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MATHEMATICS

Unbounded Perturbations of Two-Dimensional Diffusion Processes with Nonlocal Boundary Conditions

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1. In [1], a special form of a generator of a strongly continuous contraction nonnegative semigroup (Feller semigroup) of operators acting on spaces of continuous functions on the interval was considered. In the multidimensional case, it was proved that a generator of a Feller semigroup is an elliptic differential operator (possibly, degenerate), whose domain consists of continuous functions satisfying a nonlocal condition which contains an integral over the closure of the domain with respect to a nonnegative Borel measure [2]. The inverse problem remains unsolved. Given an elliptic integro-differential operator whose domain is described by nonlocal boundary conditions, is the closure of this operator a generator of a Feller semigroup? In [3–7], the transversal case was considered, in which the order of nonlocal terms is less than that of local terms, and in [7, 8], the nontransversal case was handled, in which these orders coincide (see also the references in [7]).

In [7], it was assumed that the coefficients of nonlocal terms are less than 1. In this paper, we consider nontransversal nonlocal conditions on the boundary of a plane domain in the "limit case," in which the coefficients of nonlocal terms may equal 1 at some points (the Borel measure at these points is assumed to be atomic). We state sufficient conditions on the unbounded perturbations of the elliptic operator and on the Borel measure in the nonlocal condition under which the corresponding nonlocal operator is a generator of a Feller semigroup.

Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. Consider a finite set $\mathcal{K} \subset \partial G$. Suppose that $\partial G \backslash \mathcal{K} = \bigcup_{i=1}^{N} \Gamma_i$, where the G_i are open (in the topology of ∂G) curves of class C^∞. We assume that, in a neighborhood of each point $g \in \mathcal{K}$, the domain G coincides with a plane angle.

For any closed sets $Q \subset \bar{G}$ and $K \subset \bar{G}$ such that $Q \cap K \neq \varnothing$, we set $C_K(Q) = \{u \in C(Q)\colon u(y) = 0, y \in Q \cap K\}$ and endow this space with the maximum-norm. We define a space $H_a^k(G)$ as the completion of the set of infinitely differentiable functions vanishing near $\mathcal{K}$ endowed with the norm

$$\|u\|_{H_a^k(G)} = \left(\sum_{|\alpha| \le k} \int_G \rho^{2(a+|\alpha|-k)} |D^\alpha u(y)|^2 dy \right)^{1/2},$$

where $a \in \mathbb{R}$, $k \ge 0$ is an integer, and $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{K})$. For integer $k \ge 1$, by $H_a^{k-1/2}(\Gamma)$ we denote the space of traces of functions from $H_a^k(G)$ on the smooth curve $\Gamma \subset \bar{G}$. On the weight spaces, we define norms depending on a parameter $q > 0$ as follows. We set $\|v\|_{H_a^0(\Gamma_i)} = \left(\int_{\Gamma_i} \rho^{2a} |v(y)|^2 d\Gamma \right)^2$,

$$|||u|||_{H_a^k(G)} = (\|u\|^2_{H_a^k(G)} + q^k \|u\|^2_{H_a^0(G)})^{1/2}, \quad k \ge 0,$$

$$|||v|||_{H_a^{k-1/2}(\Gamma_i)} = (\|v\|^2_{H_a^{k-1/2}(\Gamma_i)} + q^{k-1/2} \|v\|^2_{H_a^0(\Gamma_i)})^{1/2}, \quad k \ge 1.$$

2. Consider the differential operator

$$P_0 u = \sum_{i,j=1}^{2} p_{ij}(y) u_{y_i y_j}(y) + \sum_{i=1}^{2} p_i(y) u_{y_i}(y) + p_0(y) u(y),$$

where the p_{ij}, $p_i \in C^\infty(\mathbb{R}^2)$ are real-valued functions such that $p_0 \ge 0$ and $p_{ij} = p_{ji}$ for $i, j = 1, 2$. Suppose that there exists a constant $c > 0$ for which

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$$\sum_{i,j=1}^{2} p_{ij}(y)\xi_i\xi_j \ge c|\xi|^2, \quad y \in \bar{G}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Let us introduce operators corresponding to nonlocal terms supported near the set $\mathcal{K}$. For any set $\mathcal{M}$, we denote its ε-neighborhood by $\mathcal{O}_\varepsilon(\mathcal{M})$. Consider diffeomorphisms Ω_{is} (where $i = 1, 2, \ldots, N$ and $s = 1, 2, \ldots, S_i$) of class C^∞ each of which maps some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})}$ onto a set $\Omega_{is}(\mathcal{O}_i)$ such that $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$ and $\Omega_{is}(g) \in \mathcal{K}$ for $g \in \bar{\Gamma}_i \cap \mathcal{K}$. Thus, the transformations Ω_{is} map the curves $\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ inside the domain G and the set of endpoints $\bar{\Gamma}_i \cap \mathcal{K}$ to itself.

Let us describe the structure of the transformations Ω_{is} in more detail. Let Ω_{is}^{+1} denote the transformation Ω_{is}: $\mathcal{O}_i \to \Omega_{is}(\mathcal{O}_i)$, and let $\Omega_{is}^{-1} : \Omega_{is}(\mathcal{O}_i) \to \mathcal{O}_i$ denote the inverse transformation. The set of points of the form $\Omega_{i_q s_q}^{\pm 1}(\ldots\Omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}$ (where $1 \le s_j \le S_{i_j}$ for $j = 1, 2, \ldots, q$) is called the orbit of the point $g \in \mathcal{K}$. In other words, the orbit of $g \in \mathcal{K}$ is formed by those points (from the set $\mathcal{K}$) which are obtained by successively applying the transformations $\Omega_{i_j s_j}^{\pm 1}$ to the point g. The set $\mathcal{K}$ consists of finitely many disjoint orbits, which we denote by $\mathcal{K}_\nu$, where $\nu = 1, 2, \ldots, N_0$.

Take a sufficiently small $\varepsilon > 0$ for which there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ satisfying the condition $\mathcal{O}_{\varepsilon_1}(g_j) \supset \mathcal{O}_\varepsilon(g_j)$ and the following conditions:

(i) in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$, the domain G coincides with a plane angle;

(ii) $\overline{\mathcal{O}_{\varepsilon_1}(g)} \cap \overline{\mathcal{O}_{\varepsilon_1}(h)} = \varnothing$ for $g, h \in \mathcal{K}$ such that $g \ne h$;

(iii) if $g_j = \bar{\Gamma}_i$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_\varepsilon(g_j) \subset \mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j)) \subset \mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \bar{\Gamma}_i \cap \mathcal{K}_\nu$, we fix a linear transformation Y_j: $y \mapsto y'(g_j)$ (the composition of the translation by the vector $-\overrightarrow{Og_j}$ and a rotation) which maps g_j to the origin so that $Y_j(\mathcal{O}_{\varepsilon_1}(g_j)) = \mathcal{O}_{\varepsilon_1}(0)$, $Y_j(G \cap \mathcal{O}_{\varepsilon_1}(g_j)) = K_j \cap \mathcal{O}_{\varepsilon_1}(0)$, and $Y_j(\Gamma_i \cap \mathcal{O}_{\varepsilon_1}(g_j)) = \gamma_{j\sigma} \cap \mathcal{O}_{\varepsilon_1}(0)$ ($\sigma = 1$ or 2), where K_j is a nonzero plane angle and the $\gamma_{j\sigma}$ are its sides.

Condition 1. If $g_j \in \bar{\Gamma}_i \cap \mathcal{K}_\nu$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}_\nu$, then the transformation $Y_k \circ \Omega_{is} \circ Y_j^{-1} : \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$ is the composition of a rotation and a homothety.

Consider the operators $\mathbf{B}_i u = \sum_{s=1}^{S_i} b_{is}(y)u(\Omega_{is}(y))$ for $y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ and $\mathbf{B}_i u = 0$ for $y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K})$, where the $b_{is} \in C^\infty(\mathbb{R}^2)$ are real-valued functions and $\operatorname{supp} b_{is} \subset \mathcal{O}_\varepsilon(\mathcal{K})$.

Condition 2. (i) $b_{is}(y) \ge 0$ and $\sum_{s=1}^{S_i} b_{is}(y) \le 1$ for $y \in \bar{\Gamma}_i$;

(ii) $\sum_{s=1}^{S_i} b_{is}(g) + \sum_{s=1}^{S_j} b_{js}(g) < 2$ for $g \in \bar{\Gamma}_i \cap \bar{\Gamma}_j \subset \mathcal{K}$ if $i \ne j$ and $\bar{\Gamma}_i \cap \bar{\Gamma}_j \ne \varnothing$.

Theorem 1. *If Conditions* 1 *and* 2 *hold, then there exists a* $\delta_0 > 0$ *such that, for* $k = 0, 1, 2, \ldots$, $\delta \in (0, \delta_0)$, $q > q_0(\delta) \ge 0$, *and* $\psi_i \in C_{\mathcal{K}}(\bar{\Gamma}_i) \cap H_a^{k+3/2}(\Gamma_i)$, *where* $a = k + 1 - \delta$, *the problem*

$$P_0 u - qu = 0, \quad y \in G; \quad u|_{\Gamma_i} - \mathbf{B}_i u = \psi_i(y),$$
$$y \in \Gamma_i, \quad i = 1, 2, \ldots, N$$

has a unique solution $u \in C^\infty(G) \cap C_{\mathcal{K}}(\bar{G}) \cap H_a^{k+2}(G)$; *moreover*,

$$\|u\|_{C(\bar{G})} + |||u|||_{H_a^{k+2}(G)}$$
$$\le c\sum_{i=1}^{N}(\|\psi_i\|_{C(\bar{\Gamma}_i)} + |||\psi_i|||_{H_a^{k+3/2}(\Gamma_i)}),$$

where $c > 0$ *does not depend on* q, ψ_i, *and* u.

In what follows, we assume that $\delta \in (0, \delta_0)$, an integer $k \ge 2$, and $a = k + 1 - \delta$ are fixed.

3. Consider the bounded linear operator P_1: $H_a^{k+2}(G) \to H_{a-1}^k(G)$.

Condition 3. (i) If a function $u \in H_a^{k+2}(G)$ attains its positive maximum at a point $y^0 \in G$, then $P_1 u(y^0) \le 0$.

(ii) If $u \in C(\bar{G}) \cap H_a^{k+2}(G)$, then the function $P_1 u$ is bounded on G.

(iii) For all sufficiently small $\varrho > 0$, $P_1 = P_{1\varrho}^1 + P_{1\varrho}^2$, where the operators $P_{1\varrho}^1$, $P_{1\varrho}^2$: $H_a^{k+2}(G) \to H_{a-1}^k(G)$ are such that

(iii.1) $\|P_{1\varrho}^1 u\|_{H_{a-1}^k(G)} \le c(\varrho)\|u\|_{H_a^{k+2}(G)}$, where $c(\varrho) > 0$ does not depend on u and $c(\varrho) \to 0$ as $\varrho \to 0$ and

(iii.2) the operator $P_{1\varrho}^2$ is compact.

Note that $D(P_1) \subset C^k(\overline{G}\backslash\mathcal{K}) \subset C^2(G)$ and $\mathcal{R}(P_1) \subset C^{k-2}(\overline{G}\backslash\mathcal{K}) \subset C(G)$. However, generally, $\mathcal{R}(P_1) \not\subset C(\overline{G})$.

Remark 1. The prototype of the abstract operators P_1, $P^1_{1\varrho}$, and $P^2_{1\varrho}$ is the integral operators (cf. [5, 7, 9])

$$P_1u(y)$$

$$= \int_F [u(y + z(y, \eta)) - u(y) - (\nabla u(y), z(y, \eta))] \times m(y, \eta)\pi(d\eta),$$

$$P^1_{\varrho}\ u(y)$$

$$= \int_{Z \le \varrho} [u(y + z(y, \eta)) - u(y) - (\nabla u(y), z(y, \eta))] \times m(y, \eta)\pi(d\eta),$$

$$P^2_{\varrho}\ u(y)$$

$$= \int_{Z > \varrho} [u(y + z(y, \eta)) - u(y) - (\nabla u(y), z(y, \eta))] \times m(y, \eta)\pi(d\eta),$$

where F is a space with σ-algebra $\mathcal{F}$ and Borel measure π; $y + z(y, \eta) \in \overline{G}$; $|D^\alpha_y z(y, \eta)| \le Z(\eta)$ for $y \in \overline{G}$, $\eta \in F$, and $|\alpha| \le k$, where $Z(\eta)$ is a nonnegative π-measurable bounded function; and $m(y, \eta) \ge 0$ (the functions $z(y, \eta)$, $Z(\eta)$, and $m(y, \eta)$ are also subject to certain additional constraints).

4. In this paper, we consider nonlocal conditions in the nontransversal case (a probability interpretation is given in, e.g., [5]):

$$u(y) - \int_{\overline{G}} u(\eta)\mu_i(y, d\eta) = 0, \quad y \in \Gamma_i, \qquad (1)$$
$$i = 1, 2, \ldots, N; \quad u(y) = 0, \quad y \in \mathcal{K},$$

where $\mu_i(y, \cdot)$ is a nonnegative Borel measure on $\overline{G}$ such that $\mu_i(y, \overline{G}) \le 1$ for $y \in \Gamma_i$.

Let $\delta_{is}(y, \cdot)$ be the measures defined by $\delta_{is}(y, Q) = b_{is}(y)\chi_Q(\Omega_{is}(y))$ for $y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})$ and $\delta_{is}(y, Q) = 0$ for $y \in \Gamma_i\backslash\mathcal{O}_\varepsilon(\mathcal{K})$, where $Q \subset \overline{G}$ is an arbitrary Borel set and $\chi_Q(\cdot)$ is the characteristic function of the set Q.

Consider the measures $\mu_i(y, \cdot)$ that can be represented in the form

$$\mu_i(y, \cdot) = \sum_{s=1}^{S_i} \delta_{is}(y, \cdot) + \alpha_i(y, \cdot) + \beta_i(y, \cdot), \quad y \in \Gamma_i,$$

where $\alpha_i(y, \cdot)$ and $\beta_i(y, \cdot)$ are the nonnegative Borel measures defined further on.

We set $\operatorname{spt}\alpha_i(y, \cdot) = \overline{G}\backslash \bigcup_{V \in T} \{ V \in T: \alpha_i(y, V \cap \overline{G}) = 0\}$ (T is the family of all open sets in $\mathbb{R}^2$) and

$$\mathbf{B}_{\alpha i}u(y) = \int_{\overline{G}} u(\eta)\alpha_i(y, d\eta),$$

$$\mathbf{B}_{\beta i}u(y) = \int_{\overline{G}} u(\eta)\beta_i(y, d\eta), \quad \text{for} \quad y \in \Gamma_i.$$

Let $\alpha_i(y, \cdot)$ and $\beta_i(y, \cdot)$ be measures satisfying the following conditions (cf. [7]).

Condition 4. There exist numbers $\kappa_1 > \kappa_2 > 0$ and $\sigma > 0$ such that

(i) $\operatorname{spt}\alpha_i(y, \cdot) \subset \overline{G}\backslash\mathcal{O}_{\kappa_1}(\mathcal{K})$ for $y \in \Gamma_i$ and $|||\mathbf{B}_{\alpha i}u|||_{H^{k+3/2}_a(\Gamma_i)} \le c|||u|||_{H^{k+2}_a(G\backslash\overline{\mathcal{O}_{\kappa_1}(\mathcal{K})})}$;

(ii) $\operatorname{spt}\alpha_i(y, \cdot) \subset \overline{G}_\sigma$ for $y \in \Gamma_i\backslash\mathcal{O}_{\kappa_2}(\mathcal{K})$ and $|||\mathbf{B}_{\alpha i}u|||_{H^{k+3/2}_a(\Gamma_i\backslash\overline{\mathcal{O}_{\kappa_2}(\mathcal{K})})} \le c|||u|||_{H^{k+2}_a(G_\sigma)}$, where $G_\sigma = \{y \in G: \operatorname{dist}(y, \partial G) < \sigma\}$.

We set $\mathcal{N} = \bigcup_{i=1}^{N} \{ y \in \Gamma_i: \mu_i(y, \overline{G}) = 0\} \cup \mathcal{K}$ and $\mathcal{M} = \partial G\backslash\mathcal{N}$. We assume that $\mathcal{N}$ and $\mathcal{M}$ are Borel sets.

Condition 5. $\beta_i(y, \mathcal{M}) < 1$ *for* $y \in \Gamma_i \cap \mathcal{M}$ *and* $i = 1, 2, \ldots, N$.

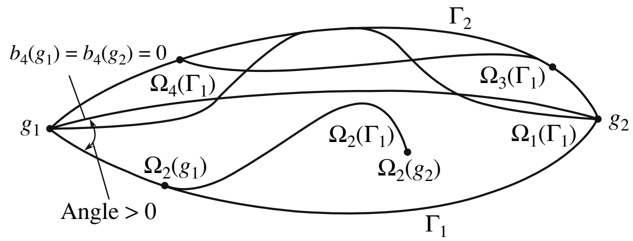
Condition 6. *For any function* $u \in C_{\mathcal{N}}(\overline{G})$, *the functions* $\mathbf{B}_{\alpha i}u$ *and* $\mathbf{B}_{\beta i}u$ *admit continuous extensions to* $\overline{\Gamma}_i$, *which belong to* $C_{\mathcal{N}}(\overline{\Gamma}_i)$.

Let us write the measures $\beta_i(y, \cdot)$ in the form $\beta_i(y, \cdot) = \beta^1_i(y, \cdot) + \beta^2_i(y, \cdot)$ where $\beta^1_i(y, \cdot)$ and $\beta^2_i(y, \cdot)$ are nonnegative Borel measures. They have the following properties. Let $\mathcal{M}_p = \mathcal{O}_p(\overline{\mathcal{M}})$ for $p > 0$. Consider a cut-off function $\hat{\zeta}_p \in C^\infty(\mathbb{R}^2)$ such that $0 \le \hat{\zeta}_p(y) \le 1$, $\hat{\zeta}_p(y) = 1$ for $y \in \mathcal{M}_{p/2}$, and $\hat{\zeta}_p(y) = 0$ for $y \notin \mathcal{M}_p$. We set $\tilde{\zeta}_p = 1 - \hat{\zeta}_p$. Let us introduce the operators

$$\hat{\mathbf{B}}^1_{\beta i}u(y) = \int_{\overline{G}} \hat{\zeta}_p(\eta)u(\eta)\beta^1_i(y, d\eta),$$

$$\tilde{\mathbf{B}}^1_{\beta i}u(y) = \int_{\overline{G}} \hat{\zeta}_p(\eta)u(\eta)\beta^1_i(y, d\eta),$$

$$\mathbf{B}^2_{\beta i}u(y) = \int_{\overline{G}} u(\eta)\beta^2_i(y, d\eta).$$



Nontransversal nonlocal conditions.

Consider the following Banach spaces with norms depending on the parameter q:

1. $H^{k+2}_{\mathcal{N},a}(G) = C_{\mathcal{N}}(\bar{G}) \cap H^{k+2}_a(G)$, $|||u|||_{H^{k+2}_{\mathcal{N},a}(G)} = \|u\|_{C_{\mathcal{N}}(\bar{G})} + |||u|||_{H^{k+2}_a(G)}$,

2. $H^{k+3/2}_{\mathcal{N},a}(\Gamma_i) = C_{\mathcal{N}}(\bar{\Gamma}_i) \cap H^{k+3/2}_a(\Gamma_i)$, $|||v|||_{H^{k+3/2}_{\mathcal{N},a}(\Gamma_i)} = \|v\|_{C_{\mathcal{N}}(\bar{\Gamma}_i)} + |||v|||_{H^{k+3/2}_a(\Gamma_i)}$.

Condition 7. For all sufficiently small $p > 0$,

(i) $|||\hat{\mathbf{B}}^1_{\beta i}|||_{H^{k+2}_{\mathcal{N},a}(G) \to H^{k+3/2}_{\mathcal{N},a}(\Gamma_i)} \to 0$ as $p \to 0$ uniformly in q;

(ii) the norms $|||\tilde{\mathbf{B}}^1_{\beta i}|||_{H^{k+2}_{\mathcal{N},a}(G) \to H^{k+3/2}_{\mathcal{N},a}(\Gamma_i)}$ are bounded uniformly in q.

Condition 8. The operators $\mathbf{B}^2_{\beta i}$: $H^{k+2}_{\mathcal{N},a}(G) \to H^{k+3/2}_{\mathcal{N},a}(\Gamma_i)$, where $i = 1, 2, \ldots, N$, are compact.

Consider the space $C_B(\bar{G}) = \{u \in C(\bar{G})$: u satisfies conditions (1)$\}$ and the unbounded operator $\mathbf{P}$: $D(\mathbf{P}) \subset C_B(\bar{G}) \to C_B(\bar{G})$ defined by

$$\mathbf{P}u = P_0 u + P_1 u,$$

$$u \in D(\mathbf{P})$$

$$= \{u \in C_B(\bar{G}) \cap H^{k+2}_a(G) \colon P_0 u + P_1 u \in C_B(\bar{G})\}.$$

Note that $D(\mathbf{P}) \subset C^2(G) \cap C_B(\bar{G})$; this follows from the inequality $k \geq 2$ and the Sobolev embedding theorem. The proof of the following theorem, which is the main result of this paper, is based on Theorem 1, the Hille–Yosida theorem, and the maximum principle.

Theorem 2. *If Conditions* 1–8 *hold, then the operator* $\mathbf{P}$ *has a closure* $\bar{\mathbf{P}}: D(\bar{\mathbf{P}}) \subset C_B(\bar{G}) \to C_B(\bar{G})$, *and* $\bar{\mathbf{P}}$ *is a generator of a Feller semigroup.*

5. Below, we give an example of nonlocal conditions satisfying the assumptions made above. Let $\partial G = \Gamma_1 \cup \Gamma_2 \cup \mathcal{K}$, where Γ_1 and Γ_2 are curves of class C^∞ open in the topology of ∂G, $\Gamma_1 \cap \Gamma_2 = \varnothing$, and $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \mathcal{K}$; the set $\mathcal{K}$ comprises two points, g_1 and g_2. For $j = 1, 2, 3, 4$, by Ω_j we denote nondegenerate transformation of class C^{k+2} defined in a neighborhood of $\bar{\Gamma}_1$ and satisfying the following conditions (see figure):

(i) $\Omega_1(\mathcal{K}) = \mathcal{K}$, $\Omega_1(\Gamma_1 \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$, $\Omega_1(\Gamma_1 \backslash \mathcal{O}_\varepsilon(\mathcal{K})) \subset G \cup \Gamma_2$, and $\Omega_1(y)$ is the composition of an argument shift, a rotation, and a homothety for $y \in \bar{\Gamma}_1 \cap \mathcal{O}_\varepsilon(\mathcal{K})$;

(ii) there exist numbers $\kappa_1 > \kappa_2 > 0$ and $\sigma > 0$ such that $\Omega_2(\bar{\Gamma}_1) \subset \bar{G} \backslash \mathcal{O}_{\kappa_1}(\mathcal{K})$ and $\Omega_2(\bar{\Gamma}_1 \backslash \mathcal{O}_{\kappa_2}(\mathcal{K})) \subset \overline{G_\sigma}$; moreover, $\Omega_2(g_1) \in \Gamma_1$ and $\Omega_2(g_2) \in G$;

(iii) $\Omega_3(\bar{\Gamma}_1) \subset G \cup \Gamma_2$ and $\Omega_3(\mathcal{K}) \subset \Gamma_2$;

(iv) $\Omega_4(\bar{\Gamma}_1) \subset G \cup \bar{\Gamma}_2$ and $\Omega_4(\mathcal{K}) = \mathcal{K}$; the angle between the tangent rays to Γ_1 and $\Omega_4(\Gamma_1)$ at the point g_j is nonzero.

Take functions $b_j \in C^{k+2}(\bar{\Gamma}_1)$ such that $b_j \geq 0$ for $j = 1, 2, 3, 4$ and consider the nonlocal boundary conditions

$$u(y) - \sum_{j=1}^{4} b_j(y) u(\Omega_j(y)) = 0, \quad y \in \Gamma_1,$$

$$u(y) = 0, \quad y \in \overline{\Gamma_2}.$$

Suppose that

$$\sum_{j=1}^{4} b_j(y) \leq 1, \quad y \in \Gamma_1;$$

$$b_2(g_1) = 0 \text{ or } \sum_{j=1}^{4} b_j(\Omega_2(g_1)) = 0;$$

$$b_2(g_2) = 0; \quad b_4(g_j) = 0.$$

Let $\zeta \in C^\infty(\mathbb{R}^2)$ be a cut-off function supported on $\mathcal{O}_\varepsilon(\mathcal{K})$, identically equal to 1 on $\mathcal{O}_{\varepsilon/2}(\mathcal{K})$, and such that $0 \leq \zeta(y) \leq 1$ for $y \in \mathbb{R}^2$. Take $y \in \Gamma_1$ and an arbitrary Borel set $Q \subset \bar{G}$. The measures

$$\delta(y, Q) = \zeta(y) b_1(y) \chi_Q(\Omega_1(y)),$$

$$\alpha(y, Q) = b_2(y) \chi_Q(\Omega_2(y)),$$

$$\beta^1(y, Q) = (1 - \zeta(y)) b_1(y) \chi_Q(\Omega_1(y))$$

$$+ \sum_{j=3,4} b_j(y) \chi_Q(\Omega_j(y)), \quad \beta^2(y, Q) = 0$$

(for simplicity, we omit the subscript 1 from the notation of measures) satisfy Conditions 1, 2, and 4–8.

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