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Smoothness of Generalized Solutions to Nonlocal Elliptic Problems on the Plane[1]

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1. The most difficult situation in the theory of elliptic problems with nonlocal boundary value conditions arises when the support of nonlocal terms intersects the boundary of the domain [1–8]. In this case, solutions may have power singularities near some points on the boundary. In this paper, we find conditions necessary and sufficient for any generalized solution $u \in W_2^1(G)$ to a nonlocal problem in a plane bounded domain G to belong to $W_2^2(G)$. We study the case in which different nonlocal conditions are set on different parts of the boundary, the coefficients of the nonlocal terms supported near the points of conjugation of boundary conditions are variable, and the nonlocal operators corresponding to the nonlocal terms supported outside the conjugation points are abstract. Both homogeneous and nonhomogeneous nonlocal conditions are investigated. We consider a nonlocal perturbation of the Dirichlet problem for an elliptic equation of order two. However, the obtained results can be generalized to elliptic equations of order $2m$ with general nonlocal conditions.

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. We introduce a set $\mathcal{K} \subset \partial G$ consisting of finitely many points. Let $\partial G \backslash \mathcal{K} = \bigcup_{i=1}^{N} \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞-curves. We assume that the domain G is a plane angle in some neighborhood of each point $g \in \mathcal{K}$. Let $\mathbf{P}$ denote a second-order differential operator with smooth complex-valued coefficients properly elliptic in $\bar{G}$.

For any closed set $\mathcal{M}$, we set $\mathcal{O}_\varepsilon(\mathcal{M}) = \{y \in \mathbb{R}^2: \operatorname{dist}(y, \mathcal{M}) < \varepsilon\}$, where $\varepsilon > 0$.

Now, we define the operators corresponding to the nonlocal conditions near the set $\mathcal{K}$. Let Ω_{is} ($i = 1, 2, \ldots, N$; $s = 1, 2, \ldots, S_i$) denote C^∞-diffeomorphisms taking a neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and $\Omega_{is}(g) \in \mathcal{K}$ for $g \in \bar{\Gamma}_i \cap \mathcal{K}$. Thus, the transformations Ω_{is} map the arcs $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ strictly inside the domain G and the endpoints $\bar{\Gamma}_i \cap \mathcal{K}$ of these arc are mapped to endpoints.

Let us specify the structure of the transformations Ω_{is} near the set $\mathcal{K}$. Let Ω_{is}: $\mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$ denote the transformation Ω_{is}^{+1} :$\Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$, and let Ω_{is}^{-1} : $\Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ be the transformation inverse to Ω_{is}. The set of points $\Omega_{i_q s_q}^{\pm 1}(\ldots \Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ ($1 \le s_j \le S_{i_j}$, $j = 1, 2, \ldots, q$) is called an orbit of the point $g \in \mathcal{K}$. In other words, the orbit of $g \in \mathcal{K}$ is formed by the points that can be obtained by successively applying the transformations $\Omega_{is}^{\pm 1}$ to g. For simplicity, we assume that the set $\mathcal{K} = \{g_1, g_2, \ldots, g_N\}$ consists of only one orbit.

Let ε be so small that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ of the points $g_j \in \mathcal{K}$ satisfying the following conditions: (i) $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$, (ii) the boundary ∂G is an angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$, (iii) $\overline{\mathcal{O}_{\varepsilon_1}(g_j)} \cap \overline{\mathcal{O}_{\varepsilon_1}(g_k)} = \varnothing$ for any $g_j, g_k \in \mathcal{K}$ with $k \ne j$, (iv) if $g_j \in \bar{\Gamma}_i$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \bar{\Gamma}_i \cap \mathcal{K}$, we fix a transformation $y \mapsto y'(g_j)$ being a composition of the shift by the vector $-\overrightarrow{O_{g_j}}$ and the rotation through some angle such that the set $\mathcal{O}_{\varepsilon_1}(g_j)$ is taken to a neighborhood $\mathcal{O}_{\varepsilon_1}(0)$ of the origin, whereas $G \cap \mathcal{O}_{\varepsilon_1}(g_j)$ and $\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)$ are taken to the intersection of the plane angle $K_j = \{y \in \mathbb{R}^2: r > 0, |\omega| < \omega_j\}$ with $\mathcal{O}_{\varepsilon_1}(0)$ and to the intersection of the side $\gamma_{j\sigma} = \{y \in \mathbb{R}^2: \omega = (-1)^\sigma \omega_j\}$ ($\sigma = 1$ or $\sigma = 2$) of the angle K_j with $\mathcal{O}_{\varepsilon_1}(0)$, respectively. Here, (ω, r) are the polar coordinates of y and $0 < \omega_j < \pi$.

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Condition 1. The transformation $\Omega_{is}(y)$, where $y \in \mathcal{O}_\varepsilon(g_j)$ and $g_j \in \overline{\Gamma}_i \cap \mathcal{K}$, corresponds to the composition of a rotation and a homothety of the vector y' in the new variables $\{y'\}$.

Let us introduce nonlocal operators $\mathbf{B}_i^1$ by the formula $\mathbf{B}_i^1 u = \sum_{s=1}^{S_i} b_{is}(y)u(\Omega_{is}(y))$ for $y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ and $\mathbf{B}_i^1 u = 0$ for $y \in \Gamma_i \setminus (\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}))$, where $b_{is} \in C^\infty(\mathbb{R}^2)$ and $\operatorname{supp} b_{is} \subset \mathcal{O}_\varepsilon(\mathcal{K})$. Since $\mathbf{B}_i^1 u = 0$ whenever $\operatorname{supp} u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_1}(\mathcal{K})}$, we say that the operators $\mathbf{B}_i^1$ correspond to the nonlocal terms supported near the set $\mathcal{K}$.

Consider the operators $\mathbf{B}_i^2$ satisfying the following condition [cf. (2.5) and (2.6) in [2] and (3.4) and (3.5) in [6]].

Condition 2. There exist numbers $\kappa_1 > \kappa_2 > 0$ and $\rho > 0$ such that the inequalities

$$\|\mathbf{B}_i^2 u\|_{W_2^{3/2}(\Gamma_i)} \le c_1 \|u\|_{W_2^2(G \setminus \overline{\mathcal{O}_{\kappa_1}(\mathcal{K})})},$$
$$\|\mathbf{B}_i^2 u\|_{W_2^{3/2}(\Gamma_i \setminus \overline{\mathcal{O}_{\kappa_2}(\mathcal{K})})} \le c_2 \|u\|_{W_2^2(G_\rho)} \quad (1)$$

hold for any $u \in W_2^2(G \setminus \overline{\mathcal{O}_{\kappa_1}(\mathcal{K})}) \cap W_2^2(G_\rho)$; here, $G_\rho = \{y \in G: \operatorname{dist}(y, \partial G) > \rho\}$, $i = 1, 2, \ldots, N$, and $c_1, c_2 > 0$.

In particular, the first inequality in (1) means that $\mathbf{B}_i^2 u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\kappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}_i^2$ correspond to the nonlocal terms supported outside the set $\mathcal{K}$. Examples of operators $\mathbf{B}_i^2$ can be found in [2, 5].

Throughout the paper, we assume that Conditions 1 and 2 hold.

Consider the nonlocal elliptic boundary value problem

$$\mathbf{P}u = f_0(y), \quad y \in G, \quad (2)$$

$$u|_{\Gamma_i} + \mathbf{B}_i^1 u + \mathbf{B}_i^2 u = f_i(y), \quad y \in \Gamma_i; \quad i = 1, 2, \ldots, N. \quad (3)$$

We set $\mathcal{W}_2^{k-1/2}(\partial G) = \prod_{i=1}^N W_2^{k-1/2}(\Gamma_i)$ for integer $k \ge 1$. For any set $X \in \mathbb{R}^2$ having nonempty interior, we use $C_0^\infty(X)$ to denote the set of functions infinitely differentiable on $\overline{X}$ and supported on X.

Definition 1. A function $u \in W_2^1(G)$ is called a generalized solution to problem (2), (3) with right-hand side $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}_2^{1/2}(\partial G)$ if u satisfies nonlocal conditions (3) [where the equalities are understood as those in $W_2^{1/2}(\Gamma_i)$] and Eq. (2) in the sense of distributions.

Now, let us write a model nonlocal problem corresponding to the points of the set (orbit) $\mathcal{K}$. Let $u_j(y)$ denote the function $u(y)$ for $y \in \mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j \in \overline{\Gamma}_i$, $y \in \mathcal{O}_\varepsilon(g_j)$, and $\Omega_{is}(y) \in \mathcal{O}_{\varepsilon_1}(g_k)$, then $u(\Omega_{is}(y))$ denotes the function $u_k(\Omega_{is}(y))$. In this case, nonlocal problem (2), (3) takes the following form in the ε-neighborhood of the set (orbit) $\mathcal{K}$:

$$\mathbf{P}u_j = f_0(y), \quad y \in \mathcal{O}_\varepsilon(g_j) \cap G,$$

$$u_j(y)|_{\mathcal{O}_\varepsilon(g_j) \cap \Gamma_i} + \sum_{s=1}^{S_i} b_{is}(y) u_k(\Omega_{is}(y))|_{\mathcal{O}_\varepsilon(g_j) \cap \Gamma_i} = \psi_i(y)$$

$$(y \in \mathcal{O}_\varepsilon(g_j) \cap \Gamma_i; \quad i \in \{1 \le i \le N: g_j \in \overline{\Gamma}_i\}; \quad j = 1, 2, \ldots, N),$$

where $\psi_i = f_i - \mathbf{B}_i^2 u$. Let $y \mapsto y'(g_j)$ be the change of variables described above. We set $K_j^\varepsilon = K_j \cap \mathcal{O}_\varepsilon(0)$ and $\gamma_{j\sigma}^\varepsilon = \gamma_{j\sigma} \cap \mathcal{O}_\varepsilon(0)$ and introduce the functions

$$U_j(y') = u_j(y(y')), \quad F_j(y') = f_0(y(y')), \quad y' \in K_j^\varepsilon,$$
$$\Psi_{j\sigma}(y') = \psi_i(y(y')), \quad y' \in \gamma_{j\sigma}^\varepsilon, \quad (4)$$

where $\sigma = 1$ ($\sigma = 2$) if the transformation $y \mapsto y'(g_j)$ takes Γ_i to the side γ_{j1} (γ_{j2}) of the angle K_j. In what follows, we write y instead of y'. Using Condition 1, we can write problem (2), (3) as

$$\mathbf{P}_j U_j = F_j(y), \quad y \in K_j^\varepsilon, \quad (5)$$

$$\mathbf{B}_{j\sigma}(U) \equiv \sum_{k,s} b_{j\sigma ks}(y) U_k(\mathcal{G}_{j\sigma ks} y) = \Psi_{j\sigma}(y), \quad y \in \gamma_{j\sigma}^\varepsilon. \quad (6)$$

Here (and in what follows, unless otherwise stated), $j, k = 1, 2, \ldots, N$; $\sigma = 1, 2$; $s = 0, 1, \ldots, S_{j\sigma k}$; $\mathbf{P}_j$ are second-order elliptic differential operators with smooth coefficients; $U = (U_1, U_2, \ldots, U_N)$; $b_{j\sigma ks}$ are smooth functions such that $b_{j\sigma j0}(y) \equiv 1$; $\mathcal{G}_{j\sigma ks}$ is the operator being the composition of rotation through an angle $\omega_{j\sigma ks}$ and a homothety with coefficient $\chi_{j\sigma ks} > 0$ in the y-plane. Moreover, $|(-1)^\sigma\omega_j + \omega_{j\sigma ks}| < \omega_k$ for $(k, s) \neq (j, 0)$ and $\omega_{j\sigma j0} = \chi_{j\sigma j0} = 1$ (i.e., $\mathcal{G}_{j\sigma j0}y \equiv y$).

Suppose that, at $y = 0$, the principal parts of the operators $\mathbf{P}_j$ in polar coordinates have the form $r^{-2}\tilde{\mathcal{P}}_j\left(\omega, \frac{\partial}{\partial\omega}, r\frac{\partial}{\partial r}\right)$. Consider the analytic operator-valued function $\tilde{\mathcal{L}}(\lambda) : \prod_j W_2^2(-\omega_j, \omega_j) \to \prod_j (L_2(-\omega_j, \omega_j) \times \mathbb{C}^2)$ defined by $\tilde{\mathcal{L}}(\lambda)\varphi = \left\{\tilde{\mathcal{P}}_j\left(\omega, \frac{\partial}{\partial\omega}, i\lambda\right)\varphi_j, \sum_{k,s}(\chi_{j\sigma ks})^{i\lambda} b_{j\sigma ks}(0) \times \varphi_k((-1)^\sigma\omega_j + \omega_{j\sigma ks})\right\}$. The basic definitions and facts concerning analytic operator-valued functions can be found in [9]. For our purposes, it is important that the spectrum of the operator $\tilde{\mathcal{L}}(\lambda)$ is discrete and, for any numbers $c_1 < c_2$, the strip $c_1 < \mathrm{Im}\lambda < c_2$ contains at most finitely many eigenvalues of the operator $\tilde{\mathcal{L}}(\lambda)$ (see [4]). The spectral properties of the operator $\tilde{\mathcal{L}}(\lambda)$ play a central role in the study of smoothness of generalized solutions.

2. Let $\lambda = \lambda_0$ be an eigenvalue of the operator $\tilde{\mathcal{L}}(\lambda)$.

Definition 2. We say that λ_0 is a proper eigenvalue if none of the eigenvectors $\varphi(\omega) = (\varphi_1(\omega), \varphi_2(\omega), \dots, \varphi_N(\omega))$ corresponding to λ_0 has an associated vector and the functions $r^{i\lambda_0}\varphi_j(\omega)$, where $j = 1, 2, \dots, N$, are polynomials in y_1 and y_2. An eigenvalue which is not proper is said to be improper.

The notion of a proper eigenvalue was originally suggested by Kondrat'ev [10] for studying "local" boundary value problems in nonsmooth domains.

Theorem 1. (i) *Suppose that the strip* $-1 \le \mathrm{Im}\lambda < 0$ *contains no eigenvalues of the operator* $\tilde{\mathcal{L}}(\lambda)$ *and* $u \in W_2^1(G)$ *is a generalized solution to problem* (2), (3) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times W_2^{3/2}(\partial G)$. *Then,* $u \in W_2^2(G)$.

(ii) *Suppose that the strip* $-1 \le \mathrm{Im}\lambda < 0$ *contains an improper eigenvalue of the operator* $\tilde{\mathcal{L}}(\lambda)$. *Then, there exists a generalized solution* $u \in W_2^1(G)$ *to problem* (2), (3) *with right-hand side* $\{f_0, 0\}$, *where* $f_0 \in L_2(G)$, *such that* $u \notin W_2^2(G)$.

It remains to study the case in which the following condition holds.

Condition 3. The strip $-1 \le \mathrm{Im}\lambda < 0$ contains the unique eigenvalue $\lambda = -i$ of the operator $\tilde{\mathcal{L}}(\lambda)$, and this eigenvalue is proper.

First, consider problem (2), (3) with nonhomogeneous nonlocal conditions.

Let $\tau_{j\sigma}$ denote the unit vector codirected with the ray $\gamma_{j\sigma}$. Consider the operators $\frac{\partial}{\partial\tau_{j\sigma}}\left(\sum_{k,s} b_{j\sigma ks}(0)U_k(\mathcal{G}_{j\sigma ks}y)\right)$. Using the chain rule, we write them as

$$\sum_{k,s}(\hat{B}_{j\sigma ks}(D_y)U_k)(\mathcal{G}_{j\sigma ks}y), \tag{7}$$

where $\hat{B}_{j\sigma ks}(D_y)$ are first-order differential operators with constant coefficients. In particular, we have $\hat{B}_{j\sigma j0}(D_y) = \frac{\partial}{\partial\tau_{j\sigma}}$, because $\mathcal{G}_{j\sigma j0}y \equiv y$. Formally replacing the nonlocal operators by the corresponding local operators in (7), we introduce the operators

$$\hat{\mathcal{B}}_{j\sigma}(D_y)U \equiv \sum_{k,s}\hat{B}_{j\sigma ks}(D_y)U_k(y). \tag{8}$$

If Condition 3 holds, then system of operators (8) is linearly dependent [11]. Let

$$\{\hat{\mathcal{B}}_{j'\sigma'}(D_y)\} \tag{9}$$

be a maximal linearly independent subsystem of system (8). In this case, any operator $\hat{\mathcal{B}}_{j\sigma}(D_y)$ not included in system (9) can be represented as

$$\hat{\mathcal{B}}_{j\sigma}(D_y) = \sum_{j',\sigma'}\beta_{j\sigma}^{j'\sigma'}\hat{\mathcal{B}}_{j'\sigma'}(D_y), \tag{10}$$

where $\beta_{j\sigma}^{j'\sigma'}$ are some constants. Let $Z_{j\sigma} \in W_2^{3/2}(\gamma_{j\sigma}^\varepsilon)$ be arbitrary functions. We set $Z_{j\sigma}^0(r) = Z_{j\sigma}(y)\big|_{y=(r\cos\omega_j,\, r(-1)^\sigma\sin\omega_j)}$. Clearly, $Z_{j\sigma}^0 \in W_2^{3/2}(0, \varepsilon)$.

Definition 3. Let $\beta_{j'\sigma'}$ be the constants involved in (10). If the relations

$$\int_0^\varepsilon r^{-1}\left|\frac{d}{dr}\left(Z_{j\sigma}^0 - \sum_{j',\sigma'}\beta_{j\sigma}^{j',\sigma'}Z_{j',\sigma'}^0\right)\right|^2 dr < \infty \tag{11}$$

hold for all indices j, σ corresponding to the operators of system (8) not included in system (9), then we say that the functions $Z_{j\sigma}$ satisfy consistency condition (11).

Let us formulate conditions which ensure that the generalized solutions are smooth. First, we show that the right-hand sides f_i in nonlocal conditions (3) cannot be arbitrary functions from the space $W_2^{3/2}(\Gamma_i)$.

Consider the change of variables $y \mapsto y'(g_j)$ described in Section 1. Let us introduce the functions

$$F_{j\sigma}(y') = f_i(y(y')), \quad y' \in \gamma_{j\sigma}^{\varepsilon}$$

[cf. functions (4)]. Let $\mathcal{S}_2^{3/2}(\partial G)$ denote the set consisting of functions $\{f_i\} \in W_2^{3/2}(\partial G)$ such that the functions $F_{j\sigma}$ satisfy consistency condition (11). The set $\mathcal{S}_2^{3/2}(\partial G)$ is not closed in the topology of $W_2^{3/2}(\partial G)$ (see [11, Lemma 3.2]).

Lemma 1. *Suppose that Condition* 3 *holds. Then there exists a function* $\{f_0, f_i\} \in L_2(G) \times \mathcal{W}_2^{3/2}(\partial G)$, *where* $\{f_i\} \notin \mathcal{S}_2^{3/2}(\partial G)$, *and a function* $u \in W_2^1(G)$ *such that u is a generalized solution to problem* (2), (3) *with right-hand side* $\{f_0, f_i\}$ *and* $u \notin W_2^2(G)$.

It follows from Lemma 1 that, if we want any generalized solution to problem (2), (3) to be smooth, then we must take right-hand sides $\{f_0, f_i\}$ from the space $L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$.

Let $v \in W_2^2(G \backslash \overline{\mathcal{O}_{\kappa_1}(\mathcal{K})})$ be an arbitrary function. Consider again the change of variables $y \mapsto y'(g_j)$ from Section 1. Let us introduce the functions

$$B_{j\sigma}^{v}(y') = (\mathbf{B}_i^2 v)(y(y')), \quad y' \in \gamma_{j\sigma}^{\varepsilon}.$$

Condition 4. For any function $v \in W_2^2(G \backslash \overline{\mathcal{O}_{\kappa_1}(\mathcal{K})})$ and any constant vector $C = (C_1, C_2, \ldots, C_N)$, the functions $B_{j\sigma}^{v}$ and $\mathbf{B}_{j\sigma}C$, respectively, satisfy consistency condition (11).

Theorem 2. *Suppose that Condition* 3 *holds. Then, the following assertions are valid.*

(i) *If Condition* 4 *holds and* $u \in W_2^1(G)$ *is a generalized solution to problem* (2), (3) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$, *then* $u \in W_2^2(G)$;

(ii) *If Condition* 4 *is violated, then there exists a generalized solution* $u \in W_2^1(G)$ *to problem* (2), (3) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$ *such that* $u \notin W_2^2(G)$.

Now, consider problem (2), (3) with more regular right-hand sides $\{f_i\} \in \mathcal{S}_2^{3/2}(\partial G)$, $f_i|_{\mathcal{K}} = 0$, in particular, with homogeneous nonlocal conditions.

Definition 4. We say that a function $v \in W_2^2(G \backslash \overline{\mathcal{O}_{\kappa_1}(\mathcal{K})})$ is admissible if there exists a constant vector $C = (C_1, C_2, \ldots, C_N)$ such that

$$B_{j\sigma}^{v}(0) + (\mathbf{B}_{j\sigma}C)(0) = 0, \quad j = 1, 2, \ldots, N, \quad \sigma = 1, 2. \tag{12}$$

Any vector C satisfying relations (12) is called an admissible vector corresponding to the function v.

The set of admissible functions is linear. Clearly, the function $v = 0$ is admissible, and the vector $C = 0$ is an admissible vector corresponding to $v = 0$. Moreover, it can be shown that any generalized solution to problem (2), (3) with homogeneous nonlocal conditions is an admissible function.

Consider the following condition (which is weaker than Condition 4).

Condition 4'. For any admissible function v and any admissible vector C corresponding to v, the functions $B_{j\sigma}^{v} + \mathbf{B}_{j\sigma}C$ satisfy consistency condition (11).

Theorem 2'. *Suppose that Condition* 3 *holds. Then, the following assertions are valid.*

(i) *If Condition* 4' *holds and* $u \in W_2^1(G)$ *is a generalized solution to problem* (2), (3) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$, $f_i|_{\mathcal{K}} = 0$, *then* $u \in W_2^2(G)$;

(ii) *If Condition* 4' *is violated, then there exists a generalized solution* $u \in W_2^1(G)$ *to problem* (2), (3) *with right-hand side* $\{f_0, f_i\} \in L_2(G) \times \mathcal{S}_2^{3/2}(\partial G)$, $f_i|_{\mathcal{K}} = 0$, *such that* $u \notin W_2^2(G)$.

The proofs of Theorems 1, 2, and 2' are based on results concerning the solvability of model nonlocal problems in plane angles in Sobolev spaces [11] and on the asymptotic behavior of solutions to these problems in weighted spaces [2, 12].

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