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ASYMPTOTICS OF SOLUTIONS FOR NONLOCAL ELLIPTIC PROBLEMS IN PLANE BOUNDED DOMAINS *

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Abstract.
The paper is devoted to the study of asymptotic behavior of solutions for nonlocal elliptic problems in weighted spaces. We deal with the most difficult case when the support of nonlocal terms intersects with boundary of a plane bounded domain. In this situation, a general form of the asymptotics is investigated, and coefficients in the asymptotics are calculated.

Key Words. Nonlocal elliptic problem, weighted space, asymptotics of solution

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Contents

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1. Introduction.

I. This work is devoted to the investigation of asymptotic behavior of solutions for nonlocal elliptic problems. Recently many mathematicians have been studying nonlocal problems. This interest is explained, on the one hand, by a significant theoretical progress in the area and, on the other hand, by a number of important applications arising in plasma theory [3], biophysics, theory of diffusion processes [7, 30, 26], modern aircraft technology (particularly, in the theory of sandwich shells and plates [26]), etc.

In the 1-dimensional case the first ones who studied nonlocal problems were A. Sommerfeld [28], Ya.D. Tamarkin [29], M. Picone [19]. In the 2-dimensional case the earliest paper devoted to nonlocal problems is due to T. Carleman [5]. T. Carleman searched for a harmonic function u in a plane bounded domain G subject to a nonlocal condition connecting the values of the unknown function in different points of boundary: $u(x)+bu\big(\omega(x)\big)=g(x)$. Here $\omega:\partial G\to\partial G$ is a nondegenerate transformation subject to the restriction $\omega(\omega(x))\equiv x$ (being referred to as Carleman's condition in the present time). Such a statement of nonlocal problems has originated further research into the area of elliptic problems with nonlocal transformations mapping a boundary onto itself and with abstract boundary conditions [31, 4, 2, 1].

In 1969, A.V. Bitsadze and A.A. Samarskii [3] considered the following nonlocal problem arising in plasma theory: find a harmonic function $u(y_1,\ y_2)$ in the rectangle $G=\{y\in\mathbb{R}^2:-1<y_1<1,\ 0<y_2<1\}$ such that it is continuous in $\bar G$ and satisfies the conditions

$$\begin{gathered} u(y_1,\ 0)=f_1(y_1),\ u(y_1,\ 1)=f_2(y_1),\ -1<y_1<1,\\ u(-1,\ y_2)=f_3(y_2),\ u(1,\ y_2)=u(0,\ y_2),\ 0<y_2<1, \end{gathered}$$

where $f_1,\ f_2,\ f_3$ are given continuous functions. We notice that this problem principally differs from the one studied by T. Carleman: now the values of the unknown function on the part of the boundary ∂G are connected with the values inside the domain G. This problem was solved in [3] by reducing to an integral Fredholm equation and using the maximum principle. In case of an arbitrary domain and general nonlocal transformations, it was formulated as an unsolved one.

The most difficult case turns out to deal with the situation when a part Υ_1 of boundary of a domain G is mapped by some nonlocal transformation Ω_1 on $\Omega_1(\Upsilon_1)$ so that $\overline{\Omega_1(\Upsilon_1)}\cap\partial G\ne\varnothing$. Various versions of such problems were considerer by S.D. Eidelman and N.V. Zhitarashu [6], K.Yu. Kishkis [14], A.K. Gushchin and V.P. Mikhailov [13], etc.

Basis of general theory for elliptic equations of order $2m$ with general nonlocal conditions was founded by A.L. Skubachevskii and his pupils. In a series of works a priori estimates were proved, a right regularizer was constructed, adjoint problems were studied, and properties of index in appropriate spaces were established; spectral properties of some problems were considered [21, 22, 23, 24, 25, 20, 16, 9, 10]; asymptotics and smoothness of solutions near some special points were investigated [22, 12]. We remark that the papers [22, 23, 24] were the first ones to deal with the case $\overline{\Omega_1(\Upsilon_1)} \cap \bar{\Upsilon}_1 \neq \varnothing$, which had not been previously considered even for the Laplace equation with nonlocal conditions in plane domains.

II. In this paper we investigate the most difficult situation mentioned above: the support of nonlocal terms can have a nonempty intersection with boundary of a domain G. In that case, power singularities for solutions near some set $\mathcal{K} \subset G$ can appear [22, 27]. Therefore it is quite natural to study such problems in special weighted spaces that take into consideration those possible singularities. (The most convenient spaces turned out to be Kondrat'ev's ones [15].) Thus we arrive at the question of asymptotics of solutions near the set $\mathcal{K}$. In the paper [22], A.L. Skubachevskii obtained a general form of an asymptotics of solutions to problems with nonlocal transformations coinciding with a rotation operator near the set $\mathcal{K}$. These theorems were applied to investigation of smoothness for generalized solutions of nonlocal elliptic problems (see [22, 27]).

In the present work we generalize the mentioned results of A.L. Skubachevskii and study the case of arbitrary nonlocal transformations, linear near the set $\mathcal{K}$. Simultaneously, we get a formula connecting the indices of one and the same nonlocal problem, but being considered in different weighted spaces.

Moreover, using the results of the paper [12] (which deals with model nonlocal problems in plane angles and in $\mathbb{R}^2 \setminus \{0\}$), we get explicit formulas for calculating the coefficients in the asymptotics of solutions. These formulas are given both in terms of eigenvectors and associated vectors of model adjoint problems and in terms of distributions from the kernel of adjoint problem in a bounded domain. The latter shows, in particular, that the values of the coefficients in the asymptotics are the functionals over the right–hand sides of the nonlocal problem under consideration. These functionals depend on the data of the problem in the whole domain, but not only in some neighborhood of the set $\mathcal{K}$.

We remark that the calculation of the coefficients in the asymptotics is both important itself and has a direct application to the question of smoothness of generalized solutions for nonlocal problems. Roughly speaking, it allows to show that a generalized solution $u \in W_2^1(G)$ to a nonlocal problem (for an elliptic 2nd order equation) with a right–hand side $f \in L_2(G)$ is smooth (i.e., $u \in W_2^2(G)$) if and only if the function f satisfies some orthogonality conditions. In a number of cases these conditions can be verified explicitly.

III. The paper is organized as follows. The statement of the problem and some assumptions concerning nonlocal transformations are given in section 2. Most of the assumptions are due to simplify computations throughout the paper. In section 3 we derive an asymptotics (with yet unknown coefficients) for solutions to nonlocal problems. Using the results of section 3, in section 4 we establish a connection between the indices of one and the same problem but being considered in different weighted spaces. In section 5 we obtain an asymptotics of solutions for adjoint nonlocal problems. This allows to get in section 6 explicit formulas for the coefficients in asymptotics of solutions to the original nonlocal problem. In section 7 we consider an example illustrating the results of sections 2–6.

2. Statement of the problem in a bounded domain.

Let $G \in \mathbb{R}^2$ be a bounded domain with a boundary $\partial G = \bigcup\limits_{\sigma=1,2} \bar{\Upsilon}_\sigma$, where Υ_σ are open (in the topology of ∂G) curves of C^∞ class such that $\Upsilon_1 \cap \Upsilon_2 = \varnothing$, $\bar{\Upsilon}_1 \cap \bar{\Upsilon}_2 = \{g_1,\ h_1\}$. We suppose that in some neighborhoods of the points g_1 and h_1 the domain G coincides with an angle.

We denote by $\mathbf{P}(y,\ D_y)$, $B_{\sigma\mu}(y,\ D_y)$, $T_{\sigma\mu}(y,\ D_y)$ differential operators of orders $2m$, $m_{\sigma\mu}$, $m_{\sigma\mu}$ respectively with complex–valued coefficients from $C^\infty(\mathbb{R}^2)$ ($m_{\sigma\mu} \le 2m-1$, $\sigma = 1,\ 2$; $\mu = 1,\ \dots,\ m$). Put also $B_\sigma(y,\ D_y) = \{B_{\sigma\mu}(y,\ D_y)\}_{\mu=1}^m$, $T_\sigma(y,\ D_y) = \{T_{\sigma\mu}(y,\ D_y)\}_{\mu=1}^m$.

Let Ω_σ ($\sigma = 1,\ 2$) be an infinitely differentiable nondegenerate transformation mapping some neighborhood $\mathcal{O}_\sigma$ of Υ_σ onto $\Omega_\sigma(\mathcal{O}_\sigma)$ such that $\Omega_\sigma(\Upsilon_\sigma) \subset G$. For definiteness, we consider the case when $\Omega_1(g_1) = g_2 \in G$, $\Omega_2(g_1) = g_1$, $\Omega_1(h_1) = h_1$, $\Omega_2(h_1) = h_2 \in G$. In this work we also assume that $g_2 \notin \overline{\Omega_2(\Upsilon_2)}$, $h_2 \notin \overline{\Omega_1(\Upsilon_1)}$. The last assumption is made in order to simplify further computations[1]. But, following [22, 24], we demand (and it

[1] If, say, $g_2 \in \overline{\Omega_2(\Upsilon_2)}$, then either $g_2 = h_2$ (in that case, an asymptotics of a solution

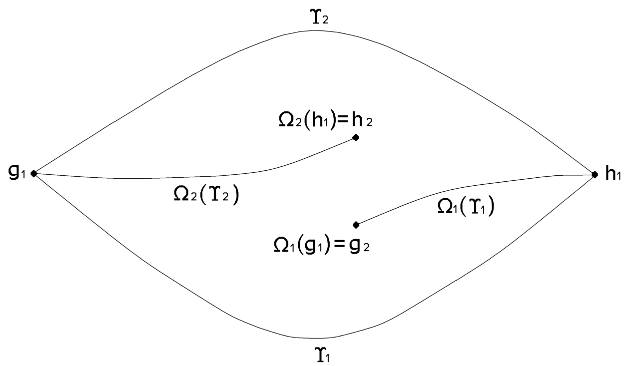

FIG. 2.1. *The domain G.*

is on principle) that following condition holds:

CONDITION 2.1. *The curves $\Omega_1(\Upsilon_1)$ and $\Omega_2(\Upsilon_2)$ are not tangent to the boundary ∂G at the "consistent points" h_1 and g_1 respectively (see Fig. 2.1).*

We also suppose for simplicity that the transformations $\Omega_\sigma(y)$ are linear near the points g_1 and h_1.

We introduce the set $\mathcal{K} = \{g_1,\ h_1,\ g_2,\ h_2\}$ and consider the nonlocal elliptic problem

$$\mathbf{P}(y,\ D_y)u = f(y) \quad (y \in G\backslash\mathcal{K}), \tag{2.1}$$

$$\mathbf{B}_\sigma(y,\ D_y)u \equiv B_\sigma(y,\ D_y)u|_{\Upsilon_\sigma} + \big(T_\sigma(y,\ D_y)u\big)\big(\Omega_\sigma(y)\big)|_{\Upsilon_\sigma} = f_\sigma(y) \\ (y \in \Upsilon_\sigma;\ \sigma = 1,\ 2). \tag{2.2}$$

Here $\big(T_\sigma(y,\ D_y)u\big)\big(\Omega_\sigma(y)\big) = T_\sigma(y',\ D_{y'})u(y')|_{y'=\Omega_\sigma(y)}$; $f_\sigma = \{f_{\sigma\mu}\}_{\mu=1}^m$.

REMARK 2.1. *The results of this paper are generalized for the case when the boundary ∂G consists of a finite number of smooth curves Υ_σ, $\sigma = 1,\ \dots,\ N$, and nonlocal conditions on each Υ_σ contain a finite number of nonlocal terms with different transformations. Moreover, these transformations can map "the consistent points" (which are g_1 and h_1 in our case) both to the boundary ∂G and inside the domain G, forming finite orbits.*

We introduce the space $H_a^l(G)$ as a completion of the set $C_0^\infty(\bar{G}\backslash\mathcal{K})$ in the norm

$$\|u\|_{H_a^l(G)} = \left(\sum_{|\alpha|\le l}\int\limits_G \rho^{2(a-l+|\alpha|)}|D^\alpha u|^2 dy\right)^{1/2}.$$

near the point g_2 will influence not only an asymptotics near g_1 but near h_1 as well) or $g_2 \in \Omega_2(\Upsilon_2)$ (in that case, one must study an asymptotics at the additional point $\Omega_2^{-1}(g_2) \in \Upsilon_2$).

Here $C_0^\infty(\bar{G}\backslash\mathcal{K})$ is the set of infinitely differentiable functions with compact supports contained in $\bar{G}\backslash\mathcal{K}$; $l \ge 0$ is an integer; $a \in \mathbb{R}$; $\rho = \rho(y) = \operatorname{dist}(y,\ \mathcal{K})$.

If, instead of the domain G, one considers an angle with a vertex g or some neighborhood of a point g, then one must put $\mathcal{K} = \{g\}$ in the definition of the weighted space.

By $H_a^{l-1/2}(\Upsilon)$ we denote the space of traces on a smooth curve $\Upsilon \subset \bar{G}$ with the norm $\|\psi\|_{H_a^{l-1/2}(\Upsilon)} = \inf \|u\|_{H_a^l(G)} \quad (u \in H_a^l(G):\ u|_\Upsilon = \psi)$.

Let us introduce the operator

$$\mathbf{L} = \{\mathbf{P}(y,\ D_y),\ \mathbf{B}_\sigma(y,\ D_y)\} :$$
$$H_a^{l+2m}(G) \to H_a^l(G,\ \Upsilon) \stackrel{def}{=} H_a^l(G) \times \prod_{\sigma=1,2} \prod_{\mu=1}^{m} H_a^{l+2m-m_{\sigma\mu}-1/2}(\Upsilon_\sigma),$$

which corresponds to nonlocal problem (2.1), (2.2).

Throughout the paper we assume that the operators $\mathbf{P}(y,\ D_y)$ and $B_\sigma(y,\ D_y)$ satisfy the following conditions (see, e.g., [17, Chapter 2, section 1]).

Condition 2.2. *For all $y \in \bar{G}$ the operator $\mathbf{P}(y,\ D_y)$ is properly elliptic.*

Condition 2.3. *For $\sigma = 1,\ 2$ and $y \in \bar{\Upsilon}_\sigma$ the system $B_\sigma(y,\ D_y) = \{B_{\sigma\mu}(y,\ D_y)\}_{\mu=1}^m$ covers the operator $\mathbf{P}(y,\ D_y)$.*

Remark that we do not impose any restrictions on the nonlocal operators $T_{\sigma\mu}(y,\ D_y)$ but the natural restriction on their orders.

3. Asymptotics of solutions for nonlocal problems.

I. In this section we obtain an asymptotics of a given solution $u \in H_a^{l+2m}(G)$ for problem (2.1), (2.2) with a right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^l(G,\ \Upsilon)$, $0 < a - a_1 < 1$.

Notice that the violation of the inequality $a - a_1 < 1$ means that $\{f,\ f_\sigma\}$ is "too regular". In that case, exact results should yield more terms in asymptotics in comparison with our case. This situation can be investigated in the way similar to [22] (see also [15, 18]). Namely, one should consider corresponding equations for a residue in the asymptotics formula and apply to them the results obtained for the case $a - a_1 < 1$. We are not going to do this here since detailed computations would lead to enormous enlargement of the paper, giving no essentially new results (with respect to the present work and to the paper [22]). The same remark is valid for the case when $u \in H_a^{l+2m}(G)$, $\{f,\ f_\sigma\} \in H_{a_1}^{l_1}(G,\ \Upsilon)$, and $l_1 \ne l$.

The asymptotics will be found with the help of eigenvalues and corresponding Jordan chains of some holomorphic operator–valued functions. Therefore let us remind some relevant definitions and facts (see [8]).

Suppose $\tilde{\mathcal{L}}(\lambda) : H_1 \to H_2$ is a holomorphic operator–valued function, $H_1,\ H_2$ are Hilbert spaces. A holomorphic at a point λ_0 vector–function $\varphi(\lambda)$ with the values in H_1 is called *a root function* of the operator $\tilde{\mathcal{L}}(\lambda)$ at λ_0 if $\varphi(\lambda_0) \neq 0$ and the vector–function $\tilde{\mathcal{L}}(\lambda)\varphi(\lambda)$ is equal to 0 at λ_0. If $\tilde{\mathcal{L}}(\lambda)$ has at least one root function at the point λ_0, then λ_0 is called *an eigenvalue* of $\tilde{\mathcal{L}}(\lambda)$. Multiplicity of zero for the vector–function $\tilde{\mathcal{L}}(\lambda)\varphi(\lambda)$ at the point λ_0 is called *a multiplicity of the root function* $\varphi(\lambda)$; the vector $\varphi^{(0)} = \varphi(\lambda_0)$ is called *an eigenvector* corresponding to the eigenvalue λ_0. Let $\varphi(\lambda)$ be a root function at the point λ_0 of multiplicity $\varkappa$, and $\varphi(\lambda) = \sum\limits_{j=0}^{\infty}(\lambda-\lambda_0)^j\varphi^{(j)}$. Then the vectors $\varphi^{(1)},\ \dots,\ \varphi^{(\varkappa-1)}$ are called *associated with the eigenvector* φ_0, and the ordered set $\varphi^{(0)},\ \dots,\ \varphi^{(\varkappa-1)}$ is called *a Jordan chain* corresponding to the eigenvalue λ_0. *Rank* of the eigenvector $\varphi^{(0)}$ ($\operatorname{rank}\varphi^{(0)}$) is the maximum of multiplicities of all root functions such that $\varphi(\lambda_0) = \varphi^{(0)}$.

Let an eigenvalue λ_0 of the operator $\tilde{\mathcal{L}}(\lambda)$ be isolated, $\dim\ker\tilde{\mathcal{L}}(\lambda_0) < \infty$, and rank of λ_0 finite. Suppose $J = \dim\ker\tilde{\mathcal{L}}(\lambda_0)$ and $\varphi^{(0,1)},\ \dots,\ \varphi^{(0,J)}$ is a system of linearly independent eigenvectors such that $\operatorname{rank}\varphi^{(0,1)}$ is the greatest of ranks of all eigenvectors corresponding to the eigenvalue λ_0, and $\operatorname{rank}\varphi^{(0,j)}$ ($j = 2,\ \dots,\ J$) is the greatest of ranks of eigenvectors from some orthogonal supplement in $\ker\tilde{\mathcal{L}}(\lambda_0)$ to the linear manifold of the vectors $\varphi^{(0,1)},\ \dots,\ \varphi^{(0,j-1)}$. The numbers $\varkappa_j = \operatorname{rank}\varphi^{(0,j)}$ are called *partial multiplicities* of the eigenvalue λ_0, and the sum $\varkappa_1+\dots+\varkappa_J$ is called *a (full) multiplicity* of λ_0. If the vectors $\varphi^{(0,j)},\ \dots,\ \varphi^{(\varkappa_j-1,j)}$ form a Jordan chain for every $j = 1,\ \dots,\ J$, then the set of vectors $\{\varphi^{(0,j)},\ \dots,\ \varphi^{(\varkappa_j-1,j)} : j = 1,\ \dots,\ J\}$ is called *a canonical system of Jordan chains* corresponding to the eigenvalue λ_0.

II. At first let us consider an asymptotics of the solution u for problem (2.1), (2.2) near the point g_2. In this case we will see that the asymptotics is defined by a model "local" problem in $\mathbb{R}^2 \setminus \{g_2\}$ with a "regular" right–hand side. Such a problem was studied in [12, section 5]. Thereafter we will consider the asymptotics near the point g_1. In that case, we will arrive at a model nonlocal problem in some angle K with a right–hand side being a sum of "regular" and "special" functions. The asymptotics of the "special" one will be defined by the asymptotics of the solution u near g_2, which is explained by the presence of the nonlocal transformation Ω_1. Then the results of [12] will be applied to this model problem.

Thus we fix a neighborhood $\mathcal{V}(g_2)$ of g_2 such that $\overline{\mathcal{V}(g_2)} \cap \partial G = \varnothing$ and $\overline{\mathcal{V}(g_2)} \cap \{h_2\} = \varnothing$. One can see that an asymptotic behavior of u in $\mathcal{V}(g_2)$ does not depend on nonlocal conditions (2.2), but is defined only by the equation

$$\mathbf{P}(y,\ D_y)u = f(y) \quad \big(y \in \mathcal{V}(g_2)\big). \tag{3.1}$$

Let $\mathcal{P}(D_y)$ be the principal homogeneous part of the operator $\mathbf{P}(g_2,\ D_y)$. Then equation (3.1) can be written in the form

$$\mathcal{P}(D_y)u(y) = \hat{f}(y) \quad \big(y \in \mathcal{V}(g_2)\big), \tag{3.2}$$

where $\hat{f}$, by virtue of the condition $0 < a - a_1 < 1$, belongs to the space $H^l_{a_1}\big(\mathcal{V}(g_2)\big)$[2]. We introduce the bounded operator

$$\mathcal{L}_2 = \mathcal{P}(D_y) : H_a^{l+2m}(\mathbb{R}^2) \to H_a^l(\mathbb{R}^2),$$

where, defining the weighted spaces, one must put $\mathcal{K} = \{g_2\}$.

We write the operator $\mathcal{P}(D_y)$ in polar coordinates with the pole at the point g_2: $\mathcal{P}(D_y) = r^{-2m}\tilde{\mathcal{P}}(\omega,\ D_\omega,\ rD_r)$, where $D_\omega = -i\dfrac{\partial}{\partial\omega},\ D_r = -i\dfrac{\partial}{\partial r}$.

Let us introduce the operator–valued function

$$\tilde{\mathcal{L}}_2(\lambda) = \tilde{\mathcal{P}}(\omega,\ D_\omega,\ \lambda) : W_{2,2\pi}^{l+2m}(0,\ 2\pi) \to W_{2,2\pi}^l(0,\ 2\pi).$$

Here $W^l_{2,2\pi}(0,\ 2\pi)$ is the closure of the set of infinitely differentiable 2π-periodic functions in $W_2^l(0,\ 2\pi)$.

The operator $\tilde{\mathcal{L}}_2(\lambda)$ is obtained from the operator $\mathcal{L}_2$ by passing to polar coordinates, followed by the Mellin transformation with respect to r:

$$\tilde{u}(\lambda) = (2\pi)^{-1/2} \int\limits_0^\infty r^{-i\lambda-1}u(r)\,dr.$$

From [22, section 1] it follows that there exists a finite–meromorphic operator–valued function $\tilde{\mathcal{R}}_2(\lambda)$ such that its poles (except, maybe, a finite number of them) are located inside a double angle of opening less than π containing the imaginary axis; moreover, if λ is not a pole of $\tilde{\mathcal{R}}_2(\lambda)$, then

[2] To show that $\hat{f} \in H^l_{a_1}\big(\mathcal{V}(g_2)\big)$, one must estimate the expressions of the two types: 1) $p_\alpha(y)D^\alpha u$, $|\alpha| \le 2m-1$, and 2) $\big(p_\alpha(y) - p_\alpha(0)\big)D^\alpha u$, $|\alpha| = 2m$, where p_α are infinitely differentiable coefficients of $\mathbf{P}(y,\ D_y)$. The 1st one is estimated by direct use of the condition $0 < a - a_1 < 1$, while the 2nd one needs additional application of Lemma 3.3′ [15]. Further, in analogous situations, we will omit these explanations.

$\tilde{\mathcal{R}}_2(\lambda)$ is inverse to the operator $\tilde{\mathcal{L}}_2(\lambda)$. Thus a number λ is a pole of $\tilde{\mathcal{R}}_2(\lambda)$ if and only if λ is an eigenvalue (of finite multiplicity) of $\tilde{\mathcal{L}}_2(\lambda)$.

If the line $\operatorname{Im}\lambda = a+1-l-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}_2(\lambda)$, then, by virtue of [22, section 1], the operator $\mathcal{L}_2$ is an isomorphism.

In order to formulate a theorem concerning an asymptotics near g_2, let us introduce some denotation. Suppose λ_2 is an eigenvalue of $\tilde{\mathcal{L}}_2(\lambda)$,

$$\{\varphi_2^{(0,\zeta)},\ \dots,\ \varphi_2^{(\varkappa_{\zeta,2}-1,\zeta)} : \zeta = 1,\ \dots,\ J_2\} \tag{3.3}$$

is a canonical system of Jordan chains of the operator $\tilde{\mathcal{L}}_2(\lambda)$ corresponding to the eigenvalue λ_2.

Consider the vector $u_2 = \{u_2^{(k,\zeta)}\}$, where

$$u_2^{(k,\zeta)}(\omega,\ r) = r^{i\lambda_2}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\varphi_2^{(k-q,\zeta)}(\omega), \tag{3.4}$$

$(\omega,\ r)$ are polar coordinates with the pole at the point g_2.

Notice that (see [12, section 5]) the vector u_2, the components $u_2^{(k,\zeta)}$ of which are defined by (3.4), satisfies the relation

$$\mathcal{L}_2 u_2 = 0. \tag{3.5}$$

THEOREM 3.1. *Let the lines* $\operatorname{Im}\lambda = a_1+1-l-2m$, $\operatorname{Im}\lambda = a+1-l-2m$ *contain no eigenvalues of* $\tilde{\mathcal{L}}_2(\lambda)$ *and the strip* $a_1+1-l-2m < \operatorname{Im}\lambda < a+1-l-2m$ *contain the only eigenvalue* λ_2 *of* $\tilde{\mathcal{L}}_2(\lambda)$. *Then*

$$u(y) = c_2u_2(y) + \hat{u}(y) \quad \big(y \in \mathcal{V}(g_2)\big)^3. \tag{3.6}$$

Here $u_2 = \{u_2^{(k,\zeta)}\}$, $u_2^{(k,\zeta)}$ *are defined by (3.4);* $c_2 = \{c_2^{(k,\zeta)}\}$ *is a vector of some constants;* $\hat{u} \in H_{a_1}^{l+2m}\big(\mathcal{V}(g_2)\big)^4$.

Proof. Introduce the cut–off function $\eta \in C^\infty(\mathbb{R}^2)$ equal to 1 in some neighborhood of the point g_2 and vanishing outside $\mathcal{V}(g_2)$. Suppose that the function ηu is defined in the whole of $\mathbb{R}^2$, being equal to 0 outside $\mathcal{V}(g_2)$. Then from (3.2) and Leibnitz's formula, it follows that

$$\mathcal{L}_2(\eta u) \in H_{a_1}^{l+2m}(\mathbb{R}^2).$$

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[3] In formula (3.6) and further the expressions such as c_2u_2 are calculated in the following way: $c_2u_2 = \sum\limits_{\zeta=1}^{J_2}\sum\limits_{k=0}^{\varkappa_{\zeta,2}-1} c_2^{(k,\zeta)}u_2^{(k,\zeta)}$.

[4] The results of this work are evidently generalized for the case when the strip $a_1+1-l-2m < \operatorname{Im}\lambda < a+1-l-2m$ contains a finite number of eigenvalues of $\tilde{\mathcal{L}}_2(\lambda)$.

Now it remains only to apply Theorem 5.1 [12], which establishes the asymptotics of solutions for nonlocal problems in $\mathbb{R}^2 \setminus \{g_2\}$. $\square$

REMARK 3.1. *In fact, the assumption that the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no eigenvalues of* $\tilde{\mathcal{L}}_2(\lambda)$ *is superfluous. Theorem 3.1 remains valid even if it is violated (see Remark 5.1 [12]). But this assumption will be used for studying the adjoint nonlocal problem and for calculating the coefficients* $c_2^{(k,\zeta)}$. *However, this assumption does not lead to the loss in generality. Indeed, one can find an* ε, $0 < \varepsilon < a - a_1$, *such that the strip* $a-\varepsilon+1-l-2m \le \operatorname{Im}\lambda \le a+1-l-2m$ *contains no eigenvalues of* $\tilde{\mathcal{L}}_2(\lambda)$, *and therefore (see [12, section 5])* $u \in H^{l+2m}_{a-\varepsilon}(\mathcal{V}(g_2))$. *Hence we arrive at the situation of Theorem 3.1.*

REMARK 3.2. *From the results of [12], proof of Theorem 3.1, and Remark 3.1, it follows that if the strip* $a_1+1-l-2m \le \operatorname{Im}\lambda < a+1-l-2m$ *has no eigenvalues of* $\tilde{\mathcal{L}}_2(\lambda)$, *then* $u \in H^{l+2m}_{a_1}\big(\mathcal{V}(g_2)\big)$ *for any right–hand side* $\{f,\ f_\sigma\} \in H^l_{a_1}(G,\ \Upsilon)$.

III. Now we consider an asymptotics of the solution u for problem (2.1), (2.2) near the point g_1. Fix a neighborhood $\mathcal{V}(g_1)$ of g_1 such that

$$\overline{\mathcal{V}(g_1)} \cap \overline{\Omega_1(\Upsilon_1)} = \varnothing \quad \text{and} \quad \overline{\mathcal{V}(g_1)} \cap \{h_2\} = \varnothing. \tag{3.7}$$

Then one can see that an asymptotic behavior of the solution u is defined by the problem

$$\mathbf{P}(y,\ D_y)u = f(y) \quad (y \in \mathcal{V}(g_1) \cap G), \tag{3.8}$$

$$\begin{gathered} B_1(y,\ D_y)u|_{\mathcal{V}(g_1)\cap\Upsilon_1} = f_1(y) - \big(T_1(y,\ D_y)u\big)\big(\Omega_1(y)\big)|_{\mathcal{V}(g_1)\cap\Upsilon_1} \\ (y \in \mathcal{V}(g_1) \cap \Upsilon_1), \\ B_2(y,\ D_y)u|_{\mathcal{V}(g_1)\cap\Upsilon_2} + \big(T_2(y,\ D_y)u\big)\big(\Omega_2(y)\big)|_{\mathcal{V}(g_1)\cap\Upsilon_2} = f_2(y) \\ (y \in \mathcal{V}(g_1) \cap \Upsilon_2). \end{gathered} \tag{3.9}$$

Let $\mathcal{P}(D_y)$, $B_\sigma(D_y)$, $T_2(D_y)$ be the principal homogeneous parts of the operators $\mathbf{P}(g_1,\ D_y)$, $B_\sigma(g_1,\ D_y)$, $T_2(g_1,\ D_y)$ respectively[5]. Let $T_1(D_y)$ be the principal homogeneous part of the operator $T_1(g_2,\ D_y)$.

From now on we shall suppose that the origin coincides with the point g_1: $g_1 = 0$, and

$$\mathcal{V}(g_1) \cap G = \mathcal{V}(0) \cap K, \tag{3.10}$$

[5] Notice that earlier, in this section, we denoted by $\mathcal{P}(D_y)$ the principal homogeneous part of the operator $\mathbf{P}(g_2,\ D_y)$. To be strict we had to denote these operators by different symbols. But we do not do it since throughout the paper it will always be clear from the context whether we consider the principal homogeneous part of $\mathbf{P}(y,\ D_y)$ at g_1 or at g_2.

where K is the plane angle: $K = \{y \in \mathbb{R}^2 : \ r > 0, \ b_1 < \omega < b_2\}$ with the arms $\gamma_\sigma = \{y \in \mathbb{R}^2 : \ r > 0, \ \omega = b_\sigma\}$, $\sigma = 1, \ 2$. Here $(\omega, \ r)$ are polar coordinates with the pole at the point $g_1 = 0$, $0 < b_1 < b_2 < 2\pi$.

According to the assumptions of section 2, the transformations $\Omega_\sigma(y)$ are linear in $\mathcal{V}(g_1) = \mathcal{V}(0)$. Let $\Omega_1(y)$ $(y \in \mathcal{V}(g_1))$ be a composition of a rotation and an expansion with respect to g_1, and the shift by the vector $\overrightarrow{g_1 g_2}$. Let $\Omega_2(y)$ $(y \in \mathcal{V}(g_1))$ coincide with the linear operator $\mathcal{G}_2$ of a rotation by an angle ω_2 $(b_1 < b_2 + \omega_2 < b_2)$ and an expansion with a coefficient $\beta_2 > 0$.

Let the neighborhood $\mathcal{V}(g_1) = \mathcal{V}(0)$ be so small that $\{\Omega_1(y) : \ y \in \mathcal{V}(g_1)\} \subset \mathcal{V}(g_2)$ and relations (3.7), (3.10) are fulfilled with the set $\{\Omega_2(y) : y \in \mathcal{V}(g_1)\}$ substituted for $\mathcal{V}(g_1)$ (which is related to (3.9)). (Mention that this requirement is automatically fulfilled whenever the expansion coefficients for the transformations $\Omega_\sigma(y)$ near the point g_1 are less or equal to 1.)

Now, using asymptotics formula (3.6) for the solution u near g_2 and Leibnitz's formula, we get that problem (3.8), (3.9) in $\mathcal{V}(g_1) \cap G$ is equivalent to the following one in $\mathcal{V}(0) \cap K$:

$$\mathcal{P}(D_y)u = \hat{f}(y) \quad (y \in \mathcal{V}(0) \cap K), \tag{3.11}$$

$$\begin{gathered} B_1(D_y)u|_{\mathcal{V}(0)\cap\gamma_1} = \hat{f}_1(y) - c_2 f_{12}(y) \quad (y \in \mathcal{V}(0) \cap \gamma_1), \\ B_2(D_y)u|_{\mathcal{V}(0)\cap\gamma_2} + (T_2(D_y)u)(\mathcal{G}_2 y)|_{\mathcal{V}(0)\cap\gamma_2} = \hat{f}_2(y) \quad (y \in \mathcal{V}(0) \cap \gamma_2). \end{gathered} \tag{3.12}$$

Here $\hat{f} = f - \big(\mathbf{P}(y, \ D_y) - \mathcal{P}(D_y)\big)u$,

$$f_{12} = \big(T_1(D_y)u_2\big)\big(\Omega_1(y)\big)|_{\mathcal{V}(0)\cap\gamma_1}, \tag{3.13}$$

$$\begin{aligned} \hat{f}_1 = f_1 - \big(B_1(y, \ D_y) - B_1(D_y)\big)u|_{\mathcal{V}(0)\cap\gamma_1} - \\ \big(T_1(y, \ D_y) - T_1(D_y))u\big)\big(\Omega_1(y)\big)|_{\mathcal{V}(0)\cap\gamma_1} - \big(T_1(D_y)\hat{u}\big)\big(\Omega_1(y)\big)|_{\mathcal{V}(0)\cap\gamma_1}, \end{aligned}$$

$$\begin{aligned} \hat{f}_2 = f_2 - \big(B_2(y, \ D_y) - B_2(D_y)\big)u|_{\mathcal{V}(0)\cap\gamma_2} - \\ \big((T_2(y, \ D_y) - T_2(D_y))u\big)\big(\mathcal{G}_2 y\big)|_{\mathcal{V}(0)\cap\gamma_2}. \end{aligned}$$

Since $T_{1\mu}(D_y)$ is a homogeneous operator of order $m_{1\mu}$, from (3.13) and (3.4) it follows that the components $f_{12\mu}^{(k,\zeta)}$ of the vector $f_{12} = \{f_{12\mu}^{(k,\zeta)}\}$ are linear combinations of the functions $r^{i\lambda_2 - m_{1\mu}}(i \ln r)^q$, $0 \le q \le k$.

Moreover, by virtue of the condition $0 < a - a_1 < 1$, we have $\hat{f} \in H_{a_1}^l(\mathcal{V}(0) \cap K)$, $\hat{f}_{\sigma\mu} \in H_{a_1}^{l+2m-m_{\sigma\mu}-1/2}(\mathcal{V}(0) \cap \gamma_\sigma)$.

Thus we see that (3.11), (3.12) is a model nonlocal problem in $\mathcal{V}(0) \cap K$ with the right–hand side being the sum of the "regular" function $\{\hat{f}, \ \hat{f}_1, \ \hat{f}_2\}$

and the "special" function $\{0,\ -c_2 f_{12},\ 0\}$. The asymptotics of the vector f_{12} is defined by the asymptotics of the solution u near the point g_2, i.e., by the vector u_2 (see (3.4)).

Now we are to apply the results of [12]. Put

$$\begin{aligned} &\mathcal{B}_1(D_y)u = B_1(D_y)u|_{\gamma_1}, \\ &\mathcal{B}_2(D_y)u = B_2(D_y)u|_{\gamma_2} + (T_2(D_y)u)(\mathcal{G}_2 y)|_{\gamma_2} \end{aligned} \tag{3.14}$$

and introduce the bounded operator

$$\mathcal{L}_1 = \{\mathcal{P}(D_y),\ \mathcal{B}_\sigma(D_y)\} : \\ H_a^{l+2m}(K) \to H_a^l(K,\ \gamma) \stackrel{def}{=} H_a^l(K) \times \prod_{\sigma=1,2} \prod_{\mu=1}^{m} H_a^{l+2m-m_{\sigma\mu}-1/2}(\gamma_\sigma),$$

which corresponds to the model nonlocal problem in the angle K.

Write the operators involved into $\mathcal{L}_1$ in polar coordinates: $\mathcal{P}(D_y) = r^{-2m}\tilde{\mathcal{P}}(\omega,\ D_\omega,\ rD_r)$, $B_\sigma(D_y) = \{r^{-m_{\sigma\mu}}\tilde{B}_{\sigma\mu}(\omega,\ D_\omega,\ rD_r)\}_{\mu=1}^m$, $T_\sigma(D_y) = \{r^{-m_{\sigma\mu}}\tilde{T}_{\sigma\mu}(\omega,\ D_\omega,\ rD_r)\}_{\mu=1}^m$.

Consider the operator–valued function

$$\tilde{\mathcal{L}}_1(\lambda) : W_2^{l+2m}(b_1,\ b_2) \to W_2^l[b_1,\ b_2] \stackrel{def}{=} W_2^l(b_1,\ b_2) \times \mathbb{C}^{2m}$$

given by

$$\tilde{\mathcal{L}}_1(\lambda)\varphi = \{\tilde{\mathcal{P}}(\omega,\ D_\omega,\ \lambda)\varphi,\ \tilde{B}_{1\mu}(\omega,\ D_\omega,\ \lambda)\varphi(\omega)|_{\omega=b_1}, \\ \tilde{B}_{2\mu}(\omega,\ D_\omega,\ \lambda)\varphi(\omega)|_{\omega=b_2} + \beta_2^{i\lambda-m_{2\mu}}\tilde{T}_{2\mu}(\omega+\omega_2,\ D_\omega,\ \lambda)\varphi(\omega+\omega_2)|_{\omega=b_2}\}.$$

The operator $\tilde{\mathcal{L}}_1(\lambda)$ is obtained from the operator $\mathcal{L}_1$ by passing to polar coordinates, followed by the Mellin transformation with respect to r.

From Lemmas 2.1, 2.2 [23] it follows that there exists a finite–meromorphic operator–valued function $\tilde{\mathcal{R}}_1(\lambda)$ such that its poles (except, maybe, a finite number of them) are located inside a double angle of opening less than π containing the imaginary axis; moreover, if λ is not a pole of $\tilde{\mathcal{R}}_1(\lambda)$, then $\tilde{\mathcal{R}}_1(\lambda)$ is inverse to the operator $\tilde{\mathcal{L}}_1(\lambda)$. Thus a number λ is a pole of $\tilde{\mathcal{R}}_1(\lambda)$ if and only if λ is an eigenvalue (of finite multiplicity) of $\tilde{\mathcal{L}}_1(\lambda)$.

If the line $\operatorname{Im}\lambda = a+1-l-2m$ contains no eigenvalues of $\tilde{\mathcal{L}}_1(\lambda)$, then, by virtue of Theorem 2.1 [23], the operator $\mathcal{L}_1$ is an isomorphism.

In order to formulate a theorem concerning an asymptotics near g_1, let us introduce some denotation. Suppose λ_1 is an eigenvalue of the operator $\tilde{\mathcal{L}}_1(\lambda)$ located inside the strip $a_1+1-l-2m < \operatorname{Im}\lambda < a+1-l-2m$,

$$\{\varphi_1^{(0,\zeta)},\ \dots,\ \varphi_1^{(\varkappa_{\zeta,1}-1,\zeta)} : \zeta = 1,\ \dots,\ J_1\} \tag{3.15}$$

is a canonical system of Jordan chains of the operator $\tilde{\mathcal{L}}_1(\lambda)$ corresponding to the eigenvalue λ_1.

Consider the vector $u_1 = \{u_1^{(k,\zeta)}\}$, where

$$u_1^{(k,\zeta)}(\omega,\ r) = r^{i\lambda_1} \sum_{q=0}^{k} \frac{1}{q!}(i\ln r)^q \varphi_1^{(k-q,\zeta)}(\omega), \tag{3.16}$$

$(\omega,\ r)$ are polar coordinates with the pole at the point $g_1 = 0$.

Notice that (see Lemma 2.2 [12]) the vector u_1, the elements $u_1^{(k,\zeta)}$ of which are defined by (3.16), satisfies the relation

$$\mathcal{L}_1 u_1 = 0. \tag{3.17}$$

If λ_2 is an eigenvalue of $\tilde{\mathcal{L}}_1(\lambda)$ (i.e., $\lambda_2 = \lambda_1$), then denote by $\varkappa(\lambda_2)$ the greatest of partial multiplicities of λ_2. If λ_2 is not an eigenvalue of $\tilde{\mathcal{L}}_1(\lambda)$ (i.e., $\lambda_2 \neq \lambda_1$), put $\varkappa(\lambda_2) = 0$.

THEOREM 3.2. *Let the lines* $\operatorname{Im}\lambda = a_1+1-l-2m$, $\operatorname{Im}\lambda = a+1-l-2m$ *contain no eigenvalues of* $\tilde{\mathcal{L}}_1(\lambda)$ *and the strip* $a_1 + 1 - l - 2m < \operatorname{Im}\lambda < a + 1 - l - 2m$ *contain the only eigenvalue* λ_1 *of* $\tilde{\mathcal{L}}_1(\lambda)$. *Then*

$$u(y) = c_1 u_1(y) + c_2 u_{12}(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_1) \cap G). \tag{3.18}$$

Here $u_1 = \{u_1^{(k,\zeta)}\}$, *where* $u_1^{(k,\zeta)}$ *is defined by (3.16);* $u_{12} = \{u_{12}^{(k,\zeta)}\}$, *where* $u_{12}^{(k,\zeta)}$ *is a linear combination (which will be strictly defined in the proof below) of the functions* $r^{i\lambda_2}(i\ln r)^q\varphi_{k\zeta q}(\omega)$, $\varphi_{k\zeta q} \in W_2^{l+2m}(b_1,\ b_2)$, $0 \le q \le k+\varkappa(\lambda_2)$, $(\omega,\ r)$ *are polar coordinates with the pole at* g_1*;* $c_1 = \{c_1^{(k,\zeta)}\}$ *is a vector of some constants;* c_2 *is the vector of constants appearing in (3.6);* $\hat{u} \in H_{a_1}^{l+2m}(\mathcal{V}(g_1) \cap G)$[6].

Proof. Let $u_{12} = \{u_{12}^{(k,\zeta)}\}$ be a particular solution (which is defined by Lemma 4.3 [12]) for the problem

$$\mathcal{P}(D_y)u_{12} = 0 \quad (y \in K), \tag{3.19}$$

$$\mathcal{B}_1(D_y)u_{12} = -f_{12}, \quad \mathcal{B}_2(D_y)u_{12} = 0, \tag{3.20}$$

Here $f_{12} = \{f_{12\mu}^{(k,\zeta)}\}$ is defined by (3.13). We remind that each element $f_{12\mu}^{(k,\zeta)}$ is the linear combination of the functions $r^{i\lambda_2 - m_{1\mu}}(i\ln r)^q$, $0 \le q \le k$. Therefore, by Lemma 4.3 [12], the particular solution u_{12} has the form described in the formulation of the theorem. Moreover, each element $u_{12}^{(k,\zeta)}$ of the vector

[6] See footnote 4 on page 183.

u_{12} is uniquely defined if λ_2 is not an eigenvalue of $\tilde{\mathcal{L}}_1(\lambda)$ (i.e., $\lambda_2 \neq \lambda_1$). Otherwise (i.e., if $\lambda_2 = \lambda_1$) it is defined accurate to an arbitrary linear combination of power solutions (3.16) corresponding to the eigenvalue $\lambda_2 = \lambda_1$. Later on we shall suppose the particular solution $u_{12} = \{u_{12}^{(k,\zeta)}\}$ being fixed.

Introduce the cut–off function $\eta \in C^\infty(\mathbb{R}^2)$ equal to 1 in some neighborhood of the origin and vanishing outside $\mathcal{V}(0)$. Put $w = \eta(u - c_2 u_{12})$. Since u is a solution for problem (3.11), (3.12) and u_{12} is a solution for problem (3.19), (3.20), one can easily check (using Leibnitz's formula) that $\mathcal{L}_1 w \in H^l_{a_1}(K,\ \gamma)$. Therefore, to conclude the proof, it remains to apply Theorem 2.2 [12], which establishes the asymptotics of solutions for nonlocal problems in angles. □

Remark 3.3. *In fact, the assumption that the line* $\operatorname{Im}\lambda = a+1-l-2m$ *contains no eigenvalues of* $\tilde{\mathcal{L}}_1(\lambda)$ *is superfluous. But, using the results of the paper [12], one can show (similarly to Remark 3.1) that this assumption does not lead to the loss in generality. Therefore we remain it since it will be used for studying the adjoint nonlocal problem and for calculating the coefficients* $c_1^{(k,\zeta)}$.

Remark 3.4. *If the strip* $a_1 + 1 - l - 2m \le \operatorname{Im}\lambda < a + 1 - l - 2m$ *has no eigenvalues of* $\tilde{\mathcal{L}}_1(\lambda)$*, but still has one (say,* λ_2*) of* $\tilde{\mathcal{L}}_2(\lambda)$*, then (3.18) will assume the form* $u(y) = c_2 u_{12}(y) + \hat{u}(y)$ $(y \in \mathcal{V}(g_1) \cap G)$*. And only if the mentioned strip has neither eigenvalues of* $\tilde{\mathcal{L}}_1(\lambda)$ *nor* $\tilde{\mathcal{L}}_2(\lambda)$*, the solution* u *will be "regular" near the point* g_1*:* $u \in H^{l+2m}_{a_1}(\mathcal{V}(g_1) \cap G)$ *for any right–hand side* $\{f,\ f_\sigma\} \in H^l_{a_1}(G,\ \Upsilon)$*. (Cf. Remark 3.2.)*

Theorem 3.2 shows that the asymptotic behavior of solutions for problem (2.1), (2.2) near the point g_1 depends on the data of the problem both near the point g_1 itself and near the point $g_2 \in G$, which is connected with g_1: $g_2 = \Omega_1(g_1)$.

IV. Quite similarly to the above one can study an asymptotics of solutions for problem (2.1), (2.2) near the points h_ν in terms of the spectral properties of the operators $\tilde{\mathcal{L}}'_\nu(\lambda)$ corresponding to the points h_ν, $\nu = 1,\ 2$. The operators $\tilde{\mathcal{L}}'_\nu(\lambda)$ are introduced similarly to the operators $\tilde{\mathcal{L}}_\nu(\lambda)$.

In order not to repeat the analogous computations, we suppose that the solution u is "regular" in some neighborhoods $\mathcal{V}(h_\nu)$ of the points h_ν: $u \in H^{l+2m}_{a_1}\big(\mathcal{V}(h_\nu)\big)$, $\nu = 1,\ 2$.

Now we shall formulate the condition that summarize all our assumptions concerning the spectral properties of the operators $\tilde{\mathcal{L}}_\nu(\lambda)$ and $\tilde{\mathcal{L}}'_\nu(\lambda)$.

Condition 3.1. *Let the lines* $\operatorname{Im}\lambda = a_1 + 1 - l - 2m$ *and* $\operatorname{Im}\lambda = a+1-l-2m$ *contain no eigenvalues of the operator–valued functions* $\tilde{\mathcal{L}}_\nu(\lambda)$*,* $\tilde{\mathcal{L}}'_\nu(\lambda)$*; let the strip* $a_1 + 1 - l - 2m < \operatorname{Im}\lambda < a + 1 - l - 2m$ *contain the only* *eigenvalue* λ_ν *of* $\tilde{\mathcal{L}}_\nu(\lambda)$ *and no eigenvalues of* $\tilde{\mathcal{L}}'_\nu(\lambda)$, $\nu = 1,\ 2$.

We notice that the assumption concerning the absence of eigenvalues of $\tilde{\mathcal{L}}'_\nu(\lambda)$, $\nu = 1,\ 2$, in the strip $a_1 + 1 - l - 2m \le \operatorname{Im}\lambda \le a + 1 - l - 2m$ guarantees the regularity of solutions in the above sense (see Remarks 3.2 and 3.4).

From now on we suppose Condition 3.1 being fulfilled.

In the sequel it will be convenient to have an asymptotics formula for the solution $u \in H_a^{l+2m}(G)$ to problem (2.1), (2.2) in the whole domain G. To write this formula, we introduce infinitely smooth functions η_ν with the supports in $\mathcal{V}(g_\nu)$ such that $\eta_\nu(y) = 1$ in some neighborhoods of the points g_ν, $\nu = 1,\ 2$. Consider the vector–functions

$$U_1 = \eta_1 u_1; \quad U_2 = \eta_2 u_2 + \eta_1 u_{12}. \tag{3.21}$$

The functions U_ν are supposed to be defined in the whole domain G, vanishing outside $\mathcal{V}(g_\nu)$, $\nu = 1,\ 2$. Then Theorems 3.1 and 3.2 yield the following asymptotics of $u \in H_a^{l+2m}(G)$:

$$u \equiv \Big(c_1 U_1 + c_2 U_2\Big)\Big(\operatorname{mod} H_{a_1}^{l+2m}(G)\Big). \tag{3.22}$$

Let us remark for the sequel that the components $U_\nu^{(k,\zeta)}$ of the vector $U_\nu = \{U_\nu^{(k,\zeta)}\}$ are such that

$$\mathbf{L}U_\nu^{(k,\zeta)} \in H_{a_1}^l(G,\ \Upsilon). \tag{3.23}$$

To prove it, we firstly put $\{F,\ F_\sigma\} = \mathbf{L}U_1^{(k,\zeta)}$. Since the support of $U_1^{(k,\zeta)} = \eta_1 u_1^{(k,\zeta)}$ is contained in $\mathcal{V}(g_1) = \mathcal{V}(0)$, we have

$$\begin{gathered}
\mathbf{P}(y,\ D_y)\eta_1 u_1^{(k,\zeta)} = F(y) \quad (y \in K),\\
B_1(y,\ D_y)\eta_1 u_1^{(k,\zeta)}|_{\gamma_1} = F_1(y) \quad (y \in \gamma_1),\\
B_2(y,\ D_y)\eta_1 u_1^{(k,\zeta)}|_{\gamma_2} + \big(T_2(y,\ D_y)\eta_1 u_1^{(k,\zeta)}\big)\big(\mathcal{G}_2 y\big)|_{\gamma_2} = F_2(y) \quad (y \in \gamma_2).
\end{gathered}$$

But the vector $u_1 = \{u_1^{(k,\zeta)}\}$ satisfies (3.17). Therefore, using Leibnitz's formula, we obtain $\{F,\ F_\sigma\} \in H_{a_1}^l(G,\ \Upsilon)$.

Now put $\{F,\ F_\sigma\} = \mathbf{L}U_2^{(k,\zeta)}$. Similarly to the above we get

$$\begin{gathered}
\mathbf{P}(y,\ D_y)\eta_1 u_{12}^{(k,\zeta)} + \mathbf{P}(y,\ D_y)\eta_2 u_2^{(k,\zeta)} = F(y) \quad (y \in K),\\
B_1(y,\ D_y)\eta_1 u_{12}^{(k,\zeta)}|_{\gamma_1} + \big(T_1(y,\ D_y)\eta_2 u_2^{(k,\zeta)}\big)\big(\Omega_1(y)\big)|_{\gamma_1} = F_1(y) \quad (y \in \gamma_1),\\
B_2(y,\ D_y)\eta_1 u_{12}^{(k,\zeta)}|_{\gamma_2} + \big(T_2(y,\ D_y)\eta_1 u_{12}^{(k,\zeta)}\big)\big(\mathcal{G}_2 y\big)|_{\gamma_2} = F_2(y) \quad (y \in \gamma_2).
\end{gathered}$$

But the vector $u_2 = \{u_2^{(k,\zeta)}\}$ satisfies (3.5), and the vector $u_{12} = \{u_{12}^{(k,\zeta)}\}$ satisfies (3.19), (3.20). Therefore, using Leibnitz's formula, we again obtain $\{F,\ F_\sigma\} \in H^l_{a_1}(G,\ \Upsilon)$.

4. Index of nonlocal problems.

I. In this section we study some properties of the kernel, cokernel, and index of the operator $\mathbf{L}$ corresponding to nonlocal problem (2.1), (2.2). In particular, using the asymptotics formula (3.22), we shall obtain a formula connecting the indices of one and the same problem (2.1), (2.2), but being considered in different weighted spaces.

Let $\varkappa_\nu$ be a full multiplicity of the eigenvalue λ_ν of the operator–valued function $\tilde{\mathcal{L}}_\nu(\lambda)$: $\varkappa_\nu = \sum\limits_{\zeta=1}^{J_\nu} \varkappa_{\zeta,\nu}$. Put $\varkappa = \varkappa_1 + \varkappa_2$.

Lemma 4.1. *Homogeneous problem (2.1), (2.2) can have no more than $\varkappa$ linearly independent modulo $H^{l+2m}_{a_1}(G)$ solutions from the space $H^{l+2m}_a(G)$.*

Proof. Put the functions $U_\nu^{(k,\zeta)}$ ($\nu = 1,\ 2;\ \zeta = 1,\ \dots,\ J_\nu;\ k = 0,\ \dots,\ \varkappa_{\zeta,\nu} - 1$) in arbitrary order and denote the elements of the obtained ordered set by $\mathcal{U}_1,\ \dots,\ \mathcal{U}_\varkappa$.

Suppose $\mathcal{Z}_t \in H^{l+2m}_{a_1}(G)$, $t = 1,\ \dots d$, are linearly independent modulo $H^{l+2m}_{a_1}(G)$ solutions to homogeneous problem (2.1), (2.2), and $d > \varkappa$. Then by (3.22) we have

$$\mathcal{Z}_t \equiv \Big(\sum_{k=1}^{\varkappa} c_{tk}\mathcal{U}_k\Big) \Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G)\Big), \qquad t = 1,\ \dots,\ d, \tag{4.1}$$

where c_{tk} are some constants. Consider the equation for unknown constants $h_1,\ \dots,\ h_d$:

$$\sum_{t=1}^{d} h_t \mathcal{Z}_t = 0\Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G)\Big).$$

By virtue of (4.1) it is equivalent to

$$\sum_{k=1}^{\varkappa}\Big(\sum_{t=1}^{d} c_{tk} h_t\Big)\mathcal{U}_k = 0\Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G)\Big).$$

Since $\mathcal{U}_1,\ \dots,\ \mathcal{U}_\varkappa$ are linearly independent modulo $H_{a_1}^{l+2m}(G)$, the last equation is equivalent to the system

$$\sum_{t=1}^{d} c_{tk} h_t = 0, \qquad k = 1,\ \dots,\ \varkappa.$$

By virtue of the inequality $d > \varkappa$, this system necessarily has a nontrivial solution $(h_1,\ \dots,\ h_d) \neq 0$, while we supposed that $\mathcal{Z}_1,\ \dots,\ \mathcal{Z}_d$ were linearly independent modulo $H_{a_1}^{l+2m}(G)$. This contradiction proves the lemma. □

Consider the vector $\mathcal{U} = (\mathcal{U}_1,\ \dots,\ \mathcal{U}_\varkappa)^T$. Let $\mathcal{Z} = (\mathcal{Z}_1,\ \dots,\ \mathcal{Z}_d)^T$, $0 \le d \le \varkappa$, be a vector, components of which form a maximal set of solutions to homogeneous problem (2.1), (2.2) from the space $H_a^{l+2m}(G)$, linearly independent modulo the space $H_{a_1}^{l+2m}(G)$ (i.e., a basis modulo $H_{a_1}^{l+2m}(G)$). By virtue of (4.1), we have $\mathcal{Z} \equiv \mathbf{C}\mathcal{U} \left(\operatorname{mod} H_{a_1}^{l+2m}(G)\right)$, where $\mathbf{C}$ is a matrix of order $d \times \varkappa$. Rank of $\mathbf{C}$ equals d. Without loss in generality we assume that $\mathbf{C} = (\mathbf{C}_1,\ \mathbf{C}_2)$, where $\mathbf{C}_1$ is a nonsingular $(d \times d)$-matrix. Hence $\mathbf{C}_1^{-1}\mathcal{Z} \equiv (\mathbf{I},\ \mathbf{C}_1^{-1}\mathbf{C}_2)\mathcal{U} \left(\operatorname{mod} H_{a_1}^{l+2m}(G)\right)$, where $\mathbf{I}$ is the identity $(d \times d)$-matrix. Therefore we can suppose that

$$\mathcal{Z}_t \equiv \left(\mathcal{U}_t + \sum_{k=d+1}^{\varkappa} c_{tk}\mathcal{U}_k\right) \left(\operatorname{mod} H_{a_1}^{l+2m}(G)\right), \quad t = 1,\ \dots,\ d. \tag{4.2}$$

We shall say that basis (4.2) is canonical. From now on we fix some canonical basis.

II. Along with the operator $\mathbf{L} = \{\mathbf{P}(y,\ D_y),\ \mathbf{B}_\sigma(y,\ D_y)\} : H_{a_1}^{l+2m}(G) \to H_{a_1}^{l}(G,\ \Upsilon)$, we consider the adjoint operator $\mathbf{L}^* : H_{a_1}^{l}(G,\ \Upsilon)^* \to H_{a_1}^{l+2m}(G)^*$ given by

$$< u,\ \mathbf{L}^*\{v,\ w_\sigma\} >=< \mathbf{P}(y,\ D_y)u,\ v > + \sum_{\sigma=1,2}\sum_{\mu=1}^{m} < \mathbf{B}_{\sigma\mu}(y,\ D_y)u,\ w_{\sigma\mu} > \tag{4.3}$$

for all $u \in H_{a_1}^{l+2m}(G)$, $\{v,\ w_\sigma\} \in H_{a_1}^{l}(G,\ \Upsilon)^*$. Here and below $< \cdot,\ \cdot >$ stands for the sesquilinear form on a pair of corresponding adjoint spaces.

LEMMA 4.2. *Let d be a number of elements in basis (4.2). Then the equation $\mathbf{L}^*\{v,\ w_\sigma\} = 0$ has $\varkappa - d$ solutions from $H_{a_1}^{l}(G,\ \Upsilon)^*$, linearly independent modulo $H_a^{l}(G,\ \Upsilon)^*$.*

Proof. 1) Let $\{\varphi_t,\ \psi_{t,\sigma}\}$, $t = 1, \dots,\ q$, be some basis modulo $H_a^{l}(G,\ \Upsilon)^*$ in the space of solutions from $H_{a_1}^{l}(G,\ \Upsilon)^*$ for the equation $\mathbf{L}^*\{v,\ w_\sigma\} = 0$.

Suppose $q < \varkappa - d$. Put $\mathcal{U} = c_{d+1}\mathcal{U}_{d+1} + \dots + c_\varkappa \mathcal{U}_\varkappa$, where the vector $(c_{d+1},\ \dots,\ c_\varkappa)$ is a nontrivial solution for the q linear algebraic equations

$$< \mathbf{L}\mathcal{U},\ \{\varphi_t,\ \psi_{t,\sigma}\} >= 0, \quad t = 1,\ \dots,\ q \tag{4.4}$$

(notice that, by virtue of (3.23), $\mathbf{L}\mathcal{U} \in H^l_{a_1}(G,\ \Upsilon)$ and therefore the forms $< \mathbf{L}\mathcal{U},\ \{\varphi_t,\ \psi_{t,\sigma}\} >$ are well–defined). This system does have a nontrivial solution since $q < \varkappa - d$.

From (4.4) it follows that there exists a solution $\hat{\mathcal{U}} \in H^{l+2m}_{a_1}(G)$ for the equation $\mathbf{L}\hat{\mathcal{U}} = \mathbf{L}\mathcal{U}$. Clearly the function $\mathcal{Z} = \mathcal{U} - \hat{\mathcal{U}} \neq 0$ is a solution from $H^{l+2m}_a(G)$ for homogeneous problem (2.1), (2.2), which has the asymptotics

$$\mathcal{Z} \equiv \Big(\sum_{k=d+1}^{\varkappa} c_k \mathcal{U}_k \Big) \ \Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G) \Big). \tag{4.5}$$

We claim that the function $\mathcal{Z}$ is linearly independent of $\mathcal{Z}_1,\ \dots,\ \mathcal{Z}_d$, the elements of basis (4.2) modulo $H^{l+2m}_{a_1}(G)$. Indeed, suppose that

$$\mathcal{Z} \equiv \Big(\sum_{t=1}^{d} h_t \mathcal{Z}_t \Big) \ \Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G) \Big);$$

then, by virtue of (4.2), we have

$$\mathcal{Z} \equiv \Big(\sum_{t=1}^{d} h_t \mathcal{U}_t + \sum_{k=d+1}^{\varkappa} (\sum_{t=1}^{d} h_t c_{tk}) \mathcal{U}_k \Big) \ \Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G) \Big).$$

From this, from (4.5), and from the linear independence of the functions $\mathcal{U}_1,\ \dots,\ \mathcal{U}_\varkappa$ modulo $H^{l+2m}_{a_1}(G)$, it follows that $h_1 = \dots = h_d = 0$. However, we assumed $\mathcal{Z}_1,\ \dots,\ \mathcal{Z}_d$ were the elements of basis (4.2) modulo $H^{l+2m}_{a_1}(G)$. This contradiction proves that $q \geq \varkappa - d$.

2) Suppose $q > \varkappa - d$. Denote by $\{\Phi_h,\ \Psi_{h,\sigma}\}$, $h = 1,\ \dots, q$, a system of elements from $H^l_{a_1}(G,\ \Upsilon)$ being biorthogonal to the system $\{\varphi_t,\ \psi_{t,\sigma}\}$, $t = 1,\ \dots, q$, and orthogonal to all solutions for the equation $\mathbf{L}^*\{v,\ w_\sigma\} = 0$ from $H^l_a(G,\ \Upsilon)^*$. Then there exist solutions $u_h \in H^{l+2m}_a(G)$ for the problems $\mathbf{L}u_h = \{\Phi_h,\ \Psi_{h,\sigma}\}$, $h = 1,\ \dots,\ q$. Subtracting from u_h (if needed) a linear combination of the elements $\mathcal{Z}_1,\ \dots,\ \mathcal{Z}_d$ forming basis (4.2), one can make the relations

$$u_h \equiv \Big(\sum_{k=d+1}^{\varkappa} d_{hk} \mathcal{U}_k \Big) \ \Big(\mathrm{mod}\, H^{l+2m}_{a_1}(G) \Big), \quad h = 1,\ \dots,\ q. \tag{4.6}$$

hold.

The functions $u_1, \dots, u_q$ are linearly independent modulo $H^{l+2m}_{a_1}(G)$. Indeed, in the opposite case some linear combination of the functions u_h, $h = 1, \dots, q$, would belong to the space $H^{l+2m}_{a_1}(G)$. Then the corresponding linear combination of the functions $\mathbf{L}u_h = \{\Phi_h, \Psi_{h,\sigma}\}$, $h = 1, \dots, q$, would be orthogonal to all the vectors $\{\varphi_t, \psi_{t,\sigma}\}$, $t = 1, \dots, q$. This would contradict the choice of the functions $\{\Phi_h, \Psi_{h,\sigma}\}$, $h = 1, \dots, q$. From (4.6) and from the linear independence of the functions u_h, it follows that $q \le \varkappa - d$. Thus, we necessarily have $q = \varkappa - d$. □

III. Consider the operators

$$\mathbf{L}_a = \{\mathbf{P}(y, D_y), \mathbf{B}_\sigma(y, D_y)\} : H^{l+2m}_a(G) \to H^l_a(G, \Upsilon),$$
$$\mathbf{L}_{a_1} = \{\mathbf{P}(y, D_y), \mathbf{B}_\sigma(y, D_y)\} : H^{l+2m}_{a_1}(G) \to H^l_{a_1}(G, \Upsilon).$$

The operators $\mathbf{L}_a$ and $\mathbf{L}_{a_1}$ correspond to one and the same nonlocal problem (2.1), (2.2), but they act in the spaces with the different weight constants (a and a_1 respectively).

THEOREM 4.1. *The operators* $\mathbf{L}_a$ *and* $\mathbf{L}_{a_1}$ *are Fredholm, and the following index formula is valid:*

$$\operatorname{ind} \mathbf{L}_a = \operatorname{ind} \mathbf{L}_{a_1} + \varkappa.$$

Proof. By Theorem 3.4 [22], the operators $\mathbf{L}_a$ and $\mathbf{L}_{a_1}$ are Fredholm[7]. By Lemma 4.1, we have $\dim\ker \mathbf{L}_a = \dim\ker \mathbf{L}_{a_1} + d$. Then by Lemma 4.2, we have $\dim\ker \mathbf{L}^*_a = \dim\ker \mathbf{L}^*_{a_1} - (\varkappa - d)$. Hence $\operatorname{ind} \mathbf{L}_a = \dim\ker \mathbf{L}_a - \dim\ker \mathbf{L}^*_a = \dim\ker \mathbf{L}_{a_1} - \dim\ker \mathbf{L}^*_{a_1} + \varkappa = \operatorname{ind} \mathbf{L}_{a_1} + \varkappa$. □

REMARK 4.1. *Theorem 4.1 remains true without the assumption* $a - a_1 < 1$*, too. Indeed, one can always choose numbers* $a = a^0 > a^1 > \dots > a^M = a_1$ *such that* $0 < a^i - a^{i+1} < 1$ *and the lines* $\operatorname{Im}\lambda = a^i + 1 - l - 2m$ *do not contain eigenvalues of* $\tilde{\mathcal{L}}_\nu(\lambda)$, $\nu = 1, 2$*. Applying Theorem 4.1 subsequently to the pairs of the operators*

$$\mathbf{L}_{a_i} = \{\mathbf{P}(y, D_y), \mathbf{B}_\sigma(y, D_y)\} : H^{l+2m}_{a_i}(G) \to H^l_{a_i}(G, \Upsilon),$$
$$\mathbf{L}_{a_{i+1}} = \{\mathbf{P}(y, D_y), \mathbf{B}_\sigma(y, D_y)\} : H^{l+2m}_{a_{i+1}}(G) \to H^l_{a_{i+1}}(G, \Upsilon)$$

we get the formula $\operatorname{ind} \mathbf{L}_a = \operatorname{ind} \mathbf{L}_{a_1} + \varkappa$*, where* $\varkappa$ *is the sum of full multiplicities of all eigenvalues of* $\tilde{\mathcal{L}}_1(\lambda)$ *and* $\tilde{\mathcal{L}}_2(\lambda)$ *contained in the strip* $a_1 + 1 - l - 2m < \operatorname{Im}\lambda < a + 1 - l - 2m$.

[7] More precisely, we use the generalization of Theorem 3.4 [22] for the case of transformations Ω_1, Ω_2 consisting near g_1 and h_1 not only of a rotation but of an expansion, too; see also [24].

5. Asymptotics of solutions for adjoint nonlocal problems.

I. In this section we shall obtain an asymptotics near the set $\mathcal{K}$ for solutions to the problem, adjoint to (2.1), (2.2). The results of this section will be applied to calculating the coefficients $c_\nu^{(k,j)}$ in (3.22).

Notice that the approach to the study of adjoint nonlocal problems has been suggested by the author in [9, 10, 12]. In the papers [9, 10], the solvability and smoothness of solutions for model adjoint nonlocal problems in plane and dihedral angles were studied. The paper [12] deals with an asymptotics of solutions for model nonlocal problems in plane angles and in $\mathbb{R}^2 \setminus \{0\}$. In the present work we essentially use both the ideology of the papers [9, 10] and the results of the paper [12].

Since we suppose Condition 3.1 being fulfilled, it suffices to obtain appropriate asymptotics formulas only near the points g_1 and g_2.

Along with formula (4.3) we will use another one for the definition of the adjoint operator. To write this formula, we introduce the following denotation. For any smooth curve $\Upsilon \subset \bar{G}$ and any distribution $w \in H_{a_1}^{k-1/2}(\Upsilon)^*$ we denote by $w \cdot \delta_\Upsilon$ the distribution from $H_{a_1}^k(G)^*$ given by

$$< u,\ w \cdot \delta_\Upsilon >_G = < u|_\Upsilon,\ w >_\Upsilon \ \text{ for all } u \in H_{a_1}^k(G).\text{[8]} \tag{5.1}$$

Clearly the support of the distribution $w \cdot \delta_\Upsilon$ is contained in $\bar{\Upsilon}$. Similarly one can define a distribution $w \cdot \delta_\gamma \in H_{a_1}^k(K)^*$, where $\gamma = \{y \in \mathbb{R}^2 :\ r > 0,\ \omega = b\}$ $(b_1 \le b \le b_2)$ and $w \in H_{a_1}^{k-1/2}(\gamma)^*$.

Denote by $\mathbf{P}^*(y,\ D_y)$, $B_\sigma^*(y,\ D_y) = \{B_{\sigma\mu}^*(y,\ D_y)\}_{\mu=1}^m$, $T_\sigma^*(y,\ D_y) = \{T_{\sigma\mu}^*(y,\ D_y)\}_{\mu=1}^m$ the operators, formally adjoint to $\mathbf{P}(y,\ D_y)$, $B_\sigma(y,\ D_y) = \{B_{\sigma\mu}(y,\ D_y)\}_{\mu=1}^m$, $T_\sigma(y,\ D_y) = \{T_{\sigma\mu}(y,\ D_y)\}_{\mu=1}^m$ respectively.

For any distribution $w_{\sigma\mu} \in H_{a_1}^{l+2m-m_{\sigma\mu}-1/2}(\Upsilon_\sigma)$, we consider the distribution $w_{\sigma\mu}^\Omega \in H_{a_1}^{l+2m-m_{\sigma\mu}-1/2}\big(\Omega_\sigma(\Upsilon_\sigma)\big)^*$ given by

$$< \psi,\ w_{\sigma\mu}^\Omega >_{\Omega_\sigma(\Upsilon_\sigma)} = < \psi\big(\Omega_\sigma(\cdot)\big),\ w_{\sigma\mu} >_{\Upsilon_\sigma} \tag{5.2}$$

for all $\psi \in H_{a_1}^{l+2m-m_{\sigma\mu}-1/2}\big(\Omega_\sigma(\Upsilon_\sigma)\big)$.

We claim that the adjoint operator $\mathbf{L}^* : H_{a_1}^l(G,\ \Upsilon)^* \to H_{a_1}^{l+2m}(G)^*$ can

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[8] In this section, for clearness, we denote sesquilinear forms on the pairs of adjoint spaces $H_{a_1}^k(G)$, $H_{a_1}^k(G)^*$ and $H_{a_1}^{k-1/2}(\Upsilon)$, $H_{a_1}^{k-1/2}(\Upsilon)^*$ by $< \cdot,\ \cdot >_G$ and $< \cdot,\ \cdot >_\Upsilon$ respectively.

be defined by the formula

$$\mathbf{L}^*\{v,\ w_\sigma\} = \mathbf{P}^*(y,\ D_y)v + \sum_{\sigma=1,2} B^*_\sigma(y,\ D_y)(w_\sigma \cdot \delta_{\Upsilon_\sigma}) + T^*_\sigma(y,\ D_y)(w^\Omega_\sigma \cdot \delta_{\Omega_\sigma(\Upsilon_\sigma)}). \tag{5.3}$$

Here and further $w_\sigma = \{w_{\sigma\mu}\}_{\mu=1}^m$, $w^\Omega_\sigma = \{w^\Omega_{\sigma\mu}\}_{\mu=1}^m$,

$$B^*_\sigma(y,\ D_y)(w_\sigma \cdot \delta_{\Upsilon_\sigma}) = \sum_{\mu=1}^m B^*_{\sigma\mu}(y,\ D_y)(w_{\sigma\mu} \cdot \delta_{\Upsilon_\sigma}),$$

$$T^*_\sigma(y,\ D_y)(w^\Omega_\sigma \cdot \delta_{\Omega_\sigma(\Upsilon_\sigma)}) = \sum_{\mu=1}^m T^*_{\sigma\mu}(y,\ D_y)(w^\Omega_{\sigma\mu} \cdot \delta_{\Omega_\sigma(\Upsilon_\sigma)}).$$

Indeed, using definition (4.3) of the adjoint operator $\mathbf{L}^*$ and then relations (5.2) and (5.1), we get (omitting $(y,\ D_y)$ for short)

$$< u,\ \mathbf{L}^*\{v,\ w_\sigma\} > = < \mathbf{P}u,\ v >_G + \sum_{\sigma=1,2}\sum_{\mu=1}^m \Big(< B_{\sigma\mu}u|_{\Upsilon_\sigma},\ w_{\sigma\mu} >_{\Upsilon_\sigma} + \\ < \big(T_{\sigma\mu}u\big)\big(\Omega_\sigma(\cdot)\big)|_{\Upsilon_\sigma},\ w_{\sigma\mu} >_{\Upsilon_\sigma} \Big) = < \mathbf{P}u,\ v >_G + \\ \sum_{\sigma=1,2}\sum_{\mu=1}^m \Big(< B_{\sigma\mu}u,\ w_{\sigma\mu} \cdot \delta_{\Upsilon_\sigma} >_G + < T_{\sigma\mu}u,\ w^\Omega_{\sigma\mu} \cdot \delta_{\Omega_\sigma(\Upsilon_\sigma)} >_G \Big)$$

for all $u \in H^{l+2m}_{a_1}(G)$, which yields (5.3).

We are to study an asymptotics of a given solution $\{v,\ w_\sigma\} \in H^l_{a_1}(G,\ \Upsilon)^*$ for the problem

$$\mathbf{L}^*\{v,\ w_\sigma\} = \Psi, \tag{5.4}$$

supposing $\Psi \in H^{l+2m}_a(G)^*$.

For this purpose, parallel to the operator $\mathbf{L}^*$, we consider the auxiliary operator

$$\mathbf{L}^*_\Omega : H^l_{a_1}(G)^* \times \prod_{\sigma=1,2}\prod_{\mu=1}^m \Big(H^{l+2m-m_{\sigma\mu}-1/2}_{a_1}(\Upsilon_\sigma)^* \times \\ \times H^{l+2m-m_{\sigma\mu}-1/2}_{a_1}\big(\Omega_\sigma(\Upsilon_\sigma)\big)^* \Big) \to H^{l+2m}_{a_1}(G)^*$$

given by

$$\mathbf{L}^*_\Omega\{v,\ w_\sigma,\ w'_\sigma\} = \mathbf{P}^*(y,\ D_y)v + \\ \sum_{\sigma=1,2} \Big(B^*_\sigma(y,\ D_y)(w_\sigma \cdot \delta_{\Upsilon_\sigma}) + T^*_\sigma(y,\ D_y)(w'_\sigma \cdot \delta_{\Omega_\sigma(\Upsilon_\sigma)}) \Big). \tag{5.5}$$

Such an auxiliary operator was used in the papers [11, 10] for the study of model nonlocal problems in angles. It also turns out to be very useful in our case. On the one hand, the operator $\mathbf{L}^*_\Omega$ is not a nonlocal one since the functions w_σ and w'_σ are not connected each with another by nonlocal transformations Ω_σ in (5.5). This will allow to use Leibnitz's formula when necessary. On the other hand, solutions for the problems corresponding to $\mathbf{L}^*$ and $\mathbf{L}^*_\Omega$ are related in the following way. If $\{v,\ w_\sigma\}$ is a solution to problem (5.4), then the distribution $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ is a solution to the problem

$$\mathbf{L}^*_\Omega\{v,\ w_\sigma,\ w_\sigma^\Omega\} = \Psi, \tag{5.6}$$

where $w_\sigma^\Omega = \{w_{\sigma\mu}^\Omega\}_{\mu=1}^m$ is defined by (5.2).

So, investigating the asymptotics of the solution $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ for problem (5.6) is equivalent to investigating the asymptotics of the solution $\{v,\ w_\sigma\}$ for problem (5.4).

Our plan is this. At first we will multiply the distribution $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ by the cut–off function η_1 and get a corresponding problem near g_1. Since the operator $\mathbf{L}^*_\Omega$ is not a nonlocal one, applying Leibnitz's formula we will show that $\mathbf{L}^*_\Omega\eta_1\{v,\ w_\sigma,\ w_\sigma^\Omega\} \in H_a^{l+2m}(G)^*$. Therefore we will arrive at the model adjoint problem in the angle K with a "regular" right–hand side. Using the results of [12], we will obtain the asymptotics near g_1. Then we will multiply the distribution $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ by η_2 and get a corresponding problem near g_2. We will arrive at the model adjoint problem in $\mathbb{R}^2$. But in this case a right–hand side will be a sum of "regular" and "special" distributions. The asymptotics of the "special" one will be defined by the asymptotics of w_1 near g_1, which will have been known from the first step. Further application of the results of [12] will allow to get the asymptotics near g_2.

Thus let us multiply $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ by η_1. Notice that $\operatorname{supp}\eta_1\cap\overline{\Omega_1(\Upsilon_1)} = \varnothing$ (see Fig. 2.1) and $\operatorname{supp}(w_1\cdot\delta_{\Omega_1(\Upsilon_1)}) \subset \overline{\Omega_1(\Upsilon_1)}$. Therefore $\eta_1 w_1^\Omega\cdot\delta_{\Omega_1(\Upsilon_1)} = 0$. From this and from (5.5), it follows that

$$\begin{aligned}\mathbf{L}^*_\Omega\eta_1\{v,\ w_\sigma,\ w_\sigma^\Omega\} = \mathbf{P}^*(y,\ D_y)\eta_1 v + B_1^*(y,\ D_y)(\eta_1 w_1\cdot\delta_{\Upsilon_1}) +\\ B_2^*(y,\ D_y)(\eta_1 w_2\cdot\delta_{\Upsilon_2}) + T_2^*(y,\ D_y)(\eta_1 w_2^\Omega\cdot\delta_{\Omega_2(\Upsilon_2)}).\end{aligned} \tag{5.7}$$

Let us show that the distribution $\eta_1\{v,\ w_\sigma\}$ satisfies the model adjoint problem in the angle K

$$\mathcal{L}_1^*\eta_1\{v,\ w_\sigma\} = \hat{\Psi}, \tag{5.8}$$

where $\mathcal{L}_1^* : H_{a_1}^l(K,\ \gamma)^* \to H_{a_1}^{l+2m}(K)^*$ is the operator, adjoint to $\mathcal{L}_1 : H_{a_1}^{l+2m}(K) \to H_{a_1}^l(K,\ \gamma)$; $\hat{\Psi} \in H_a^{l+2m}(K)^*$.

From (5.6) and Leibnitz's formula, it follows that, on the one hand,

$$\mathbf{L}^*_\Omega \eta_1\{v,\ w_\sigma,\ w^\Omega_\sigma\} = \eta_1\Psi + \hat{\Psi}_1, \tag{5.9}$$

where $\hat{\Psi}_1 \in H^{l+2m}_a(G)^*$ and $\operatorname{supp}\hat{\Psi}_1 \subset \mathcal{V}(g_1)$. On the other hand, the function $\eta_1(y) - \eta_1\big(\Omega_2^{-1}(y)\big)$ is equal to 0 near g_1 and has a support inside $\mathcal{V}(g_1)$ (here we are to suppose $\operatorname{supp}\eta_1$ is so small that $\operatorname{supp}\eta_1\big(\Omega_2^{-1}(\cdot)\big) \subset \mathcal{V}(g_1)$). Hence,

$$T^*_2(y,\ D_y)(\eta_1 w^\Omega_2 \cdot \delta_{\Omega_2(\Upsilon_2)}) - T^*_2(y,\ D_y)\big(\eta_1(\Omega_2^{-1}(\cdot))w^\Omega_2 \cdot \delta_{\Omega_2(\Upsilon_2)}\big) \in H^{l+2m}_a(G)^*$$

and has a support inside $\mathcal{V}(g_1)$. This, (5.7), and (5.9) imply

$$\begin{aligned}\mathbf{P}^*(y,\ D_y)\eta_1 v + B^*_1(y,\ D_y)(\eta_1 w_1 \cdot \delta_{\Upsilon_1}) + \\ B^*_2(y,\ D_y)(\eta_1 w_2 \cdot \delta_{\Upsilon_2}) + T^*_2(y,\ D_y)(\eta_1(\Omega_2^{-1}(\cdot))w^\Omega_2 \cdot \delta_{\Omega_2(\Upsilon_2)}) = \hat{\Psi}_2,\end{aligned}$$

where $\hat{\Psi}_2 \in H^{l+2m}_a(G)^*$ and $\operatorname{supp}\hat{\Psi}_2 \subset \mathcal{V}(g_1)$.

Let $\mathcal{P}^*(D_y),\ B^*_2(D_y),\ T^*_2(D_y)$ be the principal homogeneous parts of the operators $\mathbf{P}^*(g_1,\ D_y),\ B^*_2(g_1,\ D_y),\ T^*_2(g_1,\ D_y)$ respectively. Then, using Leibnitz's formula, we finally get

$$\begin{aligned}\mathcal{P}^*(D_y)\eta_1 v + B^*_1(D_y)(\eta_1 w_1 \cdot \delta_{\gamma_1}) + \\ B^*_2(D_y)(\eta_1 w_2 \cdot \delta_{\gamma_2}) + T^*_2(D_y)(\eta_1(\mathcal{G}_2^{-1}\cdot)w^\Omega_2 \cdot \delta_{\mathcal{G}_2(\gamma_2)}) = \hat{\Psi},\end{aligned} \tag{5.10}$$

where $\hat{\Psi} \in H^{l+2m}_a(K)^*$ and $\operatorname{supp}\hat{\Psi} \subset \mathcal{V}(0)$. Here we also took into account that near the point $g_1 = 0$ the domain G and the curves Υ_σ coincide with the angle K and the arms γ_σ respectively, while the transformation Ω_2 coincides with the linear operator $\mathcal{G}_2$. But it is easily seen that equality (5.10) is quite the same as equality (5.8). Indeed, the only not evident identity one should check is

$$< u,\ T^*_2(D_y)(\eta_1(\mathcal{G}_2^{-1}\cdot)w^\Omega_2 \cdot \delta_{\mathcal{G}_2(\gamma_2)}) >_K = < (T_2(D_y)u)(\mathcal{G}_2\cdot)|_{\gamma_2},\ \eta_1 w_2 >_{\gamma_2},$$

which follows from

$$\begin{aligned}&< u,\ T^*_2(D_y)\big(\eta_1(\mathcal{G}_2^{-1}\cdot)w^\Omega_2 \cdot \delta_{\mathcal{G}_2(\gamma_2)}\big) >_K = \\ &\quad < \eta_1(\mathcal{G}_2^{-1}\cdot)T_2(D_y)u|_{\mathcal{G}_2(\gamma_2)},\ w^\Omega_2 >_{\mathcal{G}_2(\gamma_2)} = < \eta_1(T_2(D_y)u)(\mathcal{G}_2\cdot)|_{\gamma_2},\ w_2 >_{\gamma_2}.\end{aligned}$$

Here we subsequently used (5.1) and (5.2).

Applying the results of the paper [12] to equality (5.8), we shall now obtain the asymptotics of the distribution $\eta_1\{v,\ w_\sigma\}$. Introduce some denotation. Put

$$\begin{aligned} v_1^{(k,\zeta)} &= r^{i\bar\lambda_1+2m-2}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\psi_1^{(k-q,\zeta)},\\ w_{1,\sigma\mu}^{(k,\zeta)} &= r^{i\bar\lambda_1+m_{\sigma\mu}-1}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\chi_{1,\sigma\mu}^{(k-q,\zeta)}. \end{aligned} \tag{5.11}$$

Here

$$\Big\{\{\psi_1^{(0,\zeta)},\ \chi_{1,\sigma\mu}^{(0,\zeta)}\},\ \dots,\ \{\psi_1^{(\varkappa_{\zeta,1}-1,\zeta)},\ \chi_{1,\sigma\mu}^{(\varkappa_{\zeta,1}-1,\zeta)}\}:\ \zeta=1,\ \dots,\ J_1\Big\}$$

are Jordan chains of the operator $\tilde{\mathcal{L}}_1^*(\lambda)$ (adjoint to $\tilde{\mathcal{L}}_1(\bar\lambda)$) corresponding to the eigenvalue $\bar\lambda_1$ and forming a canonical system. These chains are supposed (see Lemma 3.2 [12]) to satisfy the following condition of biorthogonality and normalization with respect to the Jordan chains (3.15):

$$\sum_{p=0}^{\nu}\sum_{q=0}^{k}\frac{1}{(\nu+k+1-p-q)!} < \partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal{L}}_1(\lambda_1)\varphi_1^{(q,\xi)},$$
$$\{\psi_1^{(p,\zeta)},\ \chi_{1,\sigma\mu}^{(p,\zeta)}\} >= \delta_{\xi,\zeta}\delta_{\varkappa_{\xi,1}-k-1,\nu}. \tag{5.12}$$

Here $\zeta,\ \xi=1,\ \dots,\ J_1$; $\nu=0,\ \dots,\ \varkappa_{\zeta,1}-1$; $k=0,\ \dots,\ \varkappa_{\xi,1}-1$; $\delta_{\xi,\zeta}$ is the Kronecker symbol.

Analogously to section 3, we introduce the vectors $w_{1,\sigma}^{(k,\zeta)}=\{w_{1,\sigma\mu}^{(k,\zeta)}\}_{\mu=1}^m$ and $\{v_1,\ w_{1,\sigma}\}=\{v_1^{(k,\zeta)},\ w_{1,\sigma}^{(k,\zeta)}\}$.

We remark that, by Lemma 3.1 [12], the distributions $\{v_1^{(k,\zeta)},\ w_{1,\sigma}^{(k,\zeta)}\}$ satisfy the homogeneous equation $\mathcal{L}_1^*\{v_1^{(k,\zeta)},\ w_{1,\sigma}^{(k,\zeta)}\}=0$.

Now from equality (5.8) and Theorem 4.2 [12] we get the following result.

THEOREM 5.1. *Let $\{v,\ w_\sigma\}\in H_{a_1}^l(G,\ \Upsilon)^*$ be a solution for equation (5.4) with a right–hand side $\Psi\in H_a^{l+2m}(G)^*$. Then the following asymptotics formula is valid:*

$$\eta_1\{v,\ w_\sigma\}\equiv d_1\eta_1\{v_1,\ w_{1,\sigma}\}\ \Big(\mathrm{mod}\, H_a^l(G,\ \Upsilon)^*\Big), \tag{5.13}$$

where $\{v_1,\ w_{1,\sigma}\}=\{v_1^{(k,\zeta)},\ w_{1,\sigma}^{(k,\zeta)}\}$ is defined by (5.11), $d_1=\{d_1^{(k,\zeta)}\}$ is a vector of some constants.

II. Now let us study the asymptotics of the solution $\{v,\ w_\sigma\}$ for equation (5.4) near the point g_2. As we mentioned above, in this case we will arrive at the model adjoint problem in $\mathbb{R}^2 \setminus \{g_2\}$. A right–hand side of the equation obtained will be a sum of "regular" and "special" distributions. The asymptotics of the latter one will be defined by the asymptotics of w_σ near g_1, which is already known (see Theorem 5.1).

We multiply $\{v,\ w_\sigma,\ w_\sigma^\Omega\}$ by η_2. Since the support of η_2 is contained in $\mathcal{V}(g_2)$ and therefore does not intersect with $\bar{\Upsilon}_1$, $\bar{\Upsilon}_2$, and $\overline{\Omega_2(\Upsilon_2)}$ (see Fig. 2.1), we have $\eta_2 w_1 \cdot \delta_{\Upsilon_1} = 0$, $\eta_2 w_2 \cdot \delta_{\Upsilon_2} = 0$, and $\eta_2 w_2^\Omega \cdot \delta_{\Omega_2(\Upsilon_2)} = 0$. Combining this with (5.5), we get

$$\mathbf{L}_\Omega^* \eta_2 \{v,\ w_\sigma,\ w_\sigma^\Omega\} = \mathbf{P}^*(y,\ D_y)\eta_2 v + T_1^*(y,\ D_y)(\eta_2 w_1^\Omega \cdot \delta_{\Omega_1(\Upsilon_1)}). \tag{5.14}$$

From (5.6) and Leibnitz's formula, it follows that $\mathbf{L}_\Omega^* \eta_2\{v,\ w_\sigma,\ w_\sigma^\Omega\} = \eta_2 \Psi + \hat{\Psi}_2$, where $\hat{\Psi}_2 \in H_a^{l+2m}(G)^*$ and $\operatorname{supp} \hat{\Psi}_2 \subset \mathcal{V}(g_2)$.

Let $\mathcal{P}^*(D_y)$, $T_1^*(D_y)$ be the principal homogeneous parts of the operators $\mathbf{P}^*(g_2,\ D_y)$, $T_1^*(g_2,\ D_y)$ respectively. Then, analogously to the above, we derive that

$$\mathcal{P}^*(D_y)\eta_2 v = -T_1^*(D_y)(\eta_2 w_1^\Omega \cdot \delta_{\Omega_1(\Upsilon_1)}) + \hat{\Psi}_3, \tag{5.15}$$

where $\hat{\Psi}_3 \in H_a^{l+2m}(G)^*$ and $\operatorname{supp} \hat{\Psi}_3 \subset \mathcal{V}(g_2)$.

Let $\mathcal{L}_2^* : H_{a_1}^l(\mathbb{R}^2)^* \to H_{a_1}^{l+2m}(\mathbb{R}^2)^*$ be the operator, adjoint to $\mathcal{L}_2 : H_{a_1}^{l+2m}(\mathbb{R}^2) \to H_{a_1}^l(\mathbb{R}^2)$.

From definition (5.2) of the distribution $w_{1\mu}^\Omega$ and from asymptotics formula (5.13), it follows that $\eta_2 w_{1\mu}^\Omega$ is a linear combination of the functions $r^{i\bar{\lambda}_1 + m_{1\mu} - 1}(i \ln r)^q$ modulo $H_a^{l+2m-m_{1\mu}-1/2}(\Upsilon_1)$, where r is a polar radius of polar coordinates with the pole at g_2. Since $T_{1\mu}^*(D_y)$ is a homogeneous operator of order $m_{1\mu}$, we can write (5.15) (taking into account (5.13)) in the form

$$\mathcal{L}_2^* \eta_2 v = -\eta_2 d_1 \Psi_{21} + \hat{\Psi}. \tag{5.16}$$

Here $\hat{\Psi} \in H_a^{l+2m}(\mathbb{R}^2)^*$ and $\operatorname{supp} \hat{\Psi} \subset \mathcal{V}(g_2)$; $\Psi_{21} = \{\Psi_{21}^{(k,\zeta)}\}$, where $\Psi_{21}^{(k,\zeta)}$ is a linear combination of the distributions $r^{i\bar{\lambda}_1 - 2}(i \ln r)^q \Psi_{21q}^{(k,\zeta)}$, $0 \le q \le k$, $\Psi_{21q}^{(k,\zeta)} \in W_{2,2\pi}^{l+2m}(0,\ 2\pi)^*$, r is a polar radius of polar coordinates with the pole at g_2[9]; d_1 is the vector of constants from (5.13).

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[9] The distribution $\eta_2 r^{i\bar{\lambda}_1-2}(i\ln r)^q \Psi_{21q}^{(k,\zeta)} \in H_{a_1}^{l+2m}(\mathbb{R}^2)^*$ is given by

$$< u,\ r^{i\bar{\lambda}_1-2}(i\ln r)^q \Psi_{21q}^{(k,\zeta)} >= \int_0^\infty < \eta_2(\cdot,\ r)u(\cdot,\ r),\ \Psi_{21q}^{(k,\zeta)} >_{(0,\ 2\pi)} \overline{r^{i\bar{\lambda}_1-2}(i\ln r)^q}\, r\, dr$$

for all $u \in H_{a_1}^{l+2m}(\mathbb{R}^2)$.

Thus we see that (5.16) is a model adjoint problem in $\mathbb{R}^2$ with the right–hand side being the sum of the "regular" distribution $\hat{\Psi}$ and the "special" distribution Ψ_{21}. The asymptotics of Ψ_{12} is defined by the asymptotics of the solution w_1 near the point g_1, i.e., by the functions $w_{1,\sigma\mu}^{(k,\zeta)}$ (see (5.11)).

Applying the results of the paper [12] to equality (5.16), we shall now obtain the asymptotics of the distribution $\eta_2 v$. Introduce some denotation. Put

$$v_2^{(k,\zeta)} = r^{i\bar{\lambda}_2+2m-2}\sum_{q=0}^{k}\frac{1}{q!}(i\ln r)^q\psi_2^{(k-q,\zeta)}. \tag{5.17}$$

Here

$$\Big\{\psi_2^{(0,\zeta)},\ \dots,\ \psi_2^{(\varkappa_{\zeta,2}-1,\zeta)}:\ \zeta=1,\ \dots,\ J_2\Big\}$$

are Jordan chains of the operator $\tilde{\mathcal{L}}_2^*(\lambda)$ (adjoint to $\tilde{\mathcal{L}}_2(\bar{\lambda})$) corresponding to the eigenvalue $\bar{\lambda}_2$ and forming a canonical system. These chains are supposed (see [12]) to satisfy the following condition of biorthogonality and normalization with respect to the Jordan chains (3.3):

$$\sum_{p=0}^{\nu}\sum_{q=0}^{k}\frac{1}{(\nu+k+1-p-q)!} < \partial_\lambda^{\nu+k+1-p-q}\tilde{\mathcal{L}}_2(\lambda_2)\varphi_2^{(q,\xi)}, \psi_2^{(p,\zeta)} >= \delta_{\xi,\zeta}\delta_{\varkappa_{\xi,2}-k-1,\nu}. \tag{5.18}$$

Here $\zeta,\ \xi=1,\ \dots,\ J_2$; $\nu=0,\ \dots,\ \varkappa_{\zeta,2}-1$; $k=0,\ \dots,\ \varkappa_{\xi,2}-1$.

Analogously to section 3, we introduce the vector $v_2=\{v_2^{(k,\zeta)}\}$.

We remark that, according to [12, section 5], the distributions $v_2^{(k,\zeta)}$ satisfy the homogeneous equation $\mathcal{L}_2^* v_2^{(k,\zeta)}=0$.

If $\bar{\lambda}_1$ is an eigenvalue of $\tilde{\mathcal{L}}_2^*(\lambda)$ (i.e., $\bar{\lambda}_1=\bar{\lambda}_2$), then denote by $\varkappa(\bar{\lambda}_1)$ the greatest of partial multiplicities of $\bar{\lambda}_1$. If $\bar{\lambda}_1$ is not an eigenvalue of $\tilde{\mathcal{L}}_2^*(\lambda)$ (i.e., $\bar{\lambda}_1\neq\bar{\lambda}_2$), put $\varkappa(\bar{\lambda}_1)=0$.

Theorem 5.2. *Let $\{v,\ w_\sigma\}\in H_{a_1}^l(G,\ \Upsilon)^*$ be a solution for equation (5.4) with a right–hand side $\Psi\in H_a^{l+2m}(G)^*$. Then the following asymptotics formula is valid:*

$$\eta_2 v\equiv\Big(d_2\eta_2 v_2+d_1\eta_2 v_{21}\Big)\Big(\operatorname{mod} H_a^l(G,\ \Upsilon)^*\Big). \tag{5.19}$$

Here $v_2=\{v_2^{(k,\zeta)}\}$ is defined by (5.17), $d_2=\{d_2^{(k,\zeta)}\}$ is a vector of some constants; $v_{21}=\{v_{21}^{(k,\zeta)}\}$, where $v_{21}^{(k,\zeta)}$ is a linear combination of the functions

$r^{i\bar\lambda_1+2m-2}(i\ln r)^q\Psi_{21q}^{(k,\zeta)}$, $0\le q\le k+\varkappa(\bar\lambda_1)$, $\Psi_{21q}^{(k,\zeta)}\in W_{2,2\pi}^l(0,\ 2\pi)^*$; *$d_1$ is the vector of constants appearing in (5.13).*

Proof. Let $v_{21}=\{v_{21}^{(k,\zeta)}\}$ be a particular solution (which is defined by Lemma 5.2 [12]) for the problem

$$\mathcal{L}_2^* v_{21} = -\Psi_{21}, \tag{5.20}$$

where $\Psi_{21}=\{\Psi_{21}^{(k,\zeta)}\}$ is a "special" distribution appearing in (5.16). We remind that each element $\Psi_{21}^{(k,\zeta)}$ is a linear combination of the distributions $r^{i\bar\lambda_1-2}(i\ln r)^q\Psi_{21q}^{(k,\zeta)}$, $0\le q\le k$. Therefore, by Lemma 5.2 [12], the particular solution v_{21} has the form described in the formulation of the theorem. Moreover, each component $v_{21}^{(k,\zeta)}$ of the vector v_{21} is uniquely defined if $\bar\lambda_1$ is not an eigenvalue of $\tilde{\mathcal{L}}_2^*(\lambda)$ (i.e., if $\bar\lambda_1\ne\bar\lambda_2$). Otherwise (i.e., if $\bar\lambda_1=\bar\lambda_2$) it is defined accurate to an arbitrary linear combination of power solutions (5.17) corresponding to the eigenvalue $\bar\lambda_1=\bar\lambda_2$. From now on we shall suppose a particular solution $v_{21}=\{v_{21}^{(k,\zeta)}\}$ being fixed.

Combining (5.16) with (5.20) and using Leibnitz's formula, one easily checks that $\mathcal{L}_2^*(\eta_2 v - d_1\eta_2 v_{21})\in H_a^{l+2m}(\mathbb{R}^2)^*$. Now the asymptotics (5.19) is resulted from Theorem 5.3 [12], which establishes the asymptotics of solutions for adjoint problems in $\mathbb{R}^2$. □

Theorem 5.2 shows that the asymptotic behavior of solutions for adjoint nonlocal problem (5.4) near the point g_2 depends on the data of the problem both near the point g_2 itself and near the point g_1, which is connected with g_2: $g_1=\Omega_1^{-1}(g_2)$.

IV. Let us write the asymptotics formula for the solution $\{v,\ w_\sigma\}\in H_{a_1}^l(G,\ \Upsilon)^*$ for adjoint nonlocal problem (5.4) in the whole domain G. Put (cf. (3.21))

$$\{V_2,\ W_{2,\sigma}\}=\eta_2\{v_2,\ 0\};\quad \{V_1,\ W_{1,\sigma}\}=\eta_1\{v_1,\ w_{1,\sigma}\}+\eta_2\{v_{21},\ 0\}. \tag{5.21}$$

Now Theorems 5.1 and 5.2 yield the following asymptotics of $\{v,\ w_\sigma\}\in H_{a_1}^l(G,\ \Upsilon)^*$ (cf. formula (3.22)):

$$\{v,\ w_\sigma\}\equiv\Big(d_1\{V_1,\ W_{1,\sigma}\}+d_2\{V_2,\ W_{2,\sigma}\}\Big)\Big(\operatorname{mod} H_a^{l+2m}(G,\ \Upsilon)^*\Big). \tag{5.22}$$

6. Calculation of the coefficients in the asymptotics formulas.

I. In this section we will calculate the coefficients $c_\nu^{(k,\zeta)}$ appearing in asymptotics (3.22).

To begin with, let us remark that the coefficients can be calculated in the following way. At first one should find $c_1^{(k,\zeta)}$. Since in the neighborhood $\mathcal{V}(g_2)$ of the point g_2 the function u has asymptotics (3.6), by Theorem 5.2 [12] we have

$$c_2^{(k,\zeta)} = < \mathcal{L}_2\eta_2 u,\ iv_2^{(\varkappa_{\zeta,2}-k-1,\zeta)} >, \tag{6.1}$$

where $v_2^{(k,\zeta)}$ is defined in (5.17). Further, by Theorem 3.2, the function $u' = u - c_2u_{12}$ (where c_2 is calculated in (6.1), u_{12} is defined in the proof of Theorem 3.2) has the following asymptotics in the neighborhood $\mathcal{V}(g_1)$ of the point g_1:

$$u'(y) = c_1u_1(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_1) \cap G). \tag{6.2}$$

Here u_1 is defined by (3.16); c_1 is to be found; $\hat{u} \in H_{a_1}^{l+2m}(\mathcal{V}(g_1) \cap G)$. From asymptotics (6.2) and Theorem 4.1 [12], it follows that

$$c_1^{(k,\zeta)} = < \mathcal{L}_1\eta_1 u',\ i\{v_1^{(\varkappa_{\zeta,1}-k-1,\zeta)},\ w_{1,\sigma}^{(\varkappa_{\zeta,1}-k-1,\zeta)}\} >, \tag{6.3}$$

where $\{v_1^{k,\zeta)},\ w_{1,\sigma}^{(k,\zeta)}\}$ is defined in (5.11).

Formulas (6.1) and (6.3) show that the value of c_1 (as well as the general form of the asymptotics near g_1) depends not only on the data of the problem near the point g_1 but also from the data near $g_2 = \Omega_1(g_1)$.

We remark that similarly to (6.1) and (6.3) one can calculate the coefficients c_ν with the help of the Green formula and so–called formally adjoint problems generated by the Green formula[10]. The corresponding technique is developed in [10, 12]. We will not recall the Green formula here, but only mention that the corresponding formulas for c_ν are immediately obtained if we use Theorems 5.4 [12] and 4.3 [12] instead of Theorems 5.2 [12] and 4.1 [12] respectively. Formally adjoint problems have the advantage that they are considered in "original" spaces, but not in adjoint ones (spaces of distributions). Therefore corresponding eigenvectors and associated vectors can be found explicitly in a number of cases.

But, anyway, both adjoint problem– and formally adjoint problem–based formulas for c_ν involve the solution u itself. Further we are to get formulas allowing to calculate the coefficients c_ν only in terms of a right–hand side $\{f,\ f_\sigma\}$ of problem (2.1), (2.2).

[10] In this case, additionally to Conditions 2.2 and 2.3, one must demand the system $\{B_{\sigma\mu}(D_y)\}_{\mu=1}^m$ to be normal on γ_σ $(\sigma = 1,\ 2)$, where $B_{\sigma\mu}(D_y)$ is the principle homogeneous part of $B_{\sigma\mu}(g_1,\ D_y)$.

II. We are supposed to calculate c_ν with the help of some special distributions from the kernel of the operator $\mathbf{L}^* : H^l_{a_1}(G,\ \Upsilon)^* \to H^{l+2m}_{a_1}(G)^*$. To begin with, assume that $\{v,\ w_\sigma\} \in H^l_{a_1}(G,\ \Upsilon)$ is an arbitrary distribution from the kernel of $\mathbf{L}^*$.

Let us calculate the value of the expression $< \mathbf{L}u,\ i\{v,\ w_\sigma\} >$.

We suppose that the following consistent condition is fulfilled. If the vector c_ν contains $c_\nu^{(k,\zeta)}$ in its tth position, then the vector d_ν has $d_\nu^{(\varkappa_{\zeta,\nu}-k-1,\zeta)}$ in its tth position. The same is true for all the other vectors related to the adjoint problem ($\{v_1,\ w_{1,\sigma}\}$, v_2, etc.).

Besides, we keep assuming that the Jordan chains corresponding to the eigenvalues λ_ν and $\bar{\lambda}_\nu$ of the operators $\tilde{\mathcal{L}}_\nu(\lambda)$ and $\tilde{\mathcal{L}}^*_\nu(\lambda)$ respectively satisfy conditions of biorthogonality and normalization (5.12) (for $\nu = 1$) and (5.18) (for $\nu = 2$).

By virtue of (3.22), we have $\mathbf{L}u = \mathbf{L}c_1U_1 + \mathbf{L}c_2U_2 + \mathbf{L}\hat{u}$, where $\hat{u} \in H^{l+2m}_{a_1}(G)$. Since $\{v,\ w_\sigma\}$ belongs to the kernel of $\mathbf{L}^* : H^l_{a_1}(G,\ \Upsilon)^* \to H^{l+2m}_{a_1}(G)^*$, we get $< \mathbf{L}\hat{u},\ i\{v,\ w_\sigma\} >= 0$. Therefore, by virtue of (3.23), we can write

$$< \mathbf{L}u,\ i\{v,\ w_\sigma\} >=< \mathbf{L}c_1U_1,\ i\{v,\ w_\sigma\} > + < \mathbf{L}c_2U_2,\ i\{v,\ w_\sigma\} > . \quad (6.4)$$

Let $\eta_\nu(\omega,\ r)$ be the function η_ν written in polar coordinates with the pole at g_ν ($\nu = 1,\ 2$). For $\varepsilon > 0$ we introduce the functions $\eta_{\nu,\varepsilon}(\omega,\ r) = \eta_\nu(\omega,\ r/\varepsilon)$.

At first let us consider the 1st term in the right–hand side of (6.4). Since the difference $\eta_1 - \eta_{1,\varepsilon}$ vanishes near g_1, we have $(\eta_1 - \eta_{1,\varepsilon})c_1u_1 \in H^{l+2m}_{a_1}(G)$. It follows from this and from (3.21) that

$$< \mathbf{L}c_1U_1,\ i\{v,\ w_\sigma\} >=< \mathbf{L}c_1\eta_{1,\varepsilon}u_1,\ i\{v,\ w_\sigma\} > . \quad (6.5)$$

Put for short $\mathcal{U}_{1,\varepsilon} = c_1\eta_{1,\varepsilon}u_1$. Since $\operatorname{supp}\mathcal{U}_{1,\varepsilon} \subset \mathcal{V}(g_1) \cap G = \mathcal{V}(0) \cap K$, we have

$$\mathbf{L}\mathcal{U}_{1,\varepsilon} = \mathcal{L}\mathcal{U}_{1,\varepsilon} + \{\mathbf{P}(y,\ D_y) - \mathcal{P}(D_y),\ \mathbf{B}_\sigma(y,\ D_y) - \mathcal{B}_\sigma(D_y)\}\mathcal{U}_{1,\varepsilon}.$$

Here $\mathcal{P}(D_y)$ is the principal homogeneous part of $\mathbf{P}(g_1,\ D_y)$; $\mathcal{B}_\sigma(D_y)$ is defined by (3.14). This and Theorem 5.1 imply that the right–hand side of (6.5) has the form

$$\begin{aligned}< \mathcal{L}_1\mathcal{U}_{1,\varepsilon},\ id_1\{v_1,\ w_{1,\sigma}\} > + < \mathcal{L}_1\mathcal{U}_{1,\varepsilon},\ i\{F,\ G_\sigma\} > +\\ < \{\mathbf{P}(y,\ D_y) - \mathcal{P}(D_y),\ \mathbf{B}_\sigma(y,\ D_y) - \mathcal{B}_\sigma(D_y)\}\mathcal{U}_{1,\varepsilon},\\ i(d_1\{v_1,\ w_{1,\sigma}\} + \{F,\ G_\sigma\}) >, \quad (6.6)\end{aligned}$$

where $\{v_1,\ w_{1,\sigma}\}$ is defined by (5.11), $\{F,\ G_\sigma\}\in H_a^l(K,\ \gamma)^*$.

By Theorem 4.1 [12], the 1st term in (6.6) is equal to $(c_1,\ d_1)$[11]. The 2nd term in (6.6) is majorized by

$$c\|\mathcal{U}_{1,\varepsilon}\|_{H_a^{l+2m}(K)}\|\{F,\ G_\sigma\}\|_{H_a^l(K,\ \gamma)^*}=\mathrm{O}(1),$$

where we use the Hardy–Littlewood symbol "O" with its usual interpretation (O(1) tends to 0 as $\varepsilon\to 0$).

By virtue of the boundedness of the imbedding operator of $H_{a_1+1}^{l+2m}(K)$ into $H_{a_1}^{l+2m-1}(K)$, Lemma 3.3′ [15], and the inequality $a<a_1+1$, the last term in (6.6) is majorized by

$$c\|\mathcal{U}_{1,\varepsilon}\|_{H_{a_1+1}^{l+2m}(K)}\|d_1\{v_1,\ w_{1,\sigma}\}+\{F,\ G_\sigma\}\|_{H_{a_1}(K,\ \gamma)^*}\le c'\|\mathcal{U}_{1,\varepsilon}\|_{H_a^{l+2m}(K)}=\mathrm{O}(1).$$

Thus, as ε tends to 0, we get

$$<\mathbf{L}c_1U_1,\ i\{v,\ w_\sigma\}>=(c_1,\ d_1). \tag{6.7}$$

Now let us consider the 2nd term in the right–hand side of (6.4). Since the functions $(\eta_2-\eta_{2,\varepsilon})c_2u_2$ and $(\eta_1-\eta_{1,\varepsilon})c_2u_{12}$ belong to the space $H_{a_1}^{l+2m}(G)$, we obtain from (3.21) that

$$<\mathbf{L}c_2U_2,\ i\{v,\ w_\sigma\}>=<\mathbf{L}c_2(\eta_{2,\varepsilon}u_2+\eta_{1,\varepsilon}u_{12}),\ i\{v,\ w_\sigma\}>. \tag{6.8}$$

Put for short $\mathcal{U}_{2,\varepsilon}=c_2\eta_{2,\varepsilon}u_2$, $\mathcal{U}_{12,\varepsilon}=c_2\eta_{1,\varepsilon}u_{12}$. Using Theorems 5.1 and 5.2, write the right–hand side of (6.8) in the form

$$\begin{aligned}&<\mathcal{L}_2\mathcal{U}_{2,\varepsilon},\ id_2v_2>+<\mathcal{L}_2\mathcal{U}_{2,\varepsilon},\ id_1v_{21}>+<\mathcal{P}(D_y)\mathcal{U}_{12,\varepsilon},\ id_1v_1>+\\ &\quad<\mathcal{B}_1(D_y)\mathcal{U}_{12,\varepsilon}+\big(T_1(D_y)\mathcal{U}_{2,\varepsilon}\big)\big(\Omega_1(y)\big)|_{\gamma_1},\ id_1w_{1,1}>+\\ &\qquad<\mathcal{B}_2(D_y)\mathcal{U}_{12,\varepsilon}\ id_1w_{1,2}>+\mathrm{O}(1).\end{aligned} \tag{6.9}$$

Here $\mathcal{P}(D_y)$, $T_1(D_y)$ are the principal homogeneous parts of $\mathbf{P}(g_1,\ D_y)$, $\mathbf{T}(g_2,\ D_y)$ respectively; $\mathcal{B}_\sigma(D_y)$, $\sigma=1,\ 2$, are defined by (3.14).

By Theorem 5.2 [12], the 1st term in (6.9) is equal to $(c_2,\ d_2)$.

Since $\mathcal{L}_2c_2u_2=0$ (see (3.5)), the 2nd term in (6.9) is equal to

$$<[\mathcal{L}_2,\ \eta_{2,\varepsilon}]c_2u_2,\ id_1v_{21}>, \tag{6.10}$$

[11] Here $(c_1,\ d_1)$ (and further $(c_2,\ d_2)$, etc) stands for the inner product of the corresponding complex vectors: $(c_1,\ d_1)=\sum\limits_{k,\zeta}c_1^{(k,\zeta)}\overline{d_1^{(\varkappa_{\zeta,1}-k-1,\zeta)}}$.

where $[\cdot,\ \cdot]$ is the commutator. Using the condition $0 < a - a_1 < 1$, one can easily check that (6.10) is equal to

$$(\hat{A}_{12}(\varepsilon)c_2,\ d_1), \tag{6.11}$$

where $\hat{A}_{12}(\varepsilon)$ is a matrix of the corresponding order, the elements of which are linear combinations of the functions $\varepsilon^{\lambda_2-\lambda_1}(i\ln\varepsilon)^q$.

Further, let us recall that the function u_{12} is a solution for problem (3.19), (3.20). Hence the sum of the 3rd, 4th, and 5th terms in (6.9) is equal to

$$\begin{aligned} &< [\mathcal{P}(D_y),\ \eta_{1,\varepsilon}]c_2u_{12},\ id_1v_1 > + < [\mathcal{B}_1(D_y),\ \eta_{1,\varepsilon}]c_2u_{12} + \\ &\qquad \big([T_1(D_y),\ \eta_{1,\varepsilon}]c_2u_2\big)\big(\Omega_1(y)\big)|_{\gamma_1},\ id_1w_{1,1} > + \\ &\qquad\qquad < [\mathcal{B}_2(D_y),\ \eta_{1,\varepsilon}]c_2u_{12},\ id_1w_{1,2} > \end{aligned} \tag{6.12}$$

and therefore is of the form (6.11). Thus we see that

$$< \mathbf{L}c_2U_2,\ i\{v,\ w_\sigma\} >= (c_2,\ d_2) + (A_{12}(\varepsilon)c_2,\ d_1) + \mathrm{O}(1), \tag{6.13}$$

where $A_{12}(\varepsilon)$ is a matrix of the corresponding order, the elements of which are linear combinations of the functions $\varepsilon^{\lambda_2-\lambda_1}(i\ln\varepsilon)^q$.

From equations (6.4), (6.7), and (6.13), it follows that

$$< \mathbf{L}u,\ i\{v,\ w_\sigma\} >= (c_2,\ d_2) + (c_1 + A_{12}(\varepsilon)c_2,\ d_1) + \mathrm{O}(1), \tag{6.14}$$

III. Keeping denotation of section 4, we will denote by $\mathcal{U}_1,\ \dots,\ \mathcal{U}_\varkappa$ the ordered set of functions $U_\nu^{(k,\zeta)}$, which are the elements of the vectors U_ν, $\nu = 1,\ 2$, defined by (3.21).

Denote by $\mathcal{V}_1,\ \dots,\mathcal{V}_\varkappa$ the set of functions $\{V_\nu^{(k,\zeta)},\ W_{\nu,\sigma}^{(k,\zeta)}\}$, which are the elements of the vectors $\{V_\nu,\ W_{\nu,\sigma}\}$, $\nu = 1,\ 2$, defined by (5.21).

Suppose the sets $\{\mathcal{U}_1,\ \dots,\mathcal{U}_\varkappa\}$ and $\{\mathcal{V}_1,\ \dots,\mathcal{V}_\varkappa\}$ are ordered consistently, i.e., the equality $\mathcal{U}_t = U_\nu^{(k,\zeta)}$ is fulfilled simultaneously with the equality $\mathcal{V}_t = \{V_\nu^{(\varkappa_{\zeta,\nu}-k-1,\zeta)},\ W_{\nu,\sigma}^{(\varkappa_{\zeta,\nu}-k-1,\zeta)}\}$.

In this work we restrict ourselves to the case when $d = 0$ in Lemma 4.2. This mean that any solution to homogeneous problem (2.1), (2.2) from the space $H_a^{l+2m}(G)$ necessarily belongs to the space $H_{a_1}^{l+2m}(G)$. In that case we will show that for any right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^l(G,\ \Upsilon)$ the coefficients in the asymptotics formula for solutions are uniquely defined. If $d > 0$, then, similarly to the case of "local" problems (see Theorem 3.6 [18, Chapter 4]), there is some freedom in choosing the coefficients of the asymptotics. Moreover, the procedure for calculation of the coefficients becomes more technically complicated (while the idea remains similar to the one we shall describe below) and will not be considered here.

So, suppose $d=0$. Then, by virtue of Lemma 4.2, there exist solutions $\mathcal{Y}_1,\ \dots,\ \mathcal{Y}_\varkappa \in H^l_{a_1}(G,\ \Upsilon)^*$ for the equation $\mathbf{L}^*\mathcal{Y}=0$, linearly independent modulo $H^l_a(G,\ \Upsilon)^*$. By (5.22) we have $\mathcal{Y}_t \equiv \sum\limits_{k=1}^{\varkappa} d_{tk}\mathcal{V}_k\ \Big(\mathrm{mod}\, H^l_a(G,\ \Upsilon)^*\Big)$, $t=1,\ \dots,\ \varkappa$. Since $\mathcal{Y}_1,\ \dots,\ \mathcal{Y}_\varkappa$ are linearly independent modulo $H^l_a(G,\ \Upsilon)^*$, the matrix $\|d_{tk}\|$ is nonsingular. Hence, without loss in generality, we can assume that

$$\mathcal{Y}_t \equiv \mathcal{V}_t\ \Big(\mathrm{mod}\, H^l_a(G,\ \Upsilon)^*\Big),\quad t=1,\ \dots,\ \varkappa. \tag{6.15}$$

Now let us prove that the elements of the matrix $A_{12}(\varepsilon)$ appearing in (6.14) have finite limits as $\varepsilon\to 0$. This limit will be denoted by A_{12}:

$$A_{12}=\lim_{\varepsilon\to 0} A_{12}(\varepsilon).$$

Let l_ν be the length of the vector c_ν (or d_ν, which is the same), $\nu=1,\ 2$. Clearly, $l_2+l_1=\varkappa$. Suppose for definiteness that the first l_2 elements in the ordered set $\{\mathcal{U}_1,\ \dots,\mathcal{U}_\varkappa\}$ ($\{\mathcal{V}_1,\ \dots,\mathcal{V}_\varkappa\}$) are components of the vector U_2 ($\{V_2,\ W_{2,\sigma}\}$) and the last l_1 ones are components of the vector U_1 ($\{V_1,\ W_{1,\sigma}\}$):

$$\{\mathcal{U}_1,\ \dots,\mathcal{U}_\varkappa\}=\{\underbrace{U_2}_{l_2},\ \underbrace{U_1}_{l_1}\}$$

$$\Big(\{\mathcal{V}_1,\ \dots,\mathcal{V}_\varkappa\}=\{\underbrace{\{V_2,\ W_{2,\sigma}\}}_{l_2},\ \underbrace{\{V_1,\ W_{1,\sigma}\}}_{l_1}\}\Big).$$

Now fix an arbitrary t from the set $\{1,\ \dots,\ l_2\}$ and an arbitrary k from the set $\{l_2+1,\ \dots,\ \varkappa\}$. Substituting in (6.14) $u=\mathcal{U}_t$ (which is a component of the vector U_2) and $\{v,\ w_\sigma\}=\mathcal{Y}_k$ (which is, by (6.15), a component of the vector $\{V_1,\ W_{1,\sigma}\}$ modulo $H^l_a(G,\ \Upsilon)^*$), we get $c_1=0$, $d_2=0$, and therefore,

$$<\mathbf{L}U_t,\ i\mathcal{Y}_k\}>=a_{tk}(\varepsilon)+\mathrm{O}(1). \tag{6.16}$$

Here $a_{tk}(\varepsilon)$ is the corresponding element of the matrix $A_{12}(\varepsilon)$. The left–hand side of (6.16) does not depend on ε. Therefore $a_{tk}(\varepsilon)$ has a finite limit as $\varepsilon\to 0$.

Thus, passing in (6.14) to the limit as $\varepsilon\to 0$, we get

$$<\mathbf{L}u,\ i\{v,\ w_\sigma\}>=(c_2,\ d_2)+(c_1+A_{12}c_2,\ d_1). \tag{6.17}$$

THEOREM 6.1. *Let $u \in H_a^{l+2m}(G)$ be a solution for problem (2.1), (2.2) with a right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^l(G,\ \Upsilon)$. Then u has the asymptotics*

$$u \equiv \Big(\sum_{t=1}^{\varkappa} c_t \mathcal{U}_t\Big) \ \Big(\operatorname{mod} H_{a_1}^{l+2m}(G)\Big). \tag{6.18}$$

The constants c_t $(t = 1,\ \dots,\ \varkappa)$ can be calculated by the formulas

$$c_t = <\{f,\ f_\sigma\},\ i\mathcal{Y}_t> \tag{6.19}$$

if $t \le l_2$ (i.e., c_t coincides with a component of the vector $\{c_2^{(k,\zeta)}\}$);

$$c_t = <\{f,\ f_\sigma\},\ i\mathcal{Y}_t - i\Big[A_{12}(\mathcal{Y}_1,\ \dots,\ \mathcal{Y}_{l_2})^T\Big]_{t-l_2} > \tag{6.20}$$

if $l_2 < t \le \varkappa$ (i.e., c_t coincides with a component of the vector $\{c_1^{(k,\zeta)}\}$). Here $[\cdot]_j$ stands for the jth component of a vector.

Proof. Substituting $\{v,\ w_\sigma\} = \mathcal{Y}_1,\ \dots,\ \{v,\ w_\sigma\} = \mathcal{Y}_\varkappa$ subsequently in (6.17), we obtain formulas (6.19) and (6.20). □

Theorem 6.1 shows that the values of the coefficients $c_\nu^{(k,\zeta)}$ are the functionals over the right–hand sides $\{f,\ f_\sigma\}$ of problem (2.1), (2.2). These functionals depend on the data of the problem in the whole domain G, but not only in the neighborhoods $\mathcal{V}(g_1)$ and $\mathcal{V}(g_2)$.

REMARK 6.1. *We remind that the elements of the matrix $A_{12}(\varepsilon)$ are linear combinations of the functions $\varepsilon^{\lambda_2-\lambda_1}(i\ln\varepsilon)^q$. Hence, if $\lambda_1 \ne \lambda_2$, then $A_{12} = 0$.*

7. Example.

I. In this section we consider an example illustrating the results of sections 2–6.

Keeping denotation and assumptions of sections 2 and 3, we consider the following nonlocal problem

$$\mathbf{P}(y,\ D_y) \equiv \sum_{|\alpha|\le 2} p_\alpha(y)\frac{\partial^{|\alpha|}u}{\partial y_1^{\alpha_1}\partial y_2^{\alpha_2}} = f(y) \quad (y \in G\setminus\mathcal{K}), \tag{7.1}$$

$$\mathbf{B}_\sigma u \equiv u(y)|_{\Upsilon_\sigma} + e_\sigma u\big(\Omega_\sigma(y)\big)|_{\Upsilon_\sigma} = f_\sigma(y) \quad (y \in \Upsilon_\sigma;\ \sigma = 1,\ 2). \tag{7.2}$$

Here $\mathbf{P}(y,\ D_y)$ is a 2nd order differential operator, properly elliptic in $\bar{G}$, with infinitely smooth coefficients $p_\alpha(y)$; $e_\sigma \in \mathbb{C}$. For clearness we assume

$$\sum_{|\alpha|=2} p_\alpha(g_\nu)\frac{\partial^{|\alpha|}u}{\partial y_1^{\alpha_1}\partial y_2^{\alpha_2}} = \Delta u, \quad \nu = 1,\ 2. \tag{7.3}$$

Let us obtain the asymptotics of a solution $u \in H_a^2(G)$ for problem (7.1), (7.2) with a right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^0(G,\ \Upsilon) \stackrel{def}{=} H_{a_1}^0(G)\times \prod\limits_{\sigma=1,2} H_{a_1}^{3/2}(\Upsilon_\sigma)$, assuming $0 < a - a_1 < 1$.

At first, according to section 3, we consider the asymptotics of the solution u in the neighborhood $\mathcal{V}(g_2)$ of the point g_2. For this purpose one must write the model equation in $\mathbb{R}^2 \setminus \{g_2\}$. Taking into account (7.3), we obtain

$$\Delta u = \hat{f}(y) \quad (y \in \mathbb{R}^2 \setminus \{g_2\}), \tag{7.4}$$

where $\hat{f} \in H_{a_1}^0\big(\mathcal{V}(g_2)\big)$.

Write equation (7.4) in polar coordinates with the pole at g_2:

$$r\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{\partial^2 u}{\partial \omega^2} = r^2 f(\omega,\ r) \quad (0 < \omega < 2\pi,\ r > 0).$$

Applying formally the Mellin transformation, we get

$$\frac{d^2\tilde{u}}{d\omega^2} - \lambda^2\tilde{u} = \tilde{F}(\lambda,\ \omega) \quad (0 < \omega < 2\pi),$$

where $\tilde{u}$ and $\tilde{F}$ are the Mellin transforms of u and $r^2 f$ with respect to r.

Introduce the corresponding operator–valued function

$$\tilde{\mathcal{L}}_2(\lambda) = \frac{d^2}{d\omega^2} - \lambda^2 : W_{2\pi}^2(0,\ 2\pi) \to L_2(0,\ 2\pi).$$

Let us suppose, additionally to Condition 3.1, that there is the only eigenvector $\varphi_2(\omega)$ corresponding to the eigenvalue λ_2 $(a_1 - 1 < \operatorname{Im}\lambda_2 < a - 1)$ of $\tilde{\mathcal{L}}_2(\lambda)$ and there are no associated vectors.

Then, by Theorem 3.1, we have

$$u(y) = c_2 u_2(y) + \hat{u}(y) \quad \big(y \in \mathcal{V}(g_2)\big). \tag{7.5}$$

Here c_2 is a scalar constant, $u_2 = r^{i\lambda_2}\varphi_2(\omega)$ is a power solution for homogeneous equation (7.4); $(\omega,\ r)$ are polar coordinates with the pole at g_2 and the polar axis being, for definiteness, tangent to the curve $\Omega_1(\Upsilon_1)$ at g_2; $\hat{u} \in H_{a_1}^2\big(\mathcal{V}(g_2)\big)$.

Now we consider the asymptotics of the solution u for problem (7.1), (7.2) in the neighborhood $\mathcal{V}(g_1)$ of the point g_1. Let $\Omega_1(y)$ $(y \in \mathcal{V}(g_1))$ be a rotation with respect to g_1 (with no expansion for simplicity) and the shift by the vector $\overrightarrow{g_1 g_2}$. Let $\Omega_2(y)$ $(y \in \mathcal{V}(g_1))$ coincide with the operator $\mathcal{G}_2$ of a rotation by an angle ω_2 $(b_1 < b_2 + \omega_2 < b_2)$ and an expansion with a coefficient $\beta_2 > 0$.

According to section 3 and assumption (7.3), the asymptotics of u in $\mathcal{V}(g_1)$ coincides with the asymptotics of a solution for the problem

$$\Delta u = \hat{f}(y) \quad (y \in \mathcal{V}(0) \cap K), \tag{7.6}$$

$$\begin{gathered} u|_{\mathcal{V}(0)\cap\gamma_1} = \hat{f}_1 - c_2 f_{12} \quad (y \in \mathcal{V}(0) \cap \gamma_1), \\ u|_{\mathcal{V}(0)\cap\gamma_2} + e_2 u(\mathcal{G}_2 y)|_{\mathcal{V}(0)\cap\gamma_2} = f_2 \quad (y \in \mathcal{V}(0) \cap \gamma_2). \end{gathered} \tag{7.7}$$

Here $\hat{f} \in H^0_{a_1}(\mathcal{V}(0) \cap K)$,

$$\begin{gathered} \hat{f}_1 = f_1 - e_1 \hat{u}\big(\Omega_1(y)\big)|_{\mathcal{V}(0)\cap\gamma_1} \in H^{3/2}_{a_1}(\mathcal{V}(0) \cap \gamma_1), \\ f_{12} = e_1 u_2\big(\Omega_1(y)\big)|_{\mathcal{V}(0)\cap\gamma_1} = e_1 r^{i\lambda_2} \varphi_2(0)^{12}. \end{gathered}$$

Similarly to the above we obtain the corresponding operator–valued function $\tilde{\mathcal{L}}_1(\lambda) : W^2_2(b_1,\ b_2) \to L_2(b_1,\ b_2) \times \mathbb{C}^2$ given by

$$\tilde{\mathcal{L}}_1(\lambda)\varphi = \Big\{ \frac{d^2\varphi}{d\omega^2} - \lambda^2\varphi,\ \varphi(\omega)|_{\omega=b_1},\ \varphi(\omega)|_{\omega=b_2} + e_2 e^{i\lambda \ln \beta_2} \varphi(\omega + \omega_2)|_{\omega=b_2} \Big\}. \tag{7.8}$$

Let us suppose, additionally to Condition 3.1, that there is the only eigenvector $\varphi_1(\omega)$ corresponding to the eigenvalue λ_1 ($a_1 - 1 < \operatorname{Im} \lambda_1 < a - 1$) of $\tilde{\mathcal{L}}_1(\lambda)$ and there are no associated vectors.

Then, by Theorem 3.18, we have

$$u(y) = c_1 u_1(y) + c_2 u_{12}(y) + \hat{u}(y) \quad (y \in \mathcal{V}(g_1)). \tag{7.9}$$

Here c_1 is some scalar constant, c_2 is the constant appearing in (7.5); $u_{12} = r^{i\lambda_2}\varphi_{12}(\omega)$ ($\varphi_{12} \in W^2_2(b_1,\ b_2)$) is a particular solution for the following problem in the angle K with the "special" right–hand side (cf. (3.19), (3.20)):

$$\Delta u = 0 \quad (y \in K), \tag{7.10}$$

$$u|_{\gamma_1} = -f_{12}, \quad u|_{\gamma_2} + e_2 u(\mathcal{G}_2 y)|_{\gamma_2} = 0; \tag{7.11}$$

$u_1 = r^{i\lambda_1}\varphi_1(\omega)$ is a solution for homogeneous problem (7.10), (7.11); $(\omega,\ r)$ are polar coordinates with the pole at the point $g_1 = 0$; $\hat{u} \in H^2_{a_1}\big(\mathcal{V}(g_1)\big)$.

To write the asymptotics in the whole domain G, we introduce the functions $U_1 = \eta_1 u_1,\ U_2 = \eta_2 u_2 + \eta_1 u_{12}$. Then (7.5) and (7.9) imply:

[12] We calculate $\varphi_2(\omega)$ for $\omega = 0$ because of the special choice of polar coordinates (see above).

Let $u \in H_a^2(G)$ be a solution for problem (7.1), (7.2) with a right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^0(G,\ \Upsilon)$, $0 < a - a_1 < 1$. Then we have

$$u \equiv \Big(c_1 U_1 + c_2 U_2\Big)\ \Big(\operatorname{mod} H_{a_1}^2(G)\Big), \tag{7.12}$$

where $c_1,\ c_2$ are some scalar constants.

II. From asymptotics formula (7.12) and Theorem 4.1 we can derive the connection between the indices of the operators

$$\mathbf{L}_a = \{\mathbf{P}(y,\ D_y),\ \mathbf{B}_\sigma\} : H_a^2(G) \to H_a^0(G,\ \Upsilon),$$
$$\mathbf{L}_{a_1} = \{\mathbf{P}(y,\ D_y),\ \mathbf{B}_\sigma\} : H_{a_1}^2(G) \to H_{a_1}^0(G,\ \Upsilon)$$

corresponding to problem (7.1), (7.2), but acting in different weighted spaces. Since the sum of full multiplicities of eigenvalues λ_1 and λ_2 is equal to 2 in our case, the connection between the indices is as follows:

$$\operatorname{ind} \mathbf{L}_a = \operatorname{ind} \mathbf{L}_{a_1} + 2.$$

III. To calculate the coefficients c_ν in formula (7.12), we will study the asymptotics of solutions for the adjoint nonlocal problem.

Consider the operator $\mathbf{L}^* : H_{a_1}^0(G,\ \Upsilon)^* \to H_{a_1}^2(G)^*$, adjoint to $\mathbf{L} = \{\mathbf{P}(y,\ D_y),\ \mathbf{B}_\sigma\} : H_{a_1}^2(G) \to H_{a_1}^0(G,\ \Upsilon)$. The operator $\mathbf{L}^*$ is given by

$$< u,\ \mathbf{L}^*\{v,\ w_\sigma\} >=< \mathbf{P}(y,\ D_y)u,\ v >_G + \sum_{\sigma=1,2} < \mathbf{B}_\sigma u,\ w_\sigma >_{\Upsilon_\sigma},$$

where $\{v,\ w_\sigma\} \in H_{a_1}^0(G,\ \Upsilon)^*$, $u \in H_{a_1}^2(G)$.

Let us study the asymptotics of a solution $\{v,\ w_\sigma\} \in H_{a_1}^0(G,\ \Upsilon)^*$ for the problem

$$\mathbf{L}^*\{v,\ w_\sigma\} = \Psi, \tag{7.13}$$

where $\Psi \in H_a^2(G)^*$.

In [12] it is shown that $\bar\lambda_\nu$ is an eigenvalue of the operator $\tilde{\mathcal{L}}_\nu^*(\lambda)$, adjoint to $\tilde{\mathcal{L}}_\nu(\bar\lambda)$. Denote by $\{\psi_1,\ \chi_{1,\sigma}\} \in L_2(b_1,\ b_2)\times\mathbb{C}^2$ ($\psi_2 \in L_2(0,\ 2\pi)$) the eigenvector of $\tilde{\mathcal{L}}_1^*(\lambda)$ ($\tilde{\mathcal{L}}_2^*(\lambda)$) corresponding to the eigenvalue $\bar\lambda_1$ ($\bar\lambda_2$). Conditions of biorthogonality and normalization (5.12) and (5.18) assume the form

$$< -2\lambda_\nu\varphi_\nu,\ \psi_\nu >= 1^{13}. \tag{7.14}$$

Put $\{v_1,\ w_{1,\sigma}\} = \{r^{i\bar\lambda_1}\psi_1,\ r^{i\bar\lambda_1-1}\chi_{1,\sigma}\}$ $\quad (v_2 = r^{i\bar\lambda_2}\psi_2)$, where $(\omega,\ r)$ are polar coordinates with the pole at g_1 (with the pole at g_2 and with the polar axis being tangent to the curve $\Omega_1(\Upsilon_1)$ at g_2).

Further, by Theorem 5.1, we have

$$\eta_1\{v,\ w_\sigma\} \equiv d_1\eta_1\{v_1,\ w_{1,\sigma}\} \Big(\operatorname{mod} H_a^0(G,\ \Upsilon)^*\Big), \tag{7.15}$$

where d_1 is some scalar constant. By Theorem 5.2, we have

$$\eta_2 v \equiv \Big(d_2\eta_2 v_2 + d_1\eta_2 v_{21}\Big) \Big(\operatorname{mod} H_a^0(G)^*\Big). \tag{7.16}$$

Here d_2 is some scalar constant, d_1 is the constant appearing in (7.15); $v_{21} = r^{i\bar\lambda_1}\psi_{21}$; $(\omega,\ r)$ are polar coordinates with the pole at g_2 and the polar axis being tangent to the curve $\Omega_1(\Upsilon_1)$ at g_2; $\psi_{21} \in L_2(0,\ 2\pi)$. Moreover, the distribution v_{21} is a particular solution for the following adjoint equation in $\mathbb{R}^2 \setminus \{g_2\}$ with the "special" right–hand side (cf. (5.20)):

$$\int\limits_{\mathbb{R}^2} \Delta u \cdot \bar v\, dy = \int\limits_0^\infty u(0,\ r)\cdot \overline{(-\bar e_1\chi_{1,1}r^{i\bar\lambda_1-1})}dr \quad \text{for all } u \in C_0^\infty(\mathbb{R}^2 \setminus \{g_2\})^{14}.$$

Put $\{V_2,\ W_{2,\sigma}\} = \eta_2\{v_2,\ 0\},\ \{V_1,\ W_{1,\sigma}\} = \eta_1\{v_1,\ w_{1,\sigma}\} + \eta_2\{v_{21},\ 0\}$.

Then (7.15) and (7.16) imply:

Let $\{v,\ w_\sigma\} \in H_{a_1}^0(G,\ \Upsilon)^*$ be a solution for problem (7.13) with a right–hand side $\Psi \in H_a^2(G)^*$. Then we have

$$\{v,\ w_\sigma\} \equiv \Big(d_1\{V_1,\ W_{1,\sigma}\} + d_2\{V_2,\ W_{2,\sigma}\}\Big) \Big(\operatorname{mod} H_a^2(G,\ \Upsilon)^*\Big), \tag{7.17}$$

where $d_1,\ d_2$ are some constants.

IV. Now let us calculate the coefficients c_ν appearing in (7.12). Formulas (6.1) and (6.3) assume the form

$$c_2 =< \Delta(\eta_2 u),\ iv_2 >_{\mathbb{R}^2},$$
$$c_1 =< \{\Delta u',\ u'|_{\gamma_1},\ u'|_{\gamma_2} + e_2 u'(\mathcal{G}_2 y)|_{\gamma_2}\},\ i\{v_1,\ w_{1,\sigma}\} >,$$

where $u' = \eta_1(u - c_2 u_{12})$.

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[13] One can show that $\lambda_\nu \neq 0$ whenever there are no associated vectors corresponding to the eigenvalue λ_ν. Hence there always exist vectors $\{\psi_1,\ \chi_{1,\sigma}\}$ and ψ_2 satisfying (7.14).

[14] We calculate $u(\omega,\ r)$ for $\omega = 0$ because of the special choice of polar coordinates (see above).

Now let us write the formulas allowing to calculate the coefficients c_ν only in terms of a right–hand side $\{f,\ f_\sigma\}$ of problem (7.1), (7.2) (i.e., independent of a solution u).

Following section 6, we assume for simplicity that any solution to homogeneous problem (7.1), (7.2) from the space $H_a^2(G)$ necessarily belongs to the space $H_{a_1}^2(G)$. Then there exist solutions $\mathcal{Y}_1,\ \mathcal{Y}_2 \in H_{a_1}^0(G,\ \Upsilon)^*$ for the equation $\mathbf{L}^*\mathcal{Y} = 0$, linearly independent modulo $H_a^0(G,\ \Upsilon)^*$ such that

$$\mathcal{Y}_\nu \equiv \{V_\nu,\ W_{\nu,\sigma}\} \Big(\operatorname{mod} H_a^0(G,\ \Upsilon)^*\Big), \quad \nu = 1,\ 2.$$

Let $\eta_{\nu,\varepsilon}$ be the functions defined in section 6.

Then from Theorem 6.1 we obtain the following result.

Let $u \in H_a^2(G)$ be a solution for problem (7.1), (7.2) with a right–hand side $\{f,\ f_\sigma\} \in H_{a_1}^0(G,\ \Upsilon)$. Then the function $u \in H_a^2(G)$ has asymptotics (7.12). The constants c_ν ($\nu = 1,\ 2$) are calculated by the formulas

$$c_2 =< \{f,\ f_\sigma\},\ i\mathcal{Y}_1 >,$$
$$c_1 =< \{f,\ f_\sigma\},\ i(\mathcal{Y}_1 - A_{12}\mathcal{Y}_2) > .$$

Here A_{12} is a scalar constant given by

$$A_{12} = \lim_{\varepsilon\to 0} < \Delta(\eta_{2,\varepsilon}u_2),\ iv_{21} > + < \{\Delta(\eta_{1,\varepsilon}u_{12}), \eta_{1,\varepsilon}u_{12}|_{\gamma_1} + \eta_{1,\varepsilon}f_{12}|_{\gamma_1},$$
$$\eta_{1,\varepsilon}u_{12}|_{\gamma_2} + e_2(\eta_{1,\varepsilon}u_{12})(\mathcal{G}_2 y)|_{\gamma_2}\},\ i\{v_1,\ w_{1,\sigma}\} >, \quad (7.18)$$

where the limit does exist.

REMARK 7.1. *The function u_2 (u_{12}) is a solution for homogeneous equation (7.4) (a solution for problem (7.10), (7.11) with the special right–hand side $\{0,\ -f_{12},\ 0\}$). Therefore, similarly to section 6, one can easily check that $A_{12} = \lim_{\varepsilon\to 0} \operatorname{const} \cdot \varepsilon^{i(\lambda_2-\lambda_1)}$. From this and from the existence of the limit in (7.18), it follows that $A_{12} = 0$ whenever $\lambda_1 \neq \lambda_2$.*

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