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# THE CONSISTENCY CONDITIONS AND THE SMOOTHNESS OF GENERALIZED SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS

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**Abstract.** We study smoothness of generalized solutions of nonlocal elliptic problems in plane bounded domains with piecewise smooth boundary. The case where the support of nonlocal terms can intersect the boundary is considered. We find conditions that are necessary and sufficient for any generalized solution to possess an appropriate smoothness (in terms of Sobolev spaces). Both homogeneous and nonhomogeneous nonlocal boundary-value conditions are studied.

## 1. Introduction

Nonlocal elliptic problems arise in various areas such as plasma theory [28], biophysics, theory of diffusion processes [10, 43, 29, 41, 11], control theory [4, 1], and so on.

In the one-dimensional case, nonlocal problems were studied since the beginning of the 20th century by Sommerfeld [39], Picone [26], Tamarkin [42], etc. In the two-dimensional case, one of the first works was due to Carleman [7], who treated the problem of finding a harmonic function, in a plane bounded domain, satisfying a nonlocal condition which connects the values of the unknown function at different points of the boundary. Further investigation of elliptic problems with transformations mapping a boundary onto itself has been carried out by Vishik [44], Browder [6], Beals [3], Antonevich [2], and others.

In 1969 Bitsadze and Samarskii [5] considered the following nonlocal problem arising in plasma theory: to find a function $u(y_1, y_2)$ harmonic on the rectangular $G = \{y \in \mathbb{R}^2 : -1 < y_1 < 1,\ 0 < y_2 < 1\}$, continuous on $\overline{G}$, and

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satisfying the relations

$$u(y_1,0)=f_1(y_1),\quad u(y_1,1)=f_2(y_1),\quad -1<y_1<1,$$
$$u(-1,y_2)=f_3(y_2),\quad u(1,y_2)=u(0,y_2),\quad 0<y_2<1,$$

where $f_1, f_2, f_3$ are given continuous functions. This problem was solved in [5] by reducing it to a Fredholm integral equation and by using the maximum principle. For arbitrary domains and for general nonlocal conditions, such a problem was formulated as an unsolved one (see also [22]). Different generalizations of nonlocal problems with transformations mapping the boundary inside the closure of a domain were studied by many authors [9, 27, 19, 18].

The most complete theory for elliptic equations of order $2m$ with general nonlocal conditions was developed by Skubachevskii and his pupils [32, 33, 34, 35, 36, 21, 37, 15, 16]: a classification with respect to types of nonlocal conditions was suggested, the Fredholm solvability in the corresponding spaces was investigated, and asymptotics of solutions near special conjugation points was obtained. One can find other relevant references and descriptions of applications in [37].

In the present paper, we consider a little-studied question concerning the smoothness of solutions for nonlocal elliptic problems. For simplicity, we study nonlocal perturbations of the Dirichlet problem for elliptic second-order equations. However, the approach we are developing is also applicable to elliptic equations of order $2m$ with general nonlocal conditions.

It appears that the most difficult situation is that where the support of nonlocal terms can intersect the boundary of a domain [33, 38]. In this case, solutions of nonlocal problems can have power-law singularities near some points of the boundary even if the right-hand side is infinitely differentiable and the boundary is infinitely smooth. It follows from our results that solutions of nonlocal problems can have power-law singularities even if the support of nonlocal terms lies strictly inside a domain. For this reason, we use special weighted spaces to study nonlocal problems. These spaces were originally proposed by Kondrat'ev [20] to study elliptic boundary-value problems in nonsmooth domains.

Note that smoothness of solutions for "local" elliptic problems in nonsmooth domains is studied rather thoroughly (see [20, 25, 30, 8] and others); here the principal difficulties are related to the presence of special singular points on the boundary of a domain. In the theory of nonlocal problems, there appear principally different difficulties: violation of smoothness of solutions is connected not only with the fact that the boundary may be nonsmooth but also with the presence of nonlocal terms in the boundary-value conditions.

Consider the following example. Let $\partial G = \Gamma_1 \cup \Gamma_2 \cup \{g, h\}$, where the $\Gamma_i$ are open (in the topology of $\partial G$) $C^\infty$-curves; $g, h$ are the end points of the curves $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$. Suppose that the domain $G$ is the plane angle of opening $\pi$ in some neighborhood of each of the points $g$ and $h$. We deliberately take a smooth domain in this example to illustrate how the nonlocal terms can affect the smoothness of solutions. Consider the following nonlocal problem in the domain $G$:

$$\Delta u = f_0(y) \quad (y \in G), \tag{1.1}$$

$$\begin{aligned} &u|_{\Gamma_1} + b_1(y)u\big(\Omega_1(y)\big)\big|_{\Gamma_1} + a(y)u\big(\Omega(y)\big)\big|_{\Gamma_1} = f_1(y) \quad (y \in \Gamma_1), \\ &u|_{\Gamma_2} + b_2(y)u\big(\Omega_2(y)\big)\big|_{\Gamma_2} = f_2(y) \quad (y \in \Gamma_2). \end{aligned} \tag{1.2}$$

Here $b_1$, $b_2$, and $a$ are real-valued $C^\infty$-functions; $\Omega_i$ ($\Omega$) are $C^\infty$-diffeomorphisms taking some neighborhood $\mathcal{O}_i$ ($\mathcal{O}_1$) of the curve $\Gamma_i$ ($\Gamma_1$) onto the set $\Omega_i(\mathcal{O}_i)$ ($\Omega(\mathcal{O}_1)$) in such a way that $\Omega_i(\Gamma_i) \subset G$, $\Omega_i(g) = g$, $\Omega_i(h) = h$, and the transformation $\Omega_i$, near the points $g, h$, is the rotation of the boundary $\Gamma_i$ through the angle $\pi/2$ inwards to the domain $G$ (respectively, $\Omega(\Gamma_1) \subset G$), $\overline{\Omega(\Gamma_1)} \cap \{g, h\} = \varnothing$, and the approach of the curve $\Omega(\overline{\Gamma_1})$ to the boundary $\partial G$ can be arbitrary, cf. [33, 35]), see Figure 1.1.

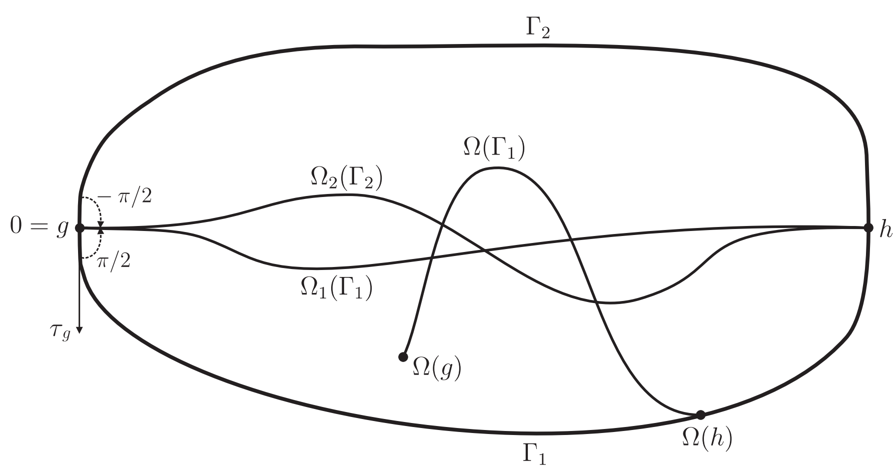


FIGURE 1.1. Domain $G$ with boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \{g, h\}$.

We say that $g$ and $h$ are the *points of conjugation of nonlocal conditions* because they divide the curves on which different nonlocal conditions are set. The closure of the set

$$\bigcup_{i=1,2}\{y\in\Omega_i(\Gamma_i):\ b_i(\Omega_i^{-1}(y))\neq 0\}\cup\{y\in\Omega(\Gamma_1):\ a(\Omega^{-1}(y))\neq 0\}$$

is referred to as the *support of nonlocal terms*. It is clear that, if $b_1(y)=a(y)=0$ for $y\in\Gamma_1$ and $b_2(y)=0$ for $y\in\Gamma_2$, then the support of nonlocal terms is the empty set. If, say, $b_1(y), a(y)\neq 0$ for $y\in\Gamma_1$ and $b_2(y)\neq 0$ for $y\in\Gamma_2$, then the support of nonlocal terms is the set $\overline{\Omega_1(\Gamma_1)}\cup\overline{\Omega_2(\Gamma_2)}\cup\overline{\Omega(\Gamma_1)}$.

Denote by $W^k(G)=W_2^k(G)$ the Sobolev space. We say that a function $u\in W^1(G)$ is a *generalized solution* of problem (1.1), (1.2) with right-hand side $f_0\in L_2(G)$, $f_i\in W^{1/2}(\Gamma_i)$ if $u$ satisfies Equation (1.1) in the sense of distributions and nonlocal conditions (1.2) in the sense of traces. Using the notation of problem (1.1), (1.2), we can formulate the main questions of our paper.

(1) Find a condition on the right-hand sides $f_0\in L_2(G)$, $f_i\in W^{3/2}(\Gamma_i)$ and on the coefficients $b_1$, $b_2$, and $a$ which is necessary and sufficient for *any* generalized solution of problem (1.1), (1.2) to belong to the space $W^2(G)$.
(2) The same question for homogeneous nonlocal conditions, $\{f_i\}=0$.

It is relatively easy to prove that any generalized solution of problem (1.1), (1.2) belongs to the space $W^2$ outside an arbitrarily small neighborhood of the points $g$ and $h$ (see Section 3). Clearly, the behavior of solutions near the points $g$ and $h$ is affected by the behavior of the coefficients $b_1$, $b_2$, and $a$ near these points. However, the influence of the coefficients $b_i$ is principally different from the influence of the coefficient $a$. This phenomenon is explained by the fact that the coefficients $b_i$ (for $y$ being in a small neighborhood of the points $g$ and $h$) correspond to nonlocal terms supported *near* the set $\{g,h\}$ (in the general case, such terms correspond to operators $\mathbf{B}_i^1$), whereas the coefficient $a$ corresponds to a nonlocal term supported *outside* some neighborhood of the set $\{g,h\}$ (in the general case, such terms correspond to abstract operators $\mathbf{B}_i^2$). What we give below is a scheme for the investigation of smoothness of generalized solutions near the point $g$ (this scheme is realized in Sections 2–6 for the general case and in Section 7 for the particular case of problem (1.1), (1.2)).

**Step 1.** We construct a model nonlocal problem, with a parameter, for ordinary differential equation corresponding to the point $g$. The structure of nonlocal conditions in the model problem depends only on the values of the coefficients $b_i(g)$, $i = 1, 2$ (Section 2).

**Step 2.** We consider the values $b_i(g)$ for which the band $-1 \le \operatorname{Im} \lambda < 0$ contains no eigenvalues of the model problem. In this case, any generalized solution belongs to $W^2$ near the point $g$. Note that, in this case, we impose no additional restrictions on the right-hand side or the coefficients $b_1$, $b_2$, and $a$ (Section 3 and Theorem 7.1).

**Step 3.** We consider the values $b_i(g)$ for which the band $-1 \le \operatorname{Im} \lambda < 0$ contains only the proper eigenvalue $\lambda = -i$ of the model problem (see Definition 4.1). This is the most complicated situation, which we call a "border case." In this case, any generalized solution belongs to $W^2$ near the point $g$ if and only if the coefficients $b_1$, $b_2$, and $a$ satisfy a certain consistency condition near the point $g$. The type of the consistency condition depends on whether we consider homogeneous or nonhomogeneous nonlocal conditions. In the latter case, the consistency conditions must also be imposed on the right-hand side $\{f_i\}$ (Section 4 and Theorems 7.2, 7.4, and Corollary 7.1).

**Step 4.** We consider the values $b_i(g)$ for which the band $-1 \le \operatorname{Im} \lambda < 0$ contains an improper eigenvalue of the model problem (see Definition 4.1). In this case, for any coefficient $a$, one can find right-hand sides $f_0 \in L_2(G)$, $\{f_i\} = 0$ ($f_0$ depends on the behavior of the coefficients $b_i$ near the point $g$ and does not depend on the coefficient $a$) and construct the corresponding generalized solution $u \in W^1(G)$ such that $u$ does not belong to $W^2$ near the point $g$ (Section 5 and Theorem 7.3).

It turns out that the smoothness of generalized solutions is preserved if $b_1(g) + b_2(g) \le -2$ or $b_1(g) + b_2(g) > 0$ and can be violated if $-2 < b_1(g) + b_2(g) < 0$. If $b_1(g) + b_2(g) = 0$, we have the border case. The necessary condition for the smoothness being preserved is the validity of a consistency condition imposed on the right-hand side $\{f_i\}$ (see (7.8)). Let us show that the *presence of variable coefficients in nonlocal conditions* may affect the smoothness of generalized solutions. For simplicity, we assume that $a(y) \equiv 0$. Let condition (7.10) hold; in particular, let $b_i(y)$ be constant near the point $g$. Then the smoothness of generalized solutions is preserved near the point $g$ whenever the right-hand side $\{f_i\}$ satisfies the consistency condition (7.8). However, if condition (7.10) fails (e.g., if $b_1(y) \equiv \beta_1 y_2$, $b_2(y) \equiv \beta_2 y_2$, $\beta_1 \ne \beta_2$, near the point $g = 0$, the axis $Oy_2$ being tangent to $\partial G$ at $g = 0$), then the smoothness of generalized solutions can be violated even if the right-hand side $\{f_i\}$ satisfies the consistency condition (7.8). This follows from Theorem 7.2.

Now we illustrate another phenomenon arising in the border case. Assume that $b_1(y) \equiv b_2(y) \equiv 0$. Let $a(y) = 0$ in some neighborhood of the point $h$ and $\Omega(g) \in G$. Then the *support of nonlocal terms lies strictly inside the domain* $G$. However, if $a(g) \neq 0$ or $(\partial a/\partial \tau_g)|_{y=g} \neq 0$, where $\tau_g$ denotes the unit vector tangent to $\partial G$ at the point $g$, then the smoothness of generalized solutions of problem (1.1), (1.2) (even with homogeneous nonlocal conditions, $\{f_i\} = 0$) can be violated. This follows from Corollary 4.3 (see also Section 7.2).

Note that the smoothness of generalized solutions for some particular nonlocal elliptic problems was earlier studied by Skubachevskii [33, 38]. In these papers, a nonlocal perturbation of the Dirichlet problem for the Laplace operator is treated; a condition which is necessary and sufficient for any generalized solution of a problem with homogeneous nonlocal conditions to belong to the space $W^2(G)$ has been found. However, it was fundamental that the "local" Dirichlet conditions are set on a part of the boundary and the coefficients of nonlocal terms are constant.

In this paper, we suggest an approach for the study of smoothness, based on the results concerning the solvability of model nonlocal problems in plane angles in Sobolev spaces [15] and on the asymptotic behavior of solutions of such problems in weighted spaces [33, 13]. Our approach enables one to investigate the smoothness of generalized solutions when different nonlocal conditions are set on different parts of the boundary, coefficients of nonlocal terms supported near the conjugation points are variable, and nonlocal operators corresponding to nonlocal terms supported outside the conjugation points are abstract. Moreover, nonlocal boundary-value conditions can be both homogeneous and nonhomogeneous.

## 2. Setting of Nonlocal Problems in Bounded Domains

2.1. **Setting of the Problem.** Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial G$. Consider a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^{N} \Gamma_i$, where $\Gamma_i$ are open (in the topology of $\partial G$) $C^\infty$-curves. Assume that the domain $G$ is a plane angle in some neighborhood of each point $g \in \mathcal{K}$.

For any set $X$ in $\mathbb{R}^2$ having a nonempty interior, we denote by $C_0^\infty(X)$ the set of functions infinitely differentiable on $\overline{X}$ and compactly supported on $X$.

For an integer $k \geq 0$, denote by $W^k(G) = W_2^k(G)$ the Sobolev space with the norm

$$\|u\|_{W^k(G)} = \Big( \sum_{|\alpha| \leq k} \int_G |D^\alpha u(y)|^2 \, dy \Big)^{1/2}$$

(set $W^0(G) = L_2(G)$ for $k = 0$). For an integer $k \geq 1$, we introduce the space $W^{k-1/2}(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{G}$ with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|u\|_{W^k(G)} \quad (u \in W^k(G): \ u|_\Gamma = \psi). \tag{2.1}$$

Along with Sobolev spaces, we will use weighted spaces (the Kondrar'ev spaces). Let us introduce these spaces. Let $Q = \{y \in \mathbb{R}^2 : \ r > 0, \ |\omega| < \omega_0\}$, $Q = \{y \in \mathbb{R}^2 : \ 0 < r < d, \ |\omega| < \omega_0\}$, $0 < \omega_0 < \pi$, $d > 0$, or $Q = G$. We denote by $\mathcal{M}$ the set $\{0\}$ in the first and second cases and the set $\mathcal{K}$ in the third case. Introduce the space $H_a^k(Q) = H_a^k(Q, \mathcal{M})$ as a completion of the set $C_0^\infty(\overline{Q} \setminus \mathcal{M})$ with respect to the norm

$$\|u\|_{H_a^k(Q)} = \Big( \sum_{|\alpha| \leq k} \int_Q \rho^{2(a-k+|\alpha|)} |D^\alpha u(y)|^2 dy \Big)^{1/2},$$

where $a \in \mathbb{R}$, $k \geq 0$ is an integer, and $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{M})$. For an integer $k \geq 1$, denote by $H_a^{k-1/2}(\Gamma)$ the set of traces on a smooth curve $\Gamma \subset \overline{Q}$ with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Gamma)} = \inf \|u\|_{H_a^k(Q)} \quad (u \in H_a^k(Q): \ u|_\Gamma = \psi). \tag{2.2}$$

For an integer $k \geq 1$, we also set

$$\mathcal{W}^{k-1/2}(\partial G) = \prod_{i=1}^N W^{k-1/2}(\Gamma_i), \qquad \mathcal{H}_a^{k-1/2}(\partial G) = \prod_{i=1}^N H_a^{k-1/2}(\Gamma_i).$$

Consider the operator

$$\mathbf{P}u = \sum_{i,k=1}^2 p_{ik}(y) u_{y_i y_k} + \sum_{k=1}^2 p_k(y) u_{y_k} + p_0(y) u,$$

where $p_{ik}$, $i, k = 1, 2$, and $p_k$, $k = 0, 1, 2$, are complex-valued $C^\infty$-coefficients. We assume throughout the paper that the operator $\mathbf{P}$ is *properly elliptic* on $\overline{G}$ (see, e.g., [24, Chapter 2, Section 1]).

For any closed set $\mathcal{M}$, we denote its $\varepsilon$-neighborhood by $\mathcal{O}_\varepsilon(\mathcal{M})$, i.e.,

$$\mathcal{O}_\varepsilon(\mathcal{M}) = \{y \in \mathbb{R}^2 : \ \operatorname{dist}(y, \mathcal{M}) < \varepsilon\}, \qquad \varepsilon > 0.$$

Now we introduce operators corresponding to nonlocal terms supported near the set $\mathcal{K}$. Let $\Omega_{is}$ ($i = 1, \dots, N$; $s = 1, \dots, S_i$) be $C^\infty$-diffeomorphisms taking some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and

$$\Omega_{is}(g) \in \mathcal{K} \quad \text{for} \quad g \in \overline{\Gamma_i} \cap \mathcal{K}. \tag{2.3}$$

Thus, the transformations $\Omega_{is}$ take the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ strictly inside the domain $G$ and the set of their end points $\overline{\Gamma_i} \cap \mathcal{K}$ to itself.

Let us specify the structure of the transformations $\Omega_{is}$ near the set $\mathcal{K}$. Denote by $\Omega_{is}^{+1}$ the transformation $\Omega_{is} : \mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$ and by $\Omega_{is}^{-1} : \Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ the inverse transformation. The set of points $\Omega_{i_q s_q}^{\pm 1}(\dots \Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ $(1 \le s_j \le S_{i_j},\ j = 1, \dots, q)$ is said to be an *orbit* of the point $g \in \mathcal{K}$ and denoted by $\operatorname{Orb}(g)$. In other words, the orbit $\operatorname{Orb}(g)$ is formed by the points (of the set $\mathcal{K}$) that can be obtained by consecutively applying the transformations $\Omega_{i_j s_j}^{\pm 1}$ to the point $g$.

It is clear that either $\operatorname{Orb}(g) = \operatorname{Orb}(g')$ or $\operatorname{Orb}(g) \cap \operatorname{Orb}(g') = \varnothing$ for any $g, g' \in \mathcal{K}$. In what follows, we assume that the set $\mathcal{K}$ consists of one orbit only (the results we will obtain are easy to generalize for the case in which $\mathcal{K}$ consists of finitely many disjoint orbits, see Section 6). The set (orbit) $\mathcal{K}$ consists of $N$ points. We denote these points by $g_j$, $j = 1, \dots, N$.

Take a sufficiently small number $\varepsilon$ (see Remark 2.3 below) such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$, $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$, satisfying the following conditions:

(1) the domain $G$ is a plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$;
(2) $\overline{\mathcal{O}_{\varepsilon_1}(g_j)} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_k)} = \varnothing$ for any $g_j, g_k \in \mathcal{K}$, $k \ne j$;
(3) if $g_j \in \overline{\Gamma_i}$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}\big(\mathcal{O}_\varepsilon(g_j)\big) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$, we fix a transformation $Y_j : y \mapsto y'(g_j)$ which is a composition of the shift by the vector $-\overrightarrow{Og_j}$ and the rotation through some angle so that

$$Y_j(\mathcal{O}_{\varepsilon_1}(g_j)) = \mathcal{O}_{\varepsilon_1}(0), \qquad Y_j(G \cap \mathcal{O}_{\varepsilon_1}(g_j)) = K_j \cap \mathcal{O}_{\varepsilon_1}(0),$$

$$Y_j(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)) = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0) \quad (\sigma = 1 \text{ or } \sigma = 2),$$

where

$$K_j = \{y \in \mathbb{R}^2 :\ r > 0,\ |\omega| < \omega_j\}, \qquad \gamma_{j\sigma} = \{y \in \mathbb{R}^2 :\ r > 0,\ \omega = (-1)^\sigma \omega_j\}.$$

Here $(\omega, r)$ are the polar coordinates and $0 < \omega_j < \pi$.

Consider the following condition (see Figure 2.1).

**Condition 2.1.** *Let $g_j \in \overline{\Gamma_i} \cap \mathcal{K}$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}$; then the transformation*

$$Y_k \circ \Omega_{is} \circ Y_j^{-1} : \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$$

*is the composition of rotation and homothety.*

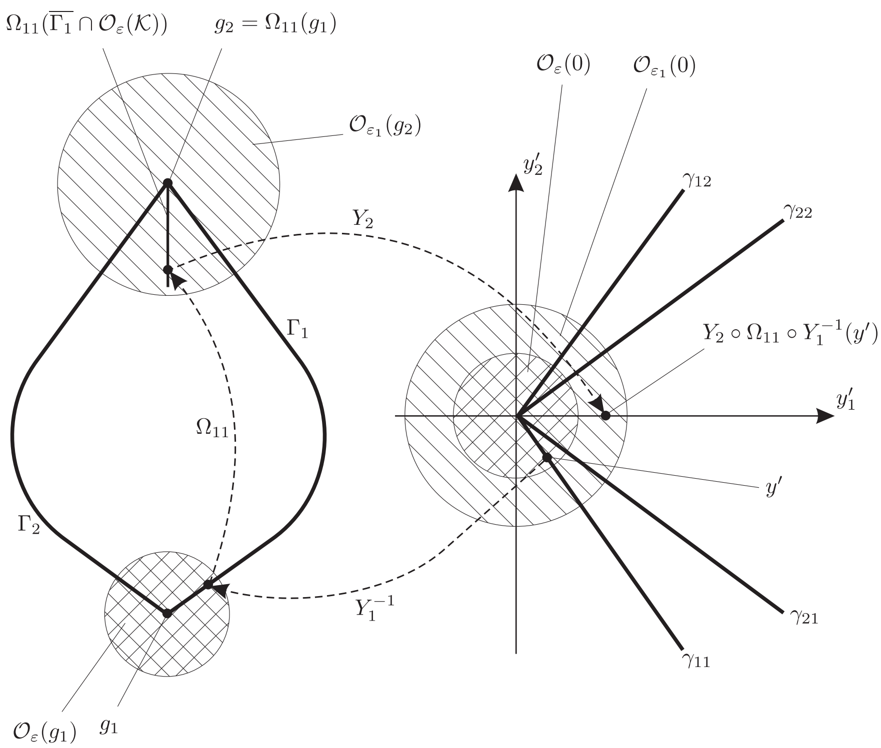


FIGURE 2.1. The transformation $Y_2 \circ \Omega_{11} \circ Y_1^{-1} : \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$ is a composition of rotation and homothety

**Remark 2.1.** Condition 2.1, together with the fact that $\Omega_{is}(\Gamma_i) \subset G$, implies that, if $g \in \Omega_{is}(\overline{\Gamma_i} \cap \mathcal{K}) \cap \overline{\Gamma_j} \cap \mathcal{K} \neq \varnothing$, then the curves $\Omega_{is}(\overline{\Gamma_i} \cap \mathcal{O}_\varepsilon(\mathcal{K}))$ and $\overline{\Gamma_j}$ intersect at a nonzero angle at the point $g$.

Introduce the nonlocal operators $\mathbf{B}_i^1$ by the formulas

$$\mathbf{B}_i^1 u = \sum_{s=1}^{S_i} b_{is}(y) u(\Omega_{is}(y)), \quad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \qquad \mathbf{B}_i^1 u = 0, \quad y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}),$$

where $b_{is} \in C^\infty(\mathbb{R}^2)$ and $\operatorname{supp} b_{is} \subset \mathcal{O}_\varepsilon(\mathcal{K})$. Since $\mathbf{B}_i^1 u = 0$ whenever $\operatorname{supp} u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_1}(\mathcal{K})}$, we say that the operators $\mathbf{B}_i^1$ *correspond to nonlocal terms supported near the set* $\mathcal{K}$.

Set $G_\rho = \{y \in G : \operatorname{dist}(y, \partial G) > \rho\}$ for $\rho > 0$. Consider operators $\mathbf{B}_i^2$ satisfying the following condition (cf. [33, 36, 15]).

**Condition 2.2.** *There exist numbers* $\varkappa_1 > \varkappa_2 > 0$ *and* $\rho > 0$ *such that*

$$\|\mathbf{B}_i^2 u\|_{W^{3/2}(\Gamma_i)} \le c_1 \|u\|_{W^2(G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})} \qquad \forall u \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}), \tag{2.4}$$

$$\|\mathbf{B}_i^2 u\|_{W^{3/2}(\Gamma_i\setminus\overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \le c_2 \|u\|_{W^2(G_\rho)} \qquad \forall u \in W^2(G_\rho), \tag{2.5}$$

*where* $i = 1, \dots, N$, *whereas* $c_1, c_2 > 0$ *do not depend on* $u$.

In particular, inequality (2.4) implies that $\mathbf{B}_i^2 u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}_i^2$ *correspond to nonlocal terms supported outside the set* $\mathcal{K}$.

We assume that Conditions 2.1 and 2.2 are fulfilled throughout Sections 2–5.

We study the following nonlocal elliptic boundary-value problem:

$$\mathbf{P}u = f_0(y) \quad (y \in G), \tag{2.6}$$

$$u|_{\Gamma_i} + \mathbf{B}_i^1 u + \mathbf{B}_i^2 u = f_i(y) \quad (y \in \Gamma_i;\ i = 1, \dots, N). \tag{2.7}$$

Note that the points $g_j$ divide the curves on which different nonlocal conditions are set; therefore, it is natural to say that $g_j$, $j = 1, \dots, N$, are the *points of conjugation of nonlocal conditions*. Problem (1.1), (1.2) is an example of an elliptic problem with nonlocal conditions (2.7) (see also Section 7).

**Definition 2.1.** A function $u \in W^1(G)$ is called a *generalized solution* of problem (2.6), (2.7) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{1/2}(\partial G)$ if $u$ satisfies nonlocal conditions (2.7) in the sense of traces and Equation (2.6) in the sense of distributions. The latter is equivalent to the validity of the integral identity

$$-\int\limits_G \sum_{i,k=1}^2 u_{y_i} \overline{(p_{ik} w)_{y_k}}\, dy + \int\limits_G \Big(\sum_{k=1}^2 p_k u_{y_k} + p_0 u\Big)\overline{w}\, dy = \int\limits_G f_0 \overline{w}\, dy$$

for all $w \in C_0^\infty(G)$.

**Remark 2.2.** Generalized solutions a priori belong to $W^1(G)$, whereas Condition 2.2 is formulated for functions from the space $W^2$ inside the domain and near a smooth part of the boundary. This formulation can be justified by the fact that any generalized solution belongs to the space $W^2$ outside an arbitrarily small neighborhood of the set $\mathcal{K}$ (see Section 3).

**Remark 2.3.** We can assume that the number $\varepsilon$ occurring in the definition of the operators $\mathbf{B}_i^1$ is sufficiently small (while $\varkappa_1, \varkappa_2, \rho$ occurring in the definition of the operators $\mathbf{B}_i^2$ can be arbitrary). Let us show that this assumption leads to no loss of generality.

Take a number $\hat{\varepsilon}$, where $0 < \hat{\varepsilon} < \varepsilon$. Set $\hat{\mathbf{B}}_i^1 u = \sum\limits_{s=1}^{S_i} \big(\hat{b}_{is}(y)u(\Omega_{is}(y))$ for $y \in \Gamma_i \cap \mathcal{O}_{\hat{\varepsilon}}(\mathcal{K})$ and $\hat{\mathbf{B}}_i^1 u = 0$ for $y \in \Gamma_i \setminus \mathcal{O}_{\hat{\varepsilon}}(\mathcal{K})$, where $\hat{b}_{is} \in C^\infty(\mathbb{R}^2)$, $\operatorname{supp} b_{is} \subset \mathcal{O}_{\hat{\varepsilon}}(\mathcal{K})$, and $\hat{b}_{is}(y) = b_{is}(y)$ for $y \in \Gamma_i \cap \mathcal{O}_{\hat{\varepsilon}/2}(\mathcal{K})$. It is clear that

$$\mathbf{B}_i^1 + \mathbf{B}_i^2 = \hat{\mathbf{B}}_i^1 + \hat{\mathbf{B}}_i^2,$$

where $\hat{\mathbf{B}}_i^2 = \mathbf{B}_i^1 - \hat{\mathbf{B}}_i^1 + \mathbf{B}_i^2$. Since $\mathbf{B}_i^1 u - \hat{\mathbf{B}}_i^1 u = 0$ near the set $\mathcal{K}$, it follows that the operator $\mathbf{B}_i^1 - \hat{\mathbf{B}}_i^1$ satisfy Condition 2.2 for appropriate $\varkappa_1, \varkappa_2, \rho$ (see [15, Section 1] for details). Thus, we see that $\varepsilon$ can be taken as small as needed. However, one must remember that the operator $\mathbf{B}_i^2$ and the values of $\varkappa_1, \varkappa_2, \rho$ may change if we change the value of $\varepsilon$.

2.2. **Model Problems.** When studying problem (2.6), (2.7), particular attention must be paid to the behavior of solutions near the set $\mathcal{K}$ of conjugation points. In this subsection, we consider corresponding model problems.

Denote by $u_j(y)$ the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma_i}$, $y \in \mathcal{O}_\varepsilon(g_j)$, $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then denote by $u_k(\Omega_{is}(y))$ the function $u(\Omega_{is}(y))$. In this case, the nonlocal problem (2.6), (2.7) acquires the following form in the neighborhood of the set (orbit) $\mathcal{K}$:

$$\mathbf{P}u_j = f_0(y) \quad (y \in \mathcal{O}_\varepsilon(g_j) \cap G),$$

$$u_j(y)|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} + \sum_{s=1}^{S_i} b_{is}(y)u_k(\Omega_{is}(y))|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i} = \psi_i(y)$$

$$\big(y \in \mathcal{O}_\varepsilon(g_j) \cap \Gamma_i;\ i \in \{1 \le i \le N:\ g_j \in \overline{\Gamma_i}\};\ j = 1, \dots, N\big),$$

where $\psi_i = f_i - \mathbf{B}_i^2 u$. Let $y \mapsto y'(g_j)$ be the change of variables described in Section 2.1. Set $K_j^\varepsilon = K_j \cap \mathcal{O}_\varepsilon(0)$, $\gamma_{j\sigma}^\varepsilon = \gamma_{j\sigma} \cap \mathcal{O}_\varepsilon(0)$ and introduce the functions

$$\begin{gathered} U_j(y') = u(y(y')), \quad F_j(y') = f_0(y(y')), \quad y' \in K_j^\varepsilon, \\ F_{j\sigma}(y') = f_i(y(y')), \quad B_{j\sigma}^u(y') = (\mathbf{B}_i^2 u)(y(y')), \\ \Psi_{j\sigma}(y') = F_{j\sigma}(y') - B_{j\sigma}^u(y'), \quad y' \in \gamma_{j\sigma}^\varepsilon, \end{gathered} \tag{2.8}$$

where $\sigma = 1$ ($\sigma = 2$) if the transformation $y \mapsto y'(g_j)$ takes $\Gamma_i$ to the side $\gamma_{j1}$ ($\gamma_{j2}$) of the angle $K_j$. Denote $y'$ by $y$ again. Then, by virtue of Condition 2.1, the problem (2.6), (2.7) acquires the form

$$\mathbf{P}_j U_j = F_j(y) \quad (y \in K_j^\varepsilon), \tag{2.9}$$

$$\mathbf{B}_{j\sigma} U \equiv \sum_{k,s} b_{j\sigma ks}(y) U_k(\mathcal{G}_{j\sigma ks} y) = \Psi_{j\sigma}(y) \quad (y \in \gamma_{j\sigma}^\varepsilon). \tag{2.10}$$

Here (and below unless otherwise stated) $j, k = 1, \dots, N$; $\sigma = 1, 2$; $s = 0, \dots, S_{j\sigma k}$; $\mathbf{P}_j$ are properly elliptic second-order differential operators with variable complex-valued $C^\infty$-coefficients,

$$\mathbf{P}_j v = \sum_{i,k=1}^{2} p_{jik}(y) v_{y_i y_k} + \sum_{k=1}^{2} p_{jk}(y) v_{y_k} + p_{j0}(y) v;$$

$U = (U_1, \dots, U_N)$; $b_{j\sigma ks}(y)$ are smooth functions, $b_{j\sigma j0}(y) \equiv 1$; $\mathcal{G}_{j\sigma ks}$ is an operator of rotation through an angle $\omega_{j\sigma ks}$ and homothetic with a coefficient $\chi_{j\sigma ks} > 0$ in the $y$-plane. Moreover,

$$|(-1)^\sigma \omega_j + \omega_{j\sigma ks}| < \omega_k \qquad \text{for} \qquad (k, s) \neq (j, 0)$$

(see Remark 2.1) and $\omega_{j\sigma j0} = 0$, $\chi_{j\sigma j0} = 1$ (i.e., $\mathcal{G}_{j\sigma j0} y \equiv y$).

Let the principal homogeneous parts of the operators $\mathbf{P}_j$ at the point $y = 0$ have the following form in the polar coordinates:

$$\sum_{i,k=1}^{2} p_{jik}(0) v_{y_i y_k} = r^{-2} \tilde{\mathcal{P}}_j(\omega, \partial/\partial\omega, r\partial/\partial r) v.$$

Consider the analytic operator-valued function

$$\tilde{\mathcal{L}}(\lambda) : \prod_j W^2(-\omega_j, \omega_j) \to \prod_j (L_2(-\omega_j, \omega_j) \times \mathbb{C}^2)$$

given by

$$\tilde{\mathcal{L}}(\lambda)\varphi = \big\{ \tilde{\mathcal{P}}_j(\omega, \partial/\partial\omega, i\lambda)\varphi_j,\ \sum_{k,s} (\chi_{j\sigma ks})^{i\lambda} b_{j\sigma ks}(0) \varphi_k((-1)^\sigma \omega_j + \omega_{j\sigma ks}) \big\}.$$

The main definitions and facts concerning analytic operator-valued functions can be found in [12]. The following assertion is of particular importance (see [34, Lemmas 2.1 and 2.2]).

**Lemma 2.1.** *The spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete. For any numbers $c_1 < c_2$, the band $c_1 < \operatorname{Im} \lambda < c_2$ contains at most finitely many eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$.*

Spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ play a crucial role in the study of smoothness of generalized solutions.

## 3. Preservation of Smoothness of Generalized Solutions

First, we study the case in which the following condition holds.

**Condition 3.1.** *The band* $-1 \le \operatorname{Im}\lambda < 0$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$*.*

The main result of this section is as follows.

**Theorem 3.1.** *Let Condition* 3.1 *hold, and let* $u \in W^1(G)$ *be a generalized solution of problem* (2.6)*,* (2.7) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$*. Then* $u \in W^2(G)$*.*

**Remark 3.1.** By Theorem 3.1, any generalized solution of problem (2.6), (2.7) belongs to $W^2(G)$ whenever Condition 3.1 holds. The right-hand sides $f_i$ in nonlocal conditions are naturally supposed to belong to the space $W^{3/2}(\Gamma_i)$. However, no additional assumptions (e.g., consistency conditions) are imposed on the behavior of the functions $f_i$ and on the behavior of the coefficients of nonlocal terms near the set $\mathcal{K}$. In fact, the functions $f_i \in W^{3/2}(\Gamma_i)$ are not quite arbitrary. For instance, if $\mathbf{B}_i^1 = 0$, $\mathbf{B}_i^2 = 0$ (i.e., we have a "local" problem), and a solution $u$ belongs to $W^2(G)$, then, by Sobolev's embedding theorem,

$$f_i(g) = f_j(g) \qquad \text{for} \qquad g \in \overline{\Gamma_i} \cap \overline{\Gamma_j} \ne \varnothing. \tag{3.1}$$

Theorem 3.1 implies that, if Condition 3.1 holds, then the existence of a *generalized* solution itself ensures the validity of relations of the same kind as (3.1). In Section 4, we will see that, if Condition 3.1 fails, then we must impose some consistency condition on the right-hand sides $f_i$ in order that any generalized solution be smooth.

Since $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$ and the operators $\mathbf{B}_i^2$ satisfy Condition 2.2, it follows from [17, Lemma 2.1] that[1]

$$u \in W^2\big(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})}\big) \qquad \forall \delta > 0. \tag{3.2}$$

Let $U_j(y') = u_j(y(y'))$, $j = 1, \dots, N$, be the functions corresponding to the set (orbit) $\mathcal{K}$ and satisfying problem (2.9), (2.10) with right-hand side $\{F_j, \Psi_{j\sigma}\}$ (see Section 2.2).

[1]See also [31].

Set

$$D_\chi = 2\max\{\chi_{j\sigma ks}\}, \qquad d_\chi = \min\{\chi_{j\sigma ks}\}/2. \tag{3.3}$$

Let $\varepsilon > 0$ be so small that $D_\chi \varepsilon < \varepsilon_1$.

Introduce the spaces of vector-valued functions

$$\mathcal{W}^k(K^\varepsilon) = \prod_j W^k(K_j^\varepsilon), \quad \mathcal{H}_a^k(K^\varepsilon) = \prod_j H_a^k(K_j^\varepsilon), \quad k \ge 0; \tag{3.4}$$

$$\mathcal{W}^{k-1/2}(\gamma^\varepsilon) = \prod_{j,\sigma} W^{k-1/2}(\gamma_{j\sigma}^\varepsilon), \quad \mathcal{H}_a^{k-1/2}(\gamma^\varepsilon) = \prod_{j,\sigma} H_a^{k-1/2}(\gamma_{j\sigma}^\varepsilon), \quad k \ge 1. \tag{3.5}$$

Similarly, one can introduce the spaces $\mathcal{W}^k(K)$, $\mathcal{H}_a^k(K)$, $\mathcal{W}^{k-1/2}(\gamma)$, and $\mathcal{H}_a^{k-1/2}(\gamma)$.

By virtue of relation (3.2),

$$U_j \in W^2(K_j^{\varepsilon_1} \cap \{|y| > \delta\}) \quad \forall \delta > 0. \tag{3.6}$$

Furthermore, it follows from the fact that $U \in \mathcal{W}^1(K^{\varepsilon_1})$ and Lemma A.1 that

$$U \in \mathcal{H}_a^1(K^{\varepsilon_1}) \subset \mathcal{H}_{a-1}^0(K^{\varepsilon_1}), \quad a > 0. \tag{3.7}$$

Finally, we have (see (2.9), (2.10)) $\{F_j\} \in \mathcal{W}^0(K^\varepsilon)$ and, by the fact that $f_i \in W^{3/2}(\Gamma_i)$, by relation (3.2), and by estimate (2.4), we have $\{\Psi_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$. Therefore, using Lemma A.1, we obtain

$$\{F_j\} \in \mathcal{H}_{1+a}^0(K^\varepsilon), \quad \{\Psi_{j\sigma}\} \in \mathcal{H}_{1+a}^{3/2}(\gamma^\varepsilon), \quad a > 0. \tag{3.8}$$

It follows from relations (3.6)–(3.8) and from Lemma A.8 that

$$U \in \mathcal{H}_{1+a}^2(K^\varepsilon), \quad a > 0. \tag{3.9}$$

To prove Theorem 3.1, it suffices to show that $U \in \mathcal{W}^2(K^\varepsilon)$.

Fix a sufficiently small number $a$, $0 < a < 1$, such that the band $a - 1 \le \operatorname{Im}\lambda \le a$ contains no nonreal eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$. The existence of such an $a$ follows from Lemma 2.1 and Condition 3.1.

Denote

$$\mathcal{P}_j v = \sum_{i,k=1}^{2} p_{jik}(0) v_{y_i y_k}, \quad \mathcal{B}_{j\sigma} U = \sum_{k,s} b_{j\sigma ks}(0) U_k(\mathcal{G}_{j\sigma ks} y).$$

**Lemma 3.1.** *Let $U \in \mathcal{W}^1(K^\varepsilon)$ be a generalized solution*[2] *of problem* (2.9), (2.10) *with right-hand side $\{F_j, \Psi_{j\sigma}\} \in \mathcal{W}^0(K^\varepsilon) \times \mathcal{W}^{3/2}(\gamma^\varepsilon)$. Then*

$$U = C + U', \tag{3.10}$$

*where $U' \in \mathcal{H}_a^2(K^\varepsilon)$, $a$ is the above number, and $C = (C_1, \dots, C_N)$ is a constant vector. The function $U'$ and the vector $C$ are uniquely defined, and the vector $C$ satisfies the relation*

$$\mathcal{B}_{j\sigma} C = \Psi_{j\sigma}(0). \tag{3.11}$$

**Proof.** 1. Write problem (2.9), (2.10) as follows:

$$\mathbf{P}_j U_j = F_j(y) \quad (y \in K_j^\varepsilon), \quad \mathbf{B}_{j\sigma} U = \Psi_{j\sigma}(0) + \Psi_{j\sigma}^0(y) \quad (y \in \gamma_{j\sigma}^\varepsilon), \tag{3.12}$$

where $\Psi_{j\sigma}^0(y) = \Psi_{j\sigma}(y) - \Psi_{j\sigma}(0)$. We claim that

$$\{F_j\} \in \mathcal{H}_a^0(K^\varepsilon), \quad \{\Psi_{j\sigma}^0\} \in \mathcal{H}_a^{3/2}(\gamma^\varepsilon). \tag{3.13}$$

Indeed, the first inclusion follows from the relation $\{F_j\} \in \mathcal{W}^0(K^\varepsilon)$, whereas the second one is from the relations $\{\Psi_{j\sigma}^0\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$ and $\Psi_{j\sigma}^0(0) = 0$ and from Lemma A.2.

2. By Lemma A.10, there exists a function

$$W = \sum_{l=0}^{\varkappa} \frac{1}{l!}(i \ln r)^l w^{(l)}(\omega) \in \mathcal{H}_{1+a}^2(K^\varepsilon) \tag{3.14}$$

such that

$$\mathcal{P}_j W_j = 0 \quad (y \in K_j), \qquad \mathcal{B}_{j\sigma} W = \Psi_{j\sigma}(0) \quad (y \in \gamma_{j\sigma}). \tag{3.15}$$

Here $\varkappa = 0$ if $\lambda = 0$ is not an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$; otherwise, $\varkappa$ equals the greatest of partial multiplicities of the eigenvalue $\lambda = 0$; $w^{(l)} \in \prod_j W^2(-\omega_j, \omega_j)$.

As we have proved before this lemma, the function $U$ satisfies (3.9). Combining this fact with relation (3.14) yields

$$U - W \in \mathcal{H}_{1+a}^2(K^\varepsilon). \tag{3.16}$$

On the other hand, Lemma A.3 implies that

$$\{\mathbf{P}_j U_j - \mathcal{P}_j U_j\} \in \mathcal{H}_a^0(K^\varepsilon), \quad \{\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon} - \mathcal{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon}\} \in \mathcal{H}_a^{3/2}(\gamma^\varepsilon). \tag{3.17}$$

It follows from (3.12), (3.13), and (3.17) that

$$\{\mathcal{P}_j(U_j - W_j)\} \in \mathcal{H}_a^0(K^\varepsilon), \quad \{\mathcal{B}_{j\sigma}(U - W)|_{\gamma_{j\sigma}^\varepsilon}\} \in \mathcal{H}_a^{3/2}(\gamma^\varepsilon). \tag{3.18}$$

[2] That is $U$ satisfies Eq. (2.9) in the sense of distributions and nonlocal conditions (2.10) in the sense of traces.

3. Applying Theorem A.1 concerning the asymptotic behavior of the function $U-W$ and using relations (3.16) and (3.18), we obtain

$$U-W=\sum_{\operatorname{Im}\lambda_n=0}\sum_{q=1}^{J_n}\sum_{m=0}^{\varkappa_{qn}-1} c_n^{(m,q)}W_n^{(m,q)}+U'. \tag{3.19}$$

Here $\{\lambda_n\}$ is a finite set of eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ lying on the line $\operatorname{Im}\lambda=0$;

$$W_n^{(m,q)}(\omega,r)=r^{i\lambda_n}\sum_{l=0}^{m}\frac{1}{l!}(i\ln r)^l\varphi_n^{(m-l,q)}(\omega),$$

$$\mathcal{B}_{j\sigma}W_n^{(m,q)}|_{\gamma_{j\sigma}}=0; \tag{3.20}$$

$\varphi_n^{(0,q)},\dots,\varphi_n^{(\varkappa_{qn}-1,q)}\in\prod_j W^2(-\omega_j,\omega_j)$ are an eigenvector and associated vectors (a Jordan chain of length $\varkappa_{qn}\ge 1$) corresponding to the eigenvalue $\lambda_n$; $c_n^{(m,q)}$ are constants; finally, $U'\in\mathcal{H}_a^2(K^\varepsilon)$.

Set

$$C=W+\sum_{n,q,m}c_n^{(m,q)}W_n^{(m,q)}.$$

It is clear that $U=C+U'$. Since $U,U'\in\mathcal{W}^1(K^\varepsilon)$, it follows that $C\in\mathcal{W}^1(K^\varepsilon)$. This relation and Lemma A.6 imply that $C$ is a constant vector. By virtue of (3.20) and (3.15),

$$\mathcal{B}_{j\sigma}C|_{\gamma_{j\sigma}^\varepsilon}=\mathcal{B}_{j\sigma}W|_{\gamma_{j\sigma}^\varepsilon}=\Psi_{j\sigma}(0).$$

Therefore, using the relation $\mathcal{B}_{j\sigma}C=\text{const}$ for $C=\text{const}$, we obtain (3.11).

4. Now suppose that the equality $U=D+V'$ holds together with (3.10), where $V'\in\mathcal{H}_a^2(K^\varepsilon)$ and $D=(D_1,\dots,D_N)$ is a constant vector. Then we have $C-D=V'-U'\in\mathcal{H}_a^2(K^\varepsilon)$, hence $C-D=0$ and $V'-U'=0$. □

**Lemma 3.2.** *Let the conditions of Lemma* 3.1 *be fulfilled, and let Condition* 3.1 *hold. Then* $U\in\mathcal{W}^2(K^\varepsilon)$.

**Proof.** 1. By Lemma 3.1, it suffices to show that $U'\in\mathcal{W}^2(K^\varepsilon)$. The function $U'$ belongs to $\mathcal{H}_a^2(K^\varepsilon)$, and, by virtue of relations (3.10) and (3.12), it is a solution of the problem

$$\mathbf{P}_jU_j'=F_j-\mathbf{P}_jC_j\ (y\in K_j^\varepsilon),\quad \mathbf{B}_{j\sigma}U'=\Psi_{j\sigma}(0)+\Psi_{j\sigma}^0(y)-\mathbf{B}_{j\sigma}C\ (y\in\gamma_{j\sigma}^\varepsilon). \tag{3.21}$$

Since $\{F_j\}\in\mathcal{W}^0(K^\varepsilon)$ and $C=\text{const}$, it follows that

$$\{F_j-\mathbf{P}_jC_j\}\in\mathcal{H}_0^0(K^\varepsilon). \tag{3.22}$$

Further,

$$\begin{gathered}\{\Psi_{j\sigma}(0)+\Psi^0_{j\sigma}(y)|_{\gamma_j^\varepsilon}-\mathbf{B}_{j\sigma}C|_{\gamma_j^\varepsilon}\}\in\mathcal{W}^{3/2}(\gamma^\varepsilon),\\ \big(\Psi_{j\sigma}(0)+\Psi^0_{j\sigma}(y)-\mathbf{B}_{j\sigma}C\big)\big|_{y=0}=0.\end{gathered}\tag{3.23}$$

The latter relation follows from the fact that $\Psi^0_{j\sigma}(0)=0$ and $\mathbf{B}_{j\sigma}C|_{y=0}=\mathcal{B}_{j\sigma}C=\Psi_{j\sigma}(0)$ (see Lemma 3.1).

2. Since the line $\operatorname{Im}\lambda=-1$ has no eigenvalues of $\tilde{\mathcal{L}}(\lambda)$ and relations (3.23) hold, it follows from Lemma A.13 that there exists a function

$$V\in\mathcal{W}^2(K)\cap\mathcal{H}^2_a(K)\tag{3.24}$$

such that

$$\{\mathbf{P}_jV_j\}\in\mathcal{H}^0_0(K^\varepsilon),\quad \{\mathbf{B}_{j\sigma}V|_{\gamma^\varepsilon_{j\sigma}}-\big(\Psi_{j\sigma}(0)+\Psi^0_{j\sigma}(y)-\mathbf{B}_{j\sigma}C\big)|_{\gamma^\varepsilon_{j\sigma}}\}\in\mathcal{H}^{3/2}_0(\gamma^\varepsilon).\tag{3.25}$$

Therefore, $U'-V\in\mathcal{H}^2_a(K^\varepsilon)$ and, due to (3.21)–(3.23) and (3.25), we have

$$\{\mathbf{P}_j(U'_j-V_j)\}\in\mathcal{H}^0_0(K^\varepsilon),\qquad \{\mathbf{B}_{j\sigma}(U'-V)|_{\gamma^\varepsilon_{j\sigma}}\}\in\mathcal{H}^{3/2}_0(\gamma^\varepsilon).$$

Further, Lemma A.3 implies that

$$\{\mathcal{P}_j(U'_j-V_j)\}\in\mathcal{H}^0_0(K^\varepsilon),\qquad \{\mathcal{B}_{j\sigma}(U'-V)|_{\gamma^\varepsilon_{j\sigma}}\}\in\mathcal{H}^{3/2}_0(\gamma^\varepsilon).$$

Since Condition 3.1 holds, we can apply Theorem A.1 concerning the asymptotic behavior of the function $U'-V$, which yields

$$U'-V\in\mathcal{H}^2_0(K^\varepsilon)\subset\mathcal{W}^2(K^\varepsilon).$$

Now the conclusion of the lemma follows from the latter relation, from (3.24), and from (3.10). □

Theorem 3.1 results from (3.2) and from Lemma 3.2.

## 4. Border Case: Consistency Conditions

4.1. **Behavior of Solutions near the Conjugation Points.** Let $\lambda=\lambda_0$ be an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

**Definition 4.1.** We say that $\lambda_0$ is a *proper eigenvalue* if none of the corresponding eigenvectors $\varphi(\omega)=(\varphi_1(\omega),\dots,\varphi_N(\omega))$ has an associated vector, while the functions $r^{i\lambda_0}\varphi_j(\omega)$, $j=1,\dots,N$, are polynomials in $y_1,y_2$. An eigenvalue which is not proper is said to be *improper*.

The notion of proper eigenvalue was originally proposed by Kondrat'ev [20] for "local" boundary-value problems in angular or conical domains.

Clearly, if $\lambda_0$ is a proper eigenvalue, then $\operatorname{Im}\lambda_0 \le 0$ and $\operatorname{Re}\lambda_0 = 0$. Therefore, the line $\operatorname{Im}\lambda = \text{const}$ can have at most one proper eigenvalue.

In this section, we suppose that the following condition holds.

**Condition 4.1.** *The band $-1 \le \operatorname{Im}\lambda < 0$ contains only the eigenvalue $\lambda = -i$ of the operator $\tilde{\mathcal{L}}(\lambda)$. This eigenvalue is a proper one.*

The principal difference between the results of this section and those of Section 3 is related to the behavior of generalized solutions near the set $\mathcal{K}$. If Condition 4.1 holds, then Lemma 3.1 remains valid. However, the conclusion of Lemma 3.2 is no longer true because Lemma A.13 (proved in [15]) is inapplicable when the line $\operatorname{Im}\lambda = -1$ contains a proper eigenvalue of $\tilde{\mathcal{L}}(\lambda)$. In this section, we make use of other results from [15]. To do this, we impose certain consistency conditions on the behavior of the functions $f_i$ and on the behavior of the coefficients of nonlocal terms near the set (orbit) $\mathcal{K}$.

Let $\tau_{j\sigma}$ be the unit vector co-directed with the ray $\gamma_{j\sigma}$. Consider the operators

$$\frac{\partial}{\partial\tau_{j\sigma}}\mathcal{B}_{j\sigma}U \equiv \frac{\partial}{\partial\tau_{j\sigma}}\Big(\sum_{k,s} b_{j\sigma ks}(0)U_k(\mathcal{G}_{j\sigma ks}y)\Big).$$

Using the chain rule, we obtain

$$\frac{\partial}{\partial\tau_{j\sigma}}\mathcal{B}_{j\sigma}U \equiv \sum_{k,s}(\hat{B}_{j\sigma ks}(D_y)U_k)(\mathcal{G}_{j\sigma ks}y), \tag{4.1}$$

where $\hat{B}_{j\sigma ks}(D_y)$ are first-order differential operators with constant coefficients. In particular, $\hat{B}_{j\sigma j0}(D_y) = \partial/\partial\tau_{j\sigma}$ because $\mathcal{G}_{j\sigma j0}y \equiv y$. Formally replacing the nonlocal operators by the corresponding local operators in (4.1), we introduce the operators

$$\hat{\mathcal{B}}_{j\sigma}(D_y)U \equiv \sum_{k,s}\hat{B}_{j\sigma ks}(D_y)U_k(y). \tag{4.2}$$

Let us prove that the system of operators (4.2) is linearly dependent if Condition 4.1 holds. Let

$$\hat{B}_{j\sigma ks}(D_y) = b_{j\sigma ks1}\frac{\partial}{\partial y_1} + b_{j\sigma ks2}\frac{\partial}{\partial y_2}, \tag{4.3}$$

where $b_{j\sigma ks1}$ and $b_{j\sigma ks2}$ are complex constants. It suffices to show that the following system of $2N$ equations for the $2N$ indeterminates $q_{k1}, q_{k2}$,

$k = 1, \dots, N$, admits a nontrivial solution:

$$\sum_{k,s} b_{j\sigma ks1} q_{k1} + b_{j\sigma ks2} q_{k2} = 0, \quad j = 1, \dots, N, \ \sigma = 1, 2. \tag{4.4}$$

Let $\varphi(\omega) = (\varphi_1(\omega), \dots, \varphi_N(\omega))$ be an eigenvector corresponding to the eigenvalue $\lambda = -i$. By Condition 4.1, the functions $Q_k(y) = r\varphi_k(\omega)$ are homogeneous polynomials of order one. Set $q_{k1} = \partial Q_k/\partial y_1$, $q_{k2} = \partial Q_k/\partial y_2$. Then, using equalities (4.3), the fact that the first derivative of a polynomial of order one is a constant, and relation (4.1), we obtain

$$\sum_{k,s} b_{j\sigma ks1} q_{k1} + b_{j\sigma ks2} q_{k2} = \sum_{k,s} \hat{B}_{j\sigma ks}(D_y) Q_k(y)$$
$$= \sum_{k,s} (\hat{B}_{j\sigma ks}(D_y) Q_k)(\mathcal{G}_{j\sigma ks} y) = \frac{\partial}{\partial \tau_{j\sigma}} \mathcal{B}_{j\sigma} Q,$$

where $Q = (Q_1, \dots, Q_N)$. Since $\lambda = -i$ is an eigenvalue of $\tilde{\mathcal{L}}(\lambda)$ and $\varphi$ is the corresponding eigenvector, it follows that $\mathcal{B}_{j\sigma} Q|_{\gamma_{j\sigma}} = 0$; hence,

$$\big(\partial(\mathcal{B}_{j\sigma} Q)/\partial \tau_{j\sigma}\big)|_{\gamma_{j\sigma}} = 0.$$

It follows from the latter relation and from the relation $\partial(\mathcal{B}_{j\sigma} Q)/\partial \tau_{j\sigma} =$ const that $\partial(\mathcal{B}_{j\sigma} Q)/\partial \tau_{j\sigma} = 0$. Thus, we have constructed a nontrivial solution of system (4.4) and, therefore, proved that system (4.2) is linearly dependent.

Let

$$\{\hat{\mathcal{B}}_{j'\sigma'}(D_y)\} \tag{4.5}$$

be a maximal linearly independent subsystem of system (4.2). In this case, any operator $\hat{\mathcal{B}}_{j\sigma}(D_y)$ which does not enter system (4.5) can be represented as follows:

$$\hat{\mathcal{B}}_{j\sigma}(D_y) = \sum_{j',\sigma'} \beta_{j\sigma}^{j'\sigma'} \hat{\mathcal{B}}_{j'\sigma'}(D_y), \tag{4.6}$$

where $\beta_{j\sigma}^{j'\sigma'}$ are some constants.

Let us introduce the notion of the consistency condition. Let $\{Z_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$ be arbitrary functions, each of which is defined on its own interval $\gamma_{j\sigma}^\varepsilon$. Consider the functions

$$Z_{j\sigma}^0(r) = Z_{j\sigma}(y)|_{y=(r\cos\omega_j,\, r(-1)^\sigma \sin\omega_j)}.$$

Each of the functions $Z_{j\sigma}^0$ belongs to $W^{3/2}(0, \varepsilon)$.

**Definition 4.2.** Let $\beta_{j\sigma}^{j'\sigma'}$ be the constants occurring in (4.6). If the relations

$$\int_0^\varepsilon r^{-1}\Big|\frac{d}{dr}\Big(Z_{j\sigma}^0-\sum_{j',\sigma'}\beta_{j\sigma}^{j'\sigma'}Z_{j'\sigma'}^0\Big)\Big|^2dr<\infty \tag{4.7}$$

hold for all indices $j,\sigma$ corresponding to the operators of system (4.2) which do not enter system (4.5), then we say that the *functions* $Z_{j\sigma}$ *satisfy the consistency condition* (4.7).

**Remark 4.1.** The relation $\{Z_{j\sigma}\}\in\mathcal{H}_0^{3/2}(\gamma^\varepsilon)$ is sufficient (but not necessary) for the functions $Z_{j\sigma}$ to satisfy the consistency condition (4.7). This follows from Lemma A.5.

**Remark 4.2.** In the paper [15], whose results we use in the present paper, the consistency condition has the form

$$\frac{\partial\mathbf{Z}_{j\sigma}}{\partial\tau_{j\sigma}}-\sum_{j',\sigma'}\beta_{j\sigma}^{j'\sigma'}\frac{\partial\mathbf{Z}_{j'\sigma'}}{\partial\tau_{j'\sigma'}}\in H_0^1(\mathbb{R}^2), \tag{4.8}$$

where $\mathbf{Z}_{j\sigma}\in W^2(\mathbb{R}^2)$ is a compactly supported extension of $Z_{j\sigma}$ to $\mathbb{R}^2$ (appropriate theorems concerning extensions of functions in angular domains can be found in [40]). Let us show that relations (4.7) are equivalent to (4.8). Denote by $\mathcal{G}_{j\sigma}$ the operator of rotation through the angle $(-1)^\sigma\omega_j$; in particular, the operator $\mathcal{G}_{j\sigma}$ takes the positive half-line $Oy_1$ onto the ray $\gamma_{j\sigma}$. Consider the functions $\mathbf{Z}_{j\sigma}^0(y)=\mathbf{Z}_{j\sigma}(\mathcal{G}_{j\sigma}y)$. It is clear that $\mathbf{Z}_{j\sigma}^0\in W^2(\mathbb{R}^2)$ and $\mathbf{Z}_{j\sigma}^0(y_1,0)=Z_{j\sigma}^0(y_1)$. Suppose that relations (4.7) hold. Then, by Lemma A.4, we have

$$\frac{\partial\mathbf{Z}_{j\sigma}^0}{\partial y_1}-\sum_{j',\sigma'}\beta_{j\sigma}^{j'\sigma'}\frac{\partial\mathbf{Z}_{j'\sigma'}^0}{\partial y_1}\in H_0^1(\mathbb{R}^2), \tag{4.9}$$

which is equivalent to

$$\frac{\partial\mathbf{Z}_{j\sigma}}{\partial\tau_{j\sigma}}\big(\mathcal{G}_{j\sigma}y\big)-\sum_{j',\sigma'}\beta_{j\sigma}^{j'\sigma'}\frac{\partial\mathbf{Z}_{j'\sigma'}}{\partial\tau_{j'\sigma'}}\big(\mathcal{G}_{j'\sigma'}y\big)\in H_0^1(\mathbb{R}^2) \tag{4.10}$$

by the chain rule. However, by Lemma A.7, we have

$$\frac{\partial\mathbf{Z}_{j\sigma}}{\partial\tau_{j\sigma}}\big(\mathcal{G}_{j\sigma}y\big)-\frac{\partial\mathbf{Z}_{j\sigma}}{\partial\tau_{j\sigma}}\big(y\big)\in H_0^1(\mathbb{R}^2) \tag{4.11}$$

for all $\mathbf{Z}_{j\sigma}\in W^2(\mathbb{R}^2)$ because $\partial\mathbf{Z}_{j\sigma}/\partial\tau_{j\sigma}\in W^1(\mathbb{R}^2)$. It follows from (4.10) and (4.11) that relations (4.8) hold.

Conversely, suppose that relations (4.8) hold. Using (4.11) again, we obtain (4.10), hence (4.9). It follows from (4.9) and from the boundedness of the trace operator in appropriate weighted spaces that

$$\frac{d}{dr}\Big(Z^0_{j\sigma}-\sum_{j',\sigma'}\beta^{j'\sigma'}_{j\sigma}Z^0_{j'\sigma'}\Big)\in H^{1/2}_0(0,\varepsilon).$$

This relation and Lemma A.5 imply (4.7).

Now we will show that the following condition is necessary and sufficient for a given generalized solution $u$ to belong to $W^2(G)$.

**Condition 4.2.** *Let $u\in W^1(G)$ be a generalized solution of problem* (2.6), (2.7)*, $\Psi_{j\sigma}$ the right-hand sides in nonlocal conditions* (2.10)*, and $C$ the constant vector appearing in Lemma* 3.1*. Then the functions $\Psi_{j\sigma}-\mathbf{B}_{j\sigma}C$ satisfy the consistency condition* (4.7)*.*

**Remark 4.3.** 1. The validity of Condition 4.2 depends, in particular, on the behavior of the function $\mathbf{B}^2_i u$ near the set (orbit) $\mathcal{K}$. Due to (2.4), the values of the function $\mathbf{B}^2_i u$ near the set $\mathcal{K}$ depend on the values of the function $u$ in $G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}$. Therefore, the smoothness of the solution $u$ near the set $\mathcal{K}$ depends on the behavior of $u$ outside $\mathcal{K}$.

2. Let us explain how the validity of Condition 4.2 depends on the behavior of the functions $u(y), f_i(y), b_{is}(y), (\mathbf{B}^2_i u)(y)$ near the set $\mathcal{K}$. On one hand, the vector $C$ appearing in Lemma 3.1 is defined by the behavior of $u(y)$ near the set $\mathcal{K}$. On the other hand, the values of $b_{is}(y)$, $y\in\mathcal{K}$, together with the operators $\mathcal{G}_{j\sigma ks}$, define the constants $\beta_{j\sigma}$ occurring in (4.6) and hence in (4.7). Finally, the derivatives of $f_i(y), (\mathbf{B}^2_i u)(y)$, and $b_{is}(y)$ near the set $\mathcal{K}$ must be consistent with each other in such a way that the absolute values of the corresponding linear combinations of the first derivatives of $\Psi_{j\sigma}-\mathbf{B}_{j\sigma}C$ are quadratically integrable, with the weight $r^{-1}$, near the origin.

Throughout this section, we suppose that the number $a$ is the same as in Section 3. The existence of such an $a$ follows from Lemma 2.1 and Condition 4.1.

**Theorem 4.1.** *Let Condition* 4.1 *hold, and let $u\in W^1(G)$ be a generalized solution of problem* (2.6), (2.7) *with right-hand side $\{f_0,f_i\}\in L_2(G)\times\mathcal{W}^{3/2}(\partial G)$. Then $u\in W^2(G)$ if and only if Condition* 4.2 *holds.*

**Proof.** 1. *Necessity.* Let $u\in W^2(G)$, and let $U=(U_1,\dots,U_N)$ be a function corresponding to the set (orbit) $\mathcal{K}$. Clearly, $U\in\mathcal{W}^2(K^\varepsilon)$. It follows from Lemma 3.1 that $U=C+U'$, where $U'\in\mathcal{H}^2_a(K^\varepsilon)$. Since we additionally have $U' = U - C \in \mathcal{W}^2(K^\varepsilon)$, it follows from Sobolev's embedding theorem that $U'(0) = 0$. This relation and Lemma A.15 imply that the functions $\Psi_{j\sigma} - \mathbf{B}_{j\sigma}C = \mathbf{B}_{j\sigma}U'$ satisfy the consistency condition (4.7).

2. *Sufficiency.* Suppose that Condition 4.2 holds. Similarly to the proof of Lemma 3.2, we infer that the function $U' \in \mathcal{H}_a^2(K^\varepsilon)$ is a solution of problem (3.21).

Using Condition 4.2 and relations (3.23), we can apply Lemma A.14, which ensures the existence of a function $V$ satisfying relations (3.24) and (3.25).

Further, similarly to the proof of Lemma 3.2, we obtain $U' - V \in \mathcal{H}_a^2(K^\varepsilon)$, $\{\mathcal{P}_j(U_j' - V_j)\} \in \mathcal{H}_0^0(K^\varepsilon)$, $\{\mathcal{B}_{j\sigma}(U' - V)|_{\gamma_{j\sigma}^\varepsilon}\} \in \mathcal{H}_0^{3/2}(\gamma^\varepsilon)$. It follows from these relations and from Lemma A.11 that all the second derivatives of the function $U' - V$ belong to $\mathcal{W}^0(K^\varepsilon)$. Combining this fact with the relations

$$U' - V \in \mathcal{H}_a^2(K^\varepsilon) \subset \mathcal{H}_{a-1}^1(K^\varepsilon) \subset \mathcal{W}^1(K^\varepsilon)$$

yields $U' - V \in \mathcal{W}^2(K^\varepsilon)$. Now the conclusion of the theorem results from (3.24) and (3.10). □

Note that Theorem 4.1 enables us to conclude whether or not a given solution $u$ is smooth near the set $\mathcal{K}$, provided that we know the asymptotics for $u$ of the same type as (3.10) near the set $\mathcal{K}$ (i.e., if we know the value of the constant[3] $C$). Theorem 4.1 shows what affects the smoothness of solutions in principle. Below, this will enable us to obtain a constructive condition which is necessary and sufficient for *any* generalized solution to belong to $W^2(G)$.

4.2. **Problem with Nonhomogeneous Nonlocal Conditions.** If any generalized solution of problem (2.6), (2.7) belongs to $W^2(G)$, then we say that *smoothness* of generalized solutions is *preserved*. If there exists a generalized solution of problem (2.6), (2.7) which does not belong to $W^2(G)$, then we say that *smoothness* of generalized solutions *can be violated*.

In this subsection, we formulate necessary and sufficient conditions for the smoothness of solutions to be preserved. First of all, we show that right-hand sides $f_i$ in nonlocal conditions (2.7) cannot be arbitrary functions from $W^{3/2}(\Gamma_i)$; they must satisfy the consistency condition (4.7).

Denote by $\mathcal{S}^{3/2}(\partial G)$ the set of functions $\{f_i\} \in \mathcal{W}^{3/2}(\partial G)$ such that the functions $F_{j\sigma}$ (see (2.8)) satisfy the consistency condition (4.7).

It follows from [15, Lemma 3.2] that the set $\mathcal{S}^{3/2}(\partial G)$ is not closed in the space $\mathcal{W}^{3/2}(\partial G)$.

[3] As for the calculation of the constant $C$, see [13, 14].

Smoothness of generalized solutions of problem (2.6), (2.7) can be violated if right-hand sides in nonlocal conditions (2.7) do not satisfy the consistency condition. The following result is valid.

**Theorem 4.2.** *Let Condition* 4.1 *hold. Then there exist a function* $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$, $\{f_i\} \notin \mathcal{S}^{3/2}(\partial G)$, *and a function* $u \in W^1(G)$ *such that* $u$ *is a generalized solution of problem* (2.6), (2.7) *with the right-hand side* $\{f_0, f_i\}$ *and* $u \notin W^2(G)$.

To prove Theorem 4.2, we preliminarily establish two auxiliary results.

**Lemma 4.1.** *Let* $f \in W^2(\mathbb{R}^2)$ *and* $f(0) = 0$. *Then there exists a sequence* $f^n \in C_0^\infty(\mathbb{R}^2)$, $n = 1, 2, \dots$, *such that* $f^n(y) = 0$ *in some neighborhood of the origin* (*depending on* $n$) *and* $f^n \to f$ *in* $W^2(\mathbb{R}^2)$.

**Proof.** As is well known, the set $C_0^\infty(\mathbb{R}^2)$ is dense in $W^2(\mathbb{R}^2)$. On the other hand, it follows from Sobolev's embedding theorem and Riesz's theorem on the general form of a linear continuous functional in a Hilbert space that the set $\{u \in W^2(\mathbb{R}^2) : u(0) = 0\}$ is a closed subspace in $W^2(\mathbb{R}^2)$ of codimension one. Therefore, by [23, Lemma 8.1], the set $C_0^\infty(\mathbb{R}^2) \cap \{u \in W^2(\mathbb{R}^2) : u(0) = 0\}$ is dense in $\{u \in W^2(\mathbb{R}^2) : u(0) = 0\}$. Hence, it suffices to prove the lemma for a function $f \in C_0^\infty(\mathbb{R}^2)$ such that $f(0) = 0$. Introduce a function $\xi \in C_0^\infty[0, \infty)$ such that $0 \le \xi(t) \le 1$, $\xi(t) = 1$ for $t < 1$, and $\xi(t) = 0$ for $t > 2$. Consider the sequence

$$\xi^n(y) = \xi\Big(-\frac{\ln r}{n}\Big),$$

where $r = |y|$. Clearly, $0 \le \xi^n(y) \le 1$, $\xi^n(y) = 0$ for $|y| < e^{-2n}$, $\xi^n(y) = 1$ for $|y| > e^{-n}$, $|\xi^n_{y_k}| \le c_1/(rn)$, $|\xi^n_{y_i y_k}| \le c_2/(r^2 n)$, where $c_1, c_2 > 0$ do not depend on $n$ and $y$.

Let us show that the sequence $\xi^n f$ converges to $f$ in $W^2(\mathbb{R}^2)$ as $n \to \infty$. Clearly,

$$\int_{\mathbb{R}^2} |f - \xi^n f|^2 dy \le \int_{|y| < e^{-n}} |f|^2 dy \to 0. \tag{4.12}$$

Further,

$$\int_{\mathbb{R}^2} |(f - \xi^n f)_{y_k}|^2 dy \le 2\Big[\int_{|y|<e^{-n}} |f_{y_k}|^2 dy + \frac{c_1^2}{n^2} \int_{e^{-2n}<|y|<e^{-n}} |f|^2 \frac{1}{r^2} dy\Big] \to 0. \tag{4.13}$$

Indeed, the first bracketed term tends to zero because $e^{-n} \to 0$, whereas the second term can be estimated from above by the following expression:

$$2\pi \max_{y\in\mathbb{R}^2} |f|^2 \frac{c_1^2}{n^2} \int_{e^{-2n}}^{e^{-n}} \frac{dr}{r} = 2\pi \max_{y\in\mathbb{R}^2} |f|^2 \frac{c_1^2}{n} \to 0.$$

Finally,

$$\int_{\mathbb{R}^2} |(f-\xi^n f)_{y_i y_k}|^2 dy \le 4\Big[ \int_{|y|<e^{-n}} |f_{y_i y_k}|^2 dy + \frac{c_1^2}{n^2} \int_{e^{-2n}<|y|<e^{-n}} (|f_{y_i}|^2 + |f_{y_k}|^2) \frac{1}{r^2} dy + \frac{c_2^2}{n^2} \int_{e^{-2n}<|y|<e^{-n}} |f|^2 \frac{1}{r^4} dy \Big] \to 0. \tag{4.14}$$

Indeed, the first and the second bracketed terms tend to zero because of the reasons similar to the above. To prove that the third term tends to zero, we recall that $f \in C_0^\infty(\mathbb{R}^2)$ and $f(0) = 0$. Therefore, by the Taylor formula, $f(y) = O(r)$ as $r \to 0$, and hence the third term can be estimated from above similarly to the second one. □

Set

$$\varepsilon' = d_\chi \min(\varepsilon, \varkappa_2), \tag{4.15}$$

where $d_\chi$ is defined in (3.3).

**Lemma 4.2.** *Let Condition* 4.1 *hold. Let a function* $\{Z_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$ *be such that* $\operatorname{supp}\{Z_{j\sigma}\} \subset \mathcal{O}_{\varepsilon/2}(0)$*,* $Z_{j\sigma}(0) = 0$*, and the functions* $Z_{j\sigma}$ *do not satisfy the consistency condition* (4.7)*. Then there exists a function* $U \in \mathcal{H}_a^2(K) \cap \mathcal{W}^1(K)$ *such that* $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$*,* $U \notin \mathcal{W}^2(K^\varepsilon)$*, and* $U$ *satisfies the relations*

$$\{\mathbf{P}_j U_j\} \in \mathcal{W}^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon} - Z_{j\sigma}\} \in \mathcal{H}_0^{3/2}(\gamma^\varepsilon). \tag{4.16}$$

**Proof.** By Lemma 4.1, there exists a sequence of vector-valued functions $\{Z_{j\sigma}^n\} \in \mathcal{W}^{3/2}(\gamma)$, $n = 1, 2, \dots$, such that $\operatorname{supp} Z_{j\sigma}^n \subset \mathcal{O}_\varepsilon(0)$, $Z_{j\sigma}^n(0) = 0$, $Z_{j\sigma}^n$ satisfy the consistency condition (4.7) (because the functions $Z_{j\sigma}^n$ vanish near the origin), and $Z_{j\sigma}^n \to Z_{j\sigma}$ in $W^{3/2}(\gamma_j)$. Now we apply Lemma 3.5 in [15], which ensures the existence of a sequence $V^n = (V_1^n, \dots, V_N^n)$ satisfying the following conditions: $V^n \in \mathcal{W}^2(K^d) \cap \mathcal{H}_0^1(K^d)$ for any $d > 0$,

$$\mathcal{P}_j V_j^n = 0 \quad (y \in K_j), \qquad \mathcal{B}_{j\sigma} V^n = Z_{j\sigma}^n(y) \quad (y \in \gamma_{j\sigma}), \tag{4.17}$$

and the sequence $V^n$ converges to some function $V \in \mathcal{H}_0^1(K^d)$ in $\mathcal{H}_0^1(K^d)$ for any $d > 0$. Passing to the limit in the first equality in (4.17) in the sense of distributions and in the second equality in $W^{1/2}(\gamma_{j\sigma}^d)$ for any $d > 0$, we obtain

$$\mathcal{P}_j V_j = 0 \quad (y \in K_j), \qquad \mathcal{B}_{j\sigma} V = Z_{j\sigma}(y) \quad (y \in \gamma_{j\sigma}). \tag{4.18}$$

In particular, it follows from these relations and from Lemma 3.1 that $V = C + V'$, where $V' \in \mathcal{H}_a^2(K^\varepsilon)$ and $C = (C_1, \dots, C_N)$ is a constant vector. Therefore, $C = V - V' \in \mathcal{H}_0^1(K^\varepsilon)$, and hence $C = 0$. Thus, we have proved that

$$V \in \mathcal{H}_a^2(K^d) \cap \mathcal{W}^1(K^d) \quad \forall d > 0. \tag{4.19}$$

Consider a cut-off function $\xi \in C_0^\infty(|y| < \varepsilon')$ equal to one near the origin. Set $U = \xi V$. Clearly, $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$ and, by virtue of (4.19),

$$U \in \mathcal{H}_a^2(K) \cap \mathcal{W}^1(K). \tag{4.20}$$

2. We claim that $U$ is the desired function. Indeed, using Leibniz' formula, relations (4.18) and Lemma A.3, we infer (4.16).

It remains to prove that $U \notin \mathcal{W}^2(K^\varepsilon)$. Assume the contrary. Let $U \in \mathcal{W}^2(K^\varepsilon)$. In this case, it follows from Sobolev's embedding theorem and from the fact that $U \in \mathcal{H}_a^2(K^\varepsilon)$ that $U(0) = 0$. Combining this fact with Lemma A.15 implies that the functions $\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon}$ satisfy the consistency condition (4.7). However, the functions $\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon} - Z_{j\sigma}$ do not satisfy the consistency condition (4.7) in that case. This contradicts (4.16) (see Remark 4.1). □

**Proof of Theorem** 4.2**.** 1. We will construct a generalized solution $u$ supported near the set $\mathcal{K}$ (so that $\mathbf{B}_i^2 u = 0$ due to (2.4)) and such that $u \notin W^2(G)$.

It was shown in the course of the proof of Lemma 3.2 in [15] that there exists a function $\{Z_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma)$ such that $\operatorname{supp} Z_{j\sigma} \subset \mathcal{O}_{\varepsilon/2}(0)$, $Z_{j\sigma}(0) = 0$, and the functions $Z_{j\sigma}$ do not satisfy the consistency condition (4.7). By Lemma 4.2, there exists a function $U \in \mathcal{H}_a^2(K) \cap \mathcal{W}^1(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{\varepsilon'}(0)$, $U \notin \mathcal{W}^2(K)$, and $U$ satisfies relations (4.16). Therefore, $\{\mathbf{P}_j U_j\} \in \mathcal{W}^0(K^\varepsilon)$, $\{\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$, and the functions $\mathbf{B}_{j\sigma} U|_{\gamma_{j\sigma}^\varepsilon}$ do not satisfy the consistency condition (4.7).

2. Introduce a function $u(y)$ such that $u(y) = U_j(y'(y))$ for $y \in \mathcal{O}_{\varepsilon'}(g_j)$ and $u(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_j)$ is the change of variables inverse to the change of variables $y \mapsto y'(g_j)$ from Section 2.1. Since $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$, it follows that $\mathbf{B}_i^2 u = 0$. Therefore, $u(y)$ is the desired generalized solution of problem (2.6), (2.7). □

Theorem 4.2 shows that, if we want *any* generalized solution of problem (2.6), (2.7) to be smooth, then we must take right-hand sides $\{f_0, f_i\}$ from the space $L_2(G) \times \mathcal{S}^{3/2}(\partial G)$.

Let $v$ be an arbitrary function from the space $W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. Consider the change of variables $y \mapsto y'(g_j)$ from Section 2.1 again and introduce the functions

$$B^v_{j\sigma}(y') = (\mathbf{B}^2_i v)(y(y')), \quad y' \in \gamma^\varepsilon_{j\sigma}$$

(cf. functions (2.8)). We prove that the following condition is necessary and sufficient for any generalized solution to be smooth.

**Condition 4.3.** (1) *For any $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, the functions $B^v_{j\sigma}$ satisfy the consistency condition* (4.7).

(2) *For any constant vector $C = (C_1, \dots, C_N)$, the functions $\mathbf{B}_{j\sigma}C|_{\gamma^\varepsilon_{j\sigma}}$ satisfy the consistency condition* (4.7).

Note that the validity of Condition 4.3, unlike Condition 4.2, does not depend on a generalized solution. It depends only on the operators $\mathbf{B}^1_i$ and $\mathbf{B}^2_i$ and on the geometry of the domain $G$ near the set (orbit) $\mathcal{K}$. This is quite natural because we study the smoothness of *all* generalized solutions in this section (while in Section 4.1, we have investigated the smoothness of a fixed solution).

**Theorem 4.3.** *Let Condition* 4.1 *hold. Then the following assertions are true.*

(1) *If Condition* 4.3 *is fulfilled and $u \in W^1(G)$ is a generalized solution of problem* (2.6), (2.7) *with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, then $u \in W^2(G)$.*

(2) *If Condition* 4.3 *fails, then there exists a right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$ and a generalized solution $u \in W^1(G)$ of problem* (2.6), (2.7) *such that $u \notin W^2(G)$.*

**Proof.** 1. *Sufficiency.* Let Condition 4.3 hold, and let $u \in W^1(G)$ be an arbitrary generalized solution of problem (2.6), (2.7) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$. By (3.2), we have $u \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. Therefore, by Condition 4.3, the functions $B^u_{j\sigma}$ satisfy the consistency condition (4.7). Let $C$ be a constant vector defined by Lemma 3.1. Using Condition 4.3 again, we see that the functions $\mathbf{B}_{j\sigma}C$ satisfy the consistency condition (4.7). Since $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$, it follows that the functions $F_{j\sigma}$ satisfy the consistency condition (4.7). Therefore, the functions $\Psi_{j\sigma} = F_{j\sigma} - B^u_{j\sigma}$ and $\mathbf{B}_{j\sigma}C$ satisfy Condition 4.2. Applying Theorem 4.1, we obtain $u \in W^2(G)$.

2. *Necessity.* Let Condition 4.3 fail. In this case, there exist a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and a constant vector $C = (C_1, \dots, C_N)$ such that the functions $B^v_{j\sigma} + \mathbf{B}_{j\sigma} C$ do not satisfy the consistency condition (4.7) (one can assume that either $v = 0$, $C \ne 0$ or $v \ne 0$, $C = 0$). Extend the function $v$ to the domain $G$ in such a way that $v(y) = 0$ for $y \in \mathcal{O}_{\varkappa_1/2}(\mathcal{K})$ and $v \in W^2(G)$.

Consider functions $F'_{j\sigma} \in C^\infty(\overline{\gamma_{j\sigma}})$ such that

$$F'_{j\sigma}(y) = B^v_{j\sigma}(0) + (\mathbf{B}_{j\sigma} C)(0), \quad |y| < \varepsilon/2, \qquad F'_{j\sigma}(y) = 0, \quad |y| > \varepsilon.$$

Since $\partial F'_{j\sigma}/\partial \tau_{j\sigma} = 0$ near the origin, it follows that the functions $F'_{j\sigma}$ satisfy the consistency condition (4.7). By construction,

$$\{F'_{j\sigma} - B^v_{j\sigma} - \mathbf{B}_{j\sigma} C|_{\gamma^\varepsilon_{j\sigma}}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon), \qquad (F'_{j\sigma} - B^v_{j\sigma} - \mathbf{B}_{j\sigma} C)|_{y=0} = 0,$$

and the functions $F'_{j\sigma} - B^v_{j\sigma} - \mathbf{B}_{j\sigma} C$ do not satisfy the consistency condition (4.7). By Lemma 4.2, there exists a function $U' \in \mathcal{H}^2_a(K) \cap \mathcal{W}^1(K)$ such that $\operatorname{supp} U' \subset \mathcal{O}_{\varepsilon'}(0)$, $U' \notin \mathcal{W}^2(K^\varepsilon)$, and

$$\{\mathbf{P}_j U'_j\} \in \mathcal{W}^0(K^\varepsilon), \tag{4.21}$$

$$\big\{\big(\mathbf{B}_{j\sigma} U' - (F'_{j\sigma} - B^v_{j\sigma} - \mathbf{B}_{j\sigma} C)\big)|_{\gamma^\varepsilon_{j\sigma}}\big\} \in \mathcal{H}^{3/2}_0(\gamma^\varepsilon).$$

One can also write the latter relation as follows:

$$\{\mathbf{B}_{j\sigma}(U' + C)|_{\gamma^\varepsilon_{j\sigma}} + B^v_{j\sigma} - F'_{j\sigma}\} \in \mathcal{H}^{3/2}_0(\gamma^\varepsilon). \tag{4.22}$$

Introduce a function $u'(y)$ such that $u'(y) = U'_j(y'(y)) + \xi_j(y) C_j$ for $y \in \mathcal{O}_{\varepsilon'}(g_j)$ and $u'(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_j)$ is the change of variables inverse to the change of variables $y \mapsto y'(g_j)$ from Section 2.1, while $\xi_j \in C^\infty_0(O_{\varepsilon'}(g_j))$, $\xi_j(y) = 1$ for $y \in \mathcal{O}_{\varepsilon'/2}(g_j)$, and $\varepsilon'$ is given by (4.15). Let us prove that the function $u = u' + v$ is the desired one. Clearly, $u \in W^1(G)$, $u \notin W^2(G)$, and $u$ satisfies relations (3.2). It follows from the fact that $v \in W^2(G)$ and from relations (4.21) that

$$\mathbf{P} u \in L_2(G).$$

Consider the functions $f_i = u|_{\Gamma_i} + \mathbf{B}^1_i u + \mathbf{B}^2_i u$. It follows from the fact that $v \in W^2(G)$, from relations (3.2), and from inequality (2.4) that $f_i \in W^{3/2}\big(\Gamma_i \setminus \overline{\mathcal{O}_\delta(\mathcal{K})}\big)$ for any $\delta > 0$. Consider the behavior of $f_i$ near the set $\mathcal{K}$. Note that $\mathbf{B}^2_i u' = 0$ by (2.4). Furthermore, $v|_{\Gamma_i} + \mathbf{B}^1_i v = 0$ for $y \in \mathcal{O}_{\varkappa_1/D_\chi}(\mathcal{K})$. Therefore,

$$f_i = u'|_{\Gamma_i} + \mathbf{B}^1_i u' + \mathbf{B}^2_i v \quad (y \in \mathcal{O}_{\varkappa_1/D_\chi}(\mathcal{K})). \tag{4.23}$$

Introduce the functions $F_{j\sigma}(y') = f_i(y(y'))$, where $y \mapsto y'(g_j)$ is the change of variables from Section 2.1. It follows from (4.23) and from (4.22) that $\{F_{j\sigma} - F'_{j\sigma}\} \in \mathcal{H}_0^{3/2}(\gamma^\varepsilon)$. Therefore, $\{F_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$ and the functions $F_{j\sigma}$, together with $F'_{j\sigma}$, satisfy the consistency condition (4.7). Hence $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$, which completes the proof. □

### 4.3. **Problem with Regular and Homogeneous Nonlocal Conditions.**

**Definition 4.3.** We say that a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ is *admissible* if there exists a constant vector $C = (C_1, \dots, C_N)$ such that

$$B^v_{j\sigma}(0) + (\mathbf{B}_{j\sigma}C)(0) = 0, \quad j = 1, \dots, N, \ \sigma = 1, 2. \tag{4.24}$$

Any vector $C$ satisfying relations (4.24) is said to be an *admissible vector corresponding to the function* $v$.

**Remark 4.4.** The set of admissible functions is linear. It is clear that the function $v = 0$ is admissible, while the vector $C = 0$ is an admissible vector corresponding to the function $v = 0$. In fact, the set of admissible functions is much wider. In particular, it contains all generalized solutions of problem (2.6), (2.7) with homogeneous nonlocal conditions for all $f \in L_2(G)$ (see the proof of Theorem 4.4 below). Therefore, this set consists of infinitely many elements due to Theorem 2.1 in [17].[4]

As for the set of admissible vectors corresponding to an admissible function $v$, it is an affine space of the form

$$\{C + \tilde{C} : \ \tilde{C} = \text{const}, \ (\mathbf{B}_{j\sigma}\tilde{C})(0) = 0\}, \tag{4.25}$$

where $C$ is a fixed admissible vector corresponding to $v$ (if the relations $(\mathbf{B}_{j\sigma}\tilde{C})(0) = 0$, $j = 1, \dots, N$, $\sigma = 1, 2$, hold for $\tilde{C} = 0$ only, then the set of admissible vectors corresponding to $v$ consists of a unique vector). Indeed, if a constant vector $D$ belongs to the set (4.25), then $(\mathbf{B}_{j\sigma}(D - C))(0) = 0$ and, therefore,

$$B^v_{j\sigma}(0) + (\mathbf{B}_{j\sigma}D)(0) = B^v_{j\sigma}(0) + (\mathbf{B}_{j\sigma}C)(0) = 0$$

due to (4.24), i.e., the vector $D$ is admissible. Conversely, if $D$ is an admissible vector corresponding to $v$, then

$$B^v_{j\sigma}(0) + (\mathbf{B}_{j\sigma}D)(0) = 0.$$

Subtracting (4.24) from this equality yields $(\mathbf{B}_{j\sigma}(D - C))(0) = 0$.

[4] Theorem 2.1 in [17] asserts that problem (2.6), (2.7) has the Fredholm property.

**Definition 4.4.** Right-hand sides $f_i$ in nonlocal conditions (2.7) are said to be *regular* if $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$ and $f_i|_{\overline{\Gamma_i}\cap\mathcal{K}} = 0$, $i = 1, \dots, N$.

In particular, right-hand sides $\{f_i\} \in \mathcal{H}_0^{3/2}(\partial G)$ are regular due to the Sobolev embedding theorem and Remark 4.1. In this subsection, we prove that the following condition is necessary and sufficient for any generalized solution of problem (2.6), (2.7) with regular $f_i$ to be smooth.

**Condition 4.4.** *For each admissible function $v$ and for each admissible vector $C$ corresponding to $v$, the functions $B^v_{j\sigma} + \mathbf{B}_{j\sigma}C$ satisfy the consistency condition* (4.7)*.*

Note that Condition 4.4 is, in general, weaker than Condition 4.3.

**Theorem 4.4.** *Let Condition* 4.1 *hold. Then the following assertions are true.*

1. *If Condition* 4.4 *is fulfilled and $u \in W^1(G)$ is a generalized solution of problem* (2.6), (2.7) *with right-hand side $\{f_0, f_i\} \in L_2(G)\times\mathcal{S}^{3/2}(\partial G)$, where $f_i$ are regular, then $u \in W^2(G)$.*
2. *If Condition* 4.4 *fails, then there exists a right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{H}_0^{3/2}(\partial G)$ and a generalized solution $u \in W^1(G)$ of problem* (2.6), (2.7) *such that $u \notin W^2(G)$.*

*Proof.* 1. *Sufficiency.* Let Condition 4.4 hold, and let $u \in W^1(G)$ be an arbitrary generalized solution of problem (2.6), (2.7) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, $f_i|_{\overline{\Gamma_i}\cap\mathcal{K}} = 0$. By (3.2), we have $u \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$.

It follows from the properties of $f_i$ that the right-hand sides in nonlocal conditions (2.10) have the form

$$\Psi_{j\sigma} = F_{j\sigma} - B^u_{j\sigma}, \tag{4.26}$$

where $F_{j\sigma} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$, $F_{j\sigma}(0) = 0$, and $F_{j\sigma}$ satisfy the consistency condition (4.7).

Further, let $U = C + U'$, where $U' \in \mathcal{H}^2_a(K^\varepsilon)$ and $C$ are the function and the constant vector defined in Lemma 3.1. It follows from (2.10) and (4.26) that

$$\mathbf{B}_{j\sigma}U' = F_{j\sigma} - (B^u_{j\sigma} + \mathbf{B}_{j\sigma}C).$$

Since $\{B^u_{j\sigma} + \mathbf{B}_{j\sigma}C|_{\gamma^\varepsilon_{j\sigma}} - F_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon)$ and $U' \in \mathcal{H}^2_a(K^\varepsilon)$, it follows that

$$\{B^u_{j\sigma} + \mathbf{B}_{j\sigma}C|_{\gamma^\varepsilon_{j\sigma}} - F_{j\sigma}\} = \{-\mathbf{B}_{j\sigma}U'\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon) \cap \mathcal{H}^{3/2}_a(\gamma^\varepsilon).$$

Therefore, $B^u_{j\sigma}(0)+(\mathbf{B}_{j\sigma}C)(0)=0$ (since $F_{j\sigma}(0)=0$ due to the above), i.e., $u$ is an admissible function and $C$ is an admissible vector corresponding to $u$. Hence, by virtue of (4.26) and by Condition 4.4, Condition 4.2 holds. Combining this fact with Theorem 4.1 implies $u\in W^2(G)$.

2. *Necessity.* Let Condition 4.4 fail. In this case, there exists a function $v\in W^2(G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and a constant vector $C=(C_1,\dots,C_N)$ such that $B^v_{j\sigma}(0)+(\mathbf{B}_{j\sigma}C)(0)=0$ and the functions $B^v_{j\sigma}+\mathbf{B}_{j\sigma}C$ do not satisfy the consistency condition (4.7).

We must find a function $u\in W^1(G)$ such that $u\notin W^2(G)$ and

$$\mathbf{P}u\in L_2(G),\qquad u|_{\Gamma_i}+\mathbf{B}^1_iu+\mathbf{B}^2_iu\in H^{3/2}_0(\Gamma_i).$$

To do this, one can repeat the proof of assertion 2 of Theorem 4.3, assuming that $v$ is the above function, $C$ is the above constant vector, and $F'_{j\sigma}(y)\equiv 0$ (which is possible due to the relation $B^v_{j\sigma}(0)+(\mathbf{B}_{j\sigma}C)(0)=0$). □

**Corollary 4.1.** *Let Condition* 4.1 *hold. If Condition* 4.4 *fails, then there exist a right-hand side* $\{f_0,f_i\}\in L_2(G)\times\mathcal{H}^{3/2}_0(\partial G)$*, where* $f_i(y)=0$ *for* $y\in\Gamma_i\cap\mathcal{O}_{\varkappa_2}(\mathcal{K})$*, and a generalized solution* $u\in W^1(G)$ *of problem* (2.6), (2.7) *such that* $u\notin W^2(G)$.

The proof of this corollary results from assertion 2 in Theorem 4.4, from the embedding $H^2_0(G)\subset W^2(G)$, and from assertion 1 of the following lemma.

**Lemma 4.3.** (1) *Let* $f_i\in H^{3/2}_0(\Gamma_i)$*,* $i=1,\dots,N$*. Then there exists a function* $u_0\in H^2_0(G)$ *such that*

$$\operatorname{supp}u_0\subset\mathcal{O}_{\varkappa_1}(\mathcal{K}),$$
$$u_0|_{\Gamma_i}=f_i(y),\quad y\in\Gamma_i\cap\mathcal{O}_{\varkappa_2}(\mathcal{K}),\ i=1,\dots,N,$$
$$\mathbf{B}^1_iu_0=\mathbf{B}^2_iu_0=0,\quad i=1,\dots,N. \tag{4.27}$$

(2) *Let* $f_i\in H^{3/2}_0(\Gamma_i)$ *and* $\operatorname{supp}f_i\subset\mathcal{O}_{\varkappa_2}(\mathcal{K})$*,* $i=1,\dots,N$*. Then there exists a function* $u_0\in H^2_0(G)$ *such that*

$$\operatorname{supp}u_0\subset\mathcal{O}_{\varkappa_2}(\mathcal{K}),$$
$$u_0|_{\Gamma_i}=f_i(y),\quad y\in\Gamma_i,\ i=1,\dots,N,$$

*and relations* (4.27) *are valid.*

**Proof.** 1. Using Lemma A.12 and a partition of unity, one can construct a function $u_0\in H^2_0(G)$ such that

$$\operatorname{supp}u_0\subset\mathcal{O}_{\varkappa_1}(\mathcal{K}), \tag{4.28}$$

$$u_0|_{\Gamma_i} = f_i(y), \quad y \in \Gamma_i \cap \mathcal{O}_{\varkappa_2}(\mathcal{K}),\ i = 1, \dots, N, \tag{4.29}$$
$$\mathbf{B}_i^1 u_0 = 0.$$

By (4.28) and (2.4), we have $\mathbf{B}_i^2 u_0 = 0$. Therefore, $u_0$ is the desired function.

2. If $\operatorname{supp} f_i \subset \mathcal{O}_{\varkappa_2}(\mathcal{K})$, we can assume that $\operatorname{supp} u_0 \subset \mathcal{O}_{\varkappa_2}(\mathcal{K})$. In this case, the equality in (4.29) holds for $y \in \Gamma_i$. □

Now we find sufficient conditions for the violation of smoothness of generalized solutions in the case of *homogeneous* nonlocal conditions. The following corollary results from assertion 2 of Theorem 4.4.

**Corollary 4.2.** *Suppose that Condition* 4.1 *holds and Condition* 4.4 *fails. Let* $\{f_0, f_i\} \in L_2(G) \times \mathcal{H}_0^{3/2}(\partial G)$ *be a function constructed in assertion* 2 *of Theorem* 4.4*, and let there exist a function* $u_0 \in W^2(G)$ *such that*

$$u_0|_{\Gamma_i} + \mathbf{B}_i^1 u_0 + \mathbf{B}_i^2 u_0 = f_i(y), \quad y \in \Gamma_i, \quad i = 1, \dots, N. \tag{4.30}$$

*Then there is a right-hand side* $\{f_0, 0\}$*, where* $f_0 \in L_2(G)$*, and a generalized solution* $u \in W^1(G)$ *of problem* (2.6)*,* (2.7) *such that* $u \notin W^2(G)$*.*

We do not have an algorithm allowing one to construct a function $u_0$ satisfying relations (4.30) in the general case of abstract operators $\mathbf{B}_i^2$. However, one can guarantee the existence of $u_0$ in some particular cases which are described in Corollaries 4.3 and 4.4 below (see also Section 7.2).

**Corollary 4.3.** *Suppose that the operators* $\mathbf{B}_i^2$ *satisfy the following condition for some* $\rho > 0$:

$$\|\mathbf{B}_i^2 v\|_{W^{3/2}(\Gamma_i)} \le c\|v\|_{W^2(G_\rho)} \qquad \text{for all} \qquad v \in W^2(G_\rho). \tag{4.31}$$

*Let Condition* 4.1 *hold, and let Condition* 4.4 *fail. Then the conclusion of Corollary* 4.2 *is true.*

The proof of this corollary results from Corollary 4.2, from the embedding $H_0^2(G) \subset W^2(G)$, and from the following lemma.

**Lemma 4.4.** *Let* $f_i \in H_0^{3/2}(\Gamma_i)$*, and let the operators* $\mathbf{B}_i^2$ *satisfy condition* (4.31)*. Then there exists a function* $u_0 \in H_0^2(G)$ *satisfying* (4.30)*.*

**Proof.** Using Lemma A.12 and a partition of unity, one can construct a function $u_0 \in H_0^2(G)$ such that

$$\operatorname{supp} u_0 \subset \overline{G} \setminus \overline{G_\rho}, \tag{4.32}$$
$$u_0|_{\Gamma_i} = f_i(y), \quad y \in \Gamma_i;\ i = 1, \dots, N.$$
$$\mathbf{B}_i^1 u_0 = 0.$$

By (4.32) and (4.31), we have $\mathbf{B}_i^2 u_0 = 0$. Therefore, $u_0$ satisfies (4.30). □

**Remark 4.5.** Condition (4.31), which is stronger than Condition 2.2, means that the operators $\mathbf{B}_i^2$ *correspond to nonlocal terms supported inside the domain* $G$.

**Corollary 4.4.** *Let Condition* 4.1 *hold. Suppose that Condition* 4.4 *fails for an admissible function* $v$ *such that*

$$\operatorname{supp}(v|_{\Gamma_i} + \mathbf{B}_i^1 v + \mathbf{B}_i^2 v) \subset \Gamma_i \cap \mathcal{O}_{\varkappa_2}(\mathcal{K}). \tag{4.33}$$

*Then the conclusion of Corollary* 4.2 *is true.*

**Proof.** If $\operatorname{supp}(v|_{\Gamma_i} + \mathbf{B}_i^1 v + \mathbf{B}_i^2 v) \subset \Gamma_i \cap \mathcal{O}_{\varkappa_2}(\mathcal{K})$, then the function $\{f_i\} \in \mathcal{H}_0^{3/2}(\partial G)$ constructed in the proof of assertion 2 of Theorem 4.4 is also supported in $\mathcal{O}_{\varkappa_2}(\mathcal{K})$. Therefore, applying assertion 2 of Lemma 4.3, we obtain a function $u_0$ satisfying (4.30). Using Corollary 4.2, we complete the proof. □

## 5. Violation of Smoothness of Generalized Solutions

It remains to study the case in which the following condition holds.

**Condition 5.1.** *The band* $-1 \le \operatorname{Im}\lambda < 0$ *contains an improper eigenvalue of the operator* $\tilde{\mathcal{L}}(\lambda)$.

In this section, we show that the smoothness of generalized solutions can be violated for any operators $\mathbf{B}_i^2$ even if nonlocal conditions (2.7) are homogeneous.

**Theorem 5.1.** *Let Condition* 5.1 *hold. Then there exists a right-hand side* $\{f_0, 0\}$*, where* $f_0 \in L_2(G)$*, and a generalized solution* $u \in W^1(G)$ *of problem* (2.6), (2.7) *such that* $u \notin W^2(G)$.

**Proof.** 1. By assertion 2 of Lemma 4.3, it suffices to find a function $u \in W^1(G)$ such that $u \notin W^2(G)$ and

$$\begin{gathered} \mathbf{P}u \in L_2(G), \qquad u|_{\Gamma_i} + \mathbf{B}_i^1 u + \mathbf{B}_i^2 u \in H_0^{3/2}(\Gamma_i), \\ \operatorname{supp}(u|_{\Gamma_i} + \mathbf{B}_i^1 u + \mathbf{B}_i^2 u) \subset \Gamma_i \cap \mathcal{O}_{\varkappa_2}(\mathcal{K}). \end{gathered} \tag{5.1}$$

Let $\lambda = \lambda_0$ be an improper eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, $-1 \le \operatorname{Im}\lambda_0 < 0$. Consider the function

$$W = r^{i\lambda_0} \sum_{l=0}^{m} \frac{1}{l!}(i \ln r)^l \varphi^{(m-l)}(\omega),$$

where $\varphi^{(0)},\dots,\varphi^{(\varkappa-1)}$ are an eigenvector and associated vectors (a Jordan chain of length $\varkappa\ge 1$) of the operator $\tilde{\mathcal{L}}(\lambda)$ corresponding to the eigenvalue $\lambda_0$. The number $m$ $(0\le m\le \varkappa-1)$ occurring in the definition of $W$ is such that the function $W$ is not a polynomial vector in $y_1, y_2$. Such an $m$ does exist because $\lambda_0$ is not a proper eigenvalue (if $\operatorname{Im}\lambda\ne -1$ or $\operatorname{Im}\lambda=-1$, $\operatorname{Re}\lambda\ne 0$, then we can take $m=0$). It follows from Lemma A.9 that

$$\mathcal{P}_j W_j = 0,\qquad \mathcal{B}_{j\sigma}W|_{\gamma_{j\sigma}}=0. \tag{5.2}$$

Consider a cut-off function $\xi\in C_0^\infty(\mathcal{O}_{\varepsilon'}(0))$ equal to one near the origin, where $\varepsilon'$ is given by (4.15). Set $U=\xi W$. Clearly, $\operatorname{supp} U\subset \mathcal{O}_{\varepsilon'}(0)$ and

$$U\in \mathcal{H}_1^2(K)\cap \mathcal{W}^1(K).$$

It follows from this relation, from (5.2), from Leibniz' formula, and from Lemma A.3 that

$$\{\mathbf{P}_j U_j\}\in \mathcal{W}^0(K^\varepsilon),\qquad \{\mathbf{B}_{j\sigma}U|_{\gamma_{j\sigma}^\varepsilon}\}\in \mathcal{H}_0^{3/2}(\gamma^\varepsilon), \tag{5.3}$$

while the relation $\operatorname{supp} U\subset \mathcal{O}_{\varepsilon'}(0)$ implies

$$\operatorname{supp} \mathbf{B}_{j\sigma}U|_{\gamma_{j\sigma}}\subset \gamma_{j\sigma}\cap \mathcal{O}_{\varkappa_2}(0). \tag{5.4}$$

Moreover, we claim that

$$U\notin \mathcal{W}^2(K). \tag{5.5}$$

Indeed, if $-1<\operatorname{Im}\lambda_0<0$, then one can directly verify the validity of (5.5); if $\operatorname{Im}\lambda_0=-1$, then (5.5) follows from Lemma A.6 and from the fact that $W$ is not a polynomial vector.

2. Consider the function $u(y)$ given by $u(y)=U_j(y'(y))$ for $y\in \mathcal{O}_{\varepsilon'}(g_j)$ and $u(y)=0$ for $y\notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y'\mapsto y(g_j)$ is the change of variables inverse to the change of variables $y\mapsto y'(g_j)$ from Section 2.1. The function $u$ is the desired one. Indeed, $u\notin W^2(G)$ due to (5.5). Furthermore, $\mathbf{B}_i^2 u = 0$ due to inequality (2.4) because $\operatorname{supp} u\subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. It follows from the equality $\mathbf{B}_i^2 u=0$ and from relations (5.3) and (5.4) that the function $u$ satisfies (5.1). □

## 6. The Case of Several Orbits

6.1. **Model Problems and Preservation of Smoothness.** In this section, we generalize the results of Sections 2–5 to the case where the set $\mathcal{K}$ consists of more than one orbit. Let

$$\mathcal{K}=\bigcup_{t=1}^{T}\mathcal{K}_t,$$

where $\mathcal{K}_1, \dots, \mathcal{K}_T$ are disjoint orbits forming the set $\mathcal{K}$ of conjugation points. Let the orbit $\mathcal{K}_t$ consists of points $g_{t,1}, \dots, g_{t,N_t}$.

Take a sufficiently small number $\varepsilon$ such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_{t,j})$, $\mathcal{O}_{\varepsilon_1}(g_{t,j}) \supset \mathcal{O}_{\varepsilon}(g_{t,j})$, satisfying the following conditions:

(1) The domain $G$ is a plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_{t,j})$;
(2) $\overline{\mathcal{O}_{\varepsilon_1}(g_{t,j})} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_{\tau,k})} = \varnothing$ for any $g_{t,j}, g_{\tau,k} \in \mathcal{K}$, $(t,j) \neq (\tau,k)$;
(3) If $g_{t,j} \in \overline{\Gamma_i}$ and $\Omega_{is}(g_{t,j}) = g_{t,k}$, then $\mathcal{O}_{\varepsilon}(g_{t,j}) \subset \mathcal{O}_i$, $\Omega_{is}\big(\mathcal{O}_{\varepsilon}(g_{t,j})\big) \subset \mathcal{O}_{\varepsilon_1}(g_{t,k})$.

For each point $g_{t,j} \in \overline{\Gamma_i} \cap \mathcal{K}$, we fix a transformation $Y_{t,j} : y \mapsto y'(g_{t,j})$ which is a composition of the shift by the vector $-\overrightarrow{Og_{t,j}}$ and the rotation through some angle so that

$$Y_{t,j}(\mathcal{O}_{\varepsilon_1}(g_{t,j})) = \mathcal{O}_{\varepsilon_1}(0), \qquad Y_{t,j}(G \cap \mathcal{O}_{\varepsilon_1}(g_{t,j})) = K_{t,j} \cap \mathcal{O}_{\varepsilon_1}(0),$$

$$Y_{t,j}(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_{t,j})) = \gamma_{t,j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0) \quad (\sigma = 1 \text{ or } \sigma = 2),$$

where

$$K_{t,j} = \{y \in \mathbb{R}^2 : \ r > 0, \ |\omega| < \omega_{t,j}\},$$
$$\gamma_{t,j\sigma} = \{y \in \mathbb{R}^2 : \ r > 0, \ \omega = (-1)^{\sigma}\omega_{t,j}\}.$$

Here $(\omega, r)$ are the polar coordinates and $0 < \omega_{t,j} < \pi$.

Consider the following condition instead of Condition 2.1.

**Condition 2.1$'$.** *Let $g_{t,j} \in \overline{\Gamma_i} \cap \mathcal{K}$ and $\Omega_{is}(g_{t,j}) = g_{t,k} \in \mathcal{K}_t$; then the transformation*

$$Y_{t,k} \circ \Omega_{is} \circ Y_{t,j}^{-1} : \mathcal{O}_{\varepsilon}(0) \to \mathcal{O}_{\varepsilon_1}(0)$$

*is the composition of rotation and homothety.*

We assume throughout this section that Conditions 2.1$'$ and 2.2 are fulfilled.

Let $y \mapsto y'(g_{t,j})$ be the above change of variables. Set

$$K_{t,j}^{\varepsilon} = K_{t,j} \cap \mathcal{O}_{\varepsilon}(0), \qquad \gamma_{t,j\sigma}^{\varepsilon} = \gamma_{t,j\sigma} \cap \mathcal{O}_{\varepsilon}(0)$$

and introduce the functions

$$\begin{aligned} U_{t,j}(y') = u(y(y')), \quad F_{t,j}(y') = f_0(y(y')), \quad y' \in K_{t,j}^{\varepsilon}, \\ F_{t,j\sigma}(y') = f_i(y(y')), \quad B_{t,j\sigma}^{u}(y') = (\mathbf{B}_i^2 u)(y(y')), \\ \Psi_{j\sigma}(y') = F_{t,j\sigma}(y') - B_{t,j\sigma}^{u}(y'), \quad y' \in \gamma_{t,j\sigma}^{\varepsilon}, \end{aligned} \tag{6.1}$$

where $\sigma = 1$ ($\sigma = 2$) if the transformation $y \mapsto y'(g_{t,j})$ takes $\Gamma_i$ to the side $\gamma_{t,j1}$ ($\gamma_{t,j2}$) of the angle $K_{t,j}$. Similarly to (2.9), (2.10), using Condition 2.1′, we obtain the following model nonlocal problem for each $t = 1, \dots, T$:

$$\mathbf{P}_{t,j} U_{t,j} = F_{t,j}(y) \quad (y \in K_{t,j}^{\varepsilon},\ j = 1, \dots N_t), \tag{6.2}$$

$$\begin{gathered} \mathbf{B}_{t,j\sigma} U_t \equiv \sum_{k=1}^{N_t} \sum_{s=1}^{S_{t,j\sigma k}} b_{t,j\sigma ks}(y) U_k(\mathcal{G}_{t,j\sigma ks} y) = \Psi_{t,j\sigma}(y) \\ (y \in \gamma_{t,j\sigma}^{\varepsilon},\ j = 1, \dots N_t,\ \sigma = 1, 2). \end{gathered} \tag{6.3}$$

Here $\mathbf{P}_{t,j}$ are properly elliptic second-order differential operators with variable complex-valued $C^\infty$-coefficients, $U_t = (U_{t,1}, \dots, U_{t,N_t})$, $b_{t,j\sigma ks}(y)$ are smooth functions, $b_{t,j\sigma j0}(y) \equiv 1$; $\mathcal{G}_{t,j\sigma ks}$ is an operator of rotation through an angle $\omega_{t,j\sigma ks}$ and homothetic with a coefficient $\chi_{t,j\sigma ks} > 0$ in the $y$-plane. Moreover,

$$|(-1)^\sigma \omega_{t,j} + \omega_{t,j\sigma ks}| < \omega_{t,k} \qquad \text{for} \qquad (k,s) \neq (j,0)$$

and $\omega_{t,j\sigma j0} = 0$, $\chi_{t,j\sigma j0} = 1$ (i.e., $\mathcal{G}_{t,j\sigma j0} y \equiv y$).

Let the principal homogeneous parts of the operators $\mathbf{P}_{t,j}$ at the point $y = 0$ have the following form in the polar coordinates: $r^{-2}\tilde{\mathcal{P}}_{t,j}(\omega, \partial/\partial\omega, r\partial/\partial r)v$. Consider the analytic operator-valued function

$$\tilde{\mathcal{L}}_t(\lambda) : \prod_{j=1}^{N_t} W^2(-\omega_{t,j}, \omega_{t,j}) \to \prod_{j=1}^{N_t} (L_2(-\omega_{t,j}, \omega_{t,j}) \times \mathbb{C}^2)$$

given by

$$\tilde{\mathcal{L}}_t(\lambda)\varphi = \Big\{ \tilde{\mathcal{P}}_{t,j}(\omega, \partial/\partial\omega, i\lambda)\varphi_j, \\ \sum_{k,s} (\chi_{t,j\sigma ks})^{i\lambda} b_{t,j\sigma ks}(0) \varphi_k((-1)^\sigma \omega_{t,j} + \omega_{t,j\sigma ks}) \Big\}.$$

First, we study the case in which the following condition holds.

**Condition 6.1.** *The band $-1 \le \operatorname{Im}\lambda < 0$ contains no eigenvalues of the operators $\tilde{\mathcal{L}}_t(\lambda)$, $t = 1, \dots, T$.*

The following result can be proved similarly to Theorem 3.1.

**Theorem 6.1.** *Let Condition* 6.1 *hold, and let $u \in W^1(G)$ be a generalized solution of problem* (2.6), (2.7) *with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$. Then $u \in W^2(G)$.*

6.2. **Border Case and Violation of Smoothness.** Now we assume that the border case occurs for some of the orbits. Let the following condition hold.

**Condition 6.2.** *The band* $-1 \le \operatorname{Im}\lambda < 0$ *contains only the eigenvalue* $\lambda = -i$ *of the operators* $\tilde{\mathcal{L}}_t(\lambda)$, $t = 1, \dots, t_1$, $t_1 \le T$, *and this eigenvalue is a proper one. If* $t_1 < T$, *then the operators* $\tilde{\mathcal{L}}_t(\lambda)$, $t = t_1 + 1, \dots, T$, *have no eigenvalues in the band* $-1 \le \operatorname{Im}\lambda < 0$.

Analogously to Section 4.1, we will introduce the notion of the consistency condition for each orbit $\mathcal{K}_t$, $t = 1, \dots, t_1$. For each $t = 1, \dots, t_1$, we denote by

$$\{\hat{\mathcal{B}}_{t,j\sigma}(D_y)\},\ j = 1, \dots, N_t,\ \sigma = 1, 2, \tag{6.4}$$

the system of operators (4.2) corresponding to the orbit $\mathcal{K}_t$. It has been proved in Section 4.1 that this system is linearly dependent. Let

$$\{\hat{\mathcal{B}}_{t,j'\sigma'}(D_y)\} \tag{6.5}$$

be a maximal linearly independent subsystem of system (6.4). In this case, any operator $\hat{\mathcal{B}}_{t,j\sigma}(D_y)$ which does not enter system (6.5) can be represented as follows:

$$\hat{\mathcal{B}}_{t,j\sigma}(D_y) = \sum_{j',\sigma'} \beta_{t,j\sigma}^{j'\sigma'} \hat{\mathcal{B}}_{t,j'\sigma'}(D_y), \tag{6.6}$$

where $\beta_{t,j\sigma}^{j'\sigma'}$ are some constants.

To introduce the notion of the consistency condition, we consider arbitrary functions $\{Z_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma_t^\varepsilon)$, each of which is defined on its own interval $\gamma_{t,j\sigma}^\varepsilon$. Consider the functions

$$Z_{j\sigma}^0(r) = Z_{j\sigma}(y)|_{y=(r\cos\omega_{t,j},\, r(-1)^\sigma \sin\omega_{t,j})}.$$

Each of the functions $Z_{j\sigma}^0$ belongs to $W^{3/2}(0, \varepsilon)$.

**Definition 6.1.** Let $\beta_{t,j\sigma}^{j'\sigma'}$ be the constants occurring in (6.6). If the relations

$$\int_0^\varepsilon r^{-1} \left| \frac{d}{dr} \Big( Z_{j\sigma}^0 - \sum_{j',\sigma'} \beta_{t,j\sigma}^{j'\sigma'} Z_{j'\sigma'}^0 \Big) \right|^2 dr < \infty \tag{6.7}$$

hold for all indices $j, \sigma$ corresponding to the operators of system (6.4) which do not enter system (6.5), then we say that the *functions* $Z_{j\sigma}$ *satisfy the consistency condition* (6.7).

Denote by $\mathcal{S}^{3/2}(\partial G)$ the set of functions $\{f_i\} \in \mathcal{W}^{3/2}(\partial G)$ such that the functions $F_{t,j\sigma}$ (see (6.1)) satisfy the consistency condition (6.7) for each $t = 1, \dots, t_1$.

The following result can be proved similarly to Theorem 4.2

**Theorem 6.2.** *Let Condition* 6.2 *hold. Then there exist a function* $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$, $\{f_i\} \notin \mathcal{S}^{3/2}(\partial G)$, *and a function* $u \in W^1(G)$ *such that* $u$ *is a generalized solution of problem* (2.6), (2.7) *with the right-hand side* $\{f_0, f_i\}$ *and* $u \notin W^2(G)$.

Now we assume that $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$ and prove that the following condition is necessary and sufficient for any generalized solution to be smooth.

**Condition 6.3.** (1) *For any* $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, *the functions* $B^v_{t,j\sigma}$ *satisfy the consistency condition* (6.7), *where* $t = 1, \dots, t_1$.

(2) *For any vector* $C_t = (C_{t,1}, \dots, C_{t,N_t})$ *with constant elements, the functions* $\mathbf{B}_{t,j\sigma} C_t|_{\gamma^\varepsilon_{t,j\sigma}}$ *satisfy the consistency condition* (6.7), *where* $t = 1, \dots, t_1$.

Set $\varepsilon' = d'_\chi \min(\varepsilon, \varkappa_2)$, where $d'_\chi = \min\{\chi_{t,j\sigma ks}\}/2$.

**Theorem 6.3.** *Let Condition* 6.2 *hold. Then the following assertions are true.*

(1) *If Condition* 6.3 *is fulfilled and* $u \in W^1(G)$ *is a generalized solution of problem* (2.6), (2.7) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, *then* $u \in W^2(G)$.

(2) *If Condition* 6.3 *fails, then there exists a right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$ *and a generalized solution* $u \in W^1(G)$ *of problem* (2.6), (2.7) *such that* $u \notin W^2(G)$.

**Proof.** The proof of this theorem is similar to the proof of Theorem 4.3. For instance, we prove assertion 2. Let Condition 6.3 be violated, e.g., for the orbit $\mathcal{K}_1$. In this case, there exist a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and a constant vector $C_1 = (C_{1,1}, \dots, C_{1,N_1})$ such that the functions $B^v_{1,j\sigma} + \mathbf{B}_{1,j\sigma} C_1$ do not satisfy the consistency condition (6.7) for $t = 1$ (one can assume that either $v = 0$, $C_1 \neq 0$ or $v \neq 0$, $C_1 = 0$). Extend the function $v$ to the domain $G$ in such a way that $v(y) = 0$ for $y \in \mathcal{O}_{\varkappa_1/2}(\mathcal{K})$ and $v \in W^2(G)$.

Consider functions $F'_{t,j\sigma} \in C^\infty(\overline{\gamma_{t,j\sigma}})$ such that

$$F'_{t,j\sigma}(y) = B^v_{t,j\sigma}(0) + (\mathbf{B}_{t,j\sigma} C_t)(0), \quad |y| < \varepsilon/2, \qquad F'_{t,j\sigma}(y) = 0, \quad |y| > \varepsilon,$$

where $t = 1, \dots, T$, $C_1$ is the above vector, and $C_2, \dots, C_T$ are arbitrary (but fixed) constant vectors.

Since the $F_{t,j\sigma}$ are constant near the origin, it follows that they satisfy the consistency condition (6.7) for each $t = 1, \dots, t_1$. By construction,

$$\{F'_{t,j\sigma} - B^v_{t,j\sigma} - \mathbf{B}_{t,j\sigma} C_t|_{\gamma^\varepsilon_{t,j\sigma}}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon), \quad (F'_{t,j\sigma} - B^v_{t,j\sigma} - \mathbf{B}_{t,j\sigma} C_t)|_{y=0} = 0,$$

where $t = 1, \dots, T$. Moreover, the functions $F'_{1,j\sigma} - B^v_{1,j\sigma} - \mathbf{B}_{1,j\sigma} C_1$ do not satisfy the consistency condition (6.7) for $t = 1$.

By Lemma 4.2, there exists a function $U'_1 \in \mathcal{H}^2_a(K_1) \cap \mathcal{W}^1(K_1)$ such that $\operatorname{supp} U'_1 \subset \mathcal{O}_{\varepsilon'}(0)$, $U'_1 \notin \mathcal{W}^2(K^\varepsilon_1)$, and

$$\{\mathbf{P}_{1,j} U'_{1,j}\} \in \mathcal{W}^0(K^\varepsilon_1), \tag{6.8}$$

$$\big\{\big(\mathbf{B}_{1,j\sigma} U'_1 - (F'_{1,j\sigma} - B^v_{1,j\sigma} - \mathbf{B}_{1,j\sigma} C_1)\big)|_{\gamma^\varepsilon_{1,j\sigma}}\big\} \in \mathcal{H}^{3/2}_0(\gamma^\varepsilon_1).$$

One can also write the latter relation as follows:

$$\{\mathbf{B}_{1,j\sigma}(U'_1 + C_1)|_{\gamma^\varepsilon_{1,j\sigma}} + B^v_{1,j\sigma} - F'_{1,j\sigma}\} \in \mathcal{H}^{3/2}_0(\gamma^\varepsilon_1). \tag{6.9}$$

Let $t = 2, \dots, t_1$. It follows from Lemma 4.2 (if the functions

$$F'_{t,j\sigma} - B^v_{t,j\sigma} - \mathbf{B}_{t,j\sigma} C_t \tag{6.10}$$

do not satisfy the consistency condition (6.7)) or from Lemma A.14 (if the functions (6.10) satisfy the consistency condition (6.7)) that there exists a function $U'_t \in \mathcal{H}^2_a(K_t) \cap \mathcal{W}^1(K_t)$ such that $\operatorname{supp} U'_t \subset \mathcal{O}_{\varepsilon'}(0)$ and

$$\{\mathbf{P}_{t,j} U'_{t,j}\} \in \mathcal{W}^0(K^\varepsilon_t), \tag{6.11}$$

$$\{\mathbf{B}_{t,j\sigma}(U'_t + C_t)|_{\gamma^\varepsilon_{t,j\sigma}} + B^v_{t,j\sigma} - F'_{t,j\sigma}\} \in \mathcal{H}^{3/2}_0(\gamma^\varepsilon_t). \tag{6.12}$$

(If Lemma A.14 has been applied, then $U'_t \in \mathcal{W}^2(K^\varepsilon_t)$.)

Finally, let $t = t_1 + 1, \dots, T$. In this case, by Lemma A.13, there exists a function $U'_t \in \mathcal{H}^2_a(K_t) \cap \mathcal{W}^2(K_t)$ such that $\operatorname{supp} U'_t \subset \mathcal{O}_{\varepsilon'}(0)$ and relations (6.11) and (6.12) hold.

Introduce a function $u'(y)$ such that $u'(y) = U'_{t,j}(y'(y)) + \xi_{t,j}(y) C_{t,j}$ for $y \in \mathcal{O}_{\varepsilon'}(g_{t,j})$ and $u'(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon'}(\mathcal{K})$, where $y' \mapsto y(g_{t,j})$ is the change of variables inverse to the change of variables $y \mapsto y'(g_{t,j})$, while $\xi_{t,j} \in C^\infty_0(O_{\varepsilon'}(g_{t,j}))$, $\xi_{t,j}(y) = 1$ for $y \in \mathcal{O}_{\varepsilon'/2}(g_{t,j})$. Similarly to the proof of assertion 2 in Theorem 4.3, using relations (6.8), (6.9), (6.11), and (6.12), one can verify that the function $u = u' + v$ is the desired one. □

Now we consider problem (2.6), (2.7) with regular and homogeneous nonlocal conditions.

**Definition 6.2.** We say that a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ is *admissible* if there exist constant vectors $C_t = (C_{t,1}, \dots, C_{t,N_t})$, $t = 1, \dots, T$, such that

$$B^v_{t,j\sigma}(0) + (\mathbf{B}_{t,j\sigma} C_t)(0) = 0, \quad j = 1, \dots, N,\ \sigma = 1, 2,\ t = 1, \dots T. \tag{6.13}$$

Vectors $C_t$, $t = 1, \dots T$, satisfying relations (6.13) are said to be *admissible vectors corresponding to the function* $v$.

**Definition 6.3.** Right-hand sides $f_i$ in nonlocal conditions (2.7) are said to be *regular* if $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$ and $f_i|_{\overline{\Gamma_i} \cap \mathcal{K}_t} = 0$, $t = 1, \dots T$ (i.e., $f_i|_{\overline{\Gamma_i} \cap \mathcal{K}} = 0$).

We prove that the following condition is necessary and sufficient for any generalized solution of problem (2.6), (2.7) with regular $f_i$ to be smooth.

**Condition 6.4.** *For each admissible function $v$ and for each admissible vector $C_t$, $t = 1, \dots, t_1$, corresponding to $v$, the functions $B^v_{t,j\sigma} + \mathbf{B}_{t,j\sigma} C_t$ satisfy the consistency condition* (6.7).

**Theorem 6.4.** *Let Condition* 6.2 *hold. Then the following assertions are true.*

(1) *If Condition* 6.4 *is fulfilled and $u \in W^1(G)$ is a generalized solution of problem* (2.6), (2.7) *with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, where $f_i$ are regular, then $u \in W^2(G)$.*

(2) *If Condition* 6.4 *fails, then there exists a right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{H}^{3/2}_0(\partial G)$ and a generalized solution $u \in W^1(G)$ of problem* (2.6), (2.7) *such that $u \notin W^2(G)$.*

**Proof.** The proof of the theorem is similar to the proof of Theorem 4.4. For instance, let us prove assertion 2. If Condition 6.4 fails, there exists a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ and constant vectors $C_t = (C_{t,1}, \dots, C_{t,N_t})$, $t = 1, \dots, T$, such that $B^v_{t,j\sigma}(0) + (\mathbf{B}_{t,j\sigma} C_t)(0) = 0$ and, e.g., the functions $B^v_{1,j\sigma} + \mathbf{B}_{1,j\sigma} C_1$ do not satisfy the consistency condition (6.7).

We must find a function $u \in W^1(G)$ such that $u \notin W^2(G)$ and

$$\mathbf{P}u \in L_2(G), \qquad u|_{\Gamma_i} + \mathbf{B}^1_i u + \mathbf{B}^2_i u \in H^{3/2}_0(\Gamma_i).$$

To do this, one can repeat the proof of assertion 2 in Theorem 6.3, assuming that $v$ is the above function, $C_t$, $t = 1, \dots, T$, are the above vectors, and $F'_{t,j\sigma}(y) \equiv 0$, $t = 1, \dots, T$ (which is possible due to the relations $B^v_{t,j\sigma}(0) + (\mathbf{B}_{t,j\sigma} C_t)(0) = 0$). □

**Remark 6.1.** It is easy to see that Corollaries 4.1–4.4 (in which Conditions 4.1 and 4.4 and Theorem 4.4 must be replaced by Conditions 6.2 and 6.4 and Theorem 6.4, respectively) are true in the case of several orbits.

It remains to study the case in which the following condition holds.

**Condition 6.5.** *There is a number $t \in \{1, \dots, T\}$ such that the band $-1 \le \operatorname{Im}\lambda < 0$ contains an improper eigenvalue of the operator $\tilde{\mathcal{L}}_t(\lambda)$.*

The proof of the following result is similar to the proof of Theorem 5.1.

**Theorem 6.5.** *Let Condition* 6.5 *hold. Then there exist a right-hand side $\{f_0, 0\}$, where $f_0 \in L_2(G)$, and a generalized solution $u \in W^1(G)$ of problem* (2.6), (2.7) *such that $u \notin W^2(G)$.*

## 7. Example

7.1. **Problem with Nonhomogeneous Nonlocal Conditions.** In this section, we apply the results of Sections 2–6 to the study of the smoothness of generalized solutions for problem (1.1), (1.2). We recall the setting of this problem.

Let $\partial G \setminus \mathcal{K} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i$ are open (in the topology of $\partial G$) $C^\infty$-curves and $\mathcal{K} = \overline{\Gamma_1} \cap \overline{\Gamma_2} = \{g, h\}$, where $g, h$ are the end points of the curves $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$. Suppose that the domain $G$ is the plane angle of opening $\pi$ in a neighborhood of each of the points $g, h$. Thus, the boundary of $G$ is infinitely smooth. We consider the following nonlocal problem in $G$:

$$\Delta u = f_0(y) \quad (y \in G), \tag{7.1}$$

$$\begin{aligned} &u|_{\Gamma_1} + b_1(y)u\big(\Omega_1(y)\big)\big|_{\Gamma_1} + a(y)u\big(\Omega(y)\big)\big|_{\Gamma_1} = f_1(y) \quad (y \in \Gamma_1),\\ &u|_{\Gamma_2} + b_2(y)u\big(\Omega_2(y)\big)\big|_{\Gamma_2} = f_2(y) \quad (y \in \Gamma_2). \end{aligned} \tag{7.2}$$

Here $b_1$, $b_2$, and $a$ are real-valued $C^\infty$-functions; $\Omega_i$ and $\Omega$ are $C^\infty$-diffeomorphisms described in the Introduction (see Figure 1.1).

Let us show that the nonlocal conditions (7.2) can be represented in the form (2.7). To do this, we take a small $\varepsilon$ such that the sets $\overline{\mathcal{O}_\varepsilon(g)}$ and $\overline{\mathcal{O}_\varepsilon(h)}$ do not intersect with the curve $\overline{\Omega(\Gamma_1)}$.

Consider a function $\zeta \in C_0^\infty(\mathbb{R}^2)$ such that $\zeta(y) = 1$ for $y \in \mathcal{O}_{\varepsilon/2}(\mathcal{K})$ and $\operatorname{supp}\zeta \subset \mathcal{O}_\varepsilon(\mathcal{K})$. Introduce the operators

$$\begin{gathered} \mathbf{B}_i^1 u = \zeta(y) b_i(y) u(\Omega_i(y))|_{\Gamma_i},\\ \mathbf{B}_1^2 u = (1 - \zeta(y)) b_1(y) u(\Omega_1(y))|_{\Gamma_1} + a(y) u(\Omega(y))|_{\Gamma_1},\\ \mathbf{B}_2^2 u = (1 - \zeta(y)) b_2(y) u(\Omega_2(y))|_{\Gamma_2}. \end{gathered}$$

In this example, the set $\mathcal{K}$ is formed by two orbits; the first orbit consists of the point $g$ and the second orbit of the point $h$. Since the support of $\zeta$ is contained in a neighborhood of the set $\mathcal{K}$, one can assume that the transformations $\Omega_i$ occurring in the definition of the operators $\mathbf{B}_i^1$ are also defined in a neighborhood of the set $\mathcal{K}$ and satisfy Condition 2.1′. Furthermore, due to the arguments of [15, Section 1.2], the operators $\mathbf{B}_i^2$ satisfy Condition 2.2 with $\varkappa_1 = \varepsilon/2$ and some $\varkappa_2 < \varkappa_1$ and $\rho$. Therefore, nonlocal conditions (7.2) can be represented in the form (2.7).

Write a model problem corresponding to the point $g$ (one can similarly write a model problem corresponding to the point $h$). To be definite, we assume that the point $g$ coincides with the origin, $g = 0$, while the axis $Oy_1$ is directed inside the domain $G$, perpendicularly to the boundary. Consider the sets

$$K^\varepsilon = \{y \in \mathbb{R}^2 :\ 0 < r < \varepsilon,\ |\omega| < \pi/2\},$$
$$\gamma_\sigma^\varepsilon = \{y \in \mathbb{R}^2 :\ 0 < r < \varepsilon,\ \omega = (-1)^\sigma \pi/2\}.$$

Take a small number $\varepsilon$ such that $\mathcal{O}_\varepsilon(0) \cap G = K^\varepsilon$. The model problem acquires the form

$$\Delta U = F(y) \quad (y \in K^\varepsilon), \tag{7.3}$$
$$U(y) + b_\sigma(y)U(\mathcal{G}_\sigma y) = \Psi_\sigma(y) \quad (y \in \gamma_\sigma^\varepsilon,\ \sigma = 1, 2). \tag{7.4}$$

Here $\mathcal{G}_\sigma = \begin{pmatrix} 0 & (-1)^\sigma \\ (-1)^{\sigma+1} & 0 \end{pmatrix}$ is the operator of rotation through the angle $(-1)^{\sigma+1}\pi/2$,

$$F(y) = f_0(y), \quad y \in K^\varepsilon, \qquad \Psi_\sigma(y) = f_\sigma(y) - B_\sigma^u(y), \quad y \in \gamma_\sigma^\varepsilon;$$

moreover,

$$B_1^u(y) = a(y)u\big(\Omega(y)\big), \quad y \in \gamma_1^{\varepsilon/2}, \qquad B_2^u(y) = 0, \quad y \in \gamma_2^{\varepsilon/2},$$

because $(1 - \zeta(y))b_\sigma(y)u(\Omega_\sigma(y)) = 0$ for $y \in \gamma_\sigma^{\varepsilon/2}$, $\sigma = 1, 2$.

The eigenvalue problem has the form

$$\varphi''(\omega) - \lambda^2\varphi(\omega) = 0 \quad (|\omega| < \pi/2), \tag{7.5}$$
$$\varphi(-\pi/2) + b_1(0)\varphi(0) = 0, \quad \varphi(\pi/2) + b_2(0)\varphi(0) = 0. \tag{7.6}$$

Set $I_1 = (-\infty, -2] \cup (0, \infty)$ and $I_2 = (-2, 0)$. Simple calculations [16, Section 9] show that the eigenvalues of problem (7.5), (7.6) are distributed in the band $-1 \le \operatorname{Im}\lambda < 0$ as follows.

**Case 1 ($b_1(0) + b_2(0) \in I_1$):** the band $-1 \le \operatorname{Im}\lambda < 0$ contains no eigenvalues.

**Case 2** **($b_1(0)+b_2(0)=0$):** the band $-1 \le \operatorname{Im}\lambda < 0$ contains the unique eigenvalue $\lambda = -i$, and this eigenvalue is proper.

**Case 3** **($b_1(0)+b_2(0) \in I_2$):** the band $-1 \le \operatorname{Im}\lambda < 0$ contains the improper eigenvalue $\lambda = 2\pi^{-1} i \arctan\big(\sqrt{4-(b_1(0)+b_2(0))^2}/(b_1(0)+b_2(0))\big)$.

Consider Case 1.

**Theorem 7.1.** *Let $b_1(0)+b_2(0) \in I_1$ and $b_1(h)+b_2(h) \in I_1$. Let $u \in W^1(G)$ be a generalized solution of problem* (7.1), (7.2) *with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}^{3/2}(\partial G)$. Then $u \in W^2(G)$.*

**Proof.** In the case under consideration, the band $-1 \le \operatorname{Im}\lambda < 0$ contains no eigenvalues of problem (7.5), (7.6) (and no eigenvalues of the analogous problem corresponding to the point $h$). Therefore, this theorem follows from Theorem 6.1. □

Note that we impose no consistency conditions on the coefficients $b_i$ and $a$ and on the right-hand sides $f_i$ in Case 1.

Consider Case 2. To be definite, we assume that $b_1(h)+b_2(h) \in I_1$. In this case, the consistency condition (6.7) is considered only near the origin. Let us find out the form of this condition in terms of problem (7.1), (7.2). Let $\tau_\sigma$ denote the vector with the coordinates $(0, (-1)^\sigma)$. Then $\partial/\partial\tau_\sigma = (-1)^\sigma \partial/\partial y_2$ and

$$\frac{\partial}{\partial \tau_1}\big(U(y) + b_1(0)U(\mathcal{G}_1 y)\big) = -U_{y_2}(y) + b_1(0)U_{y_1}(\mathcal{G}_1 y),$$

$$\frac{\partial}{\partial \tau_2}\big(U(y) + b_2(0)U(\mathcal{G}_2 y)\big) = U_{y_2}(y) + b_2(0)U_{y_1}(\mathcal{G}_2 y).$$

Therefore,

$$\hat{\mathcal{B}}_\sigma(D_y)U = (-1)^\sigma U_{y_2} + b_\sigma(0)U_{y_1}, \quad \sigma = 1, 2.$$

Since $b_1(0)+b_2(0) = 0$, it follows that the operators $\hat{\mathcal{B}}_1(D_y)$ and $\hat{\mathcal{B}}_2(D_y)$ are linearly dependent, $\hat{\mathcal{B}}_1(D_y) + \hat{\mathcal{B}}_2(D_y) = 0$. Thus, the consistency condition (6.7) for functions $Z_\sigma \in W^{3/2}(\gamma_\sigma^\varepsilon)$ acquires the form

$$\int_0^\varepsilon r^{-1}\Big|\frac{\partial Z_1}{\partial y_2}\Big|_{y=(0,-r)} - \frac{dZ_2}{dy_2}\Big|_{y=(0,r)}\Big|^2 dr < \infty. \tag{7.7}$$

Due to (7.7), the space $\mathcal{S}^{3/2}(\partial G)$ consists of the functions $\{f_i\} \in \mathcal{W}^{3/2}(\partial G)$ such that

$$\int_0^\varepsilon r^{-1}\Big|\frac{\partial f_1}{\partial y_2}\Big|_{y=(0,-r)} - \frac{\partial f_2}{\partial y_2}\Big|_{y=(0,r)}\Big|^2 dr < \infty. \tag{7.8}$$

By Theorem 6.2, the validity of the condition $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$ is necessary for any generalized solution of problem (7.1), (7.2) to belong to $W^2(G)$.

**Theorem 7.2.** *Let $b_1(0) + b_2(0) = 0$ and $b_1(h) + b_2(h) \in I_1$. Then the following assertions are true.*

(1) *If*

$$a(0) = 0, \quad \frac{\partial a}{\partial y_2}\Big|_{y=0} = 0, \tag{7.9}$$

$$\int_0^{\varepsilon} r^{-1}\Big|\frac{\partial b_1}{\partial y_2}\Big|_{y=(0,-r)} - \frac{\partial b_2}{\partial y_2}\Big|_{y=(0,r)}\Big|^2 dr < \infty, \tag{7.10}$$

*and $u \in W^1(G)$ is a generalized solution of problem* (7.1), (7.2) *with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, then $u \in W^2(G)$.*

(2) *If condition* (7.9)–(7.10) *fails, then there exists a right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$ and a generalized solution $u \in W^1(G)$ of problem* (7.1), (7.2) *such that $u \notin W^2(G)$.*

**Proof.** 1. By Theorem 6.3, it suffices to prove that condition (7.9)–(7.10) is equivalent to Condition 6.3.

For any function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, set $v_\Omega(y) = v\big(\Omega(y)\big)$, $y \in \Gamma_1$. In this case, we have

$$B_1^v(y) = a(y)v_\Omega(y), \quad y \in \gamma_1^{\varepsilon/2}, \qquad B_2^v(y) = 0, \quad y \in \gamma_2^{\varepsilon/2}.$$

Therefore, the functions $B_\sigma^v$ satisfy the consistency condition (7.7) if and only if

$$\int_0^{\varepsilon/2} r^{-1}\Big|\frac{\partial(av_\Omega)}{\partial y_2}\Big|_{y=(0,-r)}\Big|^2 dr = \int_0^{\varepsilon/2} r^{-1}\Big|\Big(\frac{\partial a}{\partial y_2}v_\Omega + a\frac{\partial v_\Omega}{\partial y_2}\Big)\Big|_{y=(0,-r)}\Big|^2 dr < \infty. \tag{7.11}$$

We take $\varepsilon/2$ instead of $\varepsilon$ as the upper limit of integration because the functions $B_\sigma^v$ look simpler in this case; clearly, this change does not affect the convergence of the integral.

Let us prove that condition (7.11) is equivalent to (7.9). Suppose that (7.11) holds. Take a function $v$ such that $v_\Omega(y) = y_2$ near the origin; then we have

$$\frac{\partial(av_\Omega)}{\partial y_2}\Big|_{y=0} = a(0).$$

Since the function $\partial(av_\Omega)/\partial y_2$ is continuous near the origin, it follows from the latter relation and from (7.11) that $a(0) = 0$. In a similar way, substituting a function $v$ such that $v_\Omega(y) = 1$ near the origin into (7.11), we obtain $(\partial a/\partial y_2)|_{y=0} = 0$.

Conversely, suppose that (7.9) holds. By virtue of smoothness of the transformation $\Omega$, we have

$$v_\Omega, \frac{\partial v_\Omega}{\partial y_2} \in W^{1/2}(\gamma_1^\varepsilon) \subset H_1^{1/2}(\gamma_1^\varepsilon)$$

for any $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. It follows from this relation, from (7.9), and from Lemma A.3 that $\partial(av_\Omega)/\partial y_2 \in H_0^{1/2}(\gamma_1^\varepsilon)$. Therefore, by Lemma A.5, relation (7.11) follows. Thus, we have proved that part 1 of Condition 6.3 is equivalent to condition (7.9).

2. Part 2 of Condition 6.3 is fulfilled if and only if the functions $C+b_1(y)C$ and $C+b_2(y)C$ satisfy the consistency condition (7.7) for any constant $C$. The latter is equivalent to (7.10). □

Thus, we see that, in Case 2, the smoothness of generalized solutions depends on the values of the first derivatives of the coefficients $b_1, b_2$ *near* the origin as well as on the values of the coefficient $a$ and its first derivative *at* the origin.

Consider Case 3.

**Theorem 7.3.** *Let $b_1(0) + b_2(0) \in I_2$ or $b_1(h) + b_2(h) \in I_2$. Then there exists a right-hand side $\{f_0, 0\}$, where $f_0 \in L_2(G)$, and a generalized solution $u \in W^1(G)$ of problem* (7.1), (7.2) *such that $u \notin W^2(G)$.*

**Proof.** The band $-1 \le \operatorname{Im}\lambda < 0$ contains an improper eigenvalue of problem (7.5), (7.6) (or an improper eigenvalue of the analogous problem corresponding to the point $h$). Therefore, this theorem follows from Theorem 6.5. □

Thus, in Case 3, the smoothness of generalized solutions can be violated irrespective of the behavior of the coefficient $a$ and of the derivatives of the coefficients $b_1, b_2$ near the point $g$.

7.2. **Problem with Regular and Homogeneous Nonlocal Conditions.** Consider problem (7.1), (7.2) with regular and homogeneous nonlocal conditions. By Theorems 7.1 and 7.3, the smoothness of generalized solutions is preserved in Case 1 and can be violated in Case 3. Case 2 (the border case) is of particular interest.

First, we study the case of regular right-hand sides. To be definite, we again assume that

$$b_1(h) + b_2(h) \in I_1.$$

**Theorem 7.4.** *Let $b_1(0) + b_2(0) = 0$ and $b_1(h) + b_2(h) \in I_1$. Then the following assertions are true.*

(1) *If*

$$a(0) = 0, \quad \frac{\partial a}{\partial y_2}\bigg|_{y=0} = 0 \tag{7.12}$$

*and $u \in W^1(G)$ is a generalized solution of problem (7.1), (7.2) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}^{3/2}(\partial G)$, where $f_i(0) = 0$, then $u \in W^2(G)$.*

(2) *If condition (7.12) fails, then there exists a right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{H}_0^{3/2}(\partial G)$, where $f_i(y) = 0$ in a neighborhood of the origin, and a generalized solution $u \in W^1(G)$ of problem (7.1), (7.2) such that $u \notin W^2(G)$.*

**Proof.** 1. By virtue of Theorem 6.4 and Corollary 4.1, it suffices to prove that condition (7.12) is equivalent to Condition 6.4.

By Definition 6.2, a function $v \in W^2(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ is admissible if there exist constants $C$ and $C_h$ such that

$$\begin{aligned} a(0)v_\Omega(0) + C + b_1(0)C = 0, \quad C + b_2(0)C = 0, \\ a(h)v_\Omega(h) + C_h + b_1(h)C_h = 0, \quad C_h + b_2(h)C_h = 0, \end{aligned} \tag{7.13}$$

where $v_\Omega(y) = v\big(\Omega(y)\big)$, $y \in \Gamma_1$.

Let $\xi \in C^\infty(\mathbb{R}^2)$ be a cut-off function such that

$$\operatorname{supp}\xi \subset \mathcal{O}_\delta(\Omega(0)), \qquad \xi(y) = 1, \quad y \in \mathcal{O}_{\delta/2}(\Omega(0)),$$

where $\delta > 0$ is so small that $\Omega(h) \notin \mathcal{O}_\delta(\Omega(0))$. Since $b_1(h) + b_2(h) \in I_1$, the consistency condition (6.7) is considered only near the origin. Therefore, if $v$ is an admissible function, $C, C_h$ are admissible constants corresponding to $v$, and Condition 6.4 holds (fails) for $v$ and $C$, then the function $\xi v$ is admissible, $C, 0$ are admissible constants corresponding to $\xi v$, and Condition 6.4 holds (respectively, fails) for $\xi v$ and $C$. Thus, it suffices to consider only functions $v$ supported in $\mathcal{O}_\delta(\Omega(0))$ (i.e., functions $v_\Omega$ supported near the origin) and assume that $C_h = 0$.

First, we study the situation in which $b_2(0) \neq -1$. In this case, according to (7.13), a function $v$ supported in $\mathcal{O}_\delta(\Omega(0))$ is admissible if and only if

$$a(0)v_\Omega(0) = 0, \tag{7.14}$$

while the corresponding set of admissible vectors (constants in our case) consists of the unique constant $C = 0$ (recall that $C_h$ is supposed to equal zero). Therefore, Condition 6.4 holds if and only if the relation

$$\int_0^{\varepsilon/2} r^{-1}\Big|\frac{\partial(av_\Omega)}{\partial y_2}\Big|_{y=(0,-r)}\Big|^2 dr = \int_0^{\varepsilon/2} r^{-1}\Big|\Big(\frac{\partial a}{\partial y_2}v_\Omega + a\frac{\partial v_\Omega}{\partial y_2}\Big)\Big|_{y=(0,-r)}\Big|^2 dr < \infty \tag{7.15}$$

holds for any $v_\Omega$ satisfying (7.14). Suppose that (7.12) is fulfilled. Then any function $v$ supported in $\mathcal{O}_\delta(\Omega(0))$ is admissible (because $a(0) = 0$), and repeating the arguments of the proof of Theorem 7.2 yields (7.15).

Conversely, suppose that (7.15) holds for any function $v_\Omega$ satisfying (7.14). Clearly, a function $v$ such that $v_\Omega(y) = y_2$ near the origin satisfies (7.14). Substituting the function $v_\Omega$ into (7.15), we obtain $a(0) = 0$ (cf. the proof of Theorem 7.2). Therefore, any function $v$ supported in $\mathcal{O}_\delta(\Omega(0))$ is admissible. Substituting $v_\Omega(y) = 1$ into (7.15), we obtain $(\partial a/\partial y_2)|_{y=0} = 0$.

2. It remains to study the situation in which $b_2(0) = -1$. This implies $b_1(0) = 1$. In this case, according to (7.13), any function $v$ supported in $\mathcal{O}_\delta(\Omega(0))$ is admissible, while the corresponding set of admissible vectors (constants in our case) consists of the unique constant $C = -a(0)v_\Omega(0)/2$ (while $C_h$ is supposed to equal zero). Therefore, Condition 6.4 holds if and only if the relation

$$\begin{aligned}\int_0^{\varepsilon/2} r^{-1}\Big|\frac{\partial(av_\Omega)}{\partial y_2}\Big|_{y=(0,-r)} + C\Big(\frac{\partial b_1}{\partial y_2}\Big|_{y=(0,-r)} - \frac{\partial b_2}{\partial y_2}\Big|_{y=(0,r)}\Big)\Big|^2 dr \\ = \int_0^{\varepsilon/2} r^{-1}\Big|\Big(\frac{\partial a}{\partial y_2}v_\Omega + a\frac{\partial v_\Omega}{\partial y_2}\Big)\Big|_{y=(0,-r)} \\ - \frac{a(0)v_\Omega(0)}{2}\Big(\frac{\partial b_1}{\partial y_2}\Big|_{y=(0,-r)} - \frac{\partial b_2}{\partial y_2}\Big|_{y=(0,r)}\Big)\Big|^2 dr < \infty\end{aligned} \tag{7.16}$$

holds for any $v$ supported in $\mathcal{O}_\delta(\Omega(0))$. Suppose that condition (7.12) is fulfilled. Then, similarly to the above, we see that (7.15) holds for any function $v_\Omega$; hence, (7.16) also holds for any $v_\Omega$ (because $a(0) = 0$).

Conversely, suppose that (7.16) is fulfilled. Consider a function $v$ such that $v_\Omega(y) = y_2$ near the origin and substitute it into (7.16). Since $v_\Omega(0) = 0$ and $(\partial v_\Omega/\partial y_2)|_{y=0} = 1$, we infer from (7.16) that $a(0) = 0$ similarly to the above. Therefore, relation (7.16) coincides with (7.15). Now, repeating the above arguments, we obtain $(\partial a/\partial y_2)|_{y=0} = 0$, which completes the proof. □

Clearly, condition (7.12) is weaker than condition (7.9)–(7.10): we impose no restrictions on the behavior of the coefficients $b_1, b_2$ in condition (7.12). The absence of those restrictions is "compensated" by the fact that nonlocal conditions are *regular*, i.e., $\{f_i\} \in \mathcal{S}^{3/2}(\partial G)$ and $f_i(0) = 0$.

Finally, we consider the case of homogeneous nonlocal conditions. In this case, assertion 1 of Theorem 7.4 implies that the validity of condition (7.12) is sufficient for any generalized solution to be smooth. We prove that this condition is also necessary in the following cases (see Figures 7.1, 7.2, and 7.3):

**Case A:** $\operatorname{supp} a(\Omega^{-1}(y))|_{\Omega(\Gamma_1)} \subset G$.

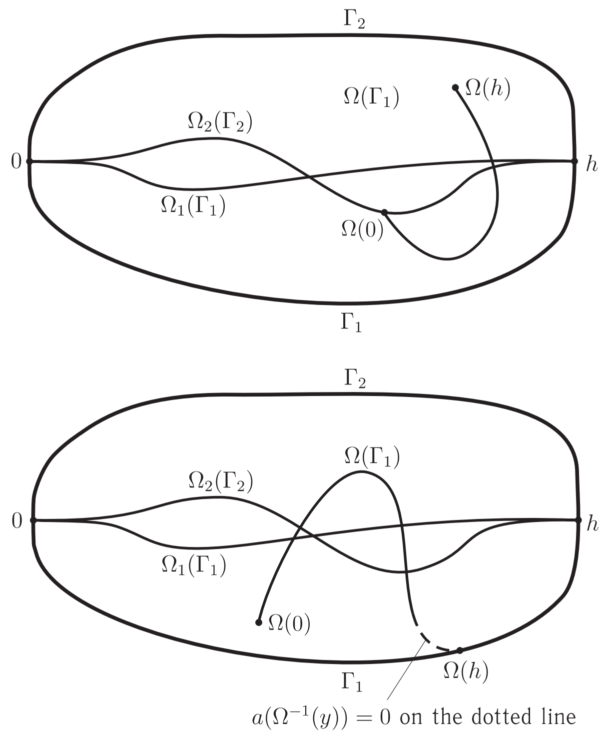


FIGURE 7.1. Case A.

**Case B:** $\Omega(0) \in G$ and $\Omega(0) \notin \Omega_1(\Gamma_1) \cup \Omega_2(\Gamma_2)$.

**Case C:** We have

$$\Omega(0) \in \Gamma_1, \qquad \Omega(\Omega(0)) \notin \Omega_1(\Gamma_1) \cup \Omega_2(\Gamma_2). \tag{7.17}$$

$$a(\Omega(0)) \neq 0. \tag{7.18}$$

**Corollary 7.1.** *Let $b_1(0) + b_2(0) = 0$ and $b_1(h) + b_2(h) \in I_1$. Suppose that either Case* A*, or Case* B*, or Case* C *takes place. If condition* (7.12) *fails, then there exists a right-hand side $\{f_0, 0\}$, where $f_0 \in L_2(G)$, and a generalized solution $u \in W^1(G)$ of problem* (7.1), (7.2) *such that $u \notin W^2(G)$.*

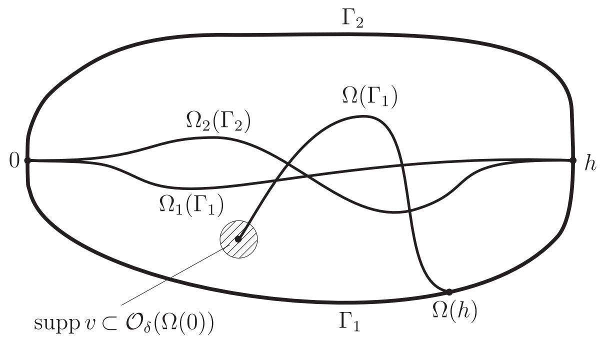


FIGURE 7.2. Case B.

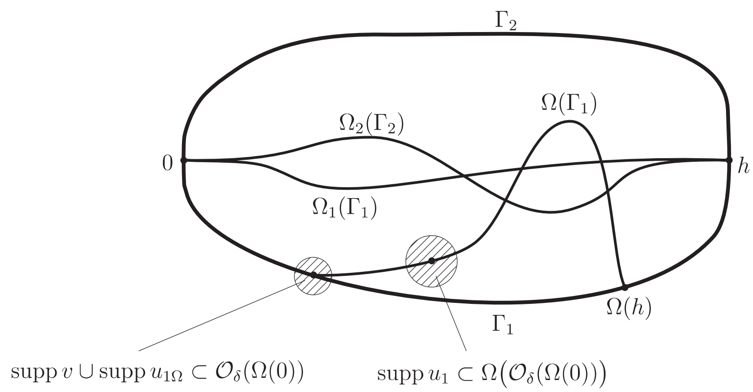


FIGURE 7.3. Case C.

**Proof.** 1. First, we assume that Case A takes place. It follows from the continuity of the transformations $\Omega_i$ and $\Omega$ that the operators $\mathbf{B}_i^2$ satisfy condition (4.31) with any $\rho$ such that $0 < \rho < \operatorname{dist}(\operatorname{supp} a(\Omega^{-1}(y))|_{\Omega(\Gamma_1)}, \partial G)$. Therefore, the conclusion of this corollary follows from Corollary 4.3 and Remark 6.1.

2. Now we assume that Case B takes place. As before, we can suppose that Condition 6.4 is violated for an admissible function $v$ supported in an arbitrarily small $\delta$-neighborhood $\mathcal{O}_\delta(\Omega(0))$ of the point $\Omega(0)$. The number $\delta$ can be chosen so small that

$$v(y)|_{\Gamma_i} \equiv 0, \qquad v(\Omega_i(y))|_{\Gamma_i} = 0, \qquad \operatorname{supp} v(\Omega(y))|_{\Gamma_1} \subset \Gamma_1 \cap \mathcal{O}_{\varkappa_2}(0).$$

Therefore, the function $v$ satisfies relations (4.33), and the conclusion of this corollary follows from Corollary 4.4 and Remark 6.1.

3. Finally, we assume that Case C takes place. Again we can suppose that Condition 6.4 is violated for an admissible function $v$ supported in $\mathcal{O}_\delta(\Omega(0))$. By virtue of relations (7.17), the number $\delta$ can be chosen so small that

$$v(\Omega_i(y))|_{\Gamma_i} \equiv 0, \tag{7.19}$$

$$\operatorname{supp} v(\Omega(y))|_{\Gamma_1} \subset \Gamma_1 \cap \mathcal{O}_{\varkappa_2}(0). \tag{7.20}$$

Let $f_i$ be the functions from assertion 2 of Theorem 4.4, constructed according to the scheme suggested in the proof of Theorem 6.4 . It follows from (7.19) and (7.20) that

$$\operatorname{supp} f_1 \subset \Gamma_1 \cap \big(\mathcal{O}_{\varkappa_2}(0) \cup \mathcal{O}_\delta(\Omega(0))\big), \qquad \operatorname{supp} f_2 \subset \Gamma_2 \cap \mathcal{O}_{\varkappa_2}(0).$$

If we construct a function $u_1 \in H_0^2(G)$ such that

$$u_1|_{\Gamma_i} + \mathbf{B}_i^1 u_1 + \mathbf{B}_i^2 u_1 = f_i(y), \quad y \in \Gamma_i \setminus \mathcal{O}_{\varkappa_2}(\mathcal{K}),\ i = 1, \dots, N, \tag{7.21}$$

$$u_1|_{\Gamma_i} + \mathbf{B}_i^1 u_1 + \mathbf{B}_i^2 u_1 = 0, \quad y \in \Gamma_i \cap \mathcal{O}_{\varkappa_2}(\mathcal{K}),\ i = 1, \dots, N, \tag{7.22}$$

then the conclusion of this corollary will follow from Lemma 4.3, Corollary 4.2, and Remark 6.1.

Let us construct the function $u_1$. To do this, we consider a function $u_{1\Omega} \in W^2\big(\mathcal{O}_\delta(\Omega(0))\big)$ supported in $\mathcal{O}_\delta(\Omega(0))$ (see Figure 7.3) such that

$$u_{1\Omega}(y) = f_1(y)/a(y), \quad y \in \Gamma_1 \cap \mathcal{O}_\delta(\Omega(0)),$$

where $\delta$ is so small that $a(y) \neq 0$ for $y \in \overline{\mathcal{O}_\delta(\Omega(0))}$ (the existence of such a $\delta$ follows from (7.18) and from the continuity of $a(y)$).

Now we set $u_1(y) = u_{1\Omega}(\Omega^{-1}(y))$ for $y \in \Omega\big(\mathcal{O}_\delta(\Omega(0))\big)$ and $u_1(y) = 0$ for $y \notin \Omega\big(\mathcal{O}_\delta(\Omega(0))\big)$. Suppose that $\delta$ is so small that

$$\Gamma_i \cap \Omega\big(\mathcal{O}_\delta(\Omega(0))\big) = \varnothing, \quad \Omega_i(\Gamma_i) \cap \Omega\big(\mathcal{O}_\delta(\Omega(0))\big) = \varnothing, \quad \mathcal{O}_\delta(\Omega(0)) \cap \mathcal{O}_{\varkappa_2}(0) = \varnothing$$

(the existence of such a $\delta$ follows from (7.17) and from the continuity of the transformation $\Omega$). Then we have

$$u_1|_{\Gamma_i} = 0, \qquad u_1(\Omega_i(y))|_{\Gamma_i} = 0,$$

$$a(y) u_1(\Omega(y)) = f_1(y), \quad y \in \Gamma_1 \setminus \mathcal{O}_{\varkappa_2}(0),$$

$$u_1(\Omega(y)) = 0, \quad y \in \Gamma_1 \cap \mathcal{O}_{\varkappa_2}(0).$$

Therefore, the function $u_1$ satisfies relations (7.21) and (7.22), and the theorem is proved. $\square$

## APPENDIX A

This appendix is included for the reader's convenience. Here we have collected some known results on weighted spaces and on properties of nonlocal operators, which are most frequently referred to in the main part of the paper.

A.1. **Properties of weighted spaces.** In this subsection, we formulate some results concerning properties of weighted spaces introduced in Section 2.1. Set

$$K = \{y \in \mathbb{R}^2 :\ r > 0,\ |\omega| < \omega_0\},$$
$$\gamma_\sigma = \{y \in \mathbb{R}^2 :\ r > 0,\ \omega = (-1)^\sigma \omega_0\} \qquad (\sigma = 1, 2).$$

**Lemma A.1** (see Lemma 4.9 in [20]). *Let a function $u \in W^k(K)$, where $k \ge 1$, be compactly supported. Then $u \in H_b^k(K)$ for any $b > k-1$.*

**Lemma A.2** (see Lemma 2.1 in [15]). *Let a function $u \in W^2(K)$ be compactly supported, and let $u(0) = 0$. Then $u \in H_b^2(K)$ for any $b > 0$.*

**Lemma A.3** (see Lemma 3.3′ in [20]). *Let a function $u \in H_b^k(K)$, where $k \ge 0$ and $b \in \mathbb{R}$, be compactly supported. Suppose that $p \in C^k(\overline{K})$ and $p(0) = 0$. Then $pu \in H_{b-1}^k(K)$.*

**Lemma A.4** (see Lemma 4.8 in [20]). *Let a function $u \in W^1(K)$ be compactly supported. Suppose that*

$$\int_{\gamma_\sigma} r^{-1}|u|^2 dr < \infty,$$

*where $\sigma = 1$ or $\sigma = 2$. Then $u \in H_0^1(K)$.*

**Lemma A.5** (see Lemma 4.18 in [20]). *Let a function $\varphi \in H_0^{1/2}(\gamma_\sigma)$, where $\sigma = 1$ or $\sigma = 2$, be compactly supported. Then*

$$\int_{\gamma_\sigma} r^{-1}|\varphi|^2 dr < \infty.$$

**Lemma A.6** (see Lemma 4.20 in [20]). *The function $r^{i\lambda_0}\Phi(\omega)\ln^s r$, where $\operatorname{Im}\lambda_0 = -(k-1)$, belongs to $W^k(K \cap \{|y| < 1\})$ if and only if it is a homogeneous polynomial in $y_1, y_2$ of order $k-1$.*

Denote by $\mathcal{G}$ the operator which is the composition of rotation about the origin and homothety.

**Lemma A.7** (see Lemma 2.2 in [15]). *Let a function $u \in W^1(\mathbb{R}^2)$ be compactly supported. Then $u(\mathcal{G}y) - u(y) \in H_0^1(\mathbb{R}^2)$.*

A.2. **Nonlocal Problems in Plane Angles in Weighted Spaces.** In this subsection and in the next one, we formulate some properties of solutions of problem (2.9), (2.10) in the spaces (3.4) and (3.5). First, we consider the case of weighted spaces.

For convenience, we rewrite the problem:

$$\mathbf{P}_j U_j = F_j(y) \quad (y \in K_j^\varepsilon),$$
$$\mathbf{B}_{j\sigma} U \equiv \sum_{k,s} b_{j\sigma ks}(y) U_k(\mathcal{G}_{j\sigma ks} y) = \Phi_{j\sigma}(y) \qquad (y \in \gamma_{j\sigma}^\varepsilon), \tag{A.1}$$

where

$$\mathbf{P}_j v = \sum_{i,k=1}^{2} p_{jik}(y) v_{y_i y_k} + \sum_{k=1}^{2} p_{jk}(y) v_{y_k} + p_{j0}(y) v$$

(see Section 2.2). Along with problem (A.1), we consider the following model problem in the unbounded angles:

$$\mathcal{P}_j U_j = F_j(y) \quad (y \in K_j),$$
$$\mathcal{B}_{j\sigma} U \equiv \sum_{k,s} b_{j\sigma ks}(0) U_k(\mathcal{G}_{j\sigma ks} y) = \Phi_{j\sigma}(y) \quad (y \in \gamma_{j\sigma}), \tag{A.2}$$

where

$$\mathcal{P}_j v = \sum_{i,k=1}^{2} p_{jik}(0) v_{y_i y_k}.$$

**Lemma A.8** (see Lemma 2.3 in [17])**.** *Let $U$ be a solution of problem* (A.1) (*or* (A.2)) *such that*

$$U_j \in W^2(K_j^{D_\chi \varepsilon} \cap \{|y| > \delta\}) \qquad \forall \delta > 0,$$
$$U \in \mathcal{H}_{b-2}^0(K^{D_\chi \varepsilon}),$$

*where $D_\chi$ is given by* (3.3) *and $b \in \mathbb{R}$. Suppose that $\{F_j\} \in \mathcal{H}_b^0(K^\varepsilon)$, $\{\Phi_{j\sigma}\} \in \mathcal{H}_b^{3/2}(\gamma^\varepsilon)$. Then $U \in \mathcal{H}_b^2(K^\varepsilon)$.*

Consider the asymptotics of solutions of problem (A.2).

**Lemma A.9** (see Lemma 2.1 in [13])**.** *The function*

$$U = r^{i\lambda_0} \sum_{l=0}^{m} \frac{1}{l!} (i \ln r)^l \varphi^{(m-l)}(\omega), \tag{A.3}$$

*is a solution of homogeneous problem* (A.2) *if and only if $\lambda_0$ is an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$ and $\varphi^{(0)}, \dots, \varphi^{(\varkappa-1)}$ is a Jordan chain corresponding to the eigenvalue $\lambda_0$; here $m \le \varkappa - 1$.*

Any solution of the same kind as (A.3) is called a *power solution*.

**Theorem A.1** (see Theorem 2.2 and Remark 2.2 in [13]). *Let* $\{F_j\} \in \mathcal{H}_b^0(K) \cap \mathcal{H}_{b'}^0(K)$ *and* $\{\Phi_{j\sigma}\} \in \mathcal{H}_b^{3/2}(\gamma) \cap \mathcal{H}_{b'}^{3/2}(\gamma)$, *where* $b > b'$. *Suppose that the line* $\operatorname{Im}\lambda = b' - 1$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$. *If* $U$ *is a solution of problem* (A.2) *belonging to the space* $\mathcal{H}_b^2(K)$, *then*

$$U = \sum_{n=1}^{n_0} \sum_{q=1}^{J_n} \sum_{m=0}^{\varkappa_{qn}-1} c_n^{(m,q)} W_n^{(m,q)}(\omega, r) + U'.$$

*Here* $\lambda_1, \dots, \lambda_{n_0}$ *are eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$ *located in the band* $b' - 1 < \operatorname{Im}\lambda < b - 1$;

$$W_n^{(m,q)}(\omega, r) = r^{i\lambda_n} \sum_{l=0}^{m} \frac{1}{l!} (i \ln r)^l \varphi_n^{(m-l,q)}(\omega)$$

*are power solutions of homogeneous problem* (A.2);

$$\{\varphi_n^{(0,q)}, \dots, \varphi_n^{(\varkappa_{qn}-1,q)} : \ q = 1, \dots, J_n\}$$

*is a canonical system of Jordan chains of the operator* $\tilde{\mathcal{L}}(\lambda)$ *corresponding to the eigenvalue* $\lambda_n$; $c_n^{(m,q)}$ *are some complex constants*; *finally,* $U'$ *is a solution of problem* (A.2) *belonging to the space* $\mathcal{H}_{b'}^2(K)$.

If the right-hand sides of problem (A.2) are of a particular form, then there exist solutions of a particular form. Let

$$F_j(\omega, r) = r^{i\lambda_0 - 2} \sum_{l=0}^{M} \frac{1}{l!} (i \ln r)^l f_j^{(l)}(\omega), \qquad \Phi_{j\sigma}(r) = r^{i\lambda_0} \sum_{l=0}^{M} \frac{1}{l!} (i \ln r)^l \psi_{j\sigma}^{(l)}, \tag{A.4}$$

where $f_j^{(l)} \in L^2(-\omega_j, \omega_j)$, $\psi_{j\sigma}^{(l)} \in \mathbb{C}$, $\lambda_0 \in \mathbb{C}$. If $\lambda_0$ is an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$, then denote by $\varkappa(\lambda_0)$ the greatest of the partial multiplicities (see [12]) of this eigenvalue; otherwise, set $\varkappa(\lambda_0) = 0$.

**Lemma A.10** (see Lemma 4.3 in [13]). *For problem* (A.2) *with right-hand side* $\{F_j, \Phi_{j\sigma}\}$ *given by* (A.4), *there exists a solution*

$$U = r^{i\lambda_0} \sum_{l=0}^{M+\varkappa(\lambda_0)} \frac{1}{l!} (i \ln r)^l u^{(l)}(\omega), \tag{A.5}$$

*where* $u^{(l)} \in \prod_j W^2(-\omega_j, \omega_j)$*. A solution of such a form is unique if* $\varkappa(\lambda_0) = 0$ *(i.e.,* $\lambda_0$ *is not an eigenvalue of* $\tilde{\mathcal{L}}(\lambda)$*). If* $\varkappa(\lambda_0) > 0$*, then the solution* (A.5) *is defined accurate to an arbitrary linear combination of power solutions* (A.3) *corresponding to the eigenvalue* $\lambda_0$*.*

Note that Theorem A.1 and Lemma A.10 were earlier proved in [33] for the case in which the operators $\mathcal{G}_{j\sigma ks}$ are rotations only (but not homothety).

The following result is a modification of Theorem A.1 for the case in which the line $\operatorname{Im}\lambda = -1$ contains the unique eigenvalue $\lambda_0 = -i$ of $\tilde{\mathcal{L}}(\lambda)$ and this eigenvalue is proper (see Definition 4.1).

**Lemma A.11** (see Lemma 3.4 in [15])**.** *Let* $U \in \mathcal{H}_b^2(K)$*, where* $b > 0$*, be a solution of problem* (A.2) *with right-hand side* $\{F_j\} \in \mathcal{H}_b^0(K) \cap \mathcal{H}_0^0(K)$*,* $\{\Phi_{j\sigma}\} \in \mathcal{H}_b^{3/2}(\gamma) \cap \mathcal{H}_0^{3/2}(\gamma)$*. Suppose that the closed band* $-1 \le \operatorname{Im}\lambda \le b-1$ *contains only the eigenvalue* $\lambda_0 = -i$ *of* $\tilde{\mathcal{L}}(\lambda)$ *and this eigenvalue is proper. Then* $D^\alpha U \in \mathcal{H}_0^0(K)$ *for* $|\alpha| = 2$*.*

Finally, we formulate the result that allows one to reduce nonlocal problems with nonhomogeneous boundary conditions to those with homogeneous boundary conditions.

**Lemma A.12** (see Lemma 8.1 in [16])**.** *For any function* $\{\Phi_{j\sigma}\} \in \mathcal{H}_b^{3/2}(\gamma)$*, there exists a function* $U \in \mathcal{H}_b^2(K)$ *such that*

$$U_j(y)|_{\gamma_{j\sigma}} = f_{j\sigma}(y), \qquad \sum_{(k,s)\ne(j,\sigma)} b_{j\sigma ks}(y) U_k(\mathcal{G}_{j\sigma ks} y)|_{\gamma_{j\sigma}} = 0 \quad (y \in \gamma_{j\sigma}).$$

A.3. **Nonlocal Problems in Plane Angles in Sobolev Spaces.** In this subsection, we formulate properties of solutions of problems (A.1) and (A.2) with right-hand sides from Sobolev spaces.

The following lemma deals with the case in which the line $\operatorname{Im}\lambda = -1$ is free of eigenvalues of $\tilde{\mathcal{L}}(\lambda)$.

**Lemma A.13** (see Lemma 2.4 and Corollary 2.1 in [15])**.** *Suppose the line* $\operatorname{Im}\lambda = -1$ *contains no eigenvalues of* $\tilde{\mathcal{L}}(\lambda)$*. Suppose that*

$$\{\Phi_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon), \qquad \Phi_{j\sigma}(0) = 0.$$

*Then there exists a compactly supported function* $V \in \mathcal{W}^2(K) \cap \mathcal{H}_b^2(K)$*, where* $b$ *is an arbitrary positive number, such that*

$$\{\mathbf{P}_j V_j\} \in \mathcal{H}_0^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma} V|_{\gamma_{j\sigma}^\varepsilon} - \Phi_{j\sigma}\} \in \mathcal{H}_0^{3/2}(\gamma^\varepsilon).$$

Now we consider the situation where the line $\operatorname{Im}\lambda = -1$ contains the unique eigenvalue $\lambda_0 = -i$ of $\tilde{\mathcal{L}}(\lambda)$ and it is proper (see Definition 4.1). In this case, we will use the following result instead of Lemma A.13.

**Lemma A.14** (see Lemma 3.3 and Corollary 3.1 in [15]). *Suppose the line* $\operatorname{Im}\lambda = -1$ *contains the unique eigenvalue* $\lambda_0 = -i$ *of* $\tilde{\mathcal{L}}(\lambda)$ *and it is proper. Suppose that*

$$\{\Phi_{j\sigma}\} \in \mathcal{W}^{3/2}(\gamma^\varepsilon), \qquad \Phi_{j\sigma}(0) = 0,$$

*and the functions* $\Phi_{j\sigma}$ *satisfy the consistency condition* (4.7). *Then there exists a compactly supported function* $V \in \mathcal{W}^2(K) \cap \mathcal{H}_b^2(K)$, *where* $b$ *is an arbitrary positive number, such that*

$$\{\mathbf{P}_j V_j\} \in \mathcal{H}_0^0(K^\varepsilon), \qquad \{\mathbf{B}_{j\sigma} V|_{\gamma_{j\sigma}^\varepsilon} - \Phi_{j\sigma}\} \in \mathcal{H}_0^{3/2}(\gamma^\varepsilon).$$

**Lemma A.15** (see Lemma 3.1 in [15]). *Suppose the line* $\operatorname{Im}\lambda = -1$ *contains the unique eigenvalue* $\lambda_0 = -i$ *of* $\tilde{\mathcal{L}}(\lambda)$ *and it is proper. Suppose that* $U \in \mathcal{W}^2(K)$ *is a compactly supported solution of problem* (A.1) (*or* (A.2)) *and* $U(0) = 0$. *Then the functions* $\Phi_{j\sigma}$ *satisfy the consistency condition* (4.7).

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