Convergence to Stationary States in Spatially Inhomogeneous Balance Laws

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ABSTRACT. We discuss the long-time behaviour of solutions to periodic scalar conservation laws with spatially inhomogeneous nonlinear source terms. In particular, we present a convergence result to stationary and time-periodic solutions. We prove the existence of discontinuous stationary solution if the characteristic equation possesses a saddle point. Numerical simulations show that this stationary state is asymptotically stable for a LWR traffic model with a source term modelling on- and off-ramps.

1. Introduction

We are interested in the long-time behaviour of a scalar conservation law with a space-dependent source term

$$\begin{aligned} u_t + f(u)_x &= g(u,x) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

with $x \in S^1 \sim \mathbb{R}/\mathbb{Z}$ and $t \geq 0$. We assume that the initial condition $u_0 \in BV(S^1, \mathbb{R})$ belongs to the space of functions with bounded variation and

- **(A1)** the flux f belongs to $C^2(\mathbb{R}, \mathbb{R})$ and is strictly convex, i.e. $f''(u) > c > 0$ for all u,
- **(A2)** the source term g is in C^1 and there exists a constant $M > 0$ such that

$$u \cdot g(u,x) < 0 \text{ for all } |u| > M \text{ and all } x \in S^1$$

- **(A3)** the flux is strictly monotone.

For definiteness, we assume $f' > 0$. We will study *admissible solutions*, i.e. those which satisfy for almost all $t > 0$ the entropy condition

$$u(x-,t) \geq u(x+,t) \tag{2}$$

where $u(x\pm,t)$ denotes $\lim_{h\searrow 0} u(x \pm h, t)$.

The long-time behaviour of solutions strongly depends on the properties of the characteristic system

$$\begin{aligned} \dot{\xi}(t) &= f'(v(t)) \\ \dot{v}(t) &= g(\xi(t), v(t)). \end{aligned} \tag{3}$$

We will impose the following non-degeneracy condition:

(A4) All periodic orbits of the characteristic system (3) are hyperbolic in the o.d.e. sense, i.e. their nontrivial Floquet exponents do not lie on the imaginary axis.

Under these assumptions there are two possibilities for the large-time behavior of entropy solutions:

Theorem 1.1 ([1]). *Consider the scalar balance law*

$$\begin{aligned} u_t + f(u)_x &= g(u, x), \quad x \in S^1 \\ u(x, 0) &= u_0(x) \end{aligned}$$

*with periodic data $u_0(x) \in BV(S^1, \mathbb{R})$. Under the assumptions **(A1)–(A4)** any entropy solution u either converges uniformly to some stationary solution, or it converges in L^1 to a discontinuous time-periodic solution. In the second case, the period of this asymptotic solution corresponds to the period of a solution of the associated characteristic equation (3).*

Note that the situation is similar to the case of spatially homogeneous source terms $g = g(u)$ treated in [4, 2, 5]. In Theorem 1.1 the periodic solutions of the characteristic equation (3) play a similar rôle as the zeroes of g in the case $g = g(u)$.

1.1. Generalized characteristics

Without restriction we may assume that admissible solutions are continuous from the left and satisfy the entropy condition (2) for all $t > 0$.

A strong tool to study their qualitative properties are generalized characteristics.

Definition 1.2. *A Lipschitz curve $x = \xi(t)$, defined on an interval $I = [a, b]$ is called* generalized characteristic *associated with the solution u of (1) if it satisfies the differential inequality*

$$\dot{\xi} \in [f_u(u(\xi(t)+, t)), f_u(u(\xi(t)-, t))] \quad \text{for almost all } t \in I.$$

A characteristic is called genuine, *if*

$$u(\xi(t)-, t) = u(\xi(t)+, t) \quad \text{for almost all } t \in [a, b].$$

If ξ is a genuine characteristic on the interval $[a, b]$ then there exists a function $v(\cdot)$ on $[a, b]$ such that $(\xi(\cdot), v(\cdot))$ is a solution of the characteristic system (3). If $v(a) = u(\xi(a), a)$, then $v(t)$ is a solution of equation (1) on the characteristic $\xi(t)$ at time t.

From Fillipov's theory of differential inclusions it follows that there is a unique generalized forward characteristic through any point $(\bar{x}, \bar{t})$ with $\bar{t} > 0$. On the other hand, through a point $(\bar{x}, \bar{t})$ there can exist many backward characteristics confined between a minimal backward characteristic ξ^- and a maximal backward characteristic ξ^+. Among the nice properties of generalized characteristics, we will make use of the fact that these minimal and maximal backward characteristics are always genuine. Moreover, two genuine characteristics can only intersect at their endpoints, in particular extremal backward characteristics do not intersect for $t > 0$. This allows to get a lot of qualitative information.

1.2. Long-time behaviour

In this section we give an outline of the proof of Theorem 1.1, for details we refer to [1].

As noted before, an important rôle for the asymptotic behaviour of solutions to (1) is played by the periodic solutions of the characteristic system. Assumptions **(A2)-(A4)** imply that there can be only a finite number of periodic solutions, none of them being contractible in the phase space $S^1 \times \mathbb{R}$ of (3).

Definition 1.3. *Let* $\{(v_i(t), \xi_i(t));\ 1 \le i \le k\}$ *be the set of all periodic orbits of (3) and denote with* $T_1 > T_2 > \ldots > T_k$ *their minimal periods.*

If **(A3)** holds, the periodic solutions of the characteristic system can be identified with the continuous stationary solutions of the balance law, more precisely:

Lemma 1.4. *There is a one-to-one correspondence between the periodic solutions* $(v_i(t), \xi_i(t))$ *of (3) and the continuous stationary solutions* $u(x,t) = a_i(x)$ *of (1).*

We will however see below that if **(A3)** is violated there may exist stationary solutions with shocks.

The first step in the proof of Theorem 1.1 consists of finding an appropriate candidate for the asymptotic profile. To this end, one shows, that after some time the solution $u(\cdot, t)$ can intersect at most one of the stationary solutions.

Lemma 1.5 ([2], Lemma 3.3). *There is a constant* $T > 0$ *depending only on the minimal periods* $T_1, T_2, \ldots, T_k$*, such that for all* $t > T$ *the set*

$$\left\{(u(\xi\pm, t), \xi);\ \xi \in S^1\right\} \cap \bigcup_{i=1}^{k} \left\{(a_i(\xi), \xi);\ \xi \in S^1\right\}$$

is either empty or a subset of $\left\{(a_n(\xi), \xi);\ \xi \in S^1\right\}$ *for precisely one* $n \in \{1, \cdots, k\}$.

It will turn out that if this stationary solution is stable as a periodic solution of the characteristic system (3), then the solution u will converge to the stationary solution a_n in L^∞. More precisely, we have the following dichotomy:

Lemma 1.6. *Either*

- **(I)** $\lim_{t\to\infty} \|u(\cdot, t) - a_{2m+1}(\cdot)\|_{L^\infty} = 0$ *for some* m *or*
- **(II)** *there is some* m *such that for all* $t > 0$ *there exists some* $x \in S^1$ *with* $u(x+, t) = a_{2m}(x)$ *or* $u(x-, t) = a_{2m}(x)$

The proof of Theorem 1.1 is complete if we can show that in case **(II)** the solution u converges to a time-periodic solution of (1). To this end we determine the set of characteristics which is defined on the infinite interval $[0, \infty)$.

Definition 1.7. *Fix $\bar{t} > 0$. Let $A(\bar{t}) \subseteq S^1$ be the set of intersections of extremal T_{2m}–periodic backward characteristics through points $(\bar{x}, \bar{t})$ with the x-axis:*

$$A(\bar{t}) := \left\{\xi^{\pm}(0; \bar{x}, \bar{t}) \in S^1;\ u(\bar{x}+, \bar{t}) = a_{2m}(\bar{x}) \text{ or } u(\bar{x}-, \bar{t}) = a_{2m}(\bar{x}), \bar{x} \in S^1\right\}$$

In particular, if we are in case **(II)** of Lemma 1.6 then $A(\bar{t})$ is non-empty for all $\bar{t} > 0$. Moreover, due to properties of genuine characteristics, $A(\bar{t})$ is compact and $A(t_1) \subseteq A(t_2)$ for $t_1 \geq t_2$. Consequently, $A_\infty := \cap_{t>0} A(t)$ is a compact, non-empty subset of S^1 and can therefore be written as

$$A_\infty = S^1 \setminus \bigcup_{n=1}^{\infty} (b_n, c_n)$$

with at most countably many disjoint open intervals (b_n, c_n).

For every $x \in A_\infty$ there is a genuine T_{2m}–periodic characteristic ξ emanating from $(x, 0)$ and defined for all $t \geq 0$. In particular, there are genuine characteristics β_n, γ_n such that $\beta_n(0) = b_n$, $\gamma_n(0) = c_n$ and $u(\beta_n(t), t) = a_{2m}(\beta_n(t))$, $u(\gamma_n(t), t) = a_{2m}(\gamma_n(t))$ for all $t > 0$.

For each n we then consider the strip

$$S_n := \left\{(x, t) \in S^1 \times \mathbb{R}^+;\ x \in [\beta_n(t), \gamma_n(t)]\right\}$$

on the cylinder $S^1 \times \mathbb{R}^+$. Outside the union of the strips S_n the solution is determined by the stationary solution a_{2m}:

$$u(x, t) = a_{2m}(x) \text{ if } (x, t) \notin \bigcup_{n=1}^{\infty} S_n$$

If we consider the backward characteristics emanating from points in S_n with large t it turns out that they intersect the line $\{t = 0\}$ very close to the boundary of S_n:

Lemma 1.8 ([2], Lemma 3.8)**.**
Let $\varepsilon > 0$. Then there exists a time $T(\varepsilon) > 0$ such that for any extremal backward characteristic ξ through a point $(\bar{x}, \bar{t}) \in S_n$ with $\bar{t} > T(\varepsilon)$

$$\xi(0) \in [b_n, b_n + \varepsilon) \cup (c_n - \varepsilon, c_n].$$

As in [2] one can show that there exists some function $\chi_n : [T(\varepsilon), \infty) \to S^1$ such that for $\bar{t} > T(\varepsilon)$

$$\begin{aligned} S_n^-(\bar{t}) &:= \left\{x \in [\beta_n(\bar{t}), \gamma_n(\bar{t})];\ \xi^-(0; x, \bar{t}) \in [b_n, b_n + \varepsilon]\right\} = [\beta_n(t), \chi_n(t))] \\ S_n^+(\bar{t}) &:= \left\{x \in [\beta_n(\bar{t}), \gamma_n(\bar{t})];\ \xi^+(0; x, \bar{t}) \in [c_n - \varepsilon, c_n]\right\} = [\chi_n(t), \gamma_n(t)] \end{aligned}$$

where χ_n is a generalized characteristic defined on $[T(\varepsilon), \infty)$. It can be shown that the solution u is discontinuous at $\chi_n(t)$, therefore the Rankine-Hugoniot jump condition holds along χ_n.

To prove that the curve χ_n converges to a T_{2m}-periodic curve as $t \to \infty$ we define the *Rankine-Hugoniot vector field*. The idea here is the following:
We extend the two characteristics β_n and γ_n for $-\infty < t \leq 0$ by (3). For any point $(\bar{x}, \bar{t}) \in S_n$ we then determine a unique value $u^-(\bar{x}, \bar{t})$ such that the minimal backward characteristic ξ^- from $(\bar{x}, \bar{t})$ with $v(\bar{t}) = u^-(\bar{x}, \bar{t})$ converges to $\beta_n(t)$ as $t \to -\infty$. Similarly, $u^+(\bar{x}, \bar{t})$ is defined as the unique value for which the maximal backward characteristic ξ^+ from $(\bar{x}, \bar{t})$ with $v(\bar{t}) = u^+(\bar{x}, \bar{t})$ converges to $\gamma_n(t)$:

$$\lim_{t \to -\infty} |\xi^+(t; \bar{x}, \bar{t}) - \gamma_n(t)| = 0$$
$$\lim_{t \to -\infty} |\xi^-(t; \bar{x}, \bar{t}) - \beta_n(t)| = 0.$$

Then the Rankine-Hugoniot vector field defined as

$$\dot{x} = s(x, t) \tag{4}$$

with

$$s(\bar{x}, \bar{t}) := \frac{f(u^+(x,t)) - f(u^-(x,t))}{u^+(x,t) - u^-(x,t)}. \tag{5}$$

describes the velocity of a (hypothetical) shock at some point $(x, t) \in S_n$ with left state u^- and right state u^+ via the Rankine-Hugoniot condition.

The following properties of the Rankine-Hugoniot vector field are derived in [1]:

Lemma 1.9. *The Rankine-Hugoniot vector field defined in equations (4) and (5) is well-defined, Lipschitz in $\bar{x}$ and continuous with respect to $\bar{t}$. Moreover, it is T_{2m}–periodic in $\bar{t}$.*

It points outside the strip S_n along the two boundary curves β_n and γ_n and possesses exactly one T_{2m}–periodic solution σ_n within the strip S_n.

The solution u and the states $u^\pm$ are connected in the following way:

Lemma 1.10. *Let u be an entropy solution of the hyperbolic balance law (1). For every $\delta > 0$ there exists some time $\tau(\delta)$ such that the following holds for $t > \tau(\delta)$:*

$$x \in S_n^+(t) \quad \Rightarrow \quad |u(x,t) - u^+(x,t)| \leq \delta$$
$$x \in S_n^-(t) \quad \Rightarrow \quad |u(x,t) - u^-(x,t)| \leq \delta.$$

Moreover, the shock curve χ_n which separates S_n^+ and S_n^- tends to the curve σ_n:

$$\lim_{t \to \infty} |\chi_n(t) - \sigma_n(t)| = 0.$$

Using these statements one can conclude that in case **(II)** of Lemma 1.6 the solution u converges to a T_{2m}-periodic solution with respect to the L^1-norm as $t \to \infty$. This completes the proof of Theorem 1.1.

As a remark, we note that along the same lines one can also prove that solutions to hyperbolic conservation laws

$$u_t + f(u, x)_x = 0, \quad x \in S^1, u \in \mathbb{R}$$

with space-dependent flux converge to stationary solutions if f is strictly monotone and convex in u.

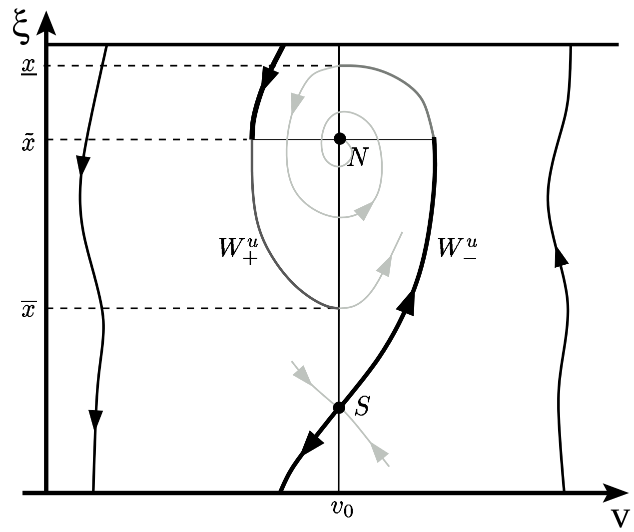

FIGURE 1. Phase portrait of the characteristic system under assumption **(A5)**. A possible stationary solution is drawn in bold.

2. Discontinuous Stationary Solutions

In many cases, assumption **(A3)** is too restrictive. We will show in this section that if **(A3)** fails, the hyperbolic balance law (1) may possess discontinuous stationary solutions. In the last chapter we present some numerical evidence that these discontinuous stationary solutions can be asymptotically stable.

If condition **(A3)** is violated, the characteristic system (3) may possess equilibria. We will assume that $f'(v_0) = 0$. Then (v_0, ξ_0) is an equilibrium if $g(v_0, \xi_0) = 0$. It is of saddle type if, and only if, $g_\xi(v_0, \xi_0) > 0$ and a spiral node if $g_v^2(v_0, \xi_0) - 4f''(v_0)g_\xi(v_0, \xi_0) < 0$.

Without **(A3)** there are still many different situations depending on the number of equilibria, their eigenvalues, relative position of invariant manifolds, existence of periodic orbits, etc.

Here we concentrate on a specific case with two equilibria and assume:

(A5) The characteristic system (3) has precisely two equilibria S and N where S is of saddle type and N is a spiral node. Moreover, both unstable manifolds of S form a heteroclinic orbit to N.

We show that under this condition there must always exist a discontinuous stationary solution:

Theorem 2.1. *Assume **(A1)** and **(A5)**. Then there exists at least one discontinuous stationary solution $u_s(x,t)$ of the hyperbolic balance law (1). This solution coincides with parts of the unstable manifold $W^u(S)$ of the saddle point S.*

Proof. Denote with $W^u_-(S)$ and $W^u_+(S)$ the two branches of the unstable manifold $W^u(S)$. Let (v_0, x_-) and (v_0, x_+) be the first intersections of $W^u_-(S)$ and $W^u_+(S)$ with the line $\{v = v_0\}$.

Note that we can parameterize $W^u_-(S)$ as $v = w^u_-(\xi)$ for $\xi \in [\underline{x}, x_0]$ and $W^u_+(S)$ as $v = w^u_+(\xi)$ for $\xi \in [x_0, \overline{x}]$. We now define

$$u_s(x) = \begin{cases} w^u_-(x) \text{ for } x \in [x_0, \tilde{x}] \\ w^u_+(x) \text{ for } x \in (\tilde{x}, x_0] \end{cases}$$

with $\tilde{x}$ such that the shock in $\tilde{x}$ is stationary i.e.

$$\nu_{shock}(\tilde{x}) = \frac{f(u_s(\tilde{x}+)) - f(u_s(\tilde{x}-))}{u_s(\tilde{x}+) - u_s(\tilde{x}-)} = 0.$$

This is clearly a stationary solution since trajectories of the characteristic system correspond to solutions of $f(u)_x = g(x, u)$ and the shock joining different of these trajectories is also stationary.

It remains to prove that such a $\tilde{x}$ always exists. First we observe that ν_{shock} is a continuous function of $\tilde{x}$ on $[\underline{x}, \overline{x}]$ because $w^u_-(\tilde{x}) \neq w^u_+(\tilde{x})$ and $f \in C^1$. Moreover

$$\nu_{shock}(\underline{x}) = \frac{f(u_s(\underline{x}+)) - f(u_s(\underline{x}-))}{u_s(\underline{x}+) - u_s(\underline{x}-)} < f'(u_s(\underline{x}-)) = f'(v_0) = 0$$

due to the convexity of f. Similarly we have

$$\nu_{shock}(\overline{x}) = \frac{f(u(\overline{x}+, 0)) - f(u(\overline{x}-, 0))}{u(\overline{x}+, 0) - u(\overline{x}-, 0)} > f'(u_s(\overline{x}+)) = f'(v_0) = 0.$$

The intermediate value theorem now gives the existence of at least one $\tilde{x}$ between $\underline{x}$ and $\overline{x}$ with $\nu_{shock}(\tilde{x}) = 0$. □

3. A traffic model with on- and off-ramps

As an application we study a traffic model of Lighthill-Whitham-Richards type [3]

$$\varrho_t + (\varrho v(\varrho))_x = g(\varrho, x)$$

on a circular road where ϱ is the car density and $v(\varrho)$ is the velocity. As usual we assume that $\rho v(\varrho)$ is a concave function of ϱ. This implies that the entropy condition reads

$$\varrho(x+, t) \geq \varrho(x-, t).$$

To model the on-/off-ramps we assume that the source term takes positive and negative values corresponding to the entry and exit of vehicles. Moreover, it vanishes for $\varrho = 0$ and $\varrho = 2$, the congested state. For small ϱ we suppose that the number of vehicles entering and leaving the freeway is approximately proportional to the traffic density on the road. However, the tendency to leave the road increases as ϱ increases while the tendency to enter the road becomes smaller.

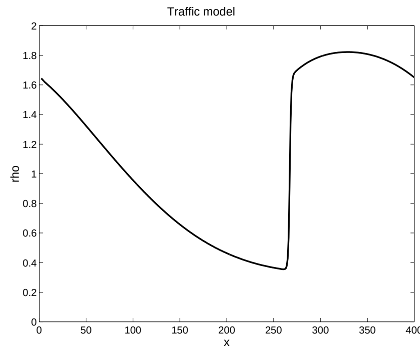

FIGURE 2. The asymptotic density profile for the traffic model

In our simulation we have chosen $\rho v(\varrho) = \frac{1}{2}\varrho(2-\varrho)$ and

$$g(\varrho, x) = \left(\frac{3}{2}\left(\cos(2\pi x) - \frac{1}{5}\right)\varrho - \frac{1}{2}\varrho^2\right)\left(1-(\varrho-1)^4\right).$$

In particular, the characteristic system for this flux and source term gives a phase portrait similar to the one in figure 1.

We have done simulations with a central scheme and found that solutions converge to a stationary state with one shock, see figure 2. The density profile agrees with parts of the unstable manifold of the saddle equilibrium in the characteristic equation joined by a shock upstream of the on-ramp. Ahead of this shock there is an acceleration wave. A similar pattern can be observed for other choices of the source term, e.g. if the positive part of g does not depend on ϱ.

References

[1] J. Ehrt and J. Härterich, Asymptotic Behavior of Spatially Inhomogeneous Balance Laws, to appear in: *J. Hyp. Diff. Eq.*

[2] H. Fan und J. K. Hale, Large-Time Behavior in Inhomogeneous Conservation Laws, *Arch. Rat. Mech. Analysis* **125** (1993) 201–216.

[3] M. J. Lighthill and G. B. Whitham, On kinematic waves II: a theory of traffic flow on long crowded roads, *Proc. Roy. Soc. Lond. A*229 (1955) 317–345.

[4] A. N. Lyberopoulos, A Poincaré-Bendixson theorem for scalar balance laws, *Proc. R. Soc. Edinb.* **A 124** (1994) 589–607.

[5] C. Sinestrari, Large time behaviour of solutions of balance laws with periodic initial data, *NoDEA* **2** (1995) 111–131.

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