Exponential Dichotomies and Transversal Homoclinic Points in Functional Differential Equations of Mixed Type

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Abstract

In this article we prove the existence of exponential dichotomies of linear functional differential equations $\dot{x}(t) = L(t)x_t$ of mixed type and extend previous results to a larger class of systems. As an application we consider a non-autonomous equation $\dot{x}(t) = f(t, x_t)$, which possesses a homoclinic solution. If a small, generic perturbation $\varepsilon g(t, x_t, \varepsilon)$ is added to the equation and if $g(\cdot, x_t, \varepsilon)$ is periodic we can prove the embedding of the Bernoulli-Shift on n symbols into the perturbed functional differential equation.

1 Introduction

Functional differential equations of mixed-type are equations of the form

$$\dot{x}(t) = f(x_t), \tag{1}$$

where $f : C^0([-a, b], \mathbb{C}^N) \to \mathbb{C}^N$ and $a, b > 0$. Here $x_t \in C^0([-a, b], \mathbb{C}^N)$ denotes the function $x_t(\theta) := x(t + \theta)$. Mixed type equations, both linear and nonlinear, occur naturally in problems of travelling waves in discrete spatial media such as lattices. Earlier papers by Bell [4] were followed by many others, see, for example [5, 6, 7, 8, 12, 13]. Mixed type equations also arise as travelling wave equations of spatially nonlocal equations of convolution type [1, 2]. In these cases a travelling-wave-ansatz leads to an equation of the form (1). Travelling waves then appear as homoclinic or heteroclinic solution of the corresponding travelling wave equation. Therefore, if one wants to understand existence and bifurcations of travelling wave solutions in lattice differential equations one has to deal with homoclinic and heteroclinic solutions of mixed type equations. There are two different techniques available for investigating homoclinic solutions. The first approach is to consider Poincaré maps. However, (1) is ill-posed and will *not* generate a semiflow (see for example [24]).

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Therefore it is not even possible to define a Poincaré map. The second approach is entirely based on Ljapunov-Schmidt reductions. The heart of this technique are *exponential dichotomies* for the linearization of (1)

$$\dot{x}(t) = L(t)x_t \tag{2}$$

along the homoclinic or heteroclinic solution, where the exact form of the linear map $L(t) \in \mathcal{L}(C^0([-a,b],\mathbb{C}^N))$ depends on the nonlinearity f in (1). However, even if the initial equation (1) is posed on $C^0([-a,b],\mathbb{C}^N)$, we will benefit from the fact to consider (2) as an equation in the phase space $L^2([-a,b],\mathbb{C}^N)$. We call a function $x(t)$ a solution of (2) on an interval $J = [c,d]$, where $c = -\infty$ and $d = \infty$ are allowed, if $x \in L^2_{loc}([c-a,b+d],\mathbb{C}^N) \cap H^1_{loc}([c,d],\mathbb{C}^N)$ and (2) is met in $L^2([c,d],\mathbb{C}^N)$. The initial value problem associated with (2) is given by

$$\dot{x}(t) = L(t)x_t, \qquad x_0 = \phi, \tag{3}$$

for some given function $\phi \in L^2 := L^2([-a,b],\mathbb{C}^N)$. *Exponential dichotomies* are projections onto t-dependent stable and unstable subspaces of the phase space $L^2([-a,b],\mathbb{C}^N)$, say $E^s(t)$ and $E^u(t)$, such that solutions $x(t)$ of (3) associated with initial values $\phi = x_{t_0}$ in the stable subspace $E^s(t_0)$ exist for $t > t_0$ and decay exponentially for $t \to \infty$. In contrast, solutions $x(t)$ associated with initial values $\phi = x_{t_0}$ in the stable unsubspace $E^u(t_0)$ solve (2) in backward direction $t < t_0$ and decay exponentially for $t \to -\infty$.

The existence of exponential dichotomies is already known for certain functional differential equations of mixed type, see [24, 16]. Since we also construct exponential dichotomies in this paper, we should clarify in which sense our results extend the previous ones.

In [24] the authors deal with mixed type equations with *discrete* forward-backward-delay, i.e. integral terms must not occur in the equation. Their approach to exponential dichotomies, which will also be adapted in this article, relies on a perturbation of an autonomous linear equation, for which exponential dichotomies have to be constructed in the first place. The authors of [24] decided to work with resolvent integrals in order to construct exponential dichotomies for autonomous equations. In respect thereof the assumptions of discrete delay is rather crucial in their approach and for technical reasons the maximal forward- respectively backward-delay has to be identical. But if one also wants to deal with more general equations, where either integral terms appear in the equation or the delay is arbitrary distributed, one has to extend the results of [24].

Exponential dichotomies also have been constructed by Mallet-Paret and Lunel in [16]. However, the authors restricted their attention to mixed type equations with discrete forward-backward delay. Although one can probably overcome this technical limitation in their approach, the crucial difference is their choice of the phase space $C^0([-a,b],\mathbb{C}^N)$. This space is not a hilbert space. However, by constructing invariant manifolds and using Melnikov theory in a forthcoming paper, we will make intensively use of properties of hilbert spaces (like the existence of smooth cut-off-functions, the existence of closed subspaces etc.) and therefore prefer to work with the phase space $L^2([-a,b],\mathbb{C}^N)$.

In this work we will extend the approach taken by Scheel et al [24] to a more general class of equations (2), which we will specify in the next section. An additional advantage of this approach compared to the one chosen in [16] is the fact, that one can relate the existence of exponential dichotomies *directly* to Fredholmproperties of the operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R}, \mathbb{C}^N) &\to L^2(\mathbb{R}, \mathbb{C}^N) \\ (\mathcal{L}x)(t) &= \dot{x}(t) - L(t)x_t. \end{aligned} \tag{4}$$

Before we state our main theorem, we need to make the following hypotheses.

Hypotheses 1
If $x \in H^1(\mathbb{R}, \mathbb{C}^N)$ is an element of the kernel of $\mathcal{L}$ or the adjoint $\mathcal{L}^*$ and $x_t = 0$ for some $t \in \mathbb{R}$, then x vanishes identically.

We can now state our first theorem.

Theorem 1
Assume that hypotheses 1 is true. Then $\mathcal{L}$ is a Fredholm operator if and only if (2) possesses exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$.

The statement of this theorem is similiar to the corresponding one in [24], but we remind the reader that we allow a bigger class of operators $\mathcal{L}$ (and we refer to section 2 for the exact assumptions on $\mathcal{L}$). Moreover, the existence of exponential dichotomies of equation (2) should naturally imply Fredholm properties of $\mathcal{L}$, though this has not been shown before. An immediate and very useful corollary of this fact is the roughness of exponential dichotomies with respect to small or compact perturbations, which is well known for ordinary differential equations, see [20].

Corollary 1
Let $\mathcal{L}$, defined by $(\mathcal{L}v)(t) = \partial_t v(t) + L(t)v_t$, be a Fredholmoperator and consider the equation

$$\dot{w}(t) = (L(t) + M(t))w_t, \tag{5}$$

where $t \mapsto M(t)$ is continuous as a map with values in $L(C^0([-a, b], \mathbb{C}^N), \mathbb{C}^N)$. Assume that $M(t), L(t)$ are of the form (6) and the operator

$$\begin{aligned} \hat{\mathcal{L}} : H^1(\mathbb{R}, \mathbb{C}^N) &\to L^2(\mathbb{R}, \mathbb{C}^N) \\ (\hat{\mathcal{L}}w)(t) &= \dot{w}(t) - (L(t) + M(t))w_t \end{aligned}$$

satisfies hypotheses (1). Then there exists an ε with the following properties. If $M(t)$ admits a decomposition in the form $M(t) = C(t) + S(t)$, such that $C(t) = 0$ for all $|t| \geq n$, where $n \in \mathbb{N}$, and $\sup_{t\in\mathbb{R}} |S(t)|_{OP} \leq \varepsilon$, where $|S(t)|_{OP}$ denotes the operator norm of $S(t) : C^0([-a, b], \mathbb{C}^N) \to C^0([-a, b], \mathbb{C}^N)$, then (5) possesses an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$. If, in addition, $\mathcal{L}$ is invertible and $M(t)$ is of the form $M(t) = C(t)$ (with $S(t) \equiv 0$), the perturbed system (5) possesses exponential dichotomies on $\mathbb{R}$ and we do need to assume that $\hat{\mathcal{L}}$ satisfies hypotheses 1.

As another corollary of the theorem we have the following.

Corollary 2
Assume that $L(t)$ is asymptotically hyperbolic and satisfies the following properties. $L(t)$ is of the form

$$L(t)\varphi(\cdot) = \sum_{k=-m}^{m} A_k(t)\varphi(r_k) + \int_{-a}^{b} p(t,\theta)\varphi(\theta)d\theta, \tag{6}$$

where $-a = r_{-m} < \ldots < r_m = b$, $p(\cdot,\cdot)$ is a continuous function with values in $\mathbb{C}^{N\times N}$, such that $p(t,\cdot)$ has compact support in $[-a,b]$ and $A_k(\cdot)$ are continuous functions with values in $\mathbb{C}^{N\times N}$, such that $\det A_{-m}(\cdot)$ and $\det A_m(\cdot)$ do nowhere vanish identically on $\mathbb{R}$. Then (2) possesses exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$.

As a first application of these results, Melnikov's method for the intersection of stable and unstable manifolds is extended to functional differential equations of mixed type. The main result is the embedding of a shift on n symbols (where $n \in \mathbb{N}$), with positive topological entropy, into the mixed type equation, provided a small generic perturbation $\varepsilon g(t, x_t, \varepsilon)$ periodic in t is added to (1). Results in this direction are known for ordinary differential equations [21], for parabolic equations [3] and elliptic equations [26].

Theorem 2
Let $f(t,\varphi)$ be smooth enough and assume in addition that f has period T in t. Suppose the system $\dot{x}(t) = f(t,x_t)$ has a T-periodic solution $x(t)$ and another solution $y(t)$ such that

a) the variational equation along $v(t)$ has an exponential dichotomy on $\mathbb{R}$ and

b) $|x(t) - y(t)|_{\mathbb{R}^N} \to 0, \qquad |t| \to \infty.$

Then there exists a compact subset Σ of $C^0([-a,b],\mathbb{C}^N)$, on which the $2m$-th iterate time map F^{2m} of the system $\dot{x}(t) = f(t,x_t)$ exists and maps Σ to Σ, if $m > 0$ is large enough. Furthermore there exists a homeomorphism $\Gamma : \mathcal{S}_2 \to \Sigma$, where S_2 denotes the space of all doubly bi-infinite sequences with entries in $\{0,1\}$, and we have the identity

$$F^{2m} \circ \Gamma = \Gamma \circ \beta,$$

where β is the Bernoulli shift on $\mathcal{S}_2$.

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