^\varepsilon(u_0, \hat{v}_\pm)\right| \leq |T_j(u_0, v_0) - T_j(u_0, \hat{v}_\pm)| + \left|T_j(u_0, \hat{v}_\pm) - T_j^\varepsilon(u_0, \hat{v}_\pm)\right| \to 0$$

as $\varepsilon \searrow 0$.
So, we have found a trajectory through $(u_0, \hat{v}_+)$ that is close to the admissible trajectory of the reduced system and takes time

$$T_+ + T_j(u_0, v_0) + \mathcal{O}(\varepsilon^{2/3}) > 1$$

to join two points on $\mathcal{C}$ and another trajectory nearby where the time is

$$T_- + T_j(u_0, v_0) + \mathcal{O}(\varepsilon^{2/3}) < 1.$$

Now consider all trajectories that pass through points (u_0, v) with v between $\hat{v}_+$ and $\hat{v}_-$. A simple continuity argument using shows that at least one of these trajectories yields an admissible solution with the intermediate value

$$T_{-1}(u_0, v) + T_j(u_0, v) = 1.$$

Thus, for any sufficiently small ε, there is an admissible solution that stays near the given type Ia solution of the reduced system. This solution intersects the singular curve exactly $j+1$ times. Thus, $z(u_x) = j+1$ and by lemma 1.4 the linearization of

the corresponding equilibrium solution to the viscous balance law possesses either $j+1$ or $j+2$ positive eigenvalues.
No uniqueness has been shown, although we believe that for sufficiently small ε there is only one solution nearby. The next chapter contains a proof for the simplest case of a type I trajectory with just one fast part. A slight modification of this proof shows uniqueness for all solutions near type I trajectories of the reduced system under the transversality hypothesis (H7).

6.2 Persistence of type II solutions

Consider a type II admissible solution of the reduced system passing through (u_0, v_0) such that $T_{-i}(u_0, v_0) + T_j(u_0, v_0) = 1$ with $i, j \geq -1$. The transversality condition (H7) allows us to find admissible trajectories of the reduced system on intervals slightly shorter resp. longer than 1 and passing near (u_0, v_0): There are $\hat{v}_+$ and $\hat{v}_-$ near v_0 and times $T_- < 1 < T_+$ such that the trajectories of the reduced system through the points $(u_0, \hat{v}_\pm)$ coincide with the trajectory through (u_0, v_0) after an initial part, and such that

$$T_{-i}(u_0, \hat{v}_\pm) + T_j(u_0, \hat{v}_\pm) = T_\pm.$$

Again the trajectories through $(u_0, \hat{v}_\pm)$ for small $\varepsilon > 0$ satisfy

$$T^\varepsilon_{-i}(u_0, \hat{v}_+) + T^\varepsilon_j(u_0, \hat{v}_+) = T_+ + \mathcal{O}(\varepsilon^{2/3}) > 1$$

and

$$T^\varepsilon_{-i}(u_0, \hat{v}_-) + T^\varepsilon_j(u_0, \hat{v}_-) = T_- + \mathcal{O}(\varepsilon^{2/3}) < 1.$$

Thus, at least one of the trajectories through points (u_0, v) with v between $\hat{v}_+$ and $\hat{v}_-$ corresponds to a solution of the original boundary value problem (12).
As above, no uniqueness of solutions for $\varepsilon > 0$ is proved with this method. So it could still happen that one admissible solution of the reduced system corresponds to more and more solutions of the boundary value problem (12) as $\varepsilon \searrow 0$.
Summarizing, we have

Proposition 6.1 *There is $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leq \varepsilon_1$ to any admissible solution of the reduced system an admissible solution exists. The trajectories of the solution of the reduced system and for $\varepsilon > 0$ are $\mathcal{O}(\varepsilon^{2/3})$-close in the (u, v)-Liénard plane.*

6.3 No further solutions

To finish the proof we have to show that for sufficiently small ε **all** admissible solutions can be found by looking at the reduced system. This will be the most difficult part of the proof, so we give a short outline first and fill in the details in a couple of lemmata later.

Due to assumptions (H6) and (H7) we can find neighborhoods of all fold points and all equilibrium points such that admissible solutions of the reduced system can only pass through these neighborhoods if they pass through the fold point resp. the equilibrium point itself. This even holds for admissible solutions of the reduced system on intervals $[0, T]$ with T near 1. Also, we will show that all admissible solutions of the reduced system on intervals of length near 1 are close to admissible solutions of the reduced system on the interval $[0, 1]$. After these preparations that still concern the reduced system only, we can now turn to admissible solutions with $\varepsilon > 0$. Given such an admissible solution for $\varepsilon > 0$, we prove that there is a fast part separating the unstable and the stable part of the trajectory (exactly as for trajectories of the reduced system). As in the proof of theorem 1.1 there is a number K such that any admissible solution can be decomposed into at most K parts (fast, slow, junction, etc.) independent of ε small.

Then, using the transition time estimates from the preceding chapter it is shown that near any given admissible solution for $\varepsilon > 0$ there is a solution of the reduced system on an interval with length near 1. This will lead to a contradiction in some cases. In other cases it will prove that the admissible solution is in an $\mathcal{O}(\varepsilon^{2/3})$-neighborhood of an admissible solution of the reduced system. Moreover, the number $z(u_x)$ can then be determined from this admissible solution of the reduced system. Recall that this number $z(u_x)$ was the most important ingredient to determine the Morse index of u.

Recall that $\overline{u}_1 < \overline{u}_2 < \ldots < \overline{u}_F$ are the locations of the fold points of f. We start by taking neighborhoods of these fold points and the stable and unstable arcs of $\mathcal{C}$ in a special way: To each fold point $(\overline{u}_k, -f(\overline{u}_k))$ and $\rho_1, \rho_2 > 0$ let N_k denote the rectangle

$$N_k := \{(u, v) \; ; \; |u - \overline{u}_k| \leq \rho_1, |v + f(\overline{u}_k)| \leq \rho_2\}.$$

The numbers ρ_1 and ρ_2 will be specified later. To the arc of $\mathcal{C}$ that joins the fold

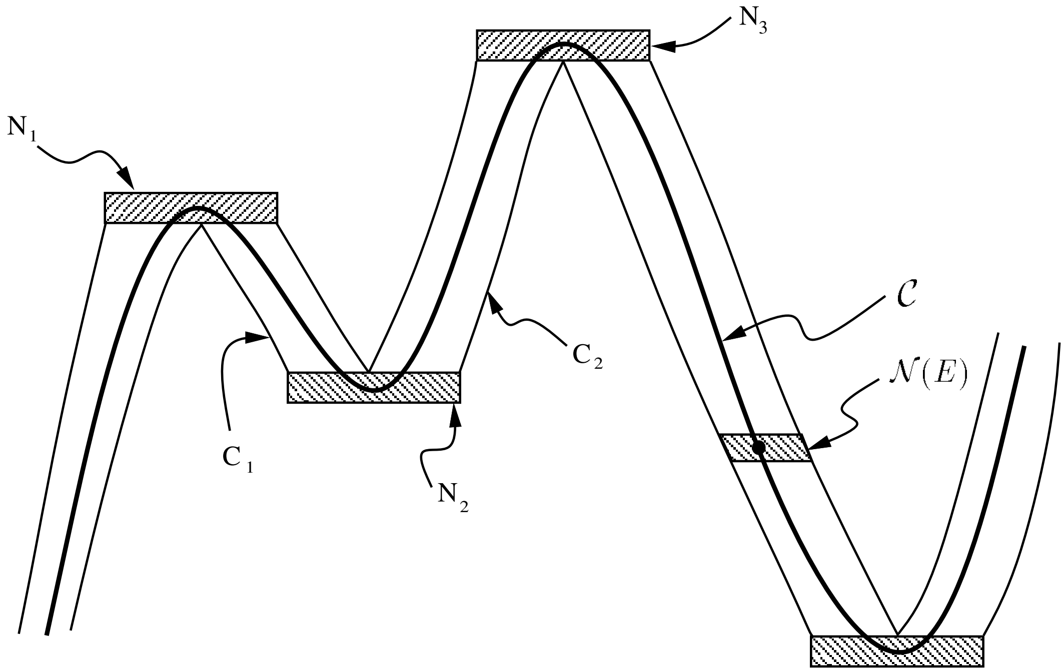


Figure 9: Choosing the neighborhoods near $\mathcal{C}$

points $(\overline{u}_k, -f(\overline{u}_k)$ and $(\overline{u}_{k+1}, -f(\overline{u}_{k+1})$ we associate a neighborhood C_k joining N_k and N_{k+1}, see figure 9. Note that there is no possibility of a direct transition from a neighborhood of an unstable arc to a neighborhood of a stable arc. Also, as can be shown in the same manner as lemma 3.5(ii), there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leq \varepsilon_1$ trajectories can leave neighborhoods of stable arcs only through the top or bottom section while neighborhoods of unstable arcs can only be entered from there.

Also for any equilibrium point E with coordinates $(u_E, -f(u_E)) \in C_l$ we define a neighborhood $\mathcal{N}(E) := C_l \cap \{|v + f(u_E)| \leq \rho_2\}$.

According to (H5), no trajectory of the fast system connects a fold point to an equilibrium point. This carries over to neighborhoods of these points and $\varepsilon > 0$: For ρ_2 small and $\varepsilon \leq \varepsilon_1$ trajectories do not connect directly some box N_k to one of the ρ_2-neighborhoods $\mathcal{N}(E)$ of an equilibrium point E.

We now show that any trajectory of the reduced system that intersects N_k can be an admissible solution on an interval with length near 1 only if it passes through the fold point itself.

Lemma 6.2 *If the numbers ρ_1, ρ_2 are chosen sufficiently small then there exists some $\eta > 0$ with the following properties:*

(i) If (u_0, v_0) is a point in N_k that does not lie in $\mathcal{C} \cup \{v = -f(\overline{u}_k)\}$ then

$$T_{-i}(u_0, v_0) + T_j(u_0, v_0) \notin [1-\eta, 1+\eta] \qquad \forall i, j \geq -1.$$

(ii) If (u_0, v_0) is a point in $\mathcal{N}(E)$ that does not lie in $\mathcal{C} \cup \{v = -f(u_E)\}$ then

$$T_{-i}(u_0, v_0) + T_j(u_0, v_0) \notin [1-\eta, 1+\eta] \qquad \forall i, j \geq -1.$$

(iii) There is no type III admissible solution on $[0, T]$ with $T \in [1-\eta, 1+\eta]$.

Proof:
(i) Assume the contrary. Then there are $i, j \geq -1$ and a sequence $(u_n, v_n)_{n=1,2,\ldots}$ of points converging to a fold or to an equilibrium such that

$$T_{-i}(u_n, v_n) + T_j(u_n, v_n) \to 1.$$

One can check now that the case that the (u_n, v_n) converge to a fold point contradicts assumption (H6). Depending from which side of $\mathcal{C}$ the points (u_n, v_n) approach the fold point, there is either an admissible type III solution of the reduced system or an admissible solution that only touches the fold point. Both cases were excluded by (H6).
(ii) If the points (u_n, v_n) approach an equilibrium point the time it takes the backward trajectory to leave a neighborhood of the equilibrium point grows exponentially. From this we conclude that $i = 1$ and

$$T_{-i}(u_n, v_n) = 0.$$

As a consequence, we have $T_j(u_n, v_n) = 1$ violating the transversality assumption (H7).
(iii) is clear since there is only a finite number of type III trajectories. Consider any interval length T such that one of these trajectories is an admissible solutions on

$[0, T]$. By (H6), $T \neq 1$, and since there are only finitely many such trajectories this implies

$$|T - 1| > \eta$$

for some $\eta > 0$.

$\square$

From now on, we assume that ρ_1 is fixed.

Part (ii) of the preceding lemma tells that all slow parts of admissible solutions of the reduced system on intervals $[0, T]$ with $T \in [1 - \eta, 1 + \eta]$ avoid the neighborhood $\mathcal{N}(E)$ of equilibrium points. Consequently, there exists a number c_g such that

$$|g(u)| \geq c_g$$

along any slow part of an admissible solutions of the reduced system on $[0, T]$. This c_g may be interpreted as the "minimal velocity" in v-direction along slow parts of admissible solution.

Recall that the transversality condition (H7) contained an inverse Lipschitz estimate for the time maps:

$$|T_j(u_0, v_1) - T_{-i}(u_0, v_1) - 1| \geq C|v_1 - v_0| \tag{15}$$

if the trajectory through (u_0, v_0) is a type II admissible solution of the reduced system and $|v_1 - v_0|$ is small.

With this estimate we can show that any admissible solution of the reduced system on an interval with length near 1 has to be near an admissible solution of the reduced system on $[0, 1]$:

Lemma 6.3 *There exists $0 < \eta_0 \leq \eta$ with the following properties:*

(i) If a type I solution of the reduced system on $[0, T]$ with $T \in [1 - \eta_0, 1 + \eta_0]$ passes through (u, v), then there is also a type I solution of the reduced system on $[0, 1]$ through (u, v).

(ii) If a type II admissible solution of the reduced system on an interval $[0, T]$ with $T \in [1 - \eta_0, 1 + \eta_0]$ passes through the point $(u, v) \notin \mathcal{C}$ and

$$T_{-i}(u, v) + T_j(u, v) = T$$

then there is a type II admissible solution of the reduced system on $[0,1]$ *nearby. More precisely, there exists some* v_0 *with* $|v - v_0| \leq C^{-1}|T-1|$ *such that*

$$T_{-i}(u, v_0) + T_j(u, v_0) = 1.$$

Proof: (i) Choose η_0 smaller than the infimum of $|1 - T_j(u,v)|$ where the infimum is taken over all j and over points (u,v) on the (finitely many) fast trajectories that connect to an equilibrium point. The type I solution on $[0,1]$ can be found easily: It coincides with the given admissible solution on $[0,T]$ except that it spends a slightly different time at the equilibrium.
(ii) Assume the contrary. Then there is a sequence $(u_n, v_n)_{n=1,2,\ldots}$ of points such that

$$T_{-i}(u_n, v_n) + T_j(u_n, v_n) \to 1$$

and

$$|v_n - v| > C^{-1}|T_{-i}(u_n, v_n) + T_j(u_n, v) - 1| \tag{16}$$

for any type II admissible solution of the reduced system that passes through a point (u_n, v). Moreover, by compactness we may suppose $(u_n, v_n) \to (u_\infty, v_\infty)$. We will find a contradiction by showing that the point (u_∞, v_∞) belongs to a type II admissible solution that does not satisfy (16).
By lemma 6.2, the trajectory of the fast system through the point (u_n, v_n) does not enter one of the boxes N_k or one of the neighborhoods $\mathcal{N}(E)$ of an equilibrium point E. By continuity (and since the time maps are locally constant in u), also the trajectory of the fast system through the point point (u_∞, v_∞) cannot intersect a box N_k or one of the $\mathcal{N}(E)$. This implies that the time maps T_{-i} and T_j are continuous at (u_∞, v_∞) and the trajectory through (u_∞, v_∞) is a type II admissible solution of the reduced system on $[0,1]$. Note that locally near (u_∞, v_∞) the time maps $T_j(u,v)$ are monotone in v. The inverse Lipschitz estimate (15) implies then that by choosing η_0 small enough one gets the desired estimate

$$|v_n - v_\infty| \leq C^{-1}|T_{-i}(u_n, v_n) + T_j(u_n, v_n) - 1|$$

in contradiction with (16).

$\square$

There is a similar classification for admissible solutions with $\varepsilon > 0$ small as for the admissible solutions of the reduced system: Any trajectory that enters a neighborhood C_k of a stable arc will either stay there forever or follow the stable arc to one of the neighborhoods N_k or N_{k+1} of a fold point. From there the trajectory has to leave a neighborhood of $\mathcal{C}$ and can reenter such a neighborhood only near another stable arc, but not near an unstable arc. Similarly, backward trajectories that have reached a neighborhood of an unstable arc of $\mathcal{C}$ cannot enter a neighborhood of the stable arcs. So, like the trajectories of the reduced system, the trajectories for $\varepsilon > 0$ can be split into an unstable and a stable part with a fast part between them.
From the proof of theorem 1.1 we know already that for small ε it takes trajectories a certain time to pass near a fold or to pass through the neighborhood of a stable or unstable arc. This implies that we can find $\varepsilon_2 \leq \varepsilon_1$ and a number $K > 0$ such that all admissible trajectories for $\varepsilon \leq \varepsilon_2$ consist of at most K parts (slow, fast, junction, etc.).
We define now

$$\eta_1 := \frac{\eta_0}{2K}$$

and fix η_1 and ρ_1.
We will choose ρ_2 and ε still smaller to achieve that any trajectory for ε small enough cannot lose more than "time" $2\eta_1$ on each part of the trajectory compared to some trajectory of the reduced system. Then the difference cannot add up to more than η_0 and we are able to use lemma 6.3.
Choose ρ_2 sufficiently small such that

$$\rho_2 \cdot c_g^{-1} \leq \eta_1 \tag{17}$$

such that a trajectory of the reduced system will need not more than time η_1 to cover some extra distance ρ_2.
Denote with

$$\varepsilon u'_{min} := \inf\{|v + f(u)| \,;\, (u,v) \in [u_{min}, u_{max}] \times [v_{min}, v_{max}] \setminus (\bigcup_k N_k \cup \bigcup_l C_l)\}$$

the minimal velocity in u-direction outside the neighborhoods N_k of the fold points and the neighborhoods C_l of the arcs of $\mathcal{C}$. Let $\varepsilon_0 \leq \varepsilon_2$ be sufficiently small such that

(i) $C\varepsilon_0^{2/3} < \eta_1$ where C from 5.1 is chosen large enough such that $C\varepsilon^{2/3}$ estimates the time from a horizontal section taken $\{v = -f(\bar{u}_k) - \rho_2\}$ to a vertical section $\{u = \bar{u}_k + \rho_1\}$.

(ii) $\varepsilon_0 u'_{min} \cdot (u_{max} - u_{min}) \leq \eta_1$.

(iii) $c_g^{-1}\left(C\varepsilon_0^{2/3} + \varepsilon_0 C_g u'_{min} \cdot (u_{max} - u_{min})\right) \leq \eta_1$.

(iv) $2\rho_1 \varepsilon_0^{1/3}/c \leq \eta_1$.

The constants c and C are the ones used in lemma 5.1 and lemma 5.2. The interpretation of these conditions is the following:
From (i) we get that a trajectory that passes near a fold point will take the same time (up to η_1) to pass through N_k as the trajectory of the reduced system that passes through the fold point. Also, the time difference on any slow part which is smaller than $C \cdot \varepsilon$ and the time difference at initial and drop parts which are smaller than $C \cdot \varepsilon \ln(\varepsilon^{-1})$ can be estimated by η_1. Condition (ii) states that a trajectory does not take longer than η_1 for a fast part. Condition (iii) accounts for the possible extra distance that solution of the reduced system have to cover since the drop points of the trajectories for $\varepsilon > 0$ and $\varepsilon = 0$ differ. The last condition (iv) ensures that any trajectory that traverses some N_k above or below the fold will not need longer than η_1.
After all these preparations, consider now an admissible trajectory for some $\varepsilon \leq \varepsilon_0$ and choose a point (u_0, v_0) on that fast part between the unstable and the stable part of the trajectory. There are $i, j \geq -1$ satisfying

$$T^{\varepsilon}_{-i}(u_0, v_0) + T^{\varepsilon}_{j}(u_0, v_0) = 1.$$

Recall that $i = -1$ is the case where the trajectory consists of a stable part only. Similarly, $j = -1$ indicates absence of a stable part.
Up to time reversal, we have to distinguish three cases:
The forward and backward trajectory may both enter a neighborhood of the singular curve $\mathcal{C}$ away from all the N_k's and the $\mathcal{N}(E)$. The second case treats the possibility that either the forward or backward trajectory enters one of the N_k, while the third case considers the possibility that either the forward or backward trajectory enters a neighborhood $\mathcal{N}(E)$ of an equilibrium point E. Note that assumption (H5) prevents

that both the forward and the backward trajectory enter into some N_k or $\mathcal{N}(E)$.

Case 1: The forward trajectory through (u_0, v_0) enters the neighborhood C_k of a stable arc away from all equilibria and the backward trajectory enters the neighborhood C_l of an unstable arc.

Then the time maps for the reduced systems trajectory through (u_0, v_0) satisfy

$$|T_{-i}(u_0, v_0) + T_j(u_0, v_0) - 1| \leq 2K \cdot \eta_1 = \eta_0. \tag{18}$$

This can be seen in the following way looking at the forward trajectory first: The first fast part does not take longer than η_1 and the trajectory enters a neighborhood of $\mathcal{C}$ at a drop point $D_1^\varepsilon(u_0, v_0)$. This drop point satisfies

$$|v_{D_1^\varepsilon(u_0,v_0)} - v_{D_1(u_0,v_0)}| \leq C\varepsilon \tag{19}$$

such that the trajectory of the reduced system (with velocity$\geq c_g$) needs at most time η_1 for the small extra portion between $v_{D_1^\varepsilon(u_0,v_0)}$ and $v_{D_1(u_0,v_0)}$. The drop part of the trajectory with $\varepsilon > 0$ may take another time η_1. Altogether the difference between the time maps for $\varepsilon = 0$ and $\varepsilon \neq 0$ may increase by $2\eta_1$ on the drop part. On the slow part the transition times for the trajectory of the reduced system and the trajectory for $\varepsilon > 0$ may differ again by η_1 and the same holds for the junction part. The next fast part takes at most time η_1 and leads to a drop point that is at a distance of at most $C\varepsilon_0^{2/3} + \varepsilon_0 C_g u'_{min} \cdot (u_{max} - u_{min})$ from the drop point of the reduced systems trajectory. Then everything can be continued in the same way. On each part the transition times differ at most by η_1 and the trajectory may have to cover an extra distance near every drop point that takes another time η_1. Applying the same arguments to the backward trajectory, yields then (18), since there are at most K parts of the trajectory which contribute at most $2\eta_1$ each to the difference between the time maps. Therefore, the trajectory of the reduced system through (u_0, v_0) is an admissible solution on an interval with length $T \in [1 - \eta_0, 1 + \eta_0]$. So, by lemma 6.3 there exists a v_{red} with $|v_{red} - v_0| \leq C^{-1} \cdot C\varepsilon^{2/3}$ and

$$T_{-i}(u_0, v_{red}) + T_j(u_0, v_{red}) = 1$$

and the given admissible solution is shown to be in an $\mathcal{O}(\varepsilon^{2/3})$-neighborhood of an admissible solution of the reduced system.

Case 2: The forward trajectory through (u_0, v_0) enters the neighborhood N_k of a fold point while the backward trajectory enters a neighborhood C_l of an unstable arc. Three subcases are treated depending on whether the trajectory leaves the box N_k via a fast part, along the stable arc of $\mathcal{C}$ or along the unstable arc of $\mathcal{C}$.

We will show that none of these cases can appear since they contradict lemma 6.2.

Case 2a: The forward trajectory leaves N_k via a fast part and reaches a stable arc of $\mathcal{C}$. Denote with (u_1, v_1) the point where the trajectory leaves N_k. We will show that for this point

$$T_{-i}(u_1, v_1) + T_j(u_1, v_1) \in [1 - \eta, 1 + \eta]. \tag{20}$$

in contrast to lemma 6.2. The reasoning is similar than in the last case. There are only two extra parts which can each contribute another η_1 to the time difference: Due to condition (iv) from the choice of ε_0, the trajectory may take this time to pass through N_k. Also the v-coordinate could change by at most ρ_2 while the trajectory passes through N_k. The trajectory of the reduced system may therefore have an extra v-distance on the next arc of $\mathcal{C}$ but by the choice (17) of ρ_2 this extra distance takes at most time η_1. Summing up the transition times of the given trajectory and the trajectory of the reduced system through (u_1, v_1) gives again a difference of less than η thus proving (20).

Case 2b: The forward trajectory continues along the stable arc of $\mathcal{C}$ that originates at the fold point.

This is only possible if the point $D_1^\varepsilon(u_0, v_0)$ where the trajectory enters the neighborhood N_k satisfies

$$v_{D_1^\varepsilon(u_0,v_0)} + f(\overline{u}_k) \leq C\varepsilon^{2/3}$$

since the trajectory must hit N_k below the point where the backwards trajectories from the unstable arc of $\mathcal{C}$ leave N_k, see figure 10.

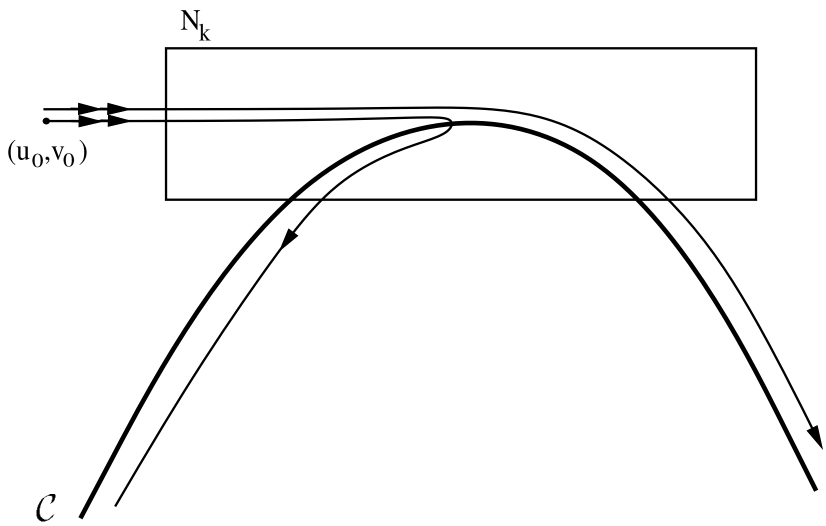


Figure 10: The situation in case 2b

Therefore, we compare the trajectory through (u_0, v_0) with $\varepsilon > 0$ and the type III trajectory of the reduced system through the fold point. In exactly the same manner as in case 2a) one gets that this type III trajectory is an admissible solution on an interval of length $T \in [1-\eta, 1+\eta]$ which contradicts lemma 6.2(iii).

Case 2c: The forward trajectory continues along the unstable arc of $\mathcal{C}$ that originates at the fold point.

This case contradicts our choice of (u_0, v_0) to be on the fast part that separates the unstable from the stable part of the admissible solution.

Case 3: The forward trajectory enters a neighborhood $\mathcal{N}(E)$ of an equilibrium point E while the backward trajectory leads to the neighborhood of an unstable arc.

In this case we have to show that the admissible trajectory corresponds to one of the type I solutions of the reduced system. We have $j = 1$ since the height $2\rho_2$ of $\mathcal{N}(E)$ was chosen so small that it takes trajectories more than time 1 to reach the next neighborhood of a fold point.

If the trajectory through (u_0, v_0) hits the curve $\mathcal{C}$ near the equilibrium point at a distance of less than $C\varepsilon^{2/3}$ then we are done because in that case the admissible trajectory is near a type Ib trajectory. In the other case that it hits $\mathcal{C}$ at a distance bigger than $C\varepsilon^{2/3}$ the normal form analysis from the appendix shows that the time from (u_0, v_0) to $\mathcal{C}$ is of order $\mathcal{O}(\varepsilon \ln \varepsilon^{-1})$. Then

$$1 - T_{-i}(u_0, v_0) = \mathcal{O}(\varepsilon \ln \varepsilon^{-1})$$

and with the same arguments that were already used in case 2 one shows that the trajectory of the reduced system through (u_0, v_0) is a type II admissible solution on $[0, T]$ with $T \in [1 - \eta, 1 + \eta]$. This contradicts lemma 6.2(ii).

All three cases together show that for $\varepsilon \leq \varepsilon_0$ admissible solutions can only be located in the (u, v)-plane near admissible solutions of the reduced system and can hence be found by studying the reduced system.
This completes the proof of theorem 1.2.

$\square$

6.4 An example: A dissipative Van der Pol oscillator

To illustrate the preceding results, a concrete example will be given in this subsection. For this example we will assume that to any admissible solution of the reduced system for sufficiently small $\varepsilon \neq 0$ there exists exactly one hyperbolic equilibrium solution of the viscous balance law. We cannot prove this hypothesis yet but believe that it is true.
Since chapter contains an example of a two-dimensional attractors, we construct in this section an example with a three-dimensional attractor. The considerations from the proof of theorem 1.2 suggest that we should chose f to have some turning points if we want to achieve a global attractor of a higher dimension. Also, it seems to be quite useful if the equilibrium equation admits periodic solutions because they can lead to solutions of the boundary value problem with a couple of oscillations and thus with a Morse index exceeding one or two.
This yields the following construction:
Choose f cubic-like with the curve v+f(u)=0 looking like in figure 11 and take as g a function with three zeroes as indicated in the figure. This is different from the

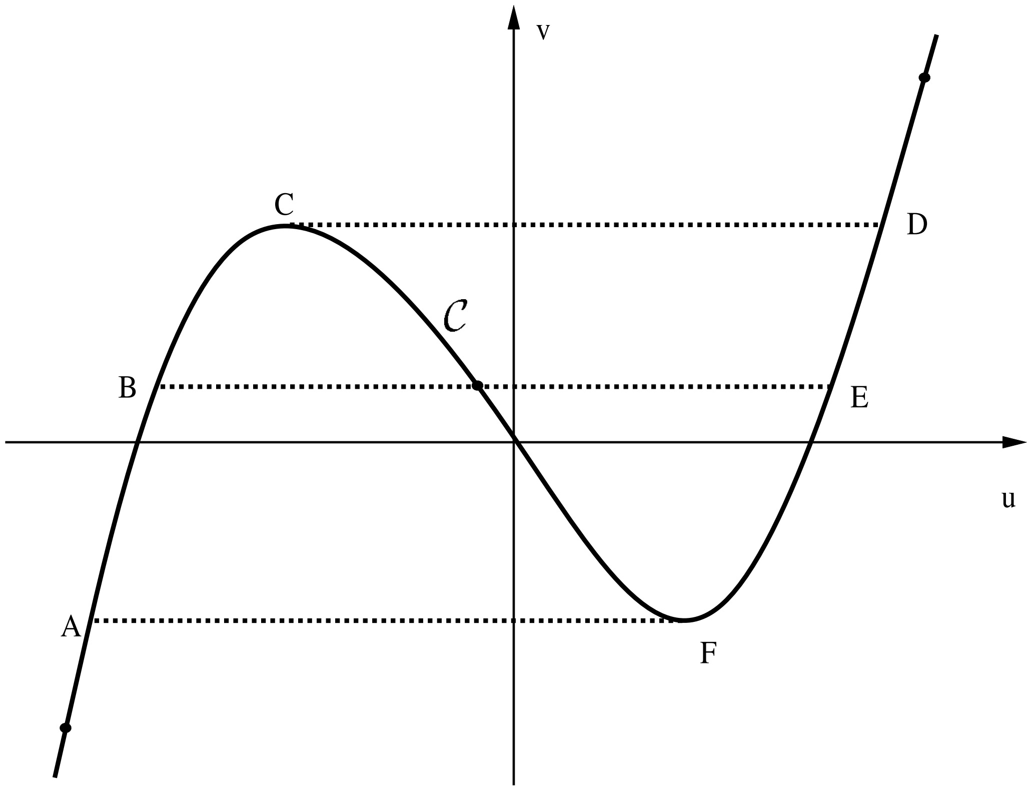


Figure 11

usual Van der Pol equation where $f(u) = u - u^3$ and $g(u) = u$, but it is an easy way to make g dissipative. As in the usual Van der Pol equation, there is a limit cycle with slow parts near AC and DF and fast transitions near CD and FA.
For the times a trajectory of the reduced system needs for the various slow parts we will assume the following:

$$T_{AC} = \int_{v_A}^{v_C} \frac{dv}{g(u_{\mathcal{C}}(v))} < 1,$$

$$T_{BC} = \int_{v_B}^{v_C} \frac{dv}{g(u_{\mathcal{C}}(v))} < 1,$$

$$T_{DF} = \int_{v_D}^{v_F} \frac{dv}{g(u_{\mathcal{C}}(v))} < 1,$$

$$T_{EF} = \int_{v_E}^{v_F} \frac{dv}{g(u_{\mathcal{C}}(v))} < 1,$$

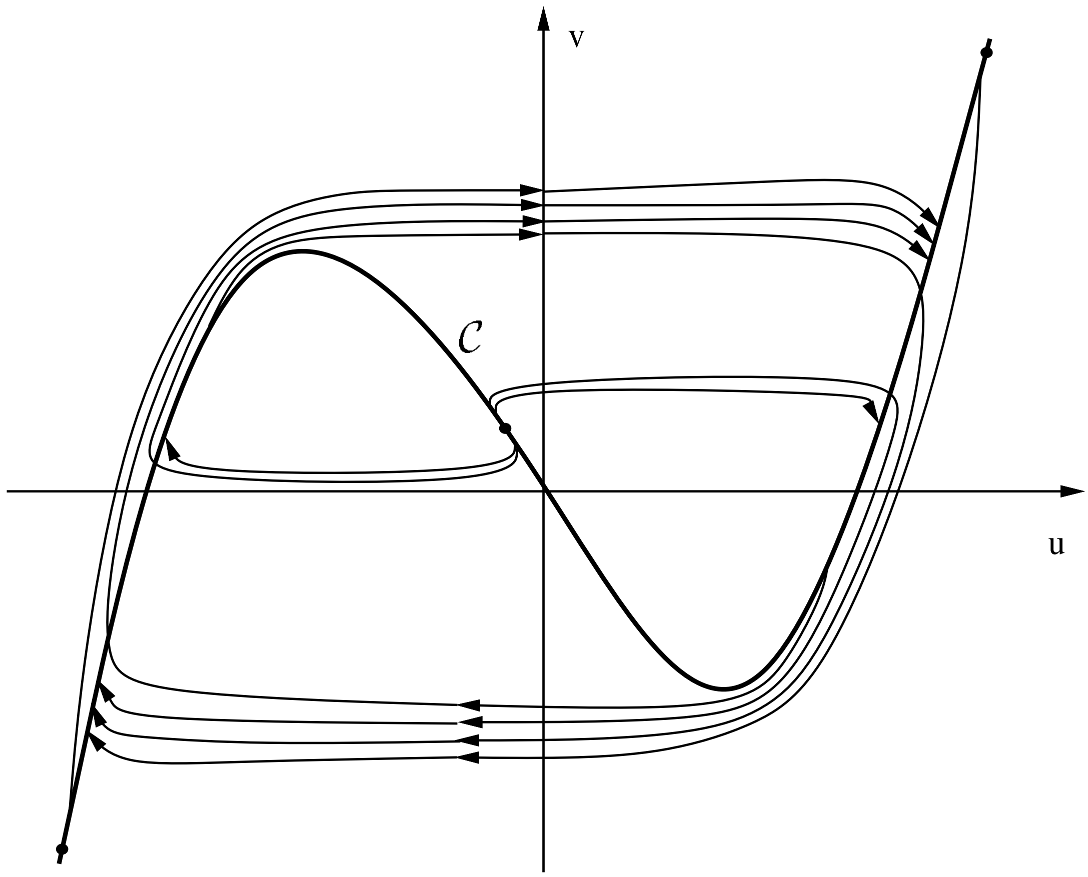


Figure 12

$$T_{BC} + T_{DF} > 1 \text{ and}$$

$$T_{EF} + T_{AC} > 1.$$

We are not going to construct a suitable g but it is obvious that this can be done. Due to the last two inequalities there are no solutions that have more than two slow parts. One can easily inspect all possible cases to find that there are 11 admissible solutions of the reduced system. We assume now that ε is small and that each of these 11 solutions corresponds to exactly one hyperbolic equilibrium solution of the viscous balance law although we cannot prove that. Their location in the Liénard plane is indicated in the next figure and the associated permutation that can be

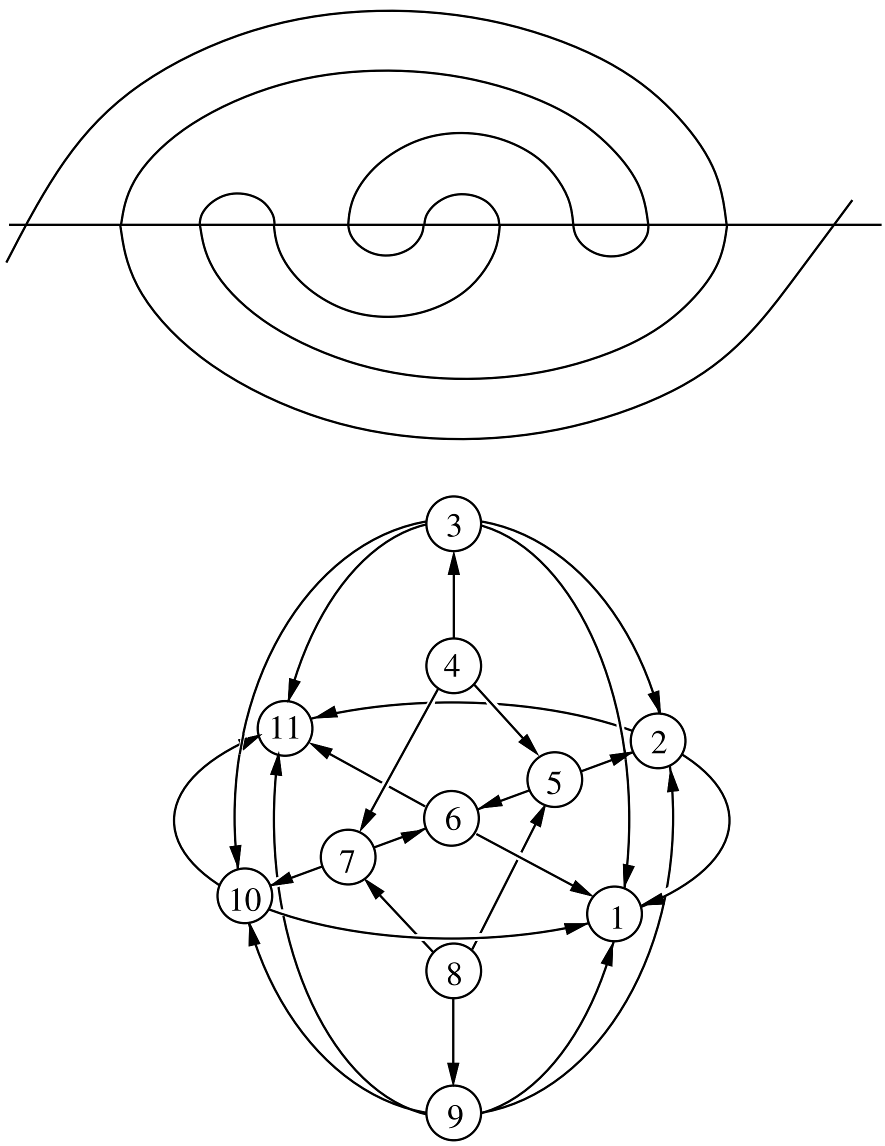


Figure 13: The Shooting curve and the connection graph

read off this figure is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 10 & 3 & 4 & 7 & 6 & 5 & 8 & 9 & 2 & 11 \end{pmatrix}$$

This gives the shooting curve shown in the upper part of figure 13.
Proposition 1.5 allows to compute the Morse indices and zero numbers of the differences from the permutation and from that information proposition 1.6 tells exactly which equilibria are connected. This is shown here in the **connection graph** where arrows indicate which equilibria are connected. The connection graph just contains arrows between equilibria whose Morse indices differ by one. So two equilibria are

connected by a heteroclinic orbit if they are connected via a sequence of arrows in the connection graph.
From this picture it is easy to guess how the attractor might look like as a three-dimensional object. Just imagine the 1-skeleton embedded in a ball that is filled with heteroclinic orbits in a rather obvious way. Nevertheless, at the moment there is no proof of a geometrical description of the global attractor.

Also, it is obvious that we could find higher-dimensional global attractors for the same f and g, simply by taking x in an interval $[0, L]$ where the interval length L is bigger than 1. In fact, we could achieve an arbitrary high dimension by choosing L larger and larger.

7 Uniqueness of equilibria

In this section we are going to prove that for ε small enough near any type I solution of the reduced equation with one fast part there is exactly one solution of the boundary value problem (12). As we have seen, there is at least one equilibrium u^ε near a type I solution of the reduced system. For symmetry reasons we will ourselves restrict again to type Ia solutions of the reduced system.

Lemma 7.1 *Consider a type Ia solution of the reduced equation with exactly one fast part beginning at an equilibrium $(u_E, -f(u_E))$.*
Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ any equilibrium solution u^ε in a neighborhood is hyperbolic with Morse index

$$\begin{aligned} i(u^\varepsilon) &= 1 \quad \text{if } g'(u_E) < 0 \\ i(u^\varepsilon) &= 2 \quad \text{if } g'(u_E) > 0. \end{aligned}$$

Corollary 7.2 *For $0 < \varepsilon \leq \varepsilon_0$ there is exactly one equilibrium solution u^ε corresponding to the admissible type Ia trajectory of the reduced system.*

Proof of the corollary: Suppose there were two equilibrium solutions u^ε and $\tilde{u}^\varepsilon$ both corresponding to the same type Ia trajectory of the reduced system. Consider now the interval $[u^\varepsilon(0), \tilde{u}^\varepsilon(0)]$. It contains only finitely many points that correspond to solutions of the boundary value problem. Otherwise there would be a non-hyperbolic equilibrium solution in contrast to the preceding lemma. So, without restriction, we may suppose that there is no other solution $\hat{u}^\varepsilon$ with $\hat{u}^\varepsilon(0) \in [u^\varepsilon(0), \tilde{u}^\varepsilon(0)]$. Then u^ε and $\tilde{u}^\varepsilon$ are adjacent on the shooting curve and their Morse indices have to differ by 1 due to lemma 1.5 and cannot be equal. This contradiction to lemma 7.1 shows that there can be only one equilibrium u^ε.

□

Proof of the lemma: We will consider the eigenvalue problem

$$\left.\begin{array}{rcl} \varepsilon w_{xx} - f'(u^\varepsilon)w_x - f''(u^\varepsilon)u_x^\varepsilon w + g'(u^\varepsilon)w & = & \lambda w \\ w_x(0) = w_x(1) & = & 0. \end{array}\right\} \tag{21}$$

at a family of equilibria u^ε of the viscous balance law with all u^ε corresponding to the same type Ia trajectory of the reduced system. The eigenvalue equation can be written as a first order system

$$\begin{array}{rcl} \varepsilon w_x & = & z + f'(u^\varepsilon)w \\ z_x & = & -(g'(u^\varepsilon) - \lambda)w \\ w_x(0) & = & w_x(1) = 0. \end{array}$$

Performing a Prüfer transformation

$$\begin{array}{rcl} \varepsilon w & = & \varrho\cos\varphi \\ z & = & -\varrho\sin\varphi \end{array}$$

leads to equations for ϱ and φ. We will only need the φ-equation

$$\varphi_x = \sin^2\varphi + \frac{g'(u^\varepsilon) - \lambda}{\varepsilon}\cos^2\varphi - \frac{f'(u^\varepsilon)}{2\varepsilon}\sin 2\varphi \tag{22}$$

to determine the Morse index of u^ε. The Neumann boundary conditions for the original eigenvalue problem show up as initial resp. terminal condition

$$\tan\varphi(x) = \frac{f'(u^\varepsilon(x))}{\varepsilon}$$

at $x = 0$ and $x = 1$.

So, if $\varphi_0(x; \varepsilon, \lambda)$ denotes the solution of (22) with initial value

$$\varphi_0(x = 0) = \arctan\frac{f'(u^\varepsilon(0))}{\varepsilon}$$

there is an easy criterium for λ to be an eigenvalue: The right boundary condition has to be satisfied and thus

$$\lambda \text{ is an eigenvalue} \iff \tan\varphi_0(1;\varepsilon,\lambda) = \frac{f'(u^\varepsilon(1))}{\varepsilon}.$$

We will use the following relation between the Morse index $i(u^\varepsilon)$ and $\varphi_0(1;\varepsilon,\lambda=0)$:

Lemma 7.3 *Let* $\varphi_1 := \arctan\left(f'(u^\varepsilon(1))/\varepsilon\right)$.
Then the following holds:

(i) If $\varphi_0(1;\varepsilon,0) - \varphi_1 \neq k\pi,\ k = 0,1,2,\ldots,$ *then the equilibrium* u^ε *is hyperbolic.*

(ii) If $\varphi_0(1;\varepsilon,0) - \varphi_1 \in ((k-1)\pi, k\pi)$*, then* $i(u^\varepsilon) = k$.

Proof : Part (i) is just a simple consequence of the characterization of eigenvalues. If there was a nontrivial solution of the eigenvalue problem with $\lambda = 0$ then the corresponding φ would satisfy $\tan\varphi = \tan\varphi_1$.
To prove part (ii) consider λ as a parameter and note that $\varphi_0(1;\varepsilon,\lambda)$ depends monotonically on λ and tends to $-\frac{\pi}{2}$ as $\lambda \to -\infty$ and to $+\infty$ as $\lambda \to +\infty$. For the eigenvalue λ_k $(k = 0,1,2,\ldots)$ we find

$$\varphi_0(1;\varepsilon,\lambda_k) = \varphi_1 + (k-1)\pi.$$

To determine the Morse index of u^ε is hence equivalent to counting how many of the numbers $\varphi_1 + k\pi$ lie between $-\frac{\pi}{2}$ and $\varphi_0(1;\varepsilon,0)$.

□

Hale and Sakamoto [HS88] have used the Prüfer transformation to compute Morse indices and eigenvalues for equilibria of the equation $u_t = \varepsilon u_{xx} + f(x,u)$. Although the details are quite different, we will see that with similar methods we can find the Morse index of u^ε.

The main idea is the following: As $z(u^\varepsilon_x) = 1$, by lemma 1.4 the Morse index $i(u^\varepsilon)$ has to be 1 or 2. Our goal is to compute $\varphi_0(1;\varepsilon,0) - \varphi_1$ accurately enough to decide whether it belongs to the interval $(0,\pi)$ or $(\pi,2\pi)$. For $x \in [0,\bar{x}]$ with $1-\bar{x}$ of order $\mathcal{O}(\varepsilon\ln\varepsilon^{-1})$ corresponding to the initial part of u^ε where u^ε is almost constant there exist invariant strips in the (x,φ)-plane. Some of them are positively invariant, some negatively. Between these strips φ_x is of order $\mathcal{O}(\frac{1}{\varepsilon})$. Besides these invariant strips we will use a comparison with the solutions of (22) to different initial values. Recall that u_x^ε solves the linearized equation (21) with Dirichlet boundary conditions and note that Dirichlet boundary conditions translate into

$$\varphi(x) \equiv \frac{\pi}{2} \ (mod\ \pi) \text{ at } x = 0, 1.$$

Hence, we already know two solutions of (22): The solution φ_- with initial value $\varphi_-(0;\varepsilon,0) = -\frac{\pi}{2}$ satisfies $\varphi_-(1;\varepsilon,0) = \frac{\pi}{2}$. Simply due to π-periodicity of equation (22) there is another solution φ_+ with initial value $\varphi_+(0;\varepsilon,0) = \frac{\pi}{2}$ and $\varphi_+(1;\varepsilon,0) = \frac{3\pi}{2}$.

The solution φ_0 with $\varphi_0(0;\varepsilon,0) = \arctan(f'(u^\varepsilon(0))/\varepsilon)$ is confined between φ_- and φ_+. As we will prove, φ_0 will after some time follow one of these solutions and end up very close to either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The only property of u^ε that we need is, that it is almost constant and that it jumps from an unstable arc of $\mathcal{C}$ to a stable arc. The latter fact shows up in the relation

$$f'(u^\varepsilon(1)) < 0 < f'(u^\varepsilon(0)).$$

The fast part and the drop part together are of length $\mathcal{O}(\varepsilon\ln(\varepsilon^{-1}))$, so we can choose $\bar{x} \in [0,1]$ (depending on ε) with

$$1 - \bar{x} = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon}) \tag{23}$$

such that

$$u_x^\varepsilon \leq C\varepsilon^2 \qquad \forall x \in [0,\bar{x}]. \tag{24}$$

and

$$f'(u^\varepsilon) > \gamma \qquad \forall x \in [0,\bar{x}]$$

for some constant $\gamma > 0$ not depending on $\varepsilon \in (0,\varepsilon_0]$. Relation (24) can be derived by means of the normal form analysis of our appendix where we show that u_x decreases at an exponential rate proportional to ε^{-1}.

We will first establish the existence of invariant strips. To this end define $\Phi_-(x)$ and $\Phi_+(x)$ as the two angles in $(-\frac{\pi}{2},\frac{\pi}{2})$ where $\varphi_x = 0$, or in other words,

$$\varepsilon \sin^2 \Phi_\pm - \frac{f'(u^\varepsilon)}{2} \sin 2\Phi_\pm + g'(u^\varepsilon)\cos^2\Phi_\pm = 0 \tag{25}$$

leading to

$$\tan\Phi_\pm(x) = \frac{f'(u^\varepsilon(x)) \pm \sqrt{f'(u^\varepsilon(x))^2 - 4\varepsilon g'(u^\varepsilon(x))}}{2\varepsilon}.$$

As both $f'(u)$ and $g'(u)$ are bounded for $u \in [u_{min}, u_{max}]$ using (23) and the asymptotic behavior of arctan near $\frac{\pi}{2}$ we get for $x\ in [0,\bar{x}]$ where $f'(u^\varepsilon(x)) > 0$:

$$\tan\Phi_-(x) = \frac{g'(u^\varepsilon(x))}{f'(u^\varepsilon(x))} + \mathcal{O}(\varepsilon),$$

$$\tan\Phi_+(x) = \frac{f'(u^\varepsilon(x))}{\varepsilon} + \mathcal{O}(1)$$

such that

$$\Phi_+(x) = \frac{\pi}{2} - f'(u^\varepsilon(x))^{-1}\varepsilon + \mathcal{O}(\varepsilon^2).$$

We show now the existence of a narrow negatively invariant strip around Φ_+.

Lemma 7.4 *Consider* (22) *for* $\lambda = 0$*, a given equilibrium* u^ε *and* $x \in [0, \bar{x}]$*. Then: There is a negatively invariant strip of width* $\mathcal{O}(\varepsilon^3)$ *around* Φ_+*.*

Proof: We have to compare $\frac{d\Phi_+}{dx}$ with φ_x at $\varphi = \Phi_+ \pm k\varepsilon^3$ for some k.
Let's start with

$$\frac{d\Phi_+}{dx} = \frac{2\varepsilon}{4\varepsilon^2 + \left(f' + \sqrt{f'^2 - 4\varepsilon g'}\right)^2} \cdot \left(f'' + \frac{f'f'' - 2\varepsilon g''}{\sqrt{f'^2 - 4\varepsilon g'}}\right) \cdot u^\varepsilon_x(x).$$

Here we have written f' as an abbreviation for $f'(u^\varepsilon(x))$, etc. .
Expanding the square root one finds easily that the first term is of order $\mathcal{O}(\varepsilon)$ while the second is of order $\mathcal{O}(1)$. Together with (24) we get

$$\left|\frac{d\Phi_+}{dx}\right| \le C\varepsilon^2 \qquad \text{for } x \in [0, \bar{x}].$$

On the other hand, applying elementary addition formulas for sines and cosines,

$$\begin{aligned}
\frac{d\varphi}{dx}(\Phi_+ \pm k\varepsilon^3) &= \sin^2(\Phi_+ \pm k\varepsilon^3) + \frac{g'}{\varepsilon}\cos^2(\Phi_+ \pm k\varepsilon^3) - \frac{f'}{2\varepsilon}\sin(2\Phi_+ \pm 2k\varepsilon^3) \\
&= \cos 2k\varepsilon^3 \left(\sin^2\Phi_+ + \frac{g'}{\varepsilon}\cos^2\Phi_+ - \frac{f'}{2\varepsilon}\sin 2\Phi_+\right) \\
&\quad + \sin^2 k\varepsilon^3 \left(1 + \frac{g'}{\varepsilon}\right) \\
&\quad - \frac{1}{2}\sin 2k\varepsilon^3 \left(\frac{g'}{\varepsilon}\sin 2\Phi_+ + \frac{f'}{\varepsilon}\cos 2\Phi_+\right).
\end{aligned}$$

The first term vanishes due to (25) and since

$$\begin{aligned} g'(u^\varepsilon)\sin 2\Phi_+ + f'(u^\varepsilon)\cos 2\Phi_+ &= f'(u^\varepsilon) - 2\varepsilon \sin^2\Phi_+ \tan\Phi_+ \\ &= -f'(u^\varepsilon) + \mathcal{O}(\varepsilon) \end{aligned}$$

the last term dominates and yields

$$\begin{aligned} \frac{d\varphi}{dx}(\Phi_+ + k\varepsilon^3) &\geq \frac{1}{2}k\gamma\cdot\varepsilon^2 \\ \frac{d\varphi}{dx}(\Phi_+ - k\varepsilon^3) &\leq -\frac{1}{2}k\gamma\cdot\varepsilon^2 \end{aligned}$$

for ε small enough and $x \in [0, \bar{x}]$. Choosing k large enough gives negative invariance for the strip around Φ_+.

□

The next lemma provides us with some larger invariant strips with the additional property that $|\varphi_x|$ is large outside these strips.

Lemma 7.5 *Consider* (22) *for* $\lambda = 0$, *a family* u^ε *of equilibria and* $x \in [0, \bar{x}]$. *Then there exists some* $\varepsilon_0 > 0$ *and constants* $\kappa, C_1, C_2, C_3 > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$

(i) the strip $[\Phi_+ - \kappa, \Phi_+ + \kappa]$ *is negatively invariant and for solutions* φ_1, φ_2 *inside this strip*

$$|\varphi_1(x_2) - \varphi_2(x_2)| \geq e^{\frac{C_1}{\varepsilon}(x_2 - x_1)}|\varphi_1(x_1) - \varphi_2(x_1)| \text{ for } 0 \leq x_1 \leq x_2 \leq \bar{x}.$$

Analogously, the strip $[\Phi_- - \kappa, \Phi_- + \kappa]$ *is positively invariant and for solutions* φ_1, φ_2 *inside this strip*

$$|\varphi_1(x_2) - \varphi_2(x_2)| \leq e^{-\frac{C_2}{\varepsilon}(x_2 - x_1)}|\varphi_1(x_1) - \varphi_2(x_1)| \text{ for } 0 \leq x_1 \leq x_2 \leq \bar{x}.$$

(ii) Outside these strips

$$|\varphi_x| \geq \frac{C_3}{\varepsilon}.$$

Proof: (i) The existence of the invariant strips is proved in a similar fashion as in the preceding lemma, so we omit the proof here although there are some minor differences.

To prove the contraction and expansion properties we denote with $H(\varphi, \varepsilon)$ the right hand side of (22) with $\lambda = 0$, i.e.

$$H(\varphi, \varepsilon) = \sin^2 \varphi + \frac{g'(u^\varepsilon)}{\varepsilon} \cos^2 \varphi - \frac{f'(u^\varepsilon)}{2\varepsilon} \sin 2\varphi.$$

and compute

$$\frac{\partial H}{\partial \varphi}(\varphi, x) = -\frac{f'(u^\varepsilon(x)}{\varepsilon} + \sin 2\varphi + 2\sin^2 \varphi \tan \varphi.$$

Evaluating this expression at $\varphi = \Phi_+$ gives

$$\frac{\partial H}{\partial \varphi}(\Phi_+, x) = \frac{f'(u^\varepsilon(x)}{\varepsilon} + \mathcal{O}(1)$$

such that

$$\frac{\partial H}{\partial \varphi}(\varphi, x) > \frac{C_1}{\varepsilon}$$

for ε small enough and $\varphi \in [\Phi_+ - \kappa, \Phi_+ + \kappa]$.
Similarly,

$$\frac{\partial H}{\partial \varphi}(\Phi_-, x) = -\frac{f'(u^\varepsilon(x))}{\varepsilon} + \mathcal{O}(1)$$

and

$$\frac{\partial H}{\partial \varphi}(\varphi, x) < -\frac{C_2}{\varepsilon}$$

for ε small enough and $\varphi \in [\Phi_- - \kappa, \Phi_- + \kappa]$.
(ii) Define

$$\begin{aligned} C_3 &:= \frac{1}{2} \inf |g'(u^\varepsilon(x)) \cos^2 \varphi - \frac{f'(u^\varepsilon(x))}{2} \sin 2\varphi| > 0 \\ \varepsilon_0 &:= \frac{1}{2} C_3 \end{aligned}$$

where the infimum is taken over the compact region that is obtained from the rectangle $[0, \bar{x}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ by removing strips

$$\{(x, \varphi);\ 0 \leq x \leq \bar{x},\ \Phi_\pm - \kappa \leq \varphi \leq \Phi_\pm + \kappa\}.$$

Then obviously

$$\varepsilon |H(\varphi, \varepsilon)| \geq C_3$$

and the proof is complete.

$\square$

The important observation is now that the initial value $\frac{f'(u^\varepsilon(0))}{\varepsilon}$ of the solution φ_0 we are mainly interested in lies above or below this negatively invariant $\mathcal{O}(\varepsilon^3)$-strip around Φ_+ depending on the sign of $g'(u^\varepsilon(0))$ which is the same as the sign of $g'(u_E)$. This follows simply from the expansion

$$\begin{aligned}\tan\Phi_+(0) &= \frac{f'(u^\varepsilon(0)) - \sqrt{f'(u^\varepsilon(0))^2 - 4\varepsilon g'(u^\varepsilon(0))}}{2\varepsilon} \\ &= \frac{f'(u^\varepsilon(0))}{\varepsilon} - \frac{g'(u^\varepsilon(0))}{f'(u^\varepsilon(0))} + \mathcal{O}(\varepsilon).\end{aligned}$$

Using the Laurent series of arctan at $\pi/2$ one gets

$$\Phi_+(0) = \frac{\pi}{2} - \frac{1}{f'(u^\varepsilon(0))}\varepsilon - \frac{g'(u^\varepsilon(0))}{f'(u^\varepsilon(0))}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{26}$$

In particular, for ε small enough and $g'(u_E) < 0$ the initial value of φ_0 lies below the negatively invariant strip. Thus φ_0 has to stay below Φ_- at least up to $x = \bar{x}$. Similarly, for $g'(u_E) < 0$ the initial value of φ_0 lies above the negatively invariant strip and φ_0 stays above this strip.
Having established the existence of invariant strips, we want to take a solution of (22) that stays inside the negatively invariant strip as a reference. Let φ_N be a solution on $[0, \bar{x}]$ inside the negatively invariant strip. Clearly φ_N can, for instance, be obtained by solving (22) backward starting with $\varphi_N(\bar{x}) = \Phi_+$.
In lemma 7.5 we have found a negatively invariant strip with the property that solutions in this strip separate with an exponential rate of C_1/ε and a positively invariant strip where solutions approach each other at an exponential rate of C_2/ε. Moreover, between these strips we have

$$|\varphi_x| \geq C_3/\varepsilon$$

where all the constants are independent of ε.
We are now able to describe the behavior of $\varphi_0(\cdot;\varepsilon,0)$.
1) $g'(u_E) < 0$ **:** In this case we compare first φ_0 and φ_N. Since φ_N lies in the $\mathcal{O}(\varepsilon^3)$-strip around Φ_+ and from (26) we have that

$$\varphi_N(0) - \varphi_0(0;\varepsilon,0) \geq C_4\varepsilon^2.$$

Thus we can find some $\tilde{x}_1$ of order $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ such that φ_0 leaves the negatively invariant strip around Φ_+ at $x = \tilde{x}_1$. Outside the strip, φ_x is large, so there is $\tilde{x}_2$ with

$$\tilde{x}_2 - \tilde{x}_1 = \mathcal{O}(\varepsilon)$$

such that φ_0 enters the positively invariant strip around Φ_- at $x = \tilde{x}_2$. The situation is depicted in figure 14.

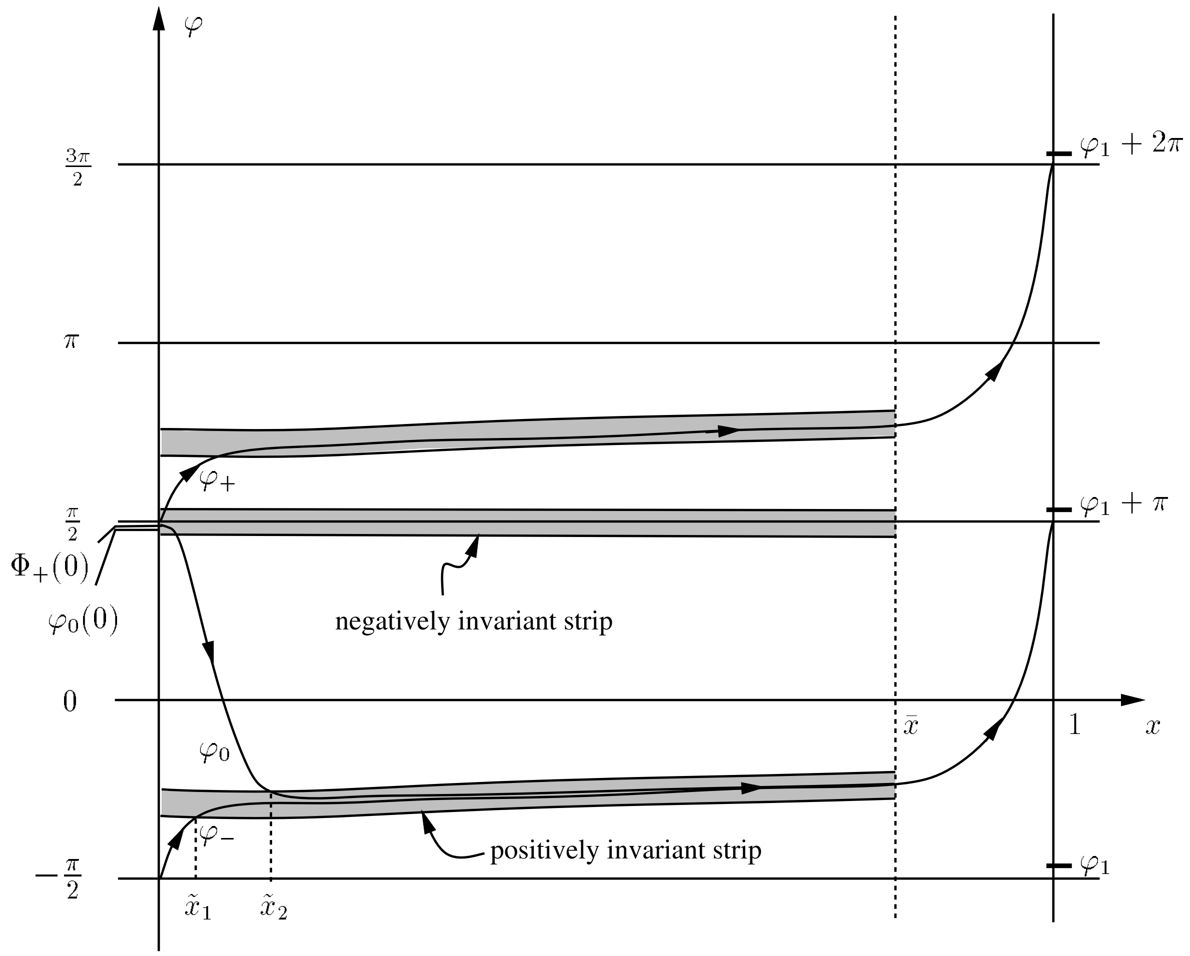


Figure 14: The case $g'(u_E) < 0$

Analogous arguments show that at $x = \tilde{x}_2$ also φ_- must have entered the positively invariant strip from below. Hence

$$|\varphi_0(\tilde{x}_2; \varepsilon, 0) - \varphi_-(\tilde{x}_2; \varepsilon, 0)| \leq 2\kappa$$

and due to the contraction property inside the positively invariant strip

$$\varphi_0(\bar{x};\varepsilon,0)-\varphi_-(\bar{x};\varepsilon,0)=\mathcal{O}(\exp(-C_2/\varepsilon)).$$

Even though φ_0 and φ_- may separate at an exponential rate of order C/ε on the interval $[x_1\bar{x},1]$ the difference between the two solutions stays small since the length of this interval is only of order $\mathcal{O}(\varepsilon\ln\frac{1}{\varepsilon})$:

$$\varphi_0(1;\varepsilon,0)-\varphi_-(1;\varepsilon,0)=\mathcal{O}(\varepsilon^{-C}\exp(-C_2/\varepsilon)).$$

Thus, the difference tends to 0 very rapidly as $\varepsilon\searrow 0$ and since we know that

$$\varphi_-(1;\varepsilon,0)=\frac{\pi}{2}$$

we can conclude that for ε sufficiently small

$$\varphi_0(1;\varepsilon,0)\le\varphi_1+\pi=\frac{\pi}{2}-\frac{1}{f'(u^\varepsilon(1))}\varepsilon+\mathcal{O}(\varepsilon)$$

since $f'(u^\varepsilon(1))<0$.
Hence,

$$\varphi_0(1;\varepsilon,0)-\varphi_1\in(0,\pi)$$

and by lemma 7.3 the solution u^ε is for all small ε a hyperbolic equilibrium solution with Morse index $i(u^\varepsilon)=1$.

2) $g'(u_E)>0$ **:** Again we compare first φ_0 and φ_N. The difference consists of the fact that φ_0 lies above the $\mathcal{O}(\varepsilon^3)$-strip that is negatively invariant and leaves the strip $[\Phi_+-\kappa,\Phi_++\kappa]$ at the top. Therefore, ϕ_0 enters the positively invariant strip $[\Phi_--\kappa+\pi,\Phi_-+\kappa+\pi]$ at some $\tilde{x}_2$ which is of order $\mathcal{O}(\varepsilon\ln\frac{1}{\varepsilon})$. The solution ϕ_+ enters this strip as well and with the same arguments as in the first case $g'(u_E)<0$ one shows that

$$\varphi_0(1;\varepsilon,0)-\varphi_+(1;\varepsilon,0)=\mathcal{O}(\varepsilon^{-C}\exp(-C_2/\varepsilon)).$$

Since

$$\varphi_+(1;\varepsilon,0)=\frac{3\pi}{2}$$

this yields

$$\varphi_0(1;\varepsilon,0)-\varphi_1\in(\pi,2\pi)$$

and by lemma 7.3 the equilibrium u^ε of the viscous balance law is hyperbolic with Morse index $i(u^\varepsilon) = 2$.

□

Remark: The same arguments can be applied with slight modifications to all solutions u^ε near type I trajectories of the reduced system. Depending on the sign of $g'(u_E)$, the solution φ_0 follows either φ_- or φ_+. On the slow parts there are also invariant strips with a strong contraction inside such that the difference between the solutions is still very small after a (rather short) junction-fast-drop-part. Then, again strong contraction applies over a time of order 1, etc.

8 Planar attractors

There is an important case in which the attractor will be two-dimensional for all sufficiently small ε: For convex f and under the generic assumption that no zero of g coincides with a zero of f' all equilibria have a Morse index of at most two. Moreover, the permutation π associated with the shooting curve $\mathcal{S}$ is the same for all small ε. This will lead to a description of $\mathcal{A}_\varepsilon$ for small ε.

Theorem 8.1 *Let $u_1, u_2, \dots, u_l$ be the zeroes of g and assume that*

(i) f is convex

(ii) $f(u_i) \neq f(u_j)$ for $1 \leq i < j \leq l$

(iii) $f'(u_i) \neq 0$ for $1 \leq i \leq l$

(iv) $g'(u_i) \neq 0$ for $1 \leq i \leq l$

Let $\bar{l}$ be the number of zeroes u_i of g such that there exists a $\tilde{u}_i$ with $f(\tilde{u}_i) = f(u_i)$ and

$$g(\tilde{u}_i)(u_i - \tilde{u}_i) > 0.$$

Then there is a ε_0, such that for all $0 < \varepsilon \leq \varepsilon_0$ the parabolic equation (5) has exactly $l + \bar{l}$ equilibrium solutions, more precisely, there are

- *$\frac{l+1}{2}$ equilibria with Morse index 0,*

- $\frac{l-1}{2} + \frac{\bar{l}}{2}$ *equilibria with Morse index* 1 *and*
- $\frac{\bar{l}}{2}$ *equilibria with Morse index* 2.

The permutation π *induced by these equilibria is the same for all* $0 < \varepsilon \leq \varepsilon_0$ *and for* $0 < \varepsilon < \varepsilon' \leq \varepsilon_0$ *the global attractors* $\mathcal{A}_\varepsilon$ *and* $\mathcal{A}_{\varepsilon'}$ *are* C^0*-equivalent.*

Proof: Obviously, there are l spatially homogenous equilibria $u \equiv u_i$, $1 \leq i \leq l$ independent of ε having Morse indices 0 and 1 alternately (see lemma 3.2). There may be also nonhomogenous equilibria with one boundary layer. Since the singular curve $\mathcal{C}$ has at most one unstable and one stable arc, there can be no equilibrium solutions with two boundary layers or an interior layer. The boundary layer solutions correspond to trajectories of the reduced system that either stay at an equilibrium on the unstable arc of $\mathcal{C}$ for time 1 and jump to the stable arc afterwards or jump from the unstable to the stable arc immediately and stay there for time 1. The condition that a jump is possible was in our notation that the opposite side is green. Recalling our definition of green, yellow and red sides, one checks immediately that $g(\tilde{u}_i)(u_i - \tilde{u}_i) > 0$ is exactly the condition for the opposite side to be a green one, or, in other words, this condition tells, that the curve $\mathcal{C}$ is not "blocked" by the slow manifold.
In the previous chapter we have shown uniqueness exactly for such solutions which are near type I solutions of the reduced system and possess exactly one fast part. Also, it is clear that the ordering of the equilibrium solutions is the same for $\varepsilon \leq \varepsilon_0$ and hence the associated permutation π is identical for all small ε.
Together with proposition 1.8 this implies the claim about C^0-equivalence of the global attractors.

$\square$

8.1 The permutation

Since for convex f the shooting curve has special properties the following simple corollary of proposition 1.5 will prove useful.

Lemma 8.1 *Let* $v_1, v_2, \ldots, v_k$ *be all the equilibria of a semilinear parabolic equation with associated permutation* π.

If v_n and v_{n+1} are two equilibrium solutions with

$$|\pi^{-1}(n+1) - \pi^{-1}(n)| = 1$$

then $z(v_{n+1} - v_m) = z(v_n - v_m)$ for all equilibria v_m with $m \neq n, n+1$.

Proof: For all m different from n and $n+1$ we have obviously

$$sign\left(\pi^{-1}(n+1) - \pi^{-1}(m)\right) = sign\left(\pi^{-1}(n) - \pi^{-1}(m)\right).$$

Consider now the difference $z(v_{n+1} - v_m) - z(v_n - v_m)$.
Using proposition 1.5, we have $z(v_{n+1} - v_m) - z(v_n - v_m) = 0$ since

$$\begin{aligned} & i(v_m) + \frac{1}{2}\left((-1)^{n+1} sign\left(\pi^{-1}(n+1) - \pi^{-1}(m)\right) - 1\right) + \sum_{j=m+1}^{n} (-1)^j sign\left(\pi^{-1}(j) - \pi^{-1}(m)\right) \\ & \quad - i(v_m) - \frac{1}{2}\left((-1)^{n} sign\left(\pi^{-1}(n) - \pi^{-1}(m)\right) - 1\right) - \sum_{j=m+1}^{n-1} (-1)^j sign\left(\pi^{-1}(j) - \pi^{-1}(m)\right) \\ & \quad = \left((-1)^{n+1} sign\left(\pi^{-1}(n+1) - \pi^{-1}(m)\right) - 1\right) + \sum_{j=n}^{n} (-1)^j sign\left(\pi^{-1}(j) - \pi^{-1}(m)\right) = 0. \end{aligned}$$

The use of this proposition makes it easier to determine with proposition 1.6 which equilibria are connected and which are not.

8.2 One last example

This last example should illustrate how the 1-skeleton of the global attractor looks like in the case of a convex f.
We choose g to possess 5 zeroes $u_1 < u_2 < u_3 < u_4 < u_5$, three on the unstable arc of $\mathcal{C}$ and two on the stable arc with

$$-f(u_1) < -f(u_5) < -f(u_2) < -f(u_3) < -f(u_4)$$

as shown in figure 15. This figure also shows the slow manifold for small nonzero ε and the dashed lines indicate the four possible solutions of the boundary value problem with a boundary layer. Each of them starts or ends near one of the equilibrium points.

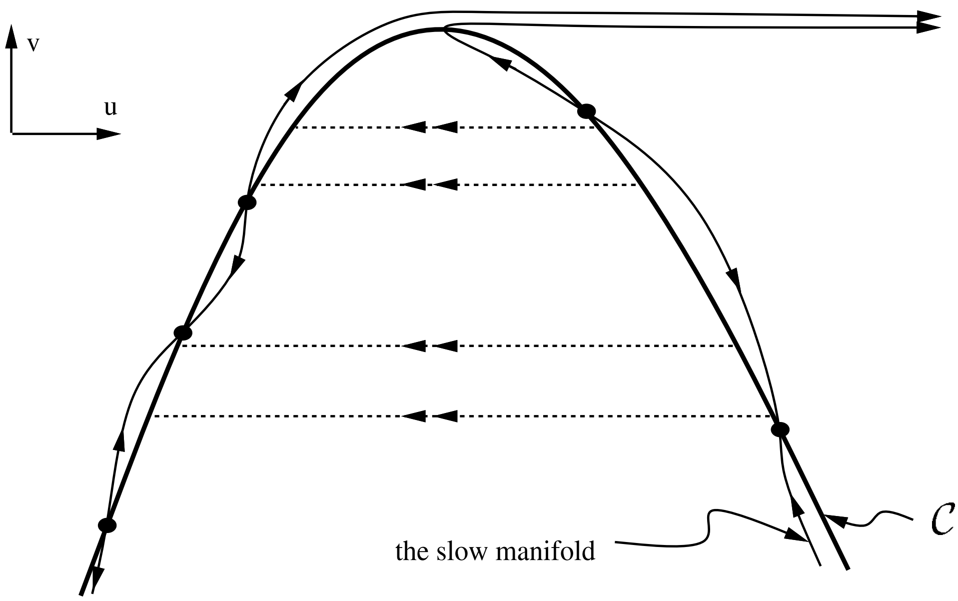


Figure 15

Altogether, there are nine equilibria for ε sufficiently small. From the shooting curve

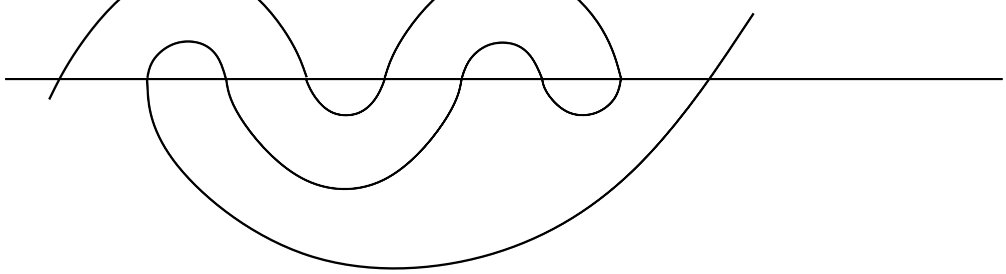


Figure 16: The shooting curvw

(or directly from the equilibria) we can read off the permutation π as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 8 & 7 & 2 & 3 & 6 & 5 & 4 & 9 \end{pmatrix}$$

giving rise to the following connection graph:

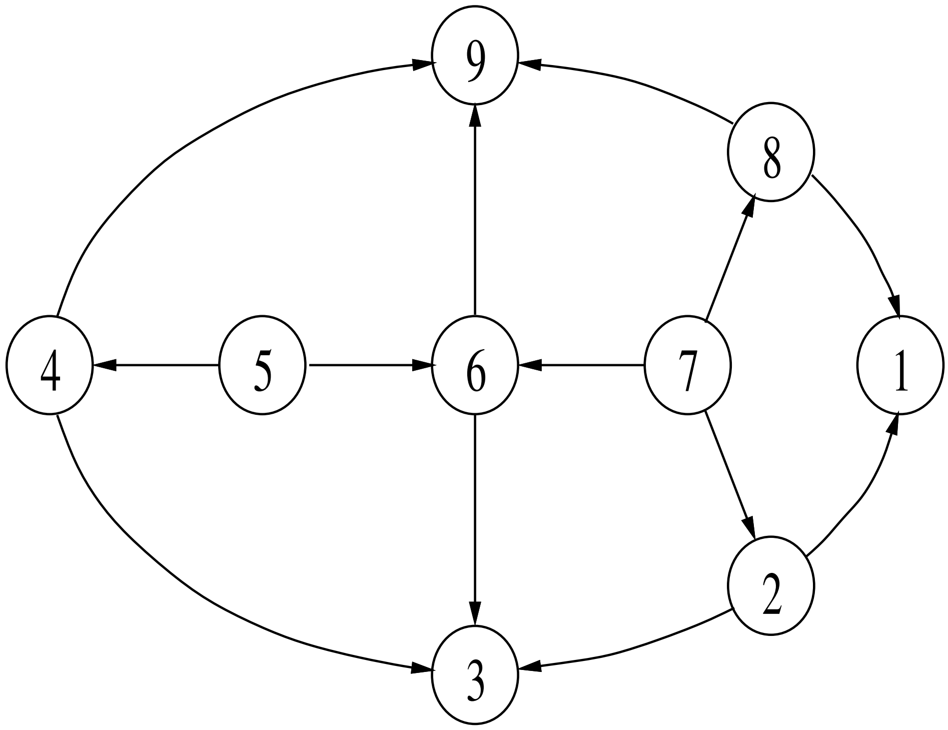


Figure 17: The connection graph

As in the example of chapter 6, one has immediately an idea how to fill this 1-skeleton with some two-dimensional "meat". Nevertheless, up to now there is no proof of a geometrical description of the global attractor.

9 Discussion

As we have already pointed out in the introduction, this paper should be considered as a first step towards the description of the global attractor $\mathcal{A}_\varepsilon$ as $\varepsilon \searrow 0$. Of course it would be more satisfactory to have uniqueness of equilibrium solution for *all* cases of theorem 1.2 and not only in case f is a convex function. This would then yield a complete description of all equilibria and the connecting orbits between different equilibria. We are optimistic that further work will soon remove the problems.

A question that is perhaps best accessible via singularly perturbed equations concerns the realization of the heteroclinic orbits. Knowing that two equilibria are connected by a heteroclinic orbit, one may be interested to know how this heteroclinic orbit looks like, i.e. the shape of the u-profile. In the general case, there is no hope at the moment, but for singularly perturbed equations, heteroclinic orbits with slowly moving transition layers have been constructed. To our knowledge, this approach has never been applied to balance laws.

Since balance laws are often considered on the whole real line one could take the interval length $0 \leq x \leq L$ as a parameter and look at the limit $L \to \infty$. Typically, one would expect that new solutions appear (by a saddle-node bifurcation) at values of L for which the transversality condition (H7) or a similar condition for type III solutions is not satisfied.

Another line of research is connected to balance laws with $x \in S^1$. Here the attractor consists not only of equilibria and heteroclinic orbits but contains also rotating waves. Although there is also a discrete Lyapunov functional, nothing similar to the permutation is available and there is by now no complete description which equilibria and rotating waves are connected by heteroclinic orbits.
For the hyperbolic balance law ($\varepsilon = 0$) and convex f some work has been done in recent years by Fan and Hale [FH95], Lyberopoulos [Lyb94] and by Sinestrari[Sin95a]. All these authors have proved a theorem of Poincaré-Bendixson type for these equations. Fan and Hale gave a description of the global attractor and some necessary conditions for heteroclinic orbits between different equilibria and rotating waves while Sinestrari was able to construct the ω-limit set for a given initial condition. Nevertheless, the connection between the global attractors for $\varepsilon = 0$ and for $\varepsilon > 0$ is not yet well understood.

And these are only some of the challenges that remain in this field...

Appendix

In the appendix we derive the asymptotic formulas given in chapter 5 as proposition 5.2 which have been used in the proof of theorem 1.2.

A A normal form

Recall that in the setting of chapter 5 for $\varepsilon = 0$ the singular curve $\mathcal{C}$ and the curve corresponding to the (left or right) boundary condition coincide. To get some estimates on the time it takes a trajectory to leave a neighborhood of the singular curve we have to find out how these two curves separate for $\varepsilon \neq 0$. To that end another choice of local coordinates will prove useful putting the vector field

$$\left.\begin{array}{rcl} \dot{u} &=& v + f(u) \\ \dot{v} &=& -\varepsilon g(u). \end{array}\right\}$$

in a nicer form. Starting from a normal form given by Takens, we will further simplify the vector field using special features of our singularly perturbed problem. There are two cases to be considered:

Case A: The normal form is computed near a point of the singular curve $\mathcal{C}$ that is not an equilibrium, e.g. where g is nonzero.

Case B: The normal form is computed near a point on $\mathcal{C}$ where g has a zero.

Let (u_0, v_0) be a point on the singular curve. Then the local normal form near $(u_0, v_0, 0)$ we will achieve in this section is given in the following lemma and will be proved in the rest of this chapter:

Lemma A.1 *If $f, g \in C^3$ with $f'(0) \neq 0$, then there is a local C^2-change of variables*

$$(\tilde{u}, \tilde{v}, \tilde{\varepsilon}) = \mathcal{T}(u, v, \varepsilon)$$

such that $\tilde{\varepsilon} = \varepsilon$ and the transformed vector field is

$$\left.\begin{array}{rcl} \dot{\tilde{u}} &=& A(\tilde{v}, \varepsilon)\, \tilde{u} \\ \dot{\tilde{v}} &=& R(\tilde{v}, \varepsilon). \end{array}\right\} \tag{27}$$

Here A and R are smooth functions of their arguments with $A(0,0) = f'(u_0)$ and $R(\tilde{v}, 0) = 0$. In case B also $R(0, \varepsilon) = 0$.

A.1 The Takens normal form

In 1971, Takens [Tak71] gave a normal form for vector fields near a nonhyperbolic equilibrium. He showed the following

Proposition A.2 *Let* 0 *be a singular point of a* C^∞*-vector field* X*. If the eigenvalues of* dX *at* 0 *satisfy a nonresonance condition ('Sternberg* $\alpha(dX(0), k)$*-condition'), then there is a* C^k*-change of coordinates such that the vector field in the new coordinates is locally in the* standard form

$$X = \sum_{i=1}^{c} X_i(x_1, ..., x_c)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{s} A_{ij}(x_1, ..., x_c)y_j\frac{\partial}{\partial y_i} + \sum_{i,j=1}^{u} B_{ij}(x_1, ..., x_c)z_j\frac{\partial}{\partial z_i}$$

where

(1) all c *eigenvalues of* $\frac{\partial X_i}{\partial x_j}$ *(in* $x_1 = ... = x_c = 0$*) have real part zero,*

(2) all s *eigenvalues of* $A_{ij}(0, ..., 0)$ *have real part* < 0 *and*

(3) all u *eigenvalues of* $B_{ij}(0, ..., 0)$ *have real part* > 0*.*

So, in the standard form the center manifold $W^c(0)$ is the linear space $\{y_1 = \ldots = y_s = z_1 = \ldots = z_u = 0\}$.

Proof of lemma A.1: To prove lemma A.1 the Takens standard form will be used near singular points of the fast system

$$\left.\begin{array}{rcl} \dot{u} & = & v + f(u) \\ \dot{v} & = & -\varepsilon g(u) \\ \dot{\varepsilon} & = & 0 \end{array}\right\} \tag{28}$$

This system was not assumed to be of class C^∞, but a look at Takens proof reveals that it is not necessary to have a C^∞-vector field. For vector fields with a finite order of differentiability a similar normal form result holds. In fact, in the first step of his proof Takens transforms the vector field to one with a finite degree of smoothness. If no resonances occur between eigenvalues this degree of smoothness determines the smoothness of the coordinate transformation $\mathcal{T}$.

At equilibrium points (u, v, ε) of (28) with $\varepsilon = 0$ the Jacobian of the fast system is

$$J = \begin{pmatrix} f'(u_0) & 1 & 0 \\ 0 & 0 & -g(u_0) \\ 0 & 0 & 0 \end{pmatrix}.$$

If $f'(u_0) \neq 0$, it has a double zero eigenvalue and one nonzero eigenvalue $f'(u_0)$. So there is only one eigenvalue with nonzero real part and all nonresonance conditions used in Takens proof will be automatically satisfied. In this case, his theorem tells that a C^{k+1}-vector field can be brought to the normal form by a C^k-change of coordinates. This is the reason why we assumed f and g to be of class C^3. Then the normal form is of class C^2 which is sufficient for us as all arguments below will not involve higher than second derivatives. By Takens' theorem a C^2-change of variables

$$(\tilde{u}, \tilde{v}, \tilde{\varepsilon}) = \mathcal{T}(u, v, \varepsilon)$$

transforms the equation to the form

$$\left.\begin{array}{rcl} \dot{\tilde{u}} &=& A(\tilde{v}, \tilde{\varepsilon})\, \tilde{u} \\ \dot{\tilde{v}} &=& R(\tilde{v}, \tilde{\varepsilon}) \\ \dot{\tilde{\varepsilon}} &=& S(\tilde{v}, \tilde{\varepsilon}). \end{array}\right\} \tag{29}$$

Note that any further C^2-smooth change of coordinates that involves only $\tilde{v}$ and $\tilde{\varepsilon}$ but leaves $\tilde{u}$ unchanged, does not alter the form of this normal form equations. We may therefore perform a linear transformation of $\tilde{v}$ and $\tilde{\varepsilon}$ and suppose that the $\tilde{v}$-direction corresponds to the zero eigenvector $(-1/f'(u_0), 1, 0)^T$, i.e. $d\mathcal{T}(u_0, v_0, 0)$ maps $(-1/f'(u_0), 1, 0)^T$ onto $(0, 1, 0)^T$. In addition, we may suppose that the eigenvector of the hyperbolic eigenvalue is mapped onto the first basis vector.
Concerning the Jacobian J, there are two cases that have to be distinguished depending on whether $g(u_0)$ is zero or not. in the first case, which was called case A above, $g(u_0) \neq 0$ and the generalized kernel of J is spanned by the two vectors

$$\begin{pmatrix} -\frac{1}{f'(u_0)} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \\ -\frac{f'(u_0)}{g(u_0)} \end{pmatrix}.$$

Since in the new coordinates the center manifold $W^c(0)$ is exactly the $(\tilde{v}, \tilde{\varepsilon})$-plane, these two vectors will be mapped by $d\mathcal{T}(u_0, v_0, 0)$ to two vectors spanning this center manifold. In the other case B, where $g(u_0) = 0$, but $g'(u_0) \neq 0$, there is an two-dimensional eigenspace to the zero eigenvalue spanned by

$$\begin{pmatrix} -\frac{1}{f'(u_0)} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

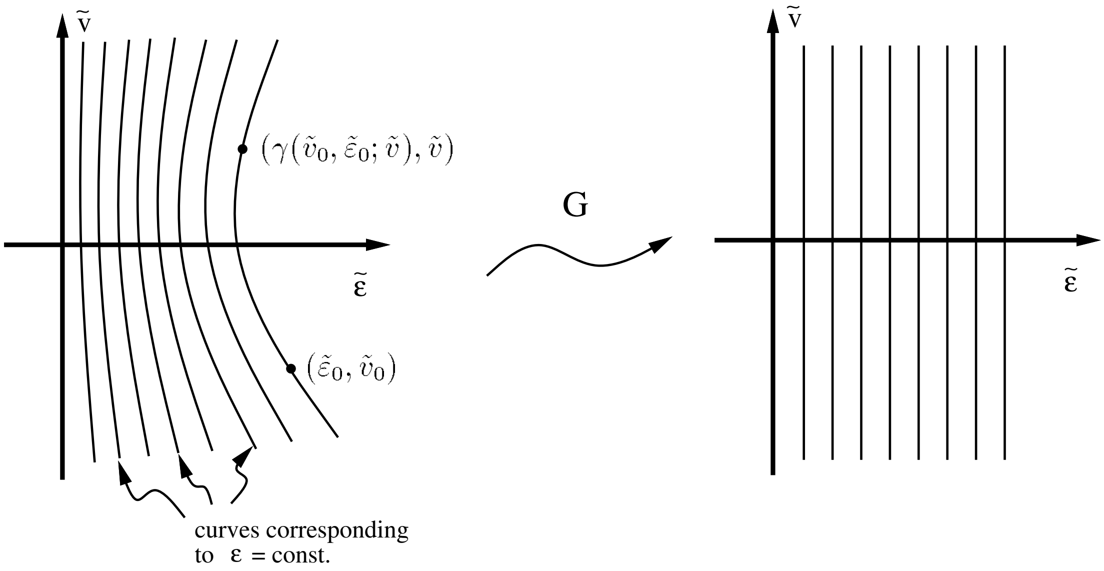


Figure 18: The foliation within $W^c(0)$

In other words, these two vectors span the tangent space of the center manifold. Another change of variables not affecting $\tilde{u}$ will lead to the result that we can suppose $S(\tilde{v}, \tilde{\varepsilon}) \equiv 0$ and $\tilde{\varepsilon} = \varepsilon$. The reason for this is the fact that for the original equation there is an invariant foliation due to the invariance of the planes $\{\varepsilon = const.\}$. As our calculation of the tangent space $T_{(u_0,v_0,0)}W^c(0)$ has shown, the (two-dimensional) leaves of this foliation intersect the center manifold transversally at $(u_0, v_0, 0)$. In fact, for the leaf corresponding to $\varepsilon = 0$ the intersection is exactly the curve $\{v + f(u) = 0\}$. The foliation is of course preserved by the coordinate change. So, in the new coordinates the leaves of this foliation are of class C^2 and their intersection with the center manifold induces at least locally a foliation of the center manifold. The $\tilde{v}$-axis $\{\tilde{\varepsilon} = 0\}$ is a leaf of this foliation in the center manifold, so locally the leaf through $(\tilde{v}_0, \tilde{\varepsilon}_0)$ is a graph over the $\tilde{v}$-axis:

$$\tilde{\varepsilon} = \gamma(\tilde{v}_0, \tilde{\varepsilon}_0; \tilde{v})$$

Straightening these fibers is achieved by the diffeomorphism

$$G(\tilde{v}, \tilde{\varepsilon}) = (\tilde{v}, \gamma(\tilde{v}, \tilde{\varepsilon}; 0)).$$

Since the fibers are invariant under the flow of (29), after this transformation, the vector field has already the form given in lemma A.1. A change of $\tilde{\varepsilon}$ assigning each of these leaves its original ε yields that we can suppose $\tilde{\varepsilon} = \varepsilon$.
The claims on $A(0,0)$ and $R(\tilde{v},0)$ are easily verified because a C^2-change of variables does not affect eigenvalues and because the $\tilde{v}$-axis consists entirely of equilibrium points.
In case B for every ε there is exactly one equilibrium point corresponding to the point (u_0, v_0) in the old coordinates. By a (last) fiber preserving transformation one can achieve that all these equilibrium points lie on the ε-axis and thus $R(0,\varepsilon) = 0$. The fact that the equilibrium point (u_0, v_0) does not depend on ε implies that $d\mathcal{T}(u_0, v_0, 0)$ maps the basis vector $(0,0,1)^T$ to itself. Altogether, we have now

$$d\mathcal{T}^{-1}(0,0,0) = \begin{pmatrix} 1 & -1/f'(u_0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

There is still some freedom since we could stretch the axes, but this is the form we will use in the next chapter.

□

B Leaving a neighborhood of $\mathcal{C}$

The normal form derived in the preceding section now serves as the basic tool to estimate the time a trajectory takes from the singular curve $\mathcal{C}$ to a section at a finite distance δ from $\mathcal{C}$. At $\varepsilon = 0$, both $\mathcal{C}$ and the manifold $\mathcal{B}$ corresponding to the boundary condition coincide. The two cases A and B differ in the way how $\mathcal{B}$ and $\mathcal{C}$ separate for positive ε.

B.1 Transition time analysis for case A

In case A the normal form near an equilibrium point $(u_0, v_0, 0)$ of (28) with $f'(u_0) \neq 0$ and $g(0) \neq 0$ is used. The point $(u_0, v_0$ is mapped onto $\tilde{u} = \tilde{v} = 0$ and for $|\tilde{u}|, |\tilde{v}| < \delta$ and $\varepsilon < \varepsilon_0$ the vector field has the form

$$\dot{\tilde{u}} = A(\tilde{v}, \varepsilon)\, \tilde{u}$$

$$\dot{\tilde{v}} \;=\; R(\tilde{v},\varepsilon)$$

The points corresponding to the boundary condition $v + f(u) = 0$ form a two-dimensional manifold $\mathcal{B}$. Since we want to perform the calculations in the new coordinates $\tilde{u}$ and $\tilde{v}$ of lemma A.1 it is important to know how $\mathcal{B}$ looks like in these coordinates. In the original coordinates, the tangent space $T_{(u_0,v_0,0)}\mathcal{B}$ in $(u_0, v_0, 0)$ is spanned by the vector $(-1, f'(u_0), 0)^T$ which lies in the (plane) center manifold $W^c(0)$ and the vector $(0,0,1)^T$ which has a component in the orthogonal complement of W^c:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \Pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{g(u_0)}{(f'(u_0))^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where Π is the orthogonal projection onto W^c. So the distance between the two manifolds $\mathcal{B}$ and $\mathcal{C}$ to first order grows proportional to ε and $\mathcal{B}$ can therefore be written as a graph

$$\tilde{u} = \varepsilon \Psi(\tilde{v},\varepsilon) \tag{31}$$

with

$$\Psi(0,0) = -\frac{g(u_0)}{(f'(u_0))^2} \neq 0$$

The geometric situation of this case is depicted in figure 19.

Using representation (31) of $\mathcal{B}$, the normal form from lemma A.1 will yield an estimate for the time t_0 from a point $(\tilde{u}_0, \tilde{v}_0)$ on $\mathcal{B}$ to a section $\Delta := \{|\tilde{u}| = \delta\}$. It depends on the sign of $\Psi(0,0)$ whether $\{\tilde{u} = \delta\}$ or $\{\tilde{u} = -\delta\}$ is the right choice but since both cases are treated in exactly the same way we restrict ourselves to the first case that corresponds to $g(u_0) < 0$.

From lemma A.1 we have $R(\tilde{v}, 0) = 0$ and therefore

$$R(\tilde{v},\varepsilon) = \varepsilon R_1(\tilde{v},\varepsilon).$$

The $\tilde{v}$-equation from (27) then reads

$$\dot{\tilde{v}} = \varepsilon R_1(\tilde{v},\varepsilon)$$

and the solution with initial value $\tilde{v}_0$ can be written as

$$\tilde{v}(s) = \tilde{v}_0 + \tilde{v}_1(\varepsilon s)$$

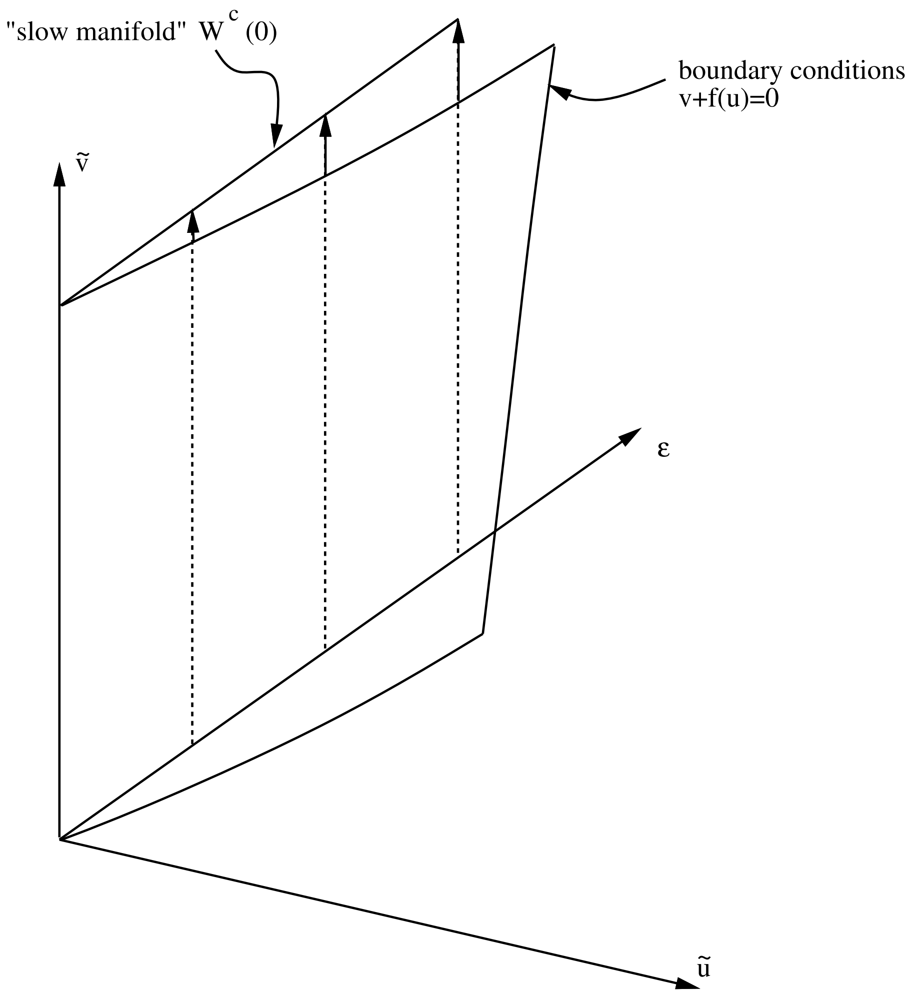


Figure 19: The geometry of case A

with

$$\tilde{v}_1(0) = 0.$$

Integrating the $\tilde{u}$-equation

$$\dot{\tilde{u}} = A(\tilde{v}, \varepsilon)\ \tilde{u}$$

gives the following condition on the time t_0:

$$\varepsilon \cdot \Psi(\tilde{v}_0, \varepsilon) \cdot \exp(\int_0^{t_0} A(\tilde{v}(s), \varepsilon)\ ds) = \delta$$

which is equivalent to

$$\int_0^{t_0} A(\tilde{v}(s),\varepsilon)\, ds = \ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right). \tag{32}$$

Because A is bounded in a neighborhood of $(0,0)$ this already shows that

$$t_0/\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right) \text{ is bounded for } |\tilde{v}_0| \le \delta, \varepsilon \le \varepsilon_0.$$

This asymptotic behaviour of t_0 suggests the scaling that is to be used in a few lines. Before, we decompose $A(\tilde{v}(s),\varepsilon)$ as

$$\begin{aligned} A(\tilde{v}(s),\varepsilon) &= A(\tilde{v}_0,0) + A(\tilde{v}_0 + \tilde{v}_1(\varepsilon s),\varepsilon) - A(\tilde{v}_0,0) \\ &= A(\tilde{v}_0,0) + A_1(\tilde{v}_0,\varepsilon,\varepsilon s) \end{aligned}$$

where A_1 is a function satisfying

$$A_1(\tilde{v}_0,0,0) = 0.$$

Using this decomposition in (32) and introducing $\sigma := \varepsilon s$ as a new integration variable yields the equation

$$\begin{aligned} & t_0 \cdot A(\tilde{v}_0,0) + \frac{1}{\varepsilon}\int_0^{\varepsilon t_0} A_1(\tilde{v}_0,\varepsilon,\sigma)\, d\sigma = \ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right) \\ \iff \quad & t_0 \cdot \frac{A(\tilde{v}_0,0)}{\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right)} + \frac{1}{\varepsilon\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right)}\int_0^{\varepsilon t_0} A_1(\tilde{v}_0,\varepsilon,\sigma)\, d\sigma = 1. \end{aligned}$$

With the new variable χ that is defined as

$$1 + \chi := t_0 \cdot \frac{A(\tilde{v}_0,0)}{\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0,\varepsilon)}\right)}$$

and compensates the asymptotic behaviour of t_0 as ε tends to 0, it is possible to define a function $\mathcal{F}(\chi,\tilde{v}_0,\varepsilon)$ in a neighborhood of $\chi = \tilde{v}_0 = \varepsilon = 0$ such that (B.1) corresponds to $\mathcal{F}(\chi,\tilde{v}_0,\varepsilon) = 0$. An application of the implicit function theorem will then yield a solution $\chi = \chi(\tilde{v}_0,\varepsilon)$ and from this solution χ it will be possible to calculate $t_0(\tilde{v}_0,\varepsilon)$.

We define thus

$$\mathcal{F}(\chi,\tilde v_0,\varepsilon)=\begin{cases}\chi+\left(-\varepsilon\ln\left(\delta^{-1}\varepsilon\Psi(\tilde v_0,\varepsilon)\right)\right)^{-1}\int_0^{-\varepsilon\ln(\delta^{-1}\varepsilon\Psi(\tilde v_0,\varepsilon))\frac{1+\chi}{A(\tilde v_0,0)}}A_1(\tilde v_0,\varepsilon,\sigma)\,d\sigma, & \varepsilon>0\\ \chi & ,\ \varepsilon=0\\ 2\chi-\mathcal{F}(\chi,\tilde v_0,-\varepsilon) & ,\ \varepsilon<0.\end{cases}$$

In other words, $\mathcal{F}(\chi,\tilde v_0,\varepsilon)-\chi$ is an odd function in ε. Of course, it remains to show that $\mathcal{F}$ is continuous. This has only to be checked at points where $\varepsilon=0$. As ε tends to 0 from above, also

$$\varepsilon\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde v_0,\varepsilon)}\right)\frac{1+\chi}{A(\tilde v_0,0)}\to 0$$

and the integral term in $\mathcal{F}$, representing the average of A_1 over an interval of that length, tends to $A_1(\tilde v_0,0,0)=0$. Thus, $\mathcal{F}$ has a one-sided limit as $\varepsilon\searrow 0$ and due to the construction of $\mathcal{F}$ for $\varepsilon<0$ it is continuous. Note that especially $\chi=\varepsilon=0$ with any $\tilde v_0$ is a solution of the equation $\mathcal{F}(\chi,\tilde v_0,\varepsilon)=0$. To apply the implicit function theorem near such a point, differentiability of $\mathcal{F}$ with respect to χ has to be proved. The derivative $\frac{\partial\mathcal{F}}{\partial\chi}$ is easily computed for $\chi,\varepsilon\neq 0$:

$$\frac{\partial\mathcal{F}}{\partial\chi}(\chi,\tilde v_0,\varepsilon)=1+\frac{1}{A(\tilde v_0,0)}A_1\left(\tilde v_0,\varepsilon,\varepsilon\ln(\frac{\delta}{\varepsilon\Psi(\tilde v_0,\varepsilon)})\frac{1+\chi}{A(\tilde v_0,0)}\right)$$

Again, $A_1(\tilde v_0,0,0)=0$ shows that $\frac{\partial\mathcal{F}}{\partial\chi}$ tends to $\frac{\partial\mathcal{F}}{\partial\chi}(\chi,\tilde v_0,0)=1$ as $\varepsilon\to 0$, and the implicit function theorem applies near every point with $\chi=\varepsilon=0$ and yields a solution $\chi=\chi(\tilde v_0,\varepsilon)$ of $\mathcal{F}(\chi,\tilde v_0,\varepsilon)=0$. Thus, the time t_0 from the curve $\mathcal{C}$ to a section $\{\tilde u=\delta\}$ is

$$t_0(\tilde v_0,\varepsilon)=\frac{1+\chi(\tilde v_0,\varepsilon)}{A(\tilde v_0,0)}\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde v_0,\varepsilon)}\right).$$

Differentiating this expression with respect to $\tilde v_0$ shows immediately that

$$\frac{\partial t_0}{\partial\tilde v_0}\cdot\left(\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde v_0,\varepsilon)}\right)\right)^{-1}$$

is bounded for $\varepsilon\neq 0$. Summarizing, we have the following:

Proposition B.1 *The time a trajectory needs from a point* $(\tilde{u}_0, \tilde{v}_0)$ *on the singular curve* $\mathcal{C}$ *to a section* $\{\tilde{u} = |\delta|\}$ *is a function* $t_0(\tilde{v}_0, \varepsilon)$ *with*

$$t_0(\tilde{v}_0, \varepsilon) = \mathcal{O}\left(\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0, \varepsilon)}\right)\right)$$

and

$$\frac{\partial t_0}{\partial \tilde{v}_0}(\tilde{v}_0, \varepsilon) = \mathcal{O}\left(\ln\left(\frac{\delta}{\varepsilon\Psi(\tilde{v}_0, \varepsilon)}\right)\right)$$

as $\varepsilon \to 0$.

Remark: All the normal form calculations were performed in the fast coordinates. Due to the scaling by a factor ε the preceding proposition implies for the original trajectory that

$$T_{initial} = \mathcal{O}(\varepsilon \ln(\frac{1}{\varepsilon})).$$

B.2 Transition time analysis for case B

In case B the normal form is calculated near an equilibrium point $(u_0, v_0, 0)$ of (28) with $f'(u_0) \neq 0$ and $g(u_0) = 0$, but $g'(u_0) \neq 0$. As in case A the first question is, how the boundary condition $v + f(u) = 0$ is transformed to the normal form coordinates. In case B the manifold $\mathcal{B}$ and the center manifold are tangent to each other, so second order terms are needed to describe the distance between the two manifolds. From the considerations at the end of chapter 4 about the normal form in case B, we know already the linear part $d\mathcal{T}^{-1}$. So we just add the second order terms of the Taylor expansion of $\mathcal{T}^{-1}$ and compare the coefficients of the vector field in old and new coordinates. To this end, the transformation $\mathcal{T}^{-1}$ is written in the form

$$\begin{aligned}
u &= \phi(\tilde{u}, \tilde{v}) \\
&= \tilde{u} + \phi_{010}\tilde{v} + \phi_{200}\tilde{u}^2 + \phi_{110}\tilde{u}\tilde{v} + \phi_{020}\tilde{v}^2 + \phi_{011}\tilde{v}\varepsilon + \phi_{101}\tilde{u}\varepsilon + \phi_{002}\varepsilon^2 \\
&\quad + o\left((|\tilde{u}| + |\tilde{v}| + |\varepsilon|)^2\right) \\
v &= \psi(\tilde{u}, \tilde{v}) \\
&= \psi_{100}\tilde{u} + \tilde{v} + \psi_{200}\tilde{u}^2 + \psi_{110}\tilde{u}\tilde{v} + \psi_{020}\tilde{v}^2 + \psi_{011}\tilde{v}\varepsilon + \psi_{101}\tilde{u}\varepsilon + \psi_{002}\varepsilon^2 \\
&\quad + o\left((|\tilde{u}| + |\tilde{v}| + |\varepsilon|)^2\right)
\end{aligned}$$

From lemma A.1 we know that in case B both $R(\tilde{v},0)=0$ and $R(0,\varepsilon)=0$ so R has the form

$$R(\tilde{v},\varepsilon)=r_{11}\varepsilon\tilde{v}+o((|\tilde{v}|+|\varepsilon|)^2).$$

Plugging this into the equation (28) one gets a very long formula that we are not going to write down here. We just note the following results from the comparison of coefficients:

Comparing the $\tilde{v}$-term of the first equation one gets $\psi_{100}=0$

From the $\tilde{v}$-term of the first equation : $\phi_{010}=-\frac{1}{f'(u_0)}$

From the $\tilde{u}^2$-term of the second equation : $\psi_{200}=0$

From the $\tilde{u}^2$-term of the first equation : $\phi_{200}=\frac{f''(u_0)}{2f'(u_0)}$

From the $\tilde{u}\tilde{v}$-term of the second equation : $\psi_{110}=0$

From the $\varepsilon\tilde{u}$-term of the second equation : $\psi_{101}=-\frac{g'(u_0)}{f'(u_0)}$

From the $\varepsilon\tilde{v}$-term of the second equation : $r_{11}=\frac{g'(u_0)}{f'(u_0)}$

With this coefficients of $\mathcal{T}$ the boundary condition $v+f(u)=0$ in the new coordinates reads

$$\mathcal{F}_{bc}(\tilde{u},\tilde{v},\varepsilon)=0$$

where

$$\mathcal{F}_{bc}(\tilde{u},\tilde{v},\varepsilon):=f'(u_0)\tilde{u}+\frac{1}{2}f''(u_0)\tilde{u}^2-\frac{f''(u_0)}{f'(u_0)}\tilde{u}\tilde{v}-\frac{g'(u_0)}{f'(u_0)}\varepsilon\tilde{u}-\frac{g'(u_0)}{f'(u_0)^2}\varepsilon\tilde{v}+o\left((|\tilde{u}|+|\tilde{v}|+|\varepsilon|)^2\right)$$

and since $f'(u_0)\neq 0$ we get by the implicit function theorem near $\tilde{u}=\tilde{v}=\varepsilon=0$ a solution of $\mathcal{F}_{bc}(\tilde{u},\tilde{v},\varepsilon)=0$ of the form $\tilde{u}=\tilde{u}_{bc}(\tilde{v},\varepsilon)$. We have

$$\frac{\partial\tilde{u}_{bc}}{\partial\varepsilon}(0,0)=\frac{\partial\tilde{u}_{bc}}{\partial\tilde{v}}(0,0)=0.$$

Important is the mixed derivative

$$\frac{\partial^2\tilde{u}_{bc}}{\partial\varepsilon\partial\tilde{v}}(0,0)=-\left(\frac{\partial\mathcal{F}_{bc}}{\partial\tilde{u}}(0,0,0)\right)^{-1}\frac{\partial^2\mathcal{F}_{bc}}{\partial\varepsilon\partial\tilde{v}}(0,0,0)=-\frac{g'(u_0)}{(f'(u_0))^3}.$$

Hence, the boundary condition $v+f(u)=0$ is transformed into

$$\tilde{u}=\varepsilon\tilde{v}\Psi(\tilde{v},\varepsilon)$$

with

$$\Psi(0,0)=-\frac{g'(u_0)}{(f'(u_0))^3}.$$

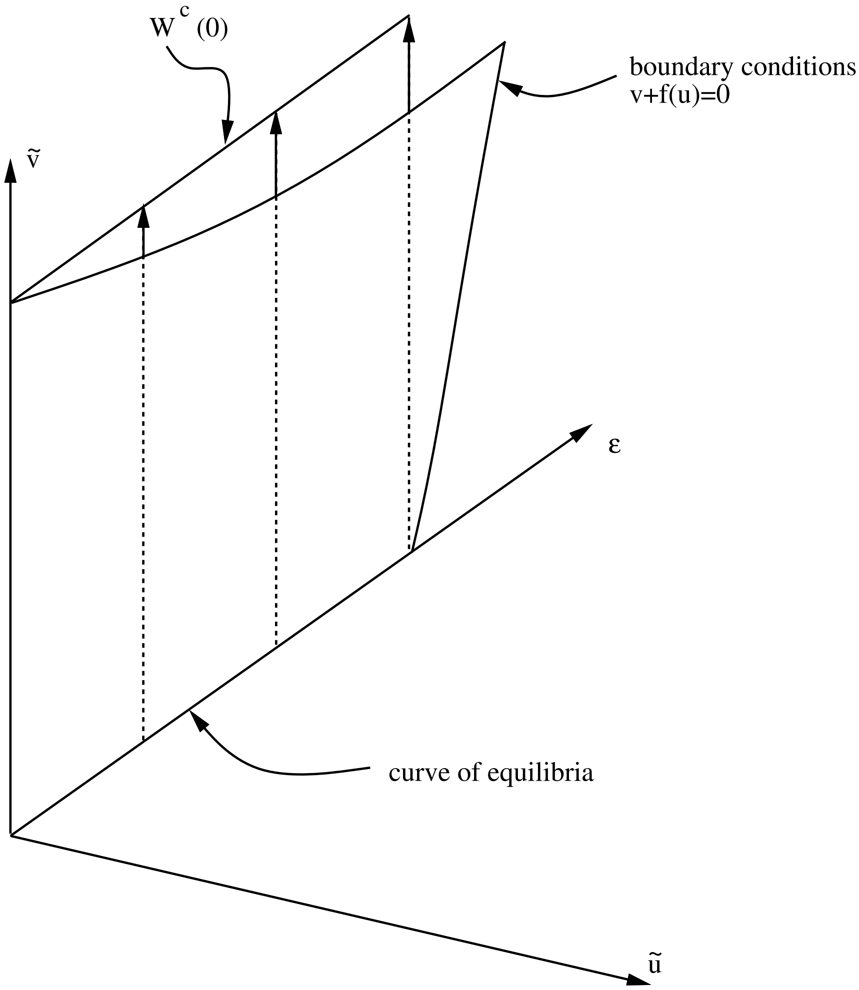


Figure 20: The geometry in case B

Since we have $R(\tilde{v},0)=0$ as well as $R(0,\varepsilon)=0$, the $\tilde{v}$-equation from (27) can be written as

$$\dot{\tilde{v}} = \varepsilon \tilde{v} R_1(\tilde{v},\varepsilon)$$

with

$$R_1(\tilde{v},\varepsilon) = \frac{g'(u_0)}{f'(u_0)} + \mathcal{O}(|\tilde{v}|+|\varepsilon|).$$

We will assume that the domain $\{|\tilde{u}|, |\tilde{v}| \leq \delta, \varepsilon \leq \varepsilon_0\}$, where the normal form is valid, is taken so small that

$$\frac{1}{2}|f'(u_0)| \leq |A(\tilde{v},\varepsilon)| \leq 2|f'(u_0)|, \tag{33}$$

$$\frac{1}{2}|g'(u_0)| \leq |R_1(\tilde{v},\varepsilon)| \leq 2|g'(u_0)| \tag{34}$$

and

$$\frac{1}{2}\left|\frac{g'(u_0)}{f'(u_0)^3}\right| \leq |\Psi(\tilde{v},\varepsilon)| \leq 2\left|\frac{g'(u_0)}{f'(u_0)^3}\right|. \tag{35}$$

We then have the (crude) estimate

$$\tilde{v}(s) \leq \tilde{v}_0 \exp(2|g'(u_0)|\varepsilon s) \tag{36}$$

as long as $\tilde{v}(s) \leq \delta$.
We can now turn to the other equation

$$\dot{\tilde{u}} = A(\tilde{v},\varepsilon)\ \tilde{u}$$

Again the time t_0 is to be determined a trajectory takes from $\mathcal{C}$ to a section $\Delta := \{\tilde{u} = \delta\}$. We have to estimate t_0 from the equation

$$\int_0^{t_0} A(\tilde{v}(s),\varepsilon)\ ds = \ln\left(\frac{\delta}{\varepsilon\tilde{v}_0\Psi(\tilde{v}_0,\varepsilon)}\right).$$

With the simple bounds on A and Ψ from (33) and (35) we get estimates for t_0:

$$\frac{1}{2}\left|f'(u_0)\ln\left(\frac{\delta f'(u_0)^3}{\varepsilon\tilde{v}_0 g'(u_0)}\right)\right| \leq t_0 \leq 2\left|f'(u_0)\ln\left(\frac{\delta f'(u_0)^3}{\varepsilon\tilde{v}_0 g'(u_0)}\right)\right|.$$

This estimates will suffice to show the next proposition.

Proposition B.2 *Fix δ small and some $T > 0$. Then for any ε small enough there exists a point $(\tilde{u}_0,\tilde{v}_0)$ on $\mathcal{B}$ such that the time for a trajectory starting in $(\tilde{u}_0,\tilde{v}_0)$ to a section $\Delta := \{|\tilde{u}| = \delta\}$ is exactly T/ε.*

Remark: Translated back into the original "slow time" x this yields a trajectory that needs exactly time T between the manifold $\mathcal{B}$ and the section Δ.

Proof: It suffices to show that for any small ε we can find a point $(\tilde{u}_1,\tilde{v}_1)$ on $\mathcal{B}$ such that a trajectory starting from this point will hit the section $\{|\tilde{u}| = \delta\}$ after some time which is strictly smaller than T/ε. Since the time a trajectory needs from $\mathcal{B}$ to Δ tends to infinity if the starting point $(\tilde{u}_0,\tilde{v}_0)$ on $\mathcal{B}$ approaches $(0,0)$, by continuity of the time map between $\mathcal{B}\setminus\{0,0\}$ and Δ the proposition follows.

Using the upper estimate for t_0, we can get a lower bound on $\tilde{v}_0$: If

$$\tilde{v}_0 \geq \frac{2\delta f'(u_0)^3}{g'(u_0)\varepsilon}\exp(-\frac{T}{2\varepsilon|f'(u_0)|})$$

then the time from $\mathcal{B}$ to Δ will be less than T/ε.
We have to check only, that for such an initial $\tilde{v}_0$ the condition $\tilde{v}(T/\varepsilon) \leq \delta$ can be satisfied for otherwise our estimates are not valid. With the estimates from (36) we have immediately that for the initial condition

$$\tilde{v}_0 = \frac{2\delta f'(u_0)^3}{\varepsilon g'(u_0)}\exp(-\frac{T}{2\varepsilon|f'(u_0)|})$$

we obtain

$$\tilde{v}(\frac{T}{\varepsilon}) \leq \frac{2\delta f'(u_0)^3}{\varepsilon g'(u_0)}\exp(-\frac{T}{2\varepsilon|f'(u_0)|}) \cdot e^{2|g'(u_0)|T}$$

which clearly tends to 0 as $\varepsilon \to 0$. Thus, for ε sufficiently small, $\tilde{v}$ remains in the domain of the normal form long enough and we are finished.

$\square$

Remark: All the transition times were calculated using sections of the special form $\{\tilde{u} = const.\}$ while the claims in chapter 5 used sections of the form $\{u = const.\}$. The different choice of sections does not affect the asymptotic estimates, since both sections are at a certain distance from the singular curve such that trajectories need only a time of order $\mathcal{O}(\varepsilon)$ between them.

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Zusammenfassung

In der vorliegenden Arbeit werden globale Attraktoren von viskosen Erhaltungsgleichungen mit Quellterm (sogenannte "balance laws") untersucht. Bei diesen Gleichungen handelt es sich um skalare parabolische Differentialgleichungen

$$u_t + (f(u))_x = g(u) + \varepsilon u_{xx} \ , \quad 0 < x < 1$$

auf dem Einheitsintervall, die von einem kleinen Parameter ε abhängen. In der vorliegenden Dissertation werden dazu Neumann-Randbedingungen gewählt.
Für festes ε besteht der globale Attraktor $\mathcal{A}_\varepsilon$ aus stationären (d.h. zeit-unabhängigen) Lösungen der parabolischen Gleichung und aus heteroklinen Orbits zwischen verschiedenen stationären Lösungen.
Im gut verstandenen sogenannten "Chafee-Infante-Fall" $f \equiv 0$ stellt sich heraus [CI74, Hen85], daß die Anzahl der stationären Lösungen und sogar die Dimension des globalen Attraktors gegen unendlich strebt für $\varepsilon \searrow 0$.
Das Verhalten für $f \not\equiv 0$ ist jedoch vollkommen anders.

Die Dissertation enthält dazu drei Hauptresultate:

In Theorem 1.1 wird bewiesen, daß schon unter schwachen Voraussetzungen an f und g die Dimension des Attraktors $\mathcal{A}_\varepsilon$ beschränkt bleibt für $\varepsilon \searrow 0$.
Theorem 1.2 zeigt eine Stabilisierung der Dimension $\dim \mathcal{A}_\varepsilon$ unter nur leicht stärkeren Voraussetzungen: Es existiert eine natürliche Zahl d, so daß für alle genügend kleinen ε gilt:

$$\dim \mathcal{A}_\varepsilon \in \{d, d+1\}.$$

Allerdings kann nicht ausgeschlossen werden, daß die Anzahl der stationären Lösungen über alle Schranken wächst, wenn ε gegen 0 strebt.
Theorem 1.3 schließt diese Lücke wenigstens für den Fall konvexer f, in dem die auftretenden globalen Attraktoren höchstens zwei-dimensional sind.

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