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MATHEMATICS

On the Existence of Periodic Solutions to Some Nonlinear Thermal Control Problems[1]

P. L. Gurevich[a, b], W. Jäger[b], and A. L. Skubachevskii[a]

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We consider the heat equation with a boundary condition involving a control function. The control function satisfies an ordinary differential equation with a right-hand side containing a nonlinear functional that provides the hysteresis phenomenon. The dependence of the functional on the "mean" temperature over the domain causes nonlocal effects. The problem under consideration occurs in the modeling of thermal control processes in chemical reactors and climate control systems. The solvability of the problem and the periodicity of its solutions are considered.

1. In chemical reactors and climate control systems, there arises the problem of temperature control inside a volume by means of some thermal elements on the boundary of the volume. We consider a mathematical model for such a thermal control process.

In our model, the temperature distribution inside the domain obeys the heat equation, while the boundary condition involves a control function. The control function satisfies an ordinary differential equation whose right-hand side is given by a nonlinear functional depending on the "mean" temperature over the domain, which provides the so-called hysteresis phenomenon.

The existence and uniqueness of solutions to two-phase Stefan problems involving a boundary hysteresis control were studied in [1–3]. In the present paper, we establish an existence and uniqueness result for the heat equation and investigate the periodicity of its solutions. We suggest the concepts of a strong periodic solution and a mean-periodic solution and show that the existence of a mean-periodic solution implies the existence of a strong periodic solution with the same period. We also give an example in which a unique mean-periodic solution (hence, a unique strong periodic solution) exists.

2. Let $Q \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with boundary Γ of class C^∞. Let $w(x, t)$ be the temperature at the point $x \in Q$ at the moment $t \geq 0$ satisfying the heat equation

$$w_t(x, t) = \Delta w(x, t) - p(x)w(x, t), \quad (x, t) \in Q_T, \quad (1)$$

with the initial condition

$$w(x, 0) = \varphi(x), \quad x \in Q, \quad (2)$$

where $Q_T = Q \times (0, T)$, $p \in C^\infty(\mathbb{R}^n)$, and $\varphi \in L_2(Q)$ are real-valued functions, and $p(x) \geq 0$.

The boundary condition contains a real-valued control function $u(t)$ (to be defined below) which regulates the temperature on the boundary, the heat flux through the boundary, or the ambient temperature:

$$-\gamma\frac{\partial w}{\partial \nu} = \sigma(x)(w(x, t) - w_e(x))$$
$$- K(x)(u(t) - u_c), \quad (x, t) \in \Gamma_T, \quad (3)$$

where $\Gamma_T = \Gamma \times (0, T)$; ν is the outward normal to Γ_T at the point (x, t); $\gamma \geq 0$; σ, w_e, $K \in C^\infty(\mathbb{R}^n)$ are real-valued functions; $\sigma(x) \geq 0$; and $u_c > 0$; if $\gamma = 0$, then we assume that $\sigma(x) \geq \sigma_0 > 0$.

For any function $v(x, t)$, we set

$$v_m(t) = \int_Q m(x)v(x, t)dx,$$

where $m \in L_\infty(Q)$ is a given function.

We assume that u is a solution of the Cauchy problem

$$u'(t) + au(t) = H(w_m, t), \quad t \in (0, T), \quad (4)$$

$$u(0) = u_0, \quad (5)$$

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[a] Department of Differential Equations and Mathematical Physics, Peoples' Friendship University of Russia, ul. Miklukho-Maklaya 6, Moscow, 117198 Russia
[b] Interdisciplinary Center for Scientific Computing of the University of Heidelberg, Im Neuenheimer Feld 368, D-69120 Heidelberg, Germany
e-mail: gurevichp@yandex.ru, jaeger@iwr.uni-heidelberg.de, skub@lector.ru

where $a > 0$, $u_0 \in \mathbb{R}$, and w is a function satisfying relations (1)–(3). The functional $H(g, t)$, $g \in C[0, T]$, is defined as follows: $H(g, 0) = 1$ if $g(0) < w_2$, $H(g, 0) = 0$ if $g(0) \ge w_2$, and

$$H(g,t) = \begin{cases} 1 \text{ if } g(t) \le w_1 \\ 1 \text{ if } w_1 < g(t) < w_2 \text{ and} \\ \exists\delta = \delta(g,t) > 0\colon H(g,\tau) = 1 \\ \text{for } t-\delta<\tau<t \\ 0 \text{ if } w_1 < g(t) < w_2 \text{ and} \\ \exists\delta = \delta(g,t) > 0\colon H(g,\tau) = 0 \\ \text{for } t-\delta<\tau<t \\ 0 \text{ if } g(t) \ge w_2 \end{cases} \qquad (6)$$

for $t > 0$, where w_1 and w_2 ($w_1 < w_2$) are fixed. Thus, if $g(t)$ is in the interval (w_1, w_2), then the value of H at the moment t is the same as its value at the moment "just before" t. If $g(t)$ achieves the lower threshold w_1 at the moment t and H has been equal to 0 just before t, then it switches to 1 (otherwise, it remains equal to 1). If $g(t)$ achieves the upper threshold w_2 at the moment t and H has been equal to 1 just before t, then it switches to 0 (otherwise, it remains equal to 0).

To be definite, we assume that

$$w_1 \le \int_Q m(x)\varphi(x)dx < w_2. \qquad (7)$$

3. Denote by $W_2^k(Q)$, $k \in \mathbb{N}$, the Sobolev space with the norm

$$\|v\|_{W_2^k(Q)} = \left(\sum_{|\alpha| \le k}\int_Q |D^\alpha v(x)|^2 dx\right)^{1/2}.$$

Let $W_\infty^1(a, b)$ ($a < b$) denote the space of absolutely continuous functions having the first derivative from $L_\infty(a, b)$ with the norm

$$\|u\|_{W_\infty^1(a,b)} = \max_{t\in[a,b]}|u(t)| + \operatorname{vrai}\sup_{t\in(a,b)}|u'(t)|. \qquad (8)$$

We denote by $W_2^{2,1}(Q \times (a, b))$, $a < b$, the anisotropic Sobolev space with the norm

$$\|w\|_{W_2^{2,1}(Q\times(a,b))} = \left(\int_a^b \|w(\cdot,t)\|^2_{W_2^2(Q)}dt + \int_a^b \|w_t(\cdot,t)\|^2_{L_2(Q)}dt\right)^{1/2}.$$

Set

$$\mathcal{W}(Q_T) = W_2^{2,1}(Q_T) \times W_\infty^1(0,T).$$

Definition 1. A pair of functions $(w, u) \in \mathcal{W}(Q_T)$ is called a strong solution of problem (1)–(5) in Q_T if w satisfies Eq. (1) a.e. in Q_T and conditions (2) and (3) in the sense of traces and u satisfies Eq. (4) a.e. in $(0, T)$ and condition (5).

Definition 2. Let a pair $(w, u) \in \mathcal{W}(Q_T)$ be a strong solution of problem (1)–(5) in Q_T. A moment $t_1 \in (0, T)$ is called a switching time if one of the following two events occurs:

1. $\exists\delta = \delta(t_1)$: $H(w_m, \tau) = 1$ for $t_1 - \delta < \tau < t_1$ and $w_m(t_1) = w_2$,

2. $\exists\delta = \delta(t_1)$: $H(w_m, \tau) = 0$ for $t_1 - \delta < \tau < t_1$ and $w_m(t_1) = w_1$.

Along with the space $\mathcal{W}(Q_T)$ of strong solutions, we introduce the set $\mathcal{V}$ of initial data. We define

$$\mathcal{V} = W_2^1(Q) \times \mathbb{R}$$

if $\gamma > 0$ and

$$\mathcal{V} = \{(\varphi, u_0) \in W_2^1(Q) \times \mathbb{R}\colon \sigma(x)(\varphi(x) - w_e(x)) - K(x)(u_0 - u_c) = 0 \ (x \in \Gamma)\}$$

if $\gamma = 0$.

Theorem 1. 1. *Let* $(\varphi, u_0) \in \mathcal{V}$ *and condition* (7) *hold. Then there exists a unique strong solution* $(w, u) \in \mathcal{W}(Q_T)$ *of problem* (1)–(5) *in* Q_T.

2. *If* t' *and* t'' *are two switching times* ($0 < t' < t'' < T$), *then*

$$t'' - t' \ge \frac{(w_2 - w_1)^2}{c\|m\|_{L_\infty(Q)}(\|\varphi\|_{W_2^1(Q)} + a_1 + u_c)^2},$$

where $a_1 = \max\left(2, \frac{1}{a}, |u_0|, 1 + a|u_0|\right)$ *and* $c > 0$ *does not depend on* $m(x)$, u_c, w_1, w_2, u_0, φ, t', *and* t''.

Assertion 2 in Theorem 1 implies that the set of switching times on the interval $(0, T)$ is finite or empty.

4. In this section, we establish the existence of a strong T-periodic solution (w, u) of problem (1), (3), (4), provided that for some initial data $(\varphi, u_0) \in \mathcal{V}$ problem (1)–(5) has a strong solution $(\tilde{w}, \tilde{u}) \in \mathcal{W}(Q_T)$ in Q_T such that $\tilde{w}_m(0) = \tilde{w}_m(T)$, $\tilde{u}(0) = \tilde{u}(T)$, and $H(\tilde{w}_m, T) = 1$.

Definition 3. A pair (w, u) is called a strong T-periodic solution of problem (1), (3), (4) if the following conditions hold for any $T_0 \ge T$:

(i) $(w, u) \in \mathcal{W}(Q_{T_0})$,

(ii) the function w satisfies the equation in (1) a.e. in Q_{T_0} and the equality in (3) on Γ_{T_0},

(iii) the function u satisfies the equation in (4) a.e. in $(0, T_0)$,

(iv) $w(\cdot, t) = w(\cdot, t + T)$, $u(t) = u(t + T)$, and $H(w_m, t) = H(w_m, t + T)$ for $t \in [0, T_0 - T]$.

Definition 4. We say that problem (1), (3), (4) possesses the mean-periodicity property if there is a pair $(\varphi, u_0) \in \mathcal{V}$ and a number $T > 0$ such that, for any $T_0 \geq T$, the strong solution $(\tilde{w}, \tilde{u}) \in \mathcal{W}(Q_{T_0})$ of problem (1)–(5) in Q_{T_0} with the initial data $(\varphi, u_0) \in \mathcal{V}$ satisfies the equalities

$$\tilde{w}_m(t) = \tilde{w}_m(t+T), \quad \tilde{u}(t) = \tilde{u}(t+T),$$

$$H(\tilde{w}_m, t) = H(\tilde{w}_m, t+T), \quad t \in [0, T_0 - T].$$

The strong solution $(\tilde{w}, \tilde{u})$ is said to be a mean-periodic solution (with period T).

Theorem 2. *Let problem* (1), (3), (4) *possess the mean-periodicity property and* $(\tilde{w}, \tilde{u})$ *be a mean-periodic solution* (*with period* T) *such that*

$$w_1 \leq \tilde{w}_m(0) < w_2.$$

If $p(x) \equiv 0$ *and* $\sigma(x) \equiv 0$, *we assume that* $m(x) \equiv m_0 =$ const.

Then there is a unique strong T-periodic solution $(w, \tilde{u})$ *of problem* (1), (3), (4) *such that* $w_m(t) = \tilde{w}_m(t)$ *for* $t \geq 0$.

Moreover,

$$\|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \quad as \quad t \to \infty.$$

Theorem 2 ensures the "conditional" existence of a strong periodic solution (i.e., under the condition that there exists a mean-periodic solution). The idea of the proof of Theorem 2 is as follows. Let $(w, u) \in \mathcal{W}(Q_{T_0})$ be a strong solution of problem (1)–(5) in Q_{T_0} with initial data $(\varphi, \tilde{u}(0)) \in \mathcal{V}$ (which exists for any $T_0 > 0$ due to Theorem 1). We introduce a nonlinear operator

$$G: \varphi(x) \mapsto w(x, T),$$

where T is a period of the mean-periodic solution $(\tilde{w}, \tilde{u})$. The domain of G consists of functions φ such that

$$(\varphi, \tilde{u}(0)) \in \mathcal{V}, \quad w_m(t) = \tilde{w}_m(t),$$

$$u(t) = \tilde{u}(t), \quad t \geq 0.$$

Then we prove that the domain of G is a nonempty closed set in $W_2^1(Q)$ and G is a contraction map. Application of the Banach fixed-point theorem yields the desired result.

Using Theorem 2, one can also prove the following result.

Corollary 1. *Let the hypotheses of Theorem* 2 *be fulfilled. Assume that* $\tilde{u}(t) \not\equiv$ const. *Then, for any* $\varphi_0 \in [w_1, w_2)$, *there is a strong T-periodic solution* (w, u) *of problem* (1), (3), (4) *such that*

$$w_m(0) = \varphi_0.$$

5. In this section, we consider a thermal control problem that possesses the mean-periodicity property. Hence, by Theorem 2, it also admits a strong periodic solution.

We consider problem (1)–(5) with $p(x) \equiv 0$, $\sigma(x) \equiv 0$, $\gamma = 1$, and $m(x) \equiv m_0$:

$$w_t(x, t) = \Delta w(x, t), \quad (x, t) \in Q_T, \tag{9}$$

$$w(x, 0) = \varphi(x), \quad x \in Q, \tag{10}$$

$$\frac{\partial w}{\partial \nu} = K(x)(u(t) - u_c), \quad (x, t) \in \Gamma_T. \tag{11}$$

The control function $u(t)$ satisfies the Cauchy problem

$$u'(t) + au(t) = H(w_m, t), \quad t \in (0, T), \tag{12}$$

$$u(0) = u_0, \tag{13}$$

where u_c, $a > 0$; $H(w_m, 0) = 1$; $H(w_m, t)$ for $t > 0$ is given by (6); and the mean temperature is given by

$$w_m(t) = m_0 \int_Q w(x, t) dx.$$

To establish the mean-periodicity property, we will show that the mean temperature $w_m(t)$ satisfies an ordinary differential equation. Integrating Eq. (9) over Q, we have

$$w'_m(t) = m_0 \int_Q \Delta w dx = m_0 \int_\Gamma \frac{\partial w}{\partial \nu} d\Gamma.$$

Combining this equality with the boundary-value condition (11), we obtain

$$w'_m(t) = k(u(t) - u_c), \quad t \in (0, T), \tag{14}$$

where $k = m_0 \int_\Gamma K(x) dx$. The initial condition for the function $w_m(t)$ has the form

$$w_m(0) = m_0 \int_Q w(x, 0) dx = m_0 \int_Q \varphi(x) dx. \tag{15}$$

The behavior of $u(t)$ and $w_m(t)$ is schematically depicted in Fig. 1 (for the case in which $m_0 \int_\Gamma K(x) d\Gamma > 0$, $u_c < u_0 < \frac{1}{a}$, and $w_1 \leq \varphi_0 < w_2$).

Analyzing Cauchy problem (12), (13) for the control function $u(t)$ simultaneously with Cauchy problem (14), (15) for the mean temperature $w_m(t)$, one can prove the following result.

Theorem 3. *Let* $m_0 \int_\Gamma K(x) d\Gamma > 0$, $u_c < \frac{1}{a}$, *and* $w_1 \leq \varphi_0 < w_2$. *Then there exists a unique initial control* $\hat{u}_0$ *on*

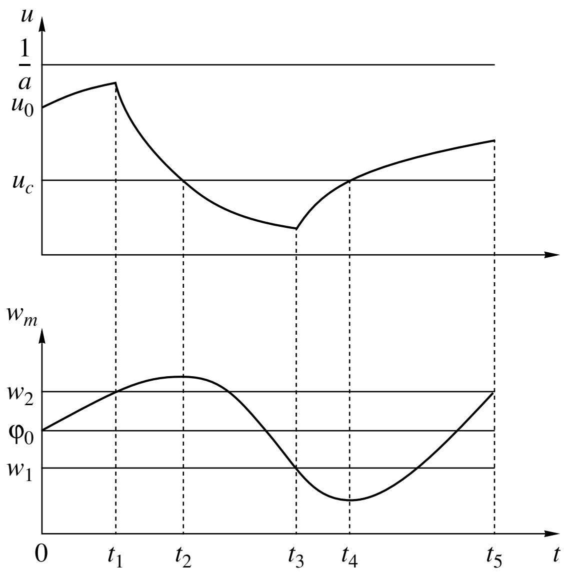

Fig. 1. The behavior of $u(t)$ and $w_m(t)$. Here, t_1, t_3, and t_5 are switching times and t_2, t_4 are the points at which $u(t) = u_c$ and which are therefore critical points for $w_m(t)$.

the interval $\left[u_c, \frac{1}{a}\right]$ *such that the solutions* $u(t)$ *and* $w_m(t)$ *of problems* (12), (13) *and* (14), (15) *with the initial data* $\hat{u}_0$ *and* φ_0, *respectively, are both periodic with the same period. This initial control* $\hat{u}_0$ *satisfies the inequalities* $u_c < \hat{u}_0 < \frac{1}{a}$. *The pair* $(w_m(t), u(t))$ *is a stable cycle in the phase plane* (w_m, u) (*see Fig*. 2).

Theorem 2 on the conditional existence of a strong periodic solution and Theorem 3 on the existence of a mean-periodic solution imply the following result.

Theorem 4. 1. *Let* $m_0\int_\Gamma K(x)d\Gamma > 0$, $u_c < \frac{1}{a}$, *and* $w_1 \le \varphi_0 < w_2$. *Then there exists a unique strong periodic solution* (w, u) *of problem* (9), (11), (12) *such that*

$$u(0) \in \left[u_c, \frac{1}{a}\right], \quad w_m(0) = \varphi_0.$$

Moreover, the initial control $u(0)$ *satisfies the inequalities* $u_c < u(0) < \frac{1}{a}$.

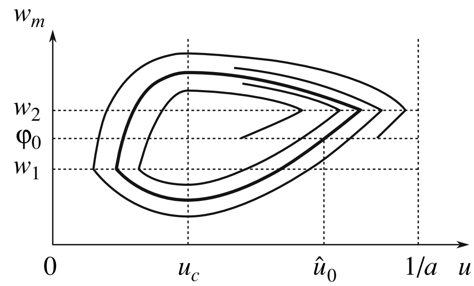

Fig. 2. A limit-cycle trajectory $(w_m(t), u(t))$.

2. *If* $(\tilde{w}, \tilde{u})$ *is a mean-periodic solution of problem* (9), (11), (12) *such that*

$$\tilde{u}(0) \in \left[u_c, \frac{1}{a}\right], \quad \tilde{w}_m(0) = \varphi_0,$$

then

$$\tilde{u}(t) \equiv u(t), \quad \tilde{w}_m(t) \equiv w_m(t),$$
$$\|\tilde{w}(\cdot, t) - w(\cdot, t)\|_{W_2^1(Q)} \to 0 \ \ as \ \ t \to \infty,$$

where (w, u) *is the strong periodic solution from assertion* 1.

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