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REVERSIBILITY, CONTINUED FRACTIONS, AND INFINITE MEANDER PERMUTATIONS OF PLANAR HOMOCLINIC OBITS IN LINEAR HYPERBOLIC ANOSOV MAPS

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– Dedicated to the memory of Leonid Pavlovich Shilnikov –

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Meander permutations have been encountered in the context of Gauss words, singularity theory, Sturm global attractors, plane Cartesian billiards, and Temperley-Lieb algebras, among others. In this spirit we attempt to investigate the difference of orderings of homoclinic orbits on the stable and unstable manifolds of a planar saddle. As an example we consider reversible linear Anosov maps on the 2-torus, and their relation to continued fraction expansions.

Keywords: Transverse homoclinic orbits, conjugation invariants, Sturm permutations

1. Motivation

Around 1840 and as an addendum to his geometria situs, Gauss studied transverse self-intersections of closed planar curves; see Fig. 1 and [Gauss, 1840]. *Gauss words* arise when we label the N points of self-intersection by $1, \ldots, N$ and then concatenate the labels in order as we traverse the closed curve. Note how each label appears exactly twice in the Gauss word. Gauss observed how, conversely, not all words with this property do arise. See [Rosenstiehl, 1999] for a more recent account.

In 1988 and related to singularity theory, Arnold studied open *meanders*: transverse intersections of a differentiable planar Jordan curve J which runs from southwest to northeast across the horizontal axis; see [Arnold, 1988a; Arnold *et al.*, 1989]. Let us close up the loose ends of J with those of the horizontal axis, left and right respectively. We then obtain all Gauss words which first traverse all labels $1, \ldots, N$ once, e.g. along the increasing horizontal axis, and then a second time along the (backwards) Jordan curve J. Orienting forward again, this gives rise to the *meander permutation* σ: the j-th intersection point along the horizontal axis becomes the $\sigma(j)$-th intersection point along J.

In 1991 and related to the scalar one-dimensional parabolic partial differential equation

$$u_t = u_{xx} + f(x, u, u_x)\,, \tag{1}$$

Fusco and Rocha introduced what we now call *Sturm permutations* [Fusco & Rocha, 1991]. Under Neumann boundary conditions $u_x = 0$ on the unit interval $0 < x < 1$, they ordered finite numbers $v_1, \ldots, v_N$ of equilibria

$$0 = v_{xx} + f(x, v, v_x) \tag{2}$$

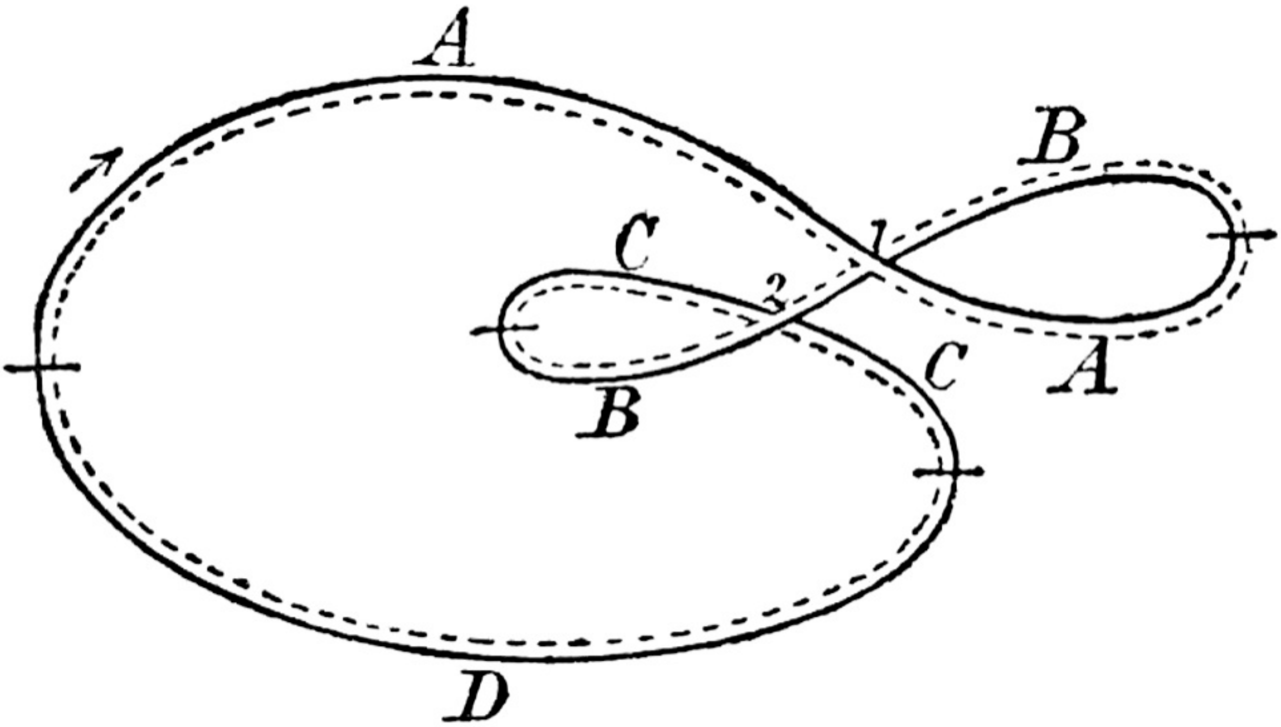

Fig. 1. *Original diagram of the Gauss word 1122; see [Gauss, 1840].*

by their boundary values such that $v_1 < v_2 < \cdots < v_N$ at $x = 0$. Then the ordering

$$v_{\sigma(1)} < v_{\sigma(2)} < \cdots < v_{\sigma(N)} \tag{3}$$

at the other boundary $x = 1$ defines the Sturm permutation σ.

Note how the Sturm permutations are meander permutations, under mild dissipativeness assumptions on f and for hyperbolic equilibria. Indeed we may study the nonautonomous ODE (2) in the v, v_x phase-plane with x as "time"-variable. The equilibria, i.e. the solutions $v = v(x)$ of the ODE boundary value problem $v_x = 0$ at $x = 0$ and 1, can then be obtained via shooting: we let the horizontal v-axis $\{v_x = 0\}$, at $x = 0$, evolve into a Jordan curve J at $x = 1$. Intersections of J with the horizontal v-axis $\{v_x = 0\}$ at $x = 0$ then represent the equilibria. Hyperbolicity of equilibria corresponds to transverse intersections. Dissipativeness of f corresponds to $\sigma(1) = 1$ and $\sigma(N) = N$.

In subsequent work by Rocha and the present author, the Sturm permutations σ developed into the crucial tool for an elaborate study of the PDE global attractors of (1) under various boundary conditions, including certain non-variational Morse-Smale cases of periodic boundary conditions; see for example [Fiedler & Rocha, 1996, 1999, 2000, 2009; Fiedler *et al.*, 2011] and the many references there. All these results are based on certain nodal properties which were first studied in 1836 by Sturm in a linear context; see [Sturm, 1836]. We therefore call the PDE global attractors of (1) *Sturm attractors*. See also [Matano, 1982] for an early nonlinear variant of Sturm nodal properties.

In the variational cases of separated boundary conditions like the Neumann condition $v_x = 0$ above, the global attractor consists of equilibria and heteroclinic orbits between them, only. In particular the Sturm permutation σ determines which equilibria are connected by a heteroclinic orbit, and which are not. As a prerequisite the PDE unstable dimensions, alias the Morse indices, of the hyperbolic equilibria $v_1, \ldots, v_N$ were determined from the "bending properties" of the shooting curve Γ in [Fusco & Rocha, 1991].

Similar bending properties, quite curiously, were studied by Bonatti et al, and later by Vago, in the context of hyperbolic diffeomorphisms of a single-rectangle Markov partition; see [Bonatti *et al.*, 1998; Vago, 2001]. For relations of closed meanders to plane Cartesian billiards see [Fiedler & Castañeda, 2012].

Another surprising link runs from meander permutations deeply into Temperley-Lieb Hopf algebras; see for example [Di Francesco *et al.*, 1997; Di Francesco & Guitter, 2005].

Viewed innocently, the second order equilibrium ODE (2) is just a forced pendulum equation. Let us assume 1-periodic forcing, i.e.

$$f(x+1, v, p) = f(x, v, p) \tag{4}$$

for all real arguments. The possibly chaotic dynamics of the time $x = 1$ diffeomorphism of (4) has received much attention; see for example [Levi, 1981] for the forced van der Pol case and [Nusse & Yorke, 1998] for beautiful illustrations of the arising complexities. On long x-intervals $0 < x < L$, for large integer $L \to \infty$, these chaotic complexities should be manifest in the sequences of associated Sturm permutations σ_L. Even the reversible area-preserving, symplectic case $f = f(x, v) = f(-x, v)$ is of significant interest. For example one may feel tempted to investigate the effects of a transverse homoclinic orbit on σ_L, even in the almost autonomous case of rapid forcing and exponentially small homoclinic splitting.

For an example of Sturm permutations in a setting which involves homoclinic orbits see [Fiedler *et al.*, 2000; Härterich & Wolfrum, 2005]. Alas, we have not obtained any systematic results in this direction so far. Instead, and as a prelude, we study transverse homoclinic orbits as such, from a permutation point of view. This is motivated by the part of the horizontal axis $\{v_x = 0\}$ which follows the unstable and stable manifolds W^+ and W^- of the homoclinic equilibrium for large forward and backward iterates $x = \pm L$ of the time $x = 1$ diffeomorphism of (2). Since this task still seemed too difficult for us, in general, we will study the permutations which arise from homoclinic orbits to $z = 0$ of linear planar *Anosov diffeomorphisms*

$$z_{n+1} = Az_n\,; \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{5}$$

Here A is a hyperbolic unimodular integer 2×2 matrix, $A \in SL_2(\mathbb{Z})$. In particular

$$|\tau| > 2\,, \qquad \det \mathrm{A} = 1\,, \tag{6}$$

where $\tau = \mathrm{tr}\,\mathrm{A} = \alpha + \delta$ denotes the trace of A. Because matrices $A \in SL_2(\mathbb{Z})$ leave the integer lattice $z \in \mathbb{Z}^2$ invariant, hyperbolic A define hyperbolic Anosov diffeomorphisms on the standard 2-torus $z \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Of course this also changes the underlying topology of the meander curves.

The stable and unstable manifolds W^- and W^+ of the trivial fixed point $z = 0$ of A, and their homoclinic intersections, are readily identified. The eigenvalues $\mu^+ = \mu$ and $\mu^- = 1/\mu$ of A satisfy the quadratic equation

$$\begin{aligned} &0 = \mu^2 - \tau\mu + 1\,; \\ &\mu = \mu^+ = \tfrac{1}{2}(\tau + \sqrt{d})\,; \\ &d{:=}\tau^2 - 4 > 0\,; \end{aligned} \tag{7}$$

with positive, nonsquare discriminant d. Indeed d is not the square of any integer because $|\tau| > 2$. Hence $\mu^\pm$ are quadratic irrationals. Also note $|\mu^+| > 1 > |\mu^-|$. Therefore $W^\pm$ are given by the spans of the eigenvectors $w^\pm$ associated to $\mu^\pm$, which we normalize to

$$w^\pm = \begin{pmatrix} \vartheta_\pm \\ 1 \end{pmatrix}. \tag{8}$$

Let us also consider A projectively, defining the fractional linear transformation

$$A\,[x] = \frac{\alpha x + \beta}{\gamma x + \delta} \tag{9}$$

for real x. Then $\vartheta = \vartheta_\pm$ are the two fixed points $\vartheta = A\,[\vartheta]$, alias the two zeros ϑ of the quadratic form $Q\,[\vartheta, 1]$, where

$$Q\,[x, y] := ax^2 + bxy + cy^2 \tag{10}$$

with coefficients, in terms of A, satisfying

$$a = \gamma; \quad b = \delta - \alpha; \quad c = -\beta\,. \tag{11}$$

The discriminant of Q is d again, of course:

$$b^2 - 4ac = d = \tau^2 - 4 > 0\,. \tag{12}$$

In particular the quadratic form Q is strictly indefinite, and the zeros $\vartheta = \vartheta_\pm$ of Q, as well as the eigenvalues $\mu^\pm$, are quadratic irrationals in the same algebraic number field $\mathbb{Q}(\sqrt{d})$.

Homoclinic intersection points $z = \pm\eta^\pm w^\pm \in W^+ \cap W^- \setminus \{0\}$ for the saddle $z = 0$ of A on the 2-torus T^2 simply arise as nonzero solutions $\eta^\pm \in \mathbb{R}$ of

$$\eta^+ w^+ + \eta^- w^- = \mathbf{m} \tag{13}$$

for suitable lattice points $\mathbf{m} \in \mathbb{Z}^2$. We call $\mathbf{m}$ a representative of the homoclinic orbit of z. Since the linearly independent eigenspace lines $W^\pm$ of irrational slopes $1/\vartheta_\pm$ both wind densely around T^2, the homoclinic points $z = z_\mathbf{m} = \eta^\pm_\mathbf{m} w^\pm$ are dense on T^2, as well, and so are their respective positions $\eta^\pm_\mathbf{m} \in \mathbb{R}$ on the lines $W^\pm$. Unfortunately the map

$$\tilde\sigma : \eta^-_\mathbf{m} \longmapsto \eta^+_\mathbf{m} \tag{14}$$

does not seem to produce a well-defined infinite permutation σ, except by some arbitrary choices of enumerations $\rho_\pm : \mathbb{N} \to \mathbb{Z}^2 \setminus \{0\}$ of the $\mathbf{m}, \eta^\pm_\mathbf{m}$ and the definition

$$\sigma := \rho_+^{-1}\,\rho_-\,. \tag{15}$$

Indeed the dense sets $\eta^\pm_\mathbf{m}$ of homoclinic points do not define *well-ordered sets*, where nonempty subsets are required to possess smallest elements. This makes it difficult to talk about "infinite permutations", a concept based on the well-ordered set $\mathbb{N}$ of the positive integers. Even if we restrict our considerations to *homoclinic orbits* instead of homoclinic points, this difficulty seems to persist, at first. For example we may consider the case $\mu^+ = \mu > 1 > 1/\mu = \mu^- > 0$ of positive eigenvalues and represent homoclinic orbits by $\mu^- \le \eta^-_\mathbf{m} < 1$. Then the associated $\eta^+_\mathbf{m} := \tilde\sigma(\eta^-_\mathbf{m})$ turn out to be a well-ordered discrete set, which comes with a natural order to define ρ_+. Still, the choice of an enumeration ρ_- of the dense set of $\eta^-_\mathbf{m}$ seems quite arbitrary.

In the following sections we address these issues as follows. In section 2 we consider arbitrary closed geodesics L through the trivial saddle $z = 0$. It is then easy to describe the class of finite meander permutations of $L \cap AL$, in the spirit of Arnold; see theorem 1. In section 3 we address infinite meander permutations associated to homoclinic orbits. The basic idea is to choose $\mathbf{m}$ in fundamental sectors of the action of A on the lattice $\mathbb{Z}^2$, such that the resulting sets of $\eta^-_\mathbf{m}$ and $\eta^+_\mathbf{m}$ are both discrete and hence well-ordered. This defines the infinite permutation σ, up to a choice of fundamental sectors. In section 4 we discuss related periodic continued fraction aspects of the eigensectors $W^\pm$ and their slopes $1/\vartheta^\pm$. Section 5 introduces integer involutions $\kappa \in GL_2(\mathbb{Z})$ and, specifically, considers κ-reversible Anosov diffeomorphisms A. We also construct κ-invariant fundamental sectors of reversible A which are tied uniquely to the reversing action. In particular we describe the interlacing of homoclinic orbits by involutive permutations σ, for reversible A; see theorem 2. Here a permutation σ of $\mathbb{N}$ is called an *involution* if $\sigma^2 = \mathrm{id}$. Equivalently, $\mathbb{N}$ decomposes into fixed points and 2-cycles of σ. Although meander involutions play a central role in Hamiltonian variants of (2), [Fiedler *et al.*, 2011], our notion of homoclinic meanders σ is quite different. As a side product of our approach section 6 develops some relations between the actions of A, of reversors κ, and of the continued fraction expansions of the eigenvector slopes $1/\vartheta_\pm$ introduced in section 4. We briefly discuss the relation of our approach to the standard arithmetic theory of quadratic forms in the concluding section 7.

2. Closed integer geodesics

Let L be an oriented closed integer geodesic on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. On $\mathbb{R}^2$ we may think of L as a line segment from $\mathbf{m} = 0$ to another lattice point $\mathbf{m}' \in \mathbb{Z}^2 \setminus \{0\}$ with co-prime components. Let $A \in GL_2(\mathbb{Z})$ be integer with determinant ± 1. Then AL is also an oriented closed integer geodesic on T^2, because A defines a bijection of the lattice $\mathbb{Z}^2$ to itself. For $AL \ne L$ we may label the N intersection points $z_0 = 0, \dots, z_{N-1}$ in order along L. This defines a *meander permutation* $\sigma \in S_N$ of the labels $\{0, \dots, N-1\}$ such that $0 = z_{\sigma(0)}, \dots, z_{\sigma(N-1)}$ is the order along AL. Of course $\sigma(0) = 0$.

Theorem 1. *Let L be a closed integer geodesic on T^2, $A \in GL_2(\mathbb{Z})$ and assume $AL \neq L$. Then the meander permutation σ defined above is* arithmetic, *i.e. there exists p co-prime to N such that*

$$\sigma(i) \equiv p\, i \qquad (\mathrm{mod}\; N) \tag{16}$$

for all $i = 0, \ldots, N-1$.

Proof. Since the components of the end point $\mathbf{m}'$ of L are co-prime we may perform an $SL_2(\mathbb{Z})$ change of coordinates such that $\mathbf{m}' = (1, 0)$, without loss of generality. Next consider the general integer matrix

$$A = \begin{pmatrix} p' & q' \\ q & \pm p \end{pmatrix} \in GL_2(\mathbb{Z}) \tag{17}$$

with det $A = \pm pp' - qq' = \pm 1$ and hence with co-prime p, q. Then AL is the line segment $s \cdot (p', q)$, $0 \leq s < 1$, and $AL \cap L$ is given by those s_j for which

$$s_j q \equiv 0 \qquad (\mathrm{mod}\; \mathbb{Z})\,. \tag{18}$$

Hence $s_j = j/q$, $j = 0, \ldots, q-1$. In particular $N = q$. The corresponding positions on L are the x-components $s_j p' = jp'/q \equiv i/q$ (mod $\mathbb{Z}$), i.e.

$$\sigma(i) = j \equiv p\, i \qquad (\mathrm{mod}\; q)\,. \tag{19}$$

Here we have used $pp' \equiv 1$ (mod q). This proves (16) and the theorem, because $q = N$. ■

3. Sectors

We consider two types of sectors in this section: eigensectors Σ and fundamental subsectors Σ^* of eigensectors. The sectors arise for hyperbolic matrices $A \in GL_2(\mathbb{Z})$. *Eigensectors* Σ are the four connected components of $\mathbb{R}^2 \setminus (W^+ \cup W^-)$ after removal of the stable and unstable eigenspaces $W^{\pm}$.

For simplicity of presentation we only consider the orientation preserving case $A \in SL_2(\mathbb{Z})$. Since $\pm A$ are projectively equivalent we restrict attention to the case

$$\mu = \mu^+ > 1 > \mu^- = 1/\mu > 0 \tag{20}$$

of positive eigenvalues $\mu^{\pm}$. In this case each of the four eigensectors is A-invariant. The other cases reduce to the present case when passing to A^2, instead.

Fundamental subsectors Σ^* of A-invariant eigensectors Σ are half-open sectors bounded by open half-lines L and AL emanating from the origin. Differently from the previous section L, and hence AL, may define closed or non-closed geodesics, i.e. the half lines L, AL may be of rational or irrational slope. By invariance of the eigensectors, any fundamental subsector Σ^* is a fundamental domain for the action of A in the eigensector Σ. In particular, any homoclinic orbit in $W^+ \cap W^- \cap \Sigma$ possesses a unique representative $\mathbf{m}$ in Σ^*. Here we may choose the boundary L, but not AL, to belong to Σ^* in the case of rational slopes. See section 5 for an example. For irrational slopes of L, AL, this choice is irrelevant.

Again we represent homoclinic points $z_{\mathbf{m}} = \pm\eta^{\pm}_{\mathbf{m}} w^{\pm} \in W^+ \cap W^- \setminus \{0\}$ by integer lattice points

$$\eta^+_{\mathbf{m}} w^+ + \eta^-_{\mathbf{m}} w^- = \mathbf{m} \tag{21}$$

in Σ; see (13). Then homoclinic orbits of A correspond to A-orbits of lattice points $\mathbf{m}$. Therefore homoclinic orbits are represented by lattice points $\mathbf{m}$ in fundamental sectors Σ^*. Let

$$P^{\pm} : \Sigma \to \mathbb{R} \tag{22}$$

be the eigenprojections of $w^{\pm}$, i.e. $P^{\pm} z = \eta^{\pm}$ for $z = \eta^+ w^+ + \eta^- w^-$. Then the two images

$$P^{\pm}(\Sigma^* \cap \mathbb{Z}^2) = \{\eta^{\pm}_{\mathbf{m}};\ \mathbf{m} \in \Sigma^*\} \tag{23}$$

of the *fundamental lattice* $\Sigma^* \cap \mathbb{Z}^2$ are well-ordered discrete subsets of a closed half line in $\mathbb{R}$, each. This follows from the fact that the boundary half-lines L, AL of Σ^* are in the interior of the eigensector Σ

which defines $P^{\pm}$. In particular the meander permutation $\sigma = \rho_+^{-1}\rho_-$ of the orbit representatives is defined uniquely through the respective orders of the well-ordered $P^{\pm}$-images

$$\eta^{\pm}_{\mathbf{m}} > 0 \tag{24}$$

of $\mathbf{m} \in \Sigma^*$. (Depending on the sector Σ, positivity requires a suitable orientation of $W^{\pm}$.) Since the slopes of L and AL are strictly between the slopes of the Σ-boundaries w^+ and w^-, the permutation σ of $\mathbb{N}$ is linearly bounded: there exists a constant $C > 1$ such that

$$C^{-1}\, i \le \sigma(i) \le C\, i\,, \tag{25}$$

for all $i \in \mathbb{N}$. Viewed in the present setting, [Bonatti *et al.*, 1998] consider a sequence of permutations by taking all lattice points $\mathbf{m}$ in the full sector Σ, not just the homoclinic orbits of $\mathbf{m} \in \Sigma^*$. To obtain finite permutations, they impose bounds on $|\eta^{\pm}_{\mathbf{m}}|$. As a drawback, the permutations then depend on those bounds and do not possess any direct limit as the bounds increase. We, in contrast, suggest a single but infinite permutation σ of $\mathbb{N}$ via a direct limit. As a drawback, our permutation depends on the choice of the boundary L of the fundamental sector Σ^*, and it is not clear to us how to systematically eliminate this dependence. Only in the next section will we study the reversible case where the additional requirement of invariance of Σ^* under the reversing involution κ will fix Σ^*, and hence the meander permutation σ, uniquely.

4. Continued fractions

We study the eigensectors Σ bounded by the stable and unstable manifolds $W^{\pm}$ of the eigenvectors $w^{\pm} = (\vartheta^{\pm}, 1)$, as introduced in the previous section. We show how the corner points $\pm z_n$, $n \in \mathbb{Z}$, of the convex hull of the lattice points in the respective eigensectors satisfy a recurrence relation

$$z_{n+1} = z_{n-1} + a_n z_n \tag{26}$$

with a p-periodic sequence of positive integer coefficients a_n:

$$a_{n+p} = a_n \tag{27}$$

for some $p \in \mathbb{N}$ and all $n \in \mathbb{Z}$. See lemma 1. Conversely any such sequence gives rise to $A \in SL_2(\mathbb{Z})$ with sectors defined by (26) (27). Although this would follow, equally, from the Lagrange proof of continued fraction periodicity for quadratic irrationals, we do not use that fact at this stage, yet, but give a direct "dynamic" proof instead; see lemma 2.

We recall that $A \in SL_2(\mathbb{Z})$ is assumed to be hyperbolic and the projective representative A of $\pm A \in PSL_2(\mathbb{Z})$ has been chosen such the eigenvalues $\mu^{\pm} = \mu^{\pm 1}$ of A are positive. Conjugating by a shear in $SL_2(\mathbb{Z})$ we may also assume

$$0 < \vartheta^+ > \vartheta^- \tag{28}$$

for the eigenvectors $w^{\pm} = (\vartheta^{\pm}, 1)$ of irrational slopes $1/\vartheta^{\pm}$ which bound the eigensector Σ. By sign symmetry $z \mapsto -z$ it is sufficient to consider the two sectors Σ_1 and Σ_2, where Σ_1 is the sector which contains $(1,0)$ and hence the positive x-axis, and Σ_2 contains the vectors $(\vartheta, 1)$ with $\vartheta^- < \vartheta < \vartheta^+$.

Consider the continued fraction expansion

$$1/\vartheta^+ = [\tilde{a}_0; \tilde{a}_1, \tilde{a}_2, \ldots] \tag{29}$$

of the positive slope of w^+, with positive integers $\tilde{a}_1, \tilde{a}_2, \ldots$ and nonnegative integer $\tilde{a}_0$. See for example [Perron, 1954; Khinchin, 1964; Arnold, 1988b]. Define $\tilde{z}_n \in \mathbb{Z}^2$ by the recursion

$$\tilde{z}_{n+1} = \tilde{z}_{n-1} + \tilde{a}_n \tilde{z}_n\,, \tag{30}$$

for integer $n \ge 0$ with $\tilde{z}_{-1} := (1,0)$ and $\tilde{z}_0 := (0,1)$. Then $\tilde{z}_n$ are the corners of the convex hull of the integer lattice points above and below W^+ in the first (i.e. nonnegative) quadrant, respectively, for even and odd $n \ge -1$. Eventually these coincide with the corners z_n of the convex hulls of the integer lattice points in the eigensectors Σ_1 and Σ_2 of A, i.e.

$$z_n = \tilde{z}_n \tag{31}$$

for all sufficiently large $n \geq n_0$. In particular this labels the corners z_n, for $n \in \mathbb{Z}$, successively along the piecewise linear paths Γ_2 and Γ_1 which constitute the boundaries of the convex lattice hulls in Σ_2 and Σ_1 respectively. Orientation is towards their positive W^+ asymptote. The vertices z_n are located in the sectors Σ_1, Σ_2, alternatingly: $z_n \in \Sigma_1$ for odd n, and $z_n \in \Sigma_2$ for even $n \in \mathbb{Z}$.

Lemma 1. *In the above setting there exists a bi-infinite sequence a_n, $n \in \mathbb{Z}$, of positive integers such that*

$$z_{n+1} = z_{n-1} + a_n z_n\,, \qquad \det(z_{n+1}, z_n) = (-1)^n \tag{32}$$

holds, for all $n \in \mathbb{Z}$. Moreover there exists a minimal integer period $p > 0$ such that

$$a_{n+p} = a_n \tag{33}$$

for all $n \in \mathbb{Z}$. In particular

$$Az_n = z_{n+kp}\,, \tag{34}$$

for all $n \in \mathbb{Z}$ and some fixed positive integer k such that kp is even.

As an obvious consequence of (31), (33) we recover the following version of Lagrange periodicity for quadratic irrationals.

Corollary 4.1. *The continued fraction expansion (29) of $1/\vartheta^+$ is asymptotically p-periodic, with*

$$\tilde{a}_n = a_n \tag{35}$$

for $n \geq n_0 + 1$.

Proof. [Proof of Lemma 1] Because A is hyperbolic with positive eigenvalues $\mu^\pm$, each of the eigensectors Σ_1, Σ_2 is A-invariant. Because $A \in SL_2(\mathbb{Z})$, the set of integer lattice points in each eigensector is also A-invariant, together with its convex hull, the boundary path Γ_1, Γ_2 and the set of corners z_n. Since $\mu^+ > 0$, A also preserves the orientations of Γ_1, Γ_2 towards their asymptote. Therefore

$$Az_n = z_{n+p'} \tag{36}$$

for all $n \in \mathbb{Z}$ and some fixed $p' > 0$. Note how eigensector invariance implies p' is even.

Actually one has to address the possibility that two different minimal periods p'_1 and p'_2 might arise in (36), one for each parity of $n \in \mathbb{Z}$, i.e. one for each eigensector Σ_1, Σ_2. This can be excluded as follows. The convergents $\tilde{z}_n$ of standard continued fractions satisfy $\det(\tilde{z}_{n+1}, \tilde{z}_n) = (-1)^n$. More precisely in fact,

$$\det(\tilde{z}_m, \tilde{z}_n) = (-1)^n \quad \Longleftrightarrow \quad m = n \pm 1\,, \tag{37}$$

for all $m, n \geq 1$. Hence (31) implies the same relation

$$\det(z_m, z_n) = (-1)^n \quad \Longleftrightarrow \quad m = n \pm 1\,, \tag{38}$$

for all $m, n \geq n_0$. But $A \in SL_2(\mathbb{Z})$ preserves this interlacing ratchet relation. This is impossible for $p'_1 \neq p'_2$.

Periodicity (36) and linearity of A immediately imply the periodic extension of (32) from $n \geq n_0 + 1$ to all $n \in \mathbb{Z}$, with positive integer coefficients a_n of even period p'. Obviously $p' = kp$ is an integer multiple of the minimal period p of the a_n. This proves lemma 1. ∎

Lemma 2. *Conversely, consider any p-periodic sequence of nonnegative integers a_n, $n \in \mathbb{Z}$, as in (33). Define z_n by the recursion (32), for all $n \in \mathbb{Z}$, with $z_{-1} := (1, 0)$ and $z_0 := (0, 1)$.*

Then there exists a hyperbolic matrix $A \in SL_2(\mathbb{Z})$ with positive eigenvalues $\mu^\pm = \mu^{\pm 1}$, $\mu > 1$, such that z_n are the corners of the convex hulls of the integer lattice points in the eigensectors Σ_1 and Σ_2 of A, for odd and even $n \in \mathbb{Z}$, respectively.

Proof. Choose any positive integer k such that $p' = kp$ is even, and define $A = (z_{p'-1}, z_{p'})$ to have first column $z_{p'-1}$ and second column $z_{p'}$. This proves the lemma. ∎

For diversion, lemmas 1 and 2 make it easy to determine the *commutator* $C(A)$ of hyperbolic $A \in SL_2(\mathbb{Z})$, i.e. the set of matrices B' commuting with $\pm A$. For a completely different proof see [Fiedler, 2005]. Note that $C(A) = C(-A)$. Both, in the groups $SL_2(\mathbb{Z})$ and $GL_2(\mathbb{Z})$, we claim that the commutator is infinite cyclic (up to $\pm$id) of the form

$$C(A) = \{\pm B^m \ ; \ m \in \mathbb{Z}\}\,. \tag{39}$$

Consider $SL_2(\mathbb{Z})$ first, with A as above. Any nontrivial element $B' \neq \pm\mathrm{id}$ of $C(A)$ in $SL_2(\mathbb{Z})$ preserves each of the eigenspaces $W^\pm$, and is hence hyperbolic itself. Passing to $\pm B'$ or their inverses, if necessary, we may assume B' to possess positive eigenvalues expanding and contracting $W^\pm$ as A does. Thus B', like A, satisfies (34) – albeit with a different integer k. In fact

$$B' z_n = z_{n+k'p} \tag{40}$$

for some even $p' = k'p > 0$. The set of such B' are the powers of the matrix B which arises from the minimal choice of k': for even p we may choose $k' = 1$, whereas $k' = 2$ for odd p. Analogously to lemma 2, the matrix B is defined explicitly by $Bz_{-1} := z_{p'-1}$ and $Bz_0 := z_{p'}$. In the group $GL_2(\mathbb{Z})$ we may always choose $k' = 1$, $p' = p$, instead. Note how $A = B^{k/k'}$ itself is an integer power of the root B in either case, since $A \in C(A)$.

The action of B', and of A, preserves the convex hull boundaries and their sets of corners z_n, in each eigensector. In particular the represented homoclinic orbits of A (and of B') remain in these sets, respectively, and k/k' counts the number of distinct such homoclinic A-orbits. To describe and distinguish all A-homoclinic orbits in a similar spirit, one may inductively remove the boundary corners of the convex hulls and study the boundary corners of the convex hull of the remaining stripped sets of lattice points, ad infinitum, until the onion of homoclinic complexities has been peeled, layer by layer.

5. Reversibility and meander involutions

By definition, the hyperbolic Anosov matrix A on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is called *reversible* if there exists a linear involution $\kappa \in GL_2(\mathbb{Z})$ such that

$$\kappa A \kappa^{-1} = A^{-1}\,; \qquad \kappa^2 = \mathrm{id}\,. \tag{41}$$

We call κ a *reversor* of A. In proposition 1 below we show that the pure and the skew *reflection types*

$$\kappa = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \kappa = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \tag{42}$$

are the only possibilities for (41) to hold, up to conjugacy in $SL_2(\mathbb{Z})$ or $GL_2(\mathbb{Z})$. We therefore fix κ as in (42). In lemma 3 we resume the thread of section 3. We define a κ-invariant fundamental subsector Σ_2^* of the eigensector Σ_2 of A. We then consider the infinite meander permutation σ of $\mathbb{N}$, associated to the homoclinic orbits represented by the well-ordered lattice points $\mathbf{m} \in \Sigma_2^*$. Theorem 2 shows that σ is an involution:

$$\sigma^2 = \mathrm{id}\,. \tag{43}$$

Proposition 1. *Let $A \in SL_2(\mathbb{Z})$ be reversible hyperbolic with reversor $\kappa \in GL_2(\mathbb{Z})$, as in (41). Then κ is $GL_2(\mathbb{Z})$-conjugate to one of the reflections (42).*

Proof. Since $\kappa A \kappa^{-1} = A^{-1}$, the reversor κ must interchange the eigenspaces W^+ and W^- of A. For suitable eigenvectors $w^\pm$, not necessarily normalized as in (8), this means

$$\kappa w^\pm = w^\mp\,. \tag{44}$$

In particular this implies the $\mathbb{Z}_2$-representation

$$\kappa v^\pm = \pm v^\pm \tag{45}$$

for the nonzero linear combinations $v^\pm := w^\pm \pm w^\mp$. Since $\kappa \in GL_2(\mathbb{Z})$, the eigenvector v^- is rational and can be normalized to possess co-prime integer components. An $SL_2(\mathbb{Z})$ basis change renders $v^- = (1, 0)$, without loss of generality. Hence

$$\kappa = \begin{pmatrix} -1 & k \\ 0 & 1 \end{pmatrix} \tag{46}$$

for some integer k. Conjugation by a lower triangular shear in $SL_2(\mathbb{Z})$ reduces k to the cases $k = 0$ or 1. The two cases are nonconjugate in $GL_2(\mathbb{Z})$ because $\frac{1}{2}(\mathrm{id} + \kappa)$ is an integer matrix only in the first case, but not in the second. This proves the proposition. ■

Let κ reverse A, as in (41). Then κ interchanges the eigenspaces $W^\pm$ of A. Hence each of the eigensectors Σ_1, Σ_2 of A contains exactly one of the (half) eigenspaces $\mathrm{Fix}(\pm\kappa)$ of the simple eigenvalues ±1 in its interior. Since $-\kappa$ reverses A if and only κ does, we may assume $\mathrm{Fix}(\kappa)$ to lie in whichever eigensector of A we may prefer. Let us select Σ_2, for definiteness, and consider the two reversors κ of (42). Since both reversors fix the y-component, we have

$$\kappa w^\pm = w^\mp \tag{47}$$

for the eigenvectors $w^\pm = (\vartheta^\pm, 1)$ normalized as in (8) and (28). Moreover $\kappa(\vartheta, 1) = (\kappa[\vartheta], 1)$ with projectivized $\kappa[\vartheta] = -\vartheta + k$, analogously to (9), and for $k = 0$ or 1. In particular $\mathrm{Fix}(\kappa)$ is represented by the fixed points $\vartheta = 0$, for the pure reflection $k = 0$, and $\vartheta = 1/2$, for the skew reflection $k = 1$. Note $\vartheta_+ + \vartheta_- = k$, in either case.

Lemma 3. *In the above setting, the eigensector Σ_2 of A contains a unique open half-line L through the origin such that*

$$\kappa L = AL\,. \tag{48}$$

In particular the sector Σ^* between the half-lines L and AL, including the boundary L but not AL, is a uniquely defined fundamental subsector of A such that the interior $\Sigma^o = \Sigma^* \setminus L$ is κ-invariant.

Proof. By reversibility (41) the product $\kappa' := \kappa A$ is also an involution in $GL_2(\mathbb{Z})$ of determinant -1. In particular κ' possesses simple eigenvalues ±1. The eigenspace $\mathrm{Fix}(\kappa')$ intersects the open sector Σ_2 because $\kappa' W^\pm = \kappa A W^\pm = W^\mp$. Let L denote the half-line of $\mathrm{Fix}(\kappa')$ in Σ_2, and the lemma is proved. ■

By construction, we note that the unbounded half lines L and AL have rational slopes and hence seem to correspond to closed geodesics on T^2. Different lattice points $\mathbf{m}$ on L, however, represent different homoclinic orbits in view of (13). To avoid duplicate representations of the same homoclinic orbit in the fundamental sector Σ^* we then have to ignore all lattice points on the opposite boundary $\kappa L = AL$. For this reason we have defined Σ^* to be the half-open sector between L and AL, including L but excluding AL.

Theorem 2. *Let $A \in SL_2(\mathbb{Z})$ be reversible hyperbolic with positive eigenvalues $\mu^\pm = \mu^{\pm1}$, $\mu > 1$, and reversor $\kappa \in GL_2(\mathbb{Z})$ as in (41). Let Σ be an eigensector of A and $\Sigma^* \subseteq \Sigma$ the fundamental subsector with boundaries L and $AL = \kappa L$ constructed in lemma 3. In particular we choose the sign $\pm\kappa$ such that the reflection axis Fix(κ) intersects Σ and Σ^*. We define the infinite meander permutation σ of the homoclinic orbits via their well-ordered representatives $\mathbf{m}$ in the fundamental lattice $\Sigma^* \cap \mathbb{Z}^2$; see (13)–(15) and (24).*

Then σ is an involution:

$$\sigma^2 = \mathrm{id}\,. \tag{49}$$

Moreover there exists a constant $C > 1$ such that any 2-cycle $(i\ j)$ of σ satisfies

$$C^{-1}\, i \le j \le C\, i\,. \tag{50}$$

Proof. We represent homoclinic orbits $\pm\eta^\pm_{\mathbf{m}} w^\pm \in W^\pm$ by points $\mathbf{m}$ in the fundamental lattice $\Sigma^* \cap \mathbb{Z}^2$, via

$$\eta^-_{\mathbf{m}} w^- + \eta^+_{\mathbf{m}} w^+ = \mathbf{m}\,; \tag{51}$$

see (13). We decompose the fundamental subsector $\Sigma^* = L \cup \Sigma^o$ into the interior Σ^o and the boundary line L. We will treat the marginal case of $\mathbf{m} \in L \cap \mathbb{Z}^2$ at the end of the proof and we outline the main case $\mathbf{m} \in \Sigma^o \cap \mathbb{Z}^2$ first.

As in (24) we define the infinite meander permutation σ via the coordinates $\eta^\pm_{\mathbf{m}} > 0$ in the well-ordered sets

$$H^\pm := \{\eta^\pm_{\mathbf{m}}\,; \quad \mathbf{m} \in \Sigma^o \cap \mathbb{Z}^2\}\,. \tag{52}$$

We scale the eigenvector w^+, and hence all $\eta^+_{\mathbf{m}}$, such that $w^+ = \kappa w^-$ or, equivalently,

$$w^{\pm} = \kappa w^{\mp} \,. \tag{53}$$

As a consequence we show below that

$$H^+ = H^- \,, \qquad \text{and} \tag{54}$$

$$\eta^{\pm}_{\mathbf{m}} = \eta^{\mp}_{\kappa\mathbf{m}} \tag{55}$$

for all $\mathbf{m} \in \mathbb{Z}^2$. Since $H^{\pm}$ is a discrete subset of $[0,\infty)$ we may identify $H^{\pm}$ with the well-ordered set $\mathbb{N}$ and represent σ by

$$\sigma(\eta^-_{\mathbf{m}}) := \eta^+_{\mathbf{m}} = \eta^-_{\kappa\mathbf{m}} \,, \tag{56}$$

for $\mathbf{m} \in \Sigma^o \cap \mathbb{Z}^2$. Therefore $\kappa^2 = \text{id}$ implies $\sigma^2 = \text{id}$, as claimed in (49), provided we show (54), (55). The boundedness claim (50) then follows from the more general observation (25).

To show (54), (55) we first observe that κ-invariance of the interior Σ^o of the fundamental sector Σ^* and (55) imply (54). To show (55) we simply apply κ to (51) for any $\mathbf{m}$, $\kappa\mathbf{m} \in \mathbb{Z}^2$. By (53) we immediately obtain

$$\eta^-_{\mathbf{m}} w^+ + \eta^+_{\mathbf{m}} w^- = \kappa\mathbf{m} = \eta^-_{\kappa\mathbf{m}} w^- + \eta^+_{\kappa\mathbf{m}} w^+ \,. \tag{57}$$

This proves claim (55), by linear independence of the eigenvectors $w^{\pm}$.

It only remains to address the marginal case $\mathbf{m} \in L \cap \mathbb{Z}^2$ of lattice points on the boundary of the fundamental lattice domain Σ^*. Since these $\mathbf{m}$ are mapped to the other boundary $\kappa L = AL$ by $\kappa\mathbf{m} = A\mathbf{m}$, they are mapped to the same orbit, under the involution κ, and are therefore fixed points of σ. Note that the only other fixed points of σ arise by the κ-fixed lattice points $\mathbf{m}$ on the diagonal $\text{Fix}(\kappa)$ of the fundamental sector Σ^*. This proves the theorem. ■

6. Continued fractions and reversibility

In this section we develop the analogue to the continued fractions approach of section 4 for hyperbolic $A \in SL_2(\mathbb{Z})$ which are reversible; see proposition 2. Again we fix the projective representative A of $\pm A \in PSL_2(\mathbb{Z})$ to have positive eigenvalues $\mu^{\pm} = \mu^{\pm 1}$, $\mu > 1$, so that A leaves each of its eigensectors invariant. We normalize eigenvectors $w^{\pm} = (\vartheta^{\pm}, 1)$ to have slopes $1/\vartheta^{\pm}$ and arrive at the recursions (32)–(34) of the periodic part a_n, $n \in \mathbb{Z}$, of the continued fraction expansion $1/\vartheta^+ = [\tilde{a}_0; \tilde{a}_1, \ldots]$ and of the corners z_n of the convex hulls of the lattice points in the eigensectors Σ_1 and Σ_2.

We fix the reversor κ in the projective class $\pm\kappa \in PGL_2(\mathbb{Z})$ to leave the eigensector Σ_2 invariant and to map Σ_1 to $-\Sigma_1$. Up to conjugacy in $SL_2(\mathbb{Z})$ or $GL_2(\mathbb{Z})$ we may thus consider κ to be one of the two reflection cases of (42). In particular

$$\begin{aligned} \kappa w^{\pm} &= w^{\mp} \\ \kappa\left[\vartheta^{\pm}\right] &= \vartheta^{\mp} \end{aligned} \tag{58}$$

and $\vartheta^+ + \vartheta^- = k$ with $k = 0$ for pure reflections and $k = 1$ for skew reflections.

Proposition 2. *Let $A \in SL_2(\mathbb{Z})$ be hyperbolic reversible with positive eigenvalues $\mu^{\pm} = \mu^{\pm 1}$, $\mu > 1$, as above, and assume the reversor κ fixes the eigensector Σ_2 of A.*

Then there exists an even integer s such that

$$\kappa z_n = (-1)^n z_{s-n} \tag{59}$$

holds for the corners z_n of the convex hulls of the lattice points in the eigensectors Σ_1, Σ_2 of A. In particular the coefficients a_n of the 2-term recursion (32) of z_n satisfy

$$a_{s-n} = a_n \tag{60}$$

for all $n \in \mathbb{Z}$. Specifically $a_{s/2}$ is even/odd, respectively, in the pure and skew reflection cases of (42).

Conversely we may start from a given coefficient sequence of positive integers a_n, $n \in \mathbb{Z}$, of minimal period $p > 0$ and with reflection symmetry (60), for some even s and even $p' = k'p$. As in lemma 2, (36) define $A \in SL_2(\mathbb{Z})$ via recursion (32) with $z_{-1} := (1,0)$ and $z_0 := (0,1)$. Define $\kappa \in GL_2(\mathbb{Z})$ such that (59) holds for two adjacent, and hence for all, $n \in \mathbb{Z}$.

Then $A \in SL_2(\mathbb{Z})$ is reversible with reversor κ. Again the parity of $a_{s/2}$ decides on the reflection class (42) of κ.

Proof. We show (59), (60) first and settle the converse afterwards.

Since the reversor $\kappa \in GL_2(\mathbb{Z})$ leaves the sector lattice $\Sigma_2 \cap \mathbb{Z}^2$ invariant, κ acts on its convex hull, the boundary, and the corners z_n, for even n. Since $\kappa w^{\pm} = w^{\mp}$ interchanges the $W^{\pm}$ boundaries of Σ_2, and preserves adjacency of z_n on the convex hull boundary, there exists an even integer s such that

$$\kappa z_n = z_{s-n} \tag{61}$$

for all even $n \in \mathbb{Z}$. Similarly there exists an even integer s' such that

$$\kappa z_n = -z_{s'-n} \tag{62}$$

for all odd $n \in \mathbb{Z}_-$ because κ interchanges $\Sigma_1 \cap \mathbb{Z}^2$ with $-\Sigma_1 \cap \mathbb{Z}^2$ for the other eigensector Σ_1. Since

$$|\det(z_{n+m}, z_n)| = 1 \quad \Longleftrightarrow \quad |m| = 1\,, \tag{63}$$

we conclude $s' = s$. This proves (59).

To prove (60) we apply κ to the recursion (26) of z_n, for $n \in \mathbb{Z}$, and use that n and $n \pm 1$ have opposite even odd parity. Therefore (59) implies

$$a_n z_{s-n} = -(z_{s-(n+1)} - z_{s-(n-1)})\,. \tag{64}$$

On the other hand (26) for $s-n$ instead of n reads

$$a_{s-n} z_{s-n} = z_{s-n+1} - z_{s-n-1}\,. \tag{65}$$

Therefore $a_n = a_{s-n}$ for all $n \in \mathbb{Z}$ and (60) is proved.

To distinguish the reflection class of κ via $a_{s/2}$ we abbreviate the reversors $\kappa' := (-1)^{s/2}\kappa$ and define $v := z_{s/2-1} + z_{s/2+1}$. Then (59) readily implies

$$\begin{aligned} \kappa' z_{s/2} &= z_{s/2}\,; \\ \kappa' v &= -v\,. \end{aligned} \tag{66}$$

In particular $z_{s/2} \in \mathbb{Z}^2$ is the axis of the reflection κ', and v is the reflection direction. To determine the $GL_2(\mathbb{Z})$ reflection type of κ' we write κ' in terms of the basis $(z_{s/2}, z_{s/2-1}) \in GL_2(\mathbb{Z})$. Since (65) for $n = s/2$ implies $z_{s/2+1} = z_{s/2-1} + a_{s/2} z_{s/2}$, we have

$$v = 2z_{s/2-1} + a_{s/2} z_{s/2} \tag{67}$$

in these coordinates. Hence (66) yields

$$\kappa' = \begin{pmatrix} 1 & -a_{s/2} \\ 0 & -1 \end{pmatrix}\,. \tag{68}$$

Since conjugation by upper triangular shears in $SL_2(\mathbb{Z})$ preserve the diagonal and the even/odd parity of the upper right entry $-a_{s/2}$ this determines the $GL_2(\mathbb{Z})$ conjugacy type of the reflection κ', and of $\kappa = (-1)^{s/2}\kappa'$, as claimed.

To prove the converse part of the proposition we just invoke lemma 2 for A. We define the reversor $\kappa = (-1)^{s/2}\kappa'$ via the fix vector $z_{s/2} \in \mathbb{Z}^2$ and the reflection vector $v := z_{s/2-1} + z_{s/2+1}$. In coordinates of $z_{s/2}$ and $z_{s/2-1}$ we have already determined $\kappa' \in GL_2(\mathbb{Z})$ in (68). By linearity of the recursion (32), and by the reversal property (60) of the coefficients a_n, we inductively obtain (59) for the reversor $\kappa := (-1)^{s/2}\kappa'$, extending from $z_{s/2}, z_{s/2-1}$ to $z_{s/2+1}, z_{s/2-2}, z_{s/2+2}$, and so on. We omit the details for brevity. Comparing (59) with $Az_n = z_{n+p'}$, for all $n \in \mathbb{Z}$, we see that $\kappa A \kappa = A^{-1}$. Indeed

$$\begin{aligned} \kappa A \kappa z_n &= (-1)^n \kappa A z_{s-n} = (-1)^n \kappa z_{p'+s-n} = \\ &= (-1)^{n+p'+s-n} z_{s-(p'+s-n)} = z_{n-p'} = A^{-1} z_n \end{aligned} \tag{69}$$

because s and $p' = k'p$ are both even. This proves the proposition. ■

We have seen how the reflection symmetry of the continued fraction coefficients a_n for the eigensectors Σ of hyperbolic matrices $A \in SL_2(\mathbb{Z})$ is equivalent to the construction of reversors κ by (59), for even s. If the coefficient reflection s in (60) can be chosen odd, then κ defined by (59) rotates through the four eigensectors $\pm\Sigma_1$, $\pm\Sigma_2$ of A. This provides the *projective reversors* $\pm\kappa \in PSL_2(\mathbb{Z})$ of $\pm A \in PSL_2(\mathbb{Z})$ which satisfy

$$\kappa A \kappa^{-1} = -A^{-1}\,; \qquad \kappa^2 = \mathrm{id}\,; \tag{70}$$

instead of (41).

Examples of both reversibilities (41) and (70) arise simultaneously when any one of them does, if and only if the minimal period p of the sequence a_n is odd. Indeed suppose $s \in \mathbb{Z}$ provides the reflection (70). Then $s' \in \mathbb{Z}$ also provides a reflection if and only if $s - s'$ is an integer multiple of the minimal period p of the sequence a_n.

Consider even minimal periods p of a_n, $n \in \mathbb{Z}$, next. Then all reflectors s are of the same parity, either even or odd. Let s be odd. In $GL_2(\mathbb{Z})$ coordinates $z_{(s\pm1)/2}$ the rotating reversor $\kappa \in SL_2(\mathbb{Z})$ then interchanges these directions and reads

$$\kappa = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{71}$$

In particular all reversors are projective and satisfy (70). For even s, in contrast, all reversors $\kappa = \pm\kappa'$ properly satisfy (41) and take the form (68). Still, s and $s + p$ are both reflectors. Suppose the continued fraction coefficients $a_{s/2}$ and $a_{(s+p)/2}$ are of opposite even/odd parity. Then the associated reversors κ_0 and κ_1 are not $GL_2(\mathbb{Z})$-conjugate and represent both reflection types of (42). Note however that the product $\kappa_0\kappa_1$ of any two reversors, be they both proper or both projective, always belongs to the commutator $C(A)$ of $GL_2(\mathbb{Z})$-matrices commuting with A; see (39).

In conclusion we see how the bi-infinite sequences $(a_n)_{n\in\mathbb{Z}}$ of positive, periodic, and integer continued fraction coefficients encode the conjugacy classes and reversibilities of hyperbolic unimodular Anosov matrices A in $SL_2(\mathbb{Z})$, $GL_2(\mathbb{Z})$, and their projective variants. Theorem 2 has identified the meander permutations σ to be involutions, for the reversible case, but we have failed to provide a detailed bridge between σ and the sequence of a_n as yet.

7. Quadratic forms

Based on helpful discussions and suggestions by Tobias Finis, we briefly recall the relation of conjugacy classes of hyperbolic Anosov matrices $A \in SL_2(\mathbb{Z})$ to the much older study of equivalence classes of binary quadratic forms

$$Q\,[x, y] := ax^2 + bxy + cy^2\,. \tag{72}$$

The material in this section is not new; see for example [Gauss, 1801; Landau, 1958; Sarnak, 1982] for some background material.

In (10)–(12) we have already derived the indefinite quadratic form Q from hyperbolic unimodular A with integer coefficients $a = \gamma$, $b = \delta - \alpha$, $c = -\beta$ and discriminant $d = b^2 - 4ac = \tau^2 - 4$, for $\tau = \mathrm{trace}\, A = \alpha + \delta$. We call quadratic forms Q *equivalent* if they differ by just a positive prefactor g, or transform into each other under an $SL_2(\mathbb{Z})$ coordinate change of (x, y) which leaves the integer lattice $(x, y) \in \mathbb{Z}^2$ invariant.

An elementary equivalence arises for B in the commutator $C(A)$ of A; see (39). The quadratic form $Q \circ B$ possesses the same discriminant as Q, which just gets multiplied by $(\det B)^2 = 1$. Since $B \in C(A)$ commutes with A, the zeros of Q, alias the eigenspaces $W^{\pm}$ of A, are also preserved. Therefore $Q \circ B = \pm Q$. By the action of B on the eigensectors of A we immediately obtain

$$Q \circ B = \det B \cdot Q\,. \tag{73}$$

In particular $A \in C(A)$ preserves the hyperbolic level sets of Q, and so does B^k for any $k \in \mathbb{Z}$ such that kp is even. A similar argument for the reversors κ in (41) of $\det \kappa = -1$, which interchange $W^{\pm}$ but also preserve the eigensectors up to sign, shows Q-invariance

$$Q \circ \kappa = Q\,. \tag{74}$$

For the continued fraction corners z_n of the eigensectors, (73) and (40) imply

$$Q\left[z_{n+p}\right] = (-1)^p Q\left[z_n\right]\,, \tag{75}$$

for all $n \in \mathbb{Z}$, in the setting of lemma 1. In the reversible case of proposition 2, analogously, (59) and (74) imply

$$Q\left[z_{s-n}\right] = Q\left[z_n\right]\,. \tag{76}$$

In particular the 2-cycles of the involutive meander permutation σ of theorem 2 each occur for homoclinic orbits in the same level set of Q. Similarly to the onion peeling at the end of section 4, we may thus decompose the homoclinic complexities of the A-homoclinic orbits via the level sets of Q, layer by layer.

To eliminate prefactors of Q, let $g = \gcd(a,b,c)$ denote the greatest common divisor of the coefficients of Q. Dividing Q by $g > 0$ we obtain an indefinite *primitive quadratic form* Q', i.e. with co-prime coefficients $(a',b',c') = g^{-1}(a,b,c)$ of discriminant $D = g^{-2}d > 0$. Conversely, an indefinite primitive quadratic form Q with given discriminant $D = b'^2 - 4a'c' > 0$ arises from some hyperbolic $A \in SL_2(\mathbb{Z})$ if, and only if

$$\tau^2 - g^2 D = 4 \tag{77}$$

possesses an integer solution τ, g. Indeed we may then define

$$\begin{aligned} \alpha &= \tfrac{1}{2}(\tau - b)\\ \beta &= -c\\ \gamma &= a\\ \delta &= \tfrac{1}{2}(\tau + b) \end{aligned} \tag{78}$$

because τ, D, and hence b, possess the same parity. In that sense, equivalence classes of primitive quadratic forms Q with given discriminant $D > 0$ give rise to $SL_2(\mathbb{Z})$ conjugacy classes of hyperbolic Anosov maps $A \in SL_2(\mathbb{Z})$ with discriminant $d = g^2 D$ and trace τ. For each integer solution (τ, g) of the Pell equation (77) with $g > 0$, one such class of A arises. Note how the choices $-\tau$ and $-Q$ give rise to the projectively equivalent matrix $-A$, for the same discriminant D.

As a corollary the periodicity of the continued fraction coefficients (33) observed in lemma 1 implies the celebrated Lagrange theorem: ϑ is a quadratic irrational if, and only if, the continued fraction expansion of ϑ (or, equivalently, of $1/\vartheta$) is asymptotically periodic. Indeed we have observed how $Q[\vartheta, 1] = 0$ if and only if $(\vartheta, 1)$ is an eigenvector of the hyperbolic matrix A, for any solution (τ, g) of Pell's equation (77). Even the existence proof for nontrivial solutions of Pell's equation, however, usually makes explicit use of Lagrange's theorem. See also the direct, but related, treatment in [Landau, 1958].

An elementary well-known example are the continued fraction expansions of $\sqrt{a} = [\tilde{a}_0; \tilde{a}_1, \ldots, \tilde{a}_p]$ for nonsquare positive integers a. Reflection symmetry κ of $Q[x,y] = ax^2 - y^2$ immediately proves $\tilde{a}_p = 2\tilde{a}_0$ and $\tilde{a}_k = \tilde{a}_{p-k}$ for $k = 1; \ldots, p-1$.

The *class numbers* $h(D)$, an object of significant study ever since they were introduced by Gauss in [Gauss, 1801], count the number of equivalence classes of (primitive) quadratic forms Q with given discriminant D. In our approach we have studied the closely related conjugacy classes of hyperbolic Anosov matrices $A \in SL_2(\mathbb{Z})$ via the p-periodic continued fraction sequences (27) of positive integers a_n, instead. Conjugacy amounts to the choices of initial vectors (z_{-1}, z_0) with determinant 1 in the lattice $\mathbb{Z}^2$, for the recursion

(26). The matrix A itself maps (z_{-1}, z_0) to $(z_{p'-1}, z_{p'})$, for some even p' which is a multiple of the minimal period p. Cyclicity of the group of automorphs of Q, for example, amounts to cyclicity of the centralizer $C(A)$ of A, which is immediate from the continued fraction sequence point of view; see (39) and also [Landau, 1958; Sarnak, 1982]. Of course we lost control of the discriminants d and D in this process, due to the focus of our approach on meander permutations. Nevertheless we continue to be intrigued by the broad aspects arising from questions of homoclinicity, as brought to us by Leonid Pavlovich Shilnikov in his life long quest.

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