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Bounded Perturbations of Two-Dimensional Diffusion Processes with Nonlocal Conditions near the Boundary

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Abstract—We study the existence of Feller semigroups arising in the theory of multidimensional diffusion processes. We study bounded perturbations of elliptic operators with boundary conditions containing an integral over the closure of the domain with respect to a nonnegative Borel measure without assuming that the measure is small. We state sufficient conditions on the measure guaranteeing that the corresponding nonlocal operator is the generator of a Feller semigroup.

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1. INTRODUCTION AND PRELIMINARIES

In [1], [2], Feller studied the general form of the generator of strongly continuous contraction nonnegative semigroup of operators acting in spaces of continuous functions on the interval, half-line, or the line. Such semigroups correspond to one-dimensional diffusion processes and are called *Feller semigroups*. In the multidimensional case, the general form of the generator of a Feller semigroup was obtained by Venttsel′ [3]. Under some additional assumptions, he proved that the generator of the corresponding Feller semigroup is an elliptic differential operator of second order (possibly, with degeneracy), whose domain of definition consists of continuous (once or twice continuously differentiable, depending on the process) functions satisfying nonlocal boundary conditions. The nonlocal summand is the integral of a function over the closure of the domain with respect to a nonnegative Borel measure $\mu(y, d\eta)$. The following problem still remains unsolved. Suppose that we are given an elliptic integro-differential operator whose domain of definition is described by nonlocal boundary conditions. Is such an operator (or its closure) the generator of a Feller semigroup?

There are two classes of nonlocal boundary conditions: the so-called *transversal* and *nontransversal* conditions. In the transversal case, the order of nonlocal terms is less than that of local ones, while, in the nontransversal case, the orders coincide (see [4], where a rigorous definition and a probabilistic interpretation are given). The transversal case was studied in [4], [5]–[9]. The more complicated nontransversal case was studied in [9]–[12].

In [11], [12], it was assumed that the coefficients of nonlocal terms decrease as the argument tends to the boundary of the domain. The papers [6], [10], deal with the boundary condition for the case in which the coefficients of nonlocal terms are less than 1. It was shown that a nonlocal problem (after reduction to the boundary) can be regarded, in a certain sense, as the perturbation of a "local" Dirichlet problem.

In the present paper, we study nontransversal nonlocal conditions on the boundary of a plane domain G, admitting the "limiting case" whenever the measure $\mu(y, \overline{G})$ is equal to 1 after the corresponding normalization (the measure cannot be greater than 1 [3]). We assume that if, for some point $y \in \partial G$, the support of the measure $\mu(y, d\eta)$ is "close" to the point y and $\mu(y, \overline{G}) = 1$, then the measure $\mu(y, d\eta)$ is atomic.

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Using the Hille–Yosida theorem and results on the solvability of elliptic equations with nonlocal conditions near the boundary [13], we isolate a class of Borel measures $\mu(y, d\eta)$ for which the corresponding nonlocal operator is the generator of a Feller semigroup.

In conclusion of this section, let us recall the notion of a Feller semigroup and of its generator and state the Hille–Yosida theorem in convenient form.

Suppose that $G \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary ∂G. Suppose that X is a closed subspace in $C(\overline{G})$ containing at least one nonnegative function.

A strongly continuous semigroup of operators $\mathbf{T}_t\colon X \to X$ is called a *Feller semigroup* X if

1) $\|\mathbf{T}_t\| \le 1$, $t \ge 0$;

2) $\mathbf{T}_t u \ge 0$ for all $t \ge 0$ and $u \in X$, $u \ge 0$.

A linear operator $\mathbf{P}\colon \mathrm{D}(\mathbf{P}) \subset X \to X$ is called the *generator* (*infinitesimal generating operator*) of a strongly continuous semigroup $\{\mathbf{T}_t\}$ if

$$\mathbf{P}u = \lim_{t\to+0} \frac{\mathbf{T}u - u}{t}, \qquad \mathrm{D}(\mathbf{P}) = \{u \in X\colon \text{the limit in } X \text{ exists}\}.$$

Theorem 1.1 (the Hille–Yosida theorem; see Theorem 9.3.1 in [9]). 1. *Let* $\mathbf{P}\colon \mathrm{D}(\mathbf{P}) \subset X \to X$ *be the generator of a Feller semigroup on* X. *Then the following assertions hold*:

(a) *the domain of definition* $\mathrm{D}(\mathbf{P})$ *is dense in* X;

(b) *for any* $q > 0$, *the operator* $q\mathbf{I} - \mathbf{P}$ *has the bounded inverse* $(q\mathbf{I} - \mathbf{P})^{-1}\colon X \to X$ *and* $\|(q\mathbf{I} - \mathbf{P})^{-1}\| \le 1/q$;

(c) *the operator* $(q\mathbf{I} - \mathbf{P})^{-1}\colon X \to X$, $q > 0$, *is nonnegative.*

2. *If* $\mathbf{P}$ *is a linear operator from* X *to* X *satisfying condition* (a) *and there exists a constant* $q_0 \ge 0$ *such that conditions* (b) *and* (c) *hold for* $q > q_0$, *then* $\mathbf{P}$ *is the generator of a Feller semigroup on* X *which is uniquely defined by the operator* $\mathbf{P}$.

2. NONLOCAL CONDITIONS NEAR THE POINTS OF CONJUGATION

Consider a set $\mathscr{K} \subset \partial G$ consisting of a finite number of points. Suppose that

$$\partial G \setminus \mathscr{K} = \bigcup_{i=1}^{N} \Gamma_i,$$

where the Γ_i are open (in the topology of ∂G) curves of class C^∞. We assume that the domain G coincides with the plane angle in some neighborhood of each point $g \in \mathscr{K}$.

For integer $k \ge 0$, denote by $W_2^k(G)$ the Sobolev space. By $W_{2,\mathrm{loc}}^k(G)$ we denote the set of functions u such that $u \in W_2^k(G')$ for any domain G', $\overline{G'} \subset G$.

Consider the differential operator

$$P_0 u = \sum_{j,k=1}^{2} p_{jk}(y) u_{y_j y_k}(y) + \sum_{j=1}^{2} p_j(y) u_{y_j}(y) + p_0(y) u(y),$$

where the $p_{jk}, p_j \in C^\infty(\mathbb{R}^2)$ are real-valued functions, with $p_{jk} = p_{kj}$, $j,k = 1,2$.

Condition 2.1. The following assertions hold:

1) there exists a constant $c > 0$ such that

$$\sum_{j,k=1}^{2} p_{jk}(y)\xi_j\xi_k \geq c|\xi|^2$$

for $y \in \overline{G}$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$;

2) $p_0(y) \leq 0$, $y \in \overline{G}$.

In what follows, we shall need the maximum principle, which will be stated as follows.

Maximum Principle 2.1 (see Theorem 9.6 in [14]). *Suppose that $D \subset \mathbb{R}^2$ is a bounded or unbounded domain, and suppose that condition 2.1 holds for the domain D. If the function $u \in C(D)$ attains a positive maximum at a point $y^0 \in D$ and, moreover,*[1] *$P_0u \in C(D)$, then $P_0u(y^0) \leq 0$.*

We introduce operators corresponding to nonlocal terms with support near the set $\mathscr{K}$. For any set $\mathscr{M}$, let $\mathscr{O}_\varepsilon(\mathscr{M})$ denote its ε-neighborhood. Suppose that Ω_{is}, $i = 1, \dots, N$, $s = 1, \dots, S_i$, are diffeomorphisms of class C^∞ mapping a neighborhood $\mathscr{O}_i$ of the curve $\overline{\Gamma_i \cap \mathscr{O}_\varepsilon(\mathscr{K})}$ onto the set $\Omega_{is}(\mathscr{O}_i)$ so that

$$\Omega_{is}(\Gamma_i \cap \mathscr{O}_\varepsilon(\mathscr{K})) \subset G \quad \text{and} \quad \Omega_{is}(g) \in \mathscr{K} \qquad \text{for} \quad g \in \overline{\Gamma}_i \cap \mathscr{K}.$$

Thus, the transformations Ω_{is} map the curves $\Gamma_i \cap \mathscr{O}_\varepsilon(\mathscr{K})$ strictly into the domain G and the set of their endpoints $\overline{\Gamma}_i \cap \mathscr{K}$ into itself.

Denote by Ω_{is}^{+1} the transformation $\Omega_{is}\colon \mathscr{O}_i \to \Omega_{is}(\mathscr{O}_i)$, and by $\Omega_{is}^{-1}\colon \Omega_{is}(\mathscr{O}_i) \to \mathscr{O}_i$ its inverse transformation. The set of point

$$\Omega_{i_q s_q}^{\pm 1}(\dots(\Omega_{i_1 s_1}^{\pm 1}(g))) \in \mathscr{K}, \quad 1 \leq s_j \leq S_{i_j}, \qquad j = 1, \dots, q,$$

is called the *orbit* of a point $g \in \mathscr{K}$. In other words, the orbit of the point g consists of points (from the set $\mathscr{K}$) that can be obtained by successive application of the transformations $\Omega_{i_j s_j}^{\pm 1}$ to the point g. The set $\mathscr{K}$ consists of a finite number of nonintersecting orbits, which are denoted by $\mathscr{K}_\nu$, $\nu = 1, \dots, N_0$.

Consider a sufficiently small $\varepsilon > 0$ for which there exist neighborhoods $\mathscr{O}_{\varepsilon_1}(g_j)$, $\mathscr{O}_{\varepsilon_1}(g_j) \supset \mathscr{O}_\varepsilon(g_j)$ satisfying the following conditions:

1) the domain G coincides with the plane angle in the neighborhood $\mathscr{O}_{\varepsilon_1}(g_j)$;

2) $\overline{\mathscr{O}_{\varepsilon_1}(g)} \cap \overline{\mathscr{O}_{\varepsilon_1}(h)} = \varnothing$ for all $g, h \in \mathscr{K}$, $g \neq h$;

3) if $g_j \in \overline{\Gamma}_i$ and $\Omega_{is}(g_j) = g_k$,, then $\mathscr{O}_\varepsilon(g_j) \subset \mathscr{O}_i$ and $\Omega_{is}(\mathscr{O}_\varepsilon(g_j)) \subset \mathscr{O}_{\varepsilon_1}(g_k)$.

For each point $g_j \in \overline{\Gamma}_i \cap \mathscr{K}_\nu$, let us fix a linear transformation $Y_j\colon y \mapsto y'(g_j)$ (the composition of operators of translation by the vector $-\overrightarrow{Og_j}$ and of rotation) mapping the point g_j into the origin so that

$$Y_j(\mathscr{O}_{\varepsilon_1}(g_j)) = \mathscr{O}_{\varepsilon_1}(0), \qquad Y_j(G \cap \mathscr{O}_{\varepsilon_1}(g_j)) = K_j \cap \mathscr{O}_{\varepsilon_1}(0),$$
$$Y_j(\Gamma_i \cap \mathscr{O}_{\varepsilon_1}(g_j)) = \gamma_{j\sigma} \cap \mathscr{O}_{\varepsilon_1}(0) \quad (\sigma = 1 \text{ or } 2)$$

where K_j is a plane nonzero angle with sides $\gamma_{j\sigma}$.

Condition 2.2. Suppose that $g_j \in \overline{\Gamma}_i \cap \mathscr{K}_\nu$ and $\Omega_{is}(g_j) = g_k \in \mathscr{K}_\nu$; then the transformation

$$Y_k \circ \Omega_{is} \circ Y_j^{-1}\colon \mathscr{O}_\varepsilon(0) \to \mathscr{O}_{\varepsilon_1}(0)$$

is the composition of operators of rotation and homothety centered at the origin.

[1] Here and elsewhere, the operator P_0 acts in the sense of distributions.

We introduce the nonlocal operators $\mathbf{B}_i$ defined by the formulas

$$\mathbf{B}_i u = \sum_{s=1}^{S_i} b_{is}(y) u(\Omega_{is}(y)), \quad y \in \Gamma_i \cap \mathscr{O}_\varepsilon(\mathscr{K}), \qquad \mathbf{B}_i u = 0, \quad y \in \Gamma_i \setminus \mathscr{O}_\varepsilon(\mathscr{K}), \tag{2.1}$$

where $b_{is} \in C^\infty(\mathbb{R}^2)$ are real-valued functions, $\operatorname{supp} b_{is} \subset \mathscr{O}_\varepsilon(\mathscr{K})$.

Condition 2.3. The following estimates hold:

1) $b_{is}(y) \geq 0$, $\sum_{s=1}^{S_i} b_{is}(y) \leq 1$, $y \in \overline{\Gamma}_i$;

2) $\sum_{s=1}^{S_i} b_{is}(g) + \sum_{s=1}^{S_j} b_{js}(g) < 2$, $g \in \overline{\Gamma}_i \cap \overline{\Gamma}_j \subset \mathscr{K}$, if $i \neq j$ and $\overline{\Gamma}_i \cap \overline{\Gamma}_j \neq \varnothing$.

Let us state some auxiliary results which will be needed in the following sections.

For all closed sets $Q \subset \overline{G}$ and $K \subset \overline{G}$ such that $Q \cap K \neq \varnothing$, we introduce the space

$$C_K(Q) = \{u \in C(Q) : u(y) = 0,\ y \in Q \cap K\} \tag{2.2}$$

with a maximum norm. Consider the space of vector functions

$$\mathscr{C}_{\mathscr{K}}(\partial G) = \prod_{i=1}^{N} C_{\mathscr{K}}(\overline{\Gamma}_i)$$

with norm

$$\|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)} = \max_{i=1,\dots,N} \max_{y \in \overline{\Gamma}_i} \|\psi_i\|_{C(\overline{\Gamma}_i)}, \qquad \text{where} \quad \psi = \{\psi_i\}, \quad \psi_i \in C_{\mathscr{K}}(\overline{\Gamma}_i).$$

Consider the nonlocal problem

$$P_0 u - qu = f_0(y), \quad y \in G, \qquad u|_{\Gamma_i} - \mathbf{B}_i u = \psi_i(y), \ \ y \in \Gamma_i, \quad i = 1, \dots, N. \tag{2.3}$$

Theorem 2.1 (see Theorem 4.1 in [13]). *Let conditions* 2.1–2.3 *hold. Then one can find a number* $q_1 > 0$ *such that, for all* $f_0 \in C(\overline{G})$, $\psi = \{\psi_i\} \in \mathscr{C}_{\mathscr{K}}(\partial G)$, *and* $q \geq q_1$, *there exists a unique solution* $u \in C_{\mathscr{K}}(\overline{G}) \cap W^2_{2,\mathrm{loc}}(G)$ *of problem* (2.3)*. Besides, if* $f_0 = 0$, *then* $u \in C_{\mathscr{K}}(\overline{G}) \cap C^\infty(G)$ *and the following estimate holds:*

$$\|u\|_{C_{\mathscr{K}}(\overline{G})} \leq c_1 \|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)}, \tag{2.4}$$

where $c_1 > 0$ *is independent of* ψ *and* q.

Suppose that $u \in C^\infty(G) \cap C_{\mathscr{K}}(\overline{G})$ is a solution of problem (2.3) with $f_0 = 0$ and $\psi = \{\psi_i\} \in \mathscr{C}_{\mathscr{K}}(\partial G)$. Denote $u = \mathbf{S}_q \psi$. By Theorem 2.1, the operator

$$\mathbf{S}_q \colon \mathscr{C}_{\mathscr{K}}(\partial G) \to C_{\mathscr{K}}(\overline{G}), \qquad q \geq q_1,$$

is bounded and $\|\mathbf{S}_q\| \leq c_1$, where $c_1 > 0$ is independent of q.

Lemma 2.1. *Let conditions* 2.1–2.3 *hold; let* Q_1 *and* Q_2 *be closed sets such that* $Q_1 \subset \partial G$, $Q_2 \subset \overline{G}$, *and* $Q_1 \cap Q_2 = \varnothing$, *and let* $q \geq q_1$. *Then, for all* $\psi \in \mathscr{C}_{\mathscr{K}}(\partial G)$ *such that* $\operatorname{supp}(\mathbf{S}_q \psi)|_{\partial G} \subset Q_1$, *the following inequality holds:*

$$\|\mathbf{S}_q \psi\|_{C(Q_2)} \leq \frac{c_2}{q} \|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)}, \qquad q \geq q_1,$$

where $c_2 > 0$ *is independent of* ψ *and* q.

Proof. Using[2] Lemma 1.3 from [2] and Theorem 2.1, we obtain

$$\|\mathbf{S}_q\psi\|_{C(Q_2)} \le \frac{k}{q}\|(\mathbf{S}_q\psi)|_{\partial G}\|_{C(\partial G)} \le \frac{k}{q}\|\mathbf{S}_q\psi\|_{C(\overline{G})} \le \frac{kc_1}{q}\|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)}, \qquad q \ge q_1, \tag{2.5}$$

where the number q_1 from Theorem 2.1 is assumed sufficiently large (so that Lemma 1.3 from [10] holds for $q \ge q_1$) and the number $k = k(q_1)$ is independent of ψ and q. □

Lemma 2.2. *Let conditions* 2.1–2.3 *hold, let Q_1 and Q_2 be the same as in Lemma* 2.1, *and let $q \ge q_1$. Also let $Q_2 \cap \mathscr{K} = \varnothing$. Then, for all $\psi \in \mathscr{C}_{\mathscr{K}}(\partial G)$ such that* $\operatorname{supp}\psi \subset Q_1$, *the following inequality holds:*

$$\|\mathbf{S}_q\psi\|_{C(Q_2)} \le \frac{c_3}{q}\|\psi\|_{\mathscr{C}_{\mathscr{K}}(Q_1)}, \qquad q \ge q_1,$$

where $c_3 > 0$ is independent of ψ and q.

Proof. 1. Consider a number $\sigma > 0$ such that

$$\operatorname{dist}(Q_1, Q_2) > 3\sigma, \qquad \operatorname{dist}(\mathscr{K}, Q_2) > 3\sigma. \tag{2.6}$$

We introduce a function $\xi \in C^\infty(\mathbb{R}^2)$ such that $0 \le \xi(y) \le 1$, $\xi(y) = 1$ if $\operatorname{dist}(y, Q_2) \le \sigma$, and $\xi(y) = 0$ if $\operatorname{dist}(y, Q_2) \ge 2\sigma$.

Consider the auxiliary problem

$$P_0 v - qv = 0, \quad y \in G, \qquad v(y) = \xi(y)u(y), \quad y \in \partial G, \tag{2.7}$$

where $u = \mathbf{S}_q\psi \in C_{\mathscr{K}}(\overline{G})$. Applying Theorem 2.1 (with $\mathbf{B}_i = 0$), we see that there exists a unique solution $v \in C^\infty(G) \cap C_{\mathscr{K}}(\overline{G})$ of problem (2.7). The maximum principle 2.1 and the definition of the function ξ imply

$$\|v\|_{C(\overline{G})} \le \|\xi u\|_{C(\partial G)} \le \max_{i=1,\dots,N} \|u|_{Q_{2,2\sigma}\cap\overline{\Gamma}_i}\|_{C(Q_{2,2\sigma}\cap\overline{\Gamma}_i)}, \tag{2.8}$$

where $Q_{2,2\sigma} = \{y \in \partial G\colon \operatorname{dist}(y, Q_2) \le 2\sigma\}$.

Since $\operatorname{supp}\psi \cap Q_{2,2\sigma} = \varnothing$, we have

$$u - \mathbf{B}_i u = 0, \qquad y \in Q_{2,2\sigma} \cap \overline{\Gamma}_i. \tag{2.9}$$

Since $\mathbf{B}_i u = 0$ for $y \notin \mathscr{O}_\varepsilon(\mathscr{K})$, it follows from (2.9) that

$$u(y) = 0, \qquad y \in [Q_{2,2\sigma} \cap \overline{\Gamma}_i] \setminus \mathscr{O}_\varepsilon(\mathscr{K}). \tag{2.10}$$

Using (2.8)–(2.10), the definition of the operators $\mathbf{B}_i$, and condition 2.3, we obtain

$$\begin{aligned} \|v\|_{C(\overline{G})} &\le \max_{i=1,\dots,N} \|u|_{Q_{2,2\sigma}\cap\overline{\Gamma}_i\cap\overline{\mathscr{O}_\varepsilon(\mathscr{K})}}\|_{C(Q_{2,2\sigma}\cap\overline{\Gamma}_i\cap\overline{\mathscr{O}_\varepsilon(\mathscr{K})})} \\ &\le \max_{i=1,\dots,N}\max_{s=1,\dots,S_i} \|u|_{\Omega_{is}(Q_{2,2\sigma}\cap\overline{\Gamma}_i\cap\overline{\mathscr{O}_\varepsilon(\mathscr{K})})}\|_{C(\Omega_{is}(Q_{2,2\sigma}\cap\overline{\Gamma}_i\cap\overline{\mathscr{O}_\varepsilon(\mathscr{K})}))}. \end{aligned} \tag{2.11}$$

Since $Q_{2,2\sigma} \cap \mathscr{K} = \varnothing$ (see (2.6)), it follows from the definition of the transformations Ω_{is} that

$$\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathscr{O}_\varepsilon(\mathscr{K})}) \subset G.$$

Therefore, using inequalities (2.11) and Lemma 2.1 in which the role of Q_1 and Q_2 is played by the sets ∂G and $\Omega_{is}(Q_{2,2\sigma} \cap \overline{\Gamma}_i \cap \overline{\mathscr{O}_\varepsilon(\mathscr{K})})$, we obtain

$$\|v\|_{C(\overline{G})} \le \frac{c_2}{q}\|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)}. \tag{2.12}$$

[2] In Lemma 1.3 from [10], it is assumed that the boundary of the domain is infinitely smooth. This assumption is used to prove the existence of classical solutions of elliptic equations with inhomogeneous boundary conditions. However, if it is known that a classical solution exists, then, in the proof of of the first inequality in (2.5), the assumption on the smoothness of the boundary can be dropped.

2. Set $w = u - v$. Obviously, the function w satisfies the relations

$$P_0 w - qw = 0, \quad y \in G, \qquad w(y) = u(y) - v(y) = 0, \quad y \in Q_{2,\sigma}.$$

Applying Lemma 2.1 (with $\mathbf{B}_i = 0$) in which the role of Q_1 is played by the set $\overline{\partial G \setminus Q_{2,\sigma}}$, and taking the relation $w|_{\partial G} = (1-\xi)u|_{\partial G}$ into account, we obtain

$$\|w\|_{C(Q_2)} \le \frac{c_2}{q}\|w|_{\partial G}\|_{C(\partial G)} \le \frac{c_2}{q}\|u\|_{C(\overline{G})}.$$

The last inequality and Theorem 2.1 imply

$$\|w\|_{C(Q_2)} \le \frac{c_2 c_1}{q}\|\psi\|_{\mathscr{C}_{\mathscr{K}}(\partial G)}.$$

Combining this estimate with (2.12), we obtain the proof. □

3. BOUNDED PERTURBATIONS OF ELLIPTIC OPERATORS AND THEIR PROPERTIES

Consider a linear operator P_1 satisfying the following property.

Condition 3.1. The operator $P_1\colon C(\overline{G}) \to C(\overline{G})$ is bounded, and if the function $u \in C(\overline{G})$ attains a positive maximum at a point $y^0 \in G$, then $P_1 u(y^0) \le 0$.

The operator P_1 plays the role of a bounded perturbation of unbounded elliptic operators in spaces of continuous functions (see [10], [6]).

The following result is a consequence of conditions 2.1 and 3.1 and the maximum principle 2.1.

Lemma 3.1. *Let conditions* 2.1 *and* 3.1 *hold. If the function* $u \in C(\overline{G})$ *attains a positive maximum at a point* $y^0 \in G$ *and* $P_0 u \in C(G)$*, then*

$$P_0 u(y^0) + P_1 u(y^0) \le 0.$$

In the present paper, we consider the following nonlocal conditions in the *nontransversal* case:

$$b(y)u(y) + \int_{\overline{G}} [u(y) - u(\eta)]\,\mu(y, d\eta) = 0, \qquad y \in \partial G, \tag{3.1}$$

where $b(y) \ge 0$ and $\mu(y, \cdot)$ is a nonnegative Borel measure on $\overline{G}$.

Set $\mathscr{N} = \{y \in \partial G\colon \mu(y, \overline{G}) = 0\}$ and $\mathscr{M} = \partial G \setminus \mathscr{N}$. We assume that $\mathscr{N}$ and $\mathscr{M}$ are Borel sets.

Condition 3.2. The set $\mathscr{K}$ is contained in $\mathscr{N}$.

We introduce the function $b_0(y) = b(y) + \mu(y, \overline{G})$.

Condition 3.3. The estimate $b_0(y) > 0$ holds for $y \in \partial G$.

In view of conditions 3.2 and 3.3, relation (3.1) can be written as

$$u(y) - \int_{\overline{G}} u(\eta)\,\mu_i(y, d\eta) = 0, \quad y \in \Gamma_i, \qquad u(y) = 0, \quad y \in \mathscr{K}, \tag{3.2}$$

where $\mu_i(y, \cdot) = \mu(y, \cdot)/b_0(y)$, $y \in \Gamma_i$. By the definition of the function $b_0(y)$, we have

$$\mu_i(y, \overline{G}) \le 1, \qquad y \in \Gamma_i. \tag{3.3}$$

For any set Q, denote by $\chi_Q(y)$ the function, equal to one on Q and zero on $\mathbb{R}^2 \setminus Q$.

Suppose that $b_{is}(y)$ and Ω_{is} are the same as above. We introduce the measures δ_{is} as follows:

$$\delta_{is}(y, Q) = \begin{cases} b_{is}(y)\chi_Q(\Omega_{is}(y)), & y \in \Gamma_i \cap \mathscr{O}_\varepsilon(\mathscr{K}), \\ 0, & y \in \Gamma_i \setminus \mathscr{O}_\varepsilon(\mathscr{K}), \end{cases}$$

where Q is an arbitrary Borel set.

We study the measures $\mu_i(y,\,\cdot\,)$ expressible as

$$\mu_i(y,\,\cdot\,) = \sum_{s=1}^{S_i} \delta_{is}(y,\,\cdot\,) + \alpha_i(y,\,\cdot\,) + \beta_i(y,\,\cdot\,), \qquad y \in \Gamma_i, \tag{3.4}$$

where $\alpha_i(y,\,\cdot\,)$ and $\beta_i(y,\,\cdot\,)$ are nonnegative Borel measures satisfying the conditions given below (see [10], [6]).

For any Borel measure $\mu(y,\,\cdot\,)$, the closed set

$$\operatorname{spt}\mu(y,\,\cdot\,) = \overline{G} \setminus \bigcup_{V\in T} \{V \in T : \mu(y, V \cap \overline{G}) = 0\}$$

(where T is the set of all open sets in $\mathbb{R}^2$) is called the *support* of the measure $\mu(y,\,\cdot\,)$.

Condition 3.4. There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\sigma > 0$ such that

1) $\operatorname{spt}\alpha_i(y,\,\cdot\,) \subset \overline{G} \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})$ for $y \in \Gamma_i$,

2) $\operatorname{spt}\alpha_i(y,\,\cdot\,) \subset \overline{G}_\sigma$ for $y \in \Gamma_i \setminus \mathscr{O}_{\varkappa_2}(\mathscr{K})$,

where $\mathscr{O}_{\varkappa_1}(\mathscr{K}) = \{y \in \mathbb{R}^2 : \operatorname{dist}(y, \mathscr{K}) < \varkappa_1\}$ and $G_\sigma = \{y \in G : \operatorname{dist}(y, \partial G) < \sigma\}$.

Condition 3.5. The estimate $\beta_i(y, \mathscr{M}) < 1$ holds for $y \in \Gamma_i \cap \mathscr{M}$, $i = 1, \dots, N$.

Remark 3.1. Condition 3.5 is weaker than similar conditions 2.2 in [10] and 3.2 in [6] in which it is required that $\mu_i(y, \mathscr{M}) < 1$ for $y \in \Gamma_i \cap \mathscr{M}$.

Remark 3.2. We can show that if conditions 3.3–3.5 hold, then

$$b(y) + \mu(y, \overline{G} \setminus \{y\}) > 0, \qquad y \in \partial G,$$

i.e., the boundary condition (3.1) is defined at each point of the boundary.

Using relations (3.4), we write the nonlocal conditions (3.2) in the form

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i, \qquad u(y) = 0, \quad y \in \mathscr{K}, \tag{3.5}$$

where the operators $\mathbf{B}_i$ are defined in (2.1),

$$\mathbf{B}_{\alpha i} u(y) = \int_{\overline{G}} u(\eta)\, \alpha_i(y, d\eta), \quad \mathbf{B}_{\beta i} u(y) = \int_{\overline{G}} u(\eta)\, \beta_i(y, d\eta), \qquad y \in \Gamma_i.$$

We introduce the space[3]

$$C_B(\overline{G}) = \{u \in C(\overline{G}) : u \text{ satisfy } (3.1)\}.$$

The definition of the space $C_B(\overline{G})$ and condition 3.2 imply[4]

$$C_B(\overline{G}) \subset C_{\mathscr{N}}(\overline{G}) \subset C_{\mathscr{K}}(G). \tag{3.6}$$

Lemma 3.2. *Let conditions* 2.1–2.3 *and* 3.1–3.5 *hold. Let the function* $u \in C_B(\overline{G})$ *attain a positive maximum at a point* $y^0 \in \overline{G}$*, and let* $P_0 u \in C(G)$*. Then there exists a a point* $y^1 \in G$ *such that*

$$u(y^1) = u(y^0) \qquad \text{and} \qquad P_0 u(y^1) + P_1 u(y^1) \le 0.$$

[3] Obviously, in the definition of the space $C_B(\overline{G})$, we can also use conditions (3.2) or (3.5).
[4] The spaces $C_{\mathscr{N}}(\,\cdot\,)$ and $C_{\mathscr{K}}(\,\cdot\,)$ are defined in (2.2).

Proof. 1. If $y^0 \in G$, then the assertion of the lemma follows from Lemma 3.1. Let $y^0 \in \partial G$. Suppose that the lemma is false, i.e., $u(y^0) > u(y)$ for all $y \in G$.

Since $u(y^0) > 0$ and $u \in C_B(\overline{G}) \subset C_{\mathscr{N}}(\overline{G})$, it follows that $y^0 \in \mathscr{M}$. Suppose that $y^0 \in \Gamma_i \cap \mathscr{M}$ for some i. If $\mu_i(y^0, G) > 0$, then, taking (3.3) into account, we obtain

$$u(y^0) - \int_{\overline{G}} u(\eta)\, \mu_i(y^0, d\eta) \geq \int_G [u(y^0) - u(\eta)]\, \mu_i(y^0, d\eta) > 0,$$

which contradicts (3.2). Therefore, $\operatorname{spt} \mu_i(y^0, \cdot\,) \subset \partial G$. Hence it follows from (3.4) and condition 3.4 (part 1) that

$$b_{is}(y^0) = 0, \qquad \operatorname{spt} \alpha_i(y^0, \cdot\,) \subset \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K}), \qquad \operatorname{spt} \beta_i(y^0, \cdot\,) \subset \partial G. \tag{3.7}$$

2. Suppose that $\alpha_i(y^0, \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})) = 0$. In this case, by (3.7),

$$\alpha_i(y^0, \overline{G}) = 0. \tag{3.8}$$

Further, from (3.4), (3.7), (3.8) and condition 3.5, we obtain

$$\mu_i(y^0, \cdot\,) = \beta_i(y^0, \cdot\,), \qquad \operatorname{spt} \beta_i(y^0, \cdot\,) \subset \partial G, \qquad \beta_i(y^0, \mathscr{M}) < 1.$$

Therefore, for $u \in C_B(\overline{G}) \subset C_{\mathscr{N}}(\overline{G})$ the following relations hold:

$$u(y^0) - \int_{\overline{G}} u(\eta)\, \mu_i(y^0, d\eta) = u(y^0) - \int_{\mathscr{M}} u(\eta)\, \beta_i(y^0, d\eta) \geq u(y^0) - u(y^0)\beta_i(y^0, \mathscr{M}) > 0,$$

which contradicts (3.2).

The resulting contradiction implies $\alpha_i(y^0, \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})) > 0$. Thus, taking condition 3.4 (part 2) into account, we obtain $y^0 \in \mathscr{O}_{\varkappa_2}(\mathscr{K})$.

3. Let us show that there exists a point

$$y' \in \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K}) \tag{3.9}$$

such that $u(y') = u(y^0)$. Assume the converse: $u(y^0) > u(y)$ for $y \in \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})$. Then, using (3.3), (3.4), and (3.7), we obtain

$$\begin{aligned} u(y^0) - \int_{\overline{G}} u(\eta)\, \mu_i(y^0, d\eta) &\geq \int_{\overline{G}} [u(y^0) - u(\eta)]\, \mu_i(y^0, d\eta) \\ &\geq \int_{\partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})} [u(y^0) - u(\eta)]\, \alpha_i(y^0, d\eta) > 0, \end{aligned} \tag{3.10}$$

because $\alpha_i(y^0, \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})) > 0$. Inequality (3.10) contradicts (3.2). Therefore, the function u attains a positive maximum at some point $y' \in \partial G \setminus \mathscr{O}_{\varkappa_1}(\mathscr{K})$. Repeating the arguments from parts 1 and 2 of the proof, we find $y' \in \mathscr{O}_{\varkappa_2}(\mathscr{K})$. But this contradicts (3.9).

Thus, we have proved the existence of a point $y^1 \in G$ such that $u(y^1) = u(y^0)$. Applying Lemma 3.1, we obtain $P_0 u(y^1) + P_1 u(y^1) \leq 0$. □

Corollary 3.1. *Let conditions* 2.1–2.3 *and* 3.1–3.5 *hold. Let* $u \in C_B(\overline{G})$ *be the solution of the equation*

$$qu(y) - P_0 u(y) - P_1 u(y) = f_0(y), \qquad y \in G,$$

where $q > 0$ *and* $f_0 \in C(\overline{G})$*. Then*

$$\|u\|_{C(\overline{G})} \leq \frac{1}{q} \|f_0\|_{C(\overline{G})}. \tag{3.11}$$

Proof. Suppose that

$$\max_{y\in\overline{G}} |u(y)| = u(y^0) > 0$$

for some $y^0 \in \overline{G}$. Then, by Lemma 3.2, there exists a point $y^1 \in G$ such that

$$u(y^1) = u(y^0) \qquad \text{and} \qquad P_0u(y^1) + P_1u(y^1) \le 0.$$

Therefore,

$$\|u\|_{C(\overline{G})} = u(y^0) = u(y^1) = \frac{1}{q}(P_0u(y^1) + P_1u(y^1) + f_0(y^1)) \le \frac{1}{q}\|f_0\|_{C(\overline{G})}. \quad \square$$

4. REDUCTION TO AN OPERATOR EQUATION ON THE BOUNDARY

In this section, we impose some additional constraints on nonlocal operators that reduce nonlocal elliptic problems to operator equations on the boundary.

Note that if $u \in C_{\mathscr{N}}(\overline{G})$, then the function $\mathbf{B}_i u$ is continuous on Γ_i and can be continued to a continuous function on $\overline{\Gamma}_i$ belonging to $C_{\mathscr{N}}(\overline{\Gamma}_i)$ (this function will also be denoted by $\mathbf{B}_i u$). Suppose that the operators $\mathbf{B}_{\alpha i}$ and $\mathbf{B}_{\beta i}$ possess a similar property.

Condition 4.1. For any $u \in C_{\mathscr{N}}(\overline{G})$, the functions $\mathbf{B}_{\alpha i}u$ and $\mathbf{B}_{\beta i}u$ can be continued to $\overline{\Gamma}_i$ so that the extended functions (also denoted by $\mathbf{B}_{\alpha i}u$ and $\mathbf{B}_{\beta i}u$, respectively,) belong to $C_{\mathscr{N}}(\overline{\Gamma}_i)$.

The following lemma immediately follows from the definition of nonlocal operators.

Lemma 4.1. *Let conditions* 2.2*,* 2.3*,* 3.2*,* 3.3*, and* 4.1 *hold. Then the operators*

$$\mathbf{B}_i,\ \mathbf{B}_{\alpha i},\ \mathbf{B}_{\beta i}:\ C_{\mathscr{N}}(\overline{G}) \to C_{\mathscr{N}}(\overline{\Gamma}_i)$$

are bounded and

$$\|\mathbf{B}_i u\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \|u\|_{C_{\mathscr{N}}(\overline{G})}, \quad \|\mathbf{B}_{\alpha i} u\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \|u\|_{C_{\mathscr{N}}(\overline{G}\setminus\mathscr{O}_{\varkappa_1}(\mathscr{K}))}, \quad \|\mathbf{B}_{\beta i} u\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \|u\|_{C_{\mathscr{N}}(\overline{G})},$$
$$\|\mathbf{B}_{\alpha i} u + \mathbf{B}_{\beta i} u\| \le \|u\|_{C_{\mathscr{N}}(\overline{G})}, \qquad \|\mathbf{B}_i u + \mathbf{B}_{\alpha i} u + \mathbf{B}_{\beta i} u\| \le \|u\|_{C_{\mathscr{N}}(\overline{G})}.$$

Consider the space of vector functions

$$\mathscr{C}_{\mathscr{N}}(\partial G) = \prod_{i=1}^{N} C_{\mathscr{N}}(\overline{\Gamma}_i)$$

with norm

$$\|\psi\|_{\mathscr{C}_{\mathscr{N}}(\partial G)} = \max_{i=1,\dots,N}\max_{y\in\overline{\Gamma}_i} \|\psi_i\|_{C(\overline{\Gamma}_i)}, \qquad \psi = \{\psi_i\}, \quad \psi_i \in C_{\mathscr{N}}(\overline{\Gamma}_i).$$

We introduce the operators

$$\mathbf{B} = \{\mathbf{B}_i\}\colon C_{\mathscr{N}}(\overline{G}) \to \mathscr{C}_{\mathscr{N}}(\partial G), \qquad \mathbf{B}_{\alpha\beta} = \{\mathbf{B}_{\alpha i} + \mathbf{B}_{\beta i}\}\colon C_{\mathscr{N}}(\overline{G}) \to \mathscr{C}_{\mathscr{N}}(\partial G). \tag{4.1}$$

Using the operator $\mathbf{S}_q$ defined in Sec. 2, we introduce the bounded operator

$$\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q\colon \mathscr{C}_{\mathscr{N}}(\partial G) \to \mathscr{C}_{\mathscr{N}}(\partial G), \qquad q \ge q_1. \tag{4.2}$$

Since $\mathbf{S}_q\psi \in C_{\mathscr{N}}(\overline{G})$ for $\psi \in \mathscr{C}_{\mathscr{N}}(\partial G)$, the operator in (4.2) is well defined.

Further, we state sufficient conditions guaranteeing the existence of the bounded operator

$$(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1}\colon \mathscr{C}_{\mathscr{N}}(\partial G) \to \mathscr{C}_{\mathscr{N}}(\partial G).$$

Let us express the measures $\beta_i(y,\cdot)$ as

$$\beta_i(y,\cdot) = \beta_i^1(y,\cdot) + \beta_i^2(y,\cdot), \tag{4.3}$$

where $\beta_i^1(y,\cdot)$ and $\beta_i^2(y,\cdot)$ are nonnegative Borel measures. Let us describe them. For each $p>0$, consider the covering of the set $\overline{\mathscr{M}}$ by p-neighborhoods of all of its points. Denote by $\mathscr{M}_p$ some finite subcovering. Obviously, $\mathscr{M}_p$ is an open Borel set. Further, for each $p>0$, consider a cut-off function $\widehat{\zeta}_p \in C^\infty(\mathbb{R}^2)$ such that $0 \le \widehat{\zeta}_p(y) \le 1$, $\widehat{\zeta}_p(y)=1$ if $y \in \mathscr{M}_{p/2}$ and $\widehat{\zeta}_p(y)=0$ if $y \notin \mathscr{M}_p$. Set $\widetilde{\zeta}_p = 1-\widehat{\zeta}_p$. We introduce the operators

$$\widehat{\mathbf{B}}^1_{\beta i}u(y) = \int_{\overline{G}} \widehat{\zeta}_p(\eta)u(\eta)\,\beta_i^1(y,d\eta), \qquad \widetilde{\mathbf{B}}^1_{\beta i}u(y) = \int_{\overline{G}} \widetilde{\zeta}_p(\eta)u(\eta)\,\beta_i^1(y,d\eta),$$

$$\mathbf{B}^2_{\beta i}u(y) = \int_{\overline{G}} u(\eta)\,\beta_i^2(y,d\eta).$$

Condition 4.2. For all $i=1,\dots,N$, we have

1) the operators $\widehat{\mathbf{B}}^1_{\beta i}, \widetilde{\mathbf{B}}^1_{\beta i}\colon C_{\mathscr{N}}(\overline{G}) \to C_{\mathscr{N}}(\overline{\Gamma}_i)$ are bounded;

2) there exists a number $p>0$ such that[5]

$$\|\widehat{\mathbf{B}}^1_{\beta i}\| < \begin{cases} \dfrac{1}{c_1} & \text{if } \alpha_j(y,\overline{G})=0 \ \ \forall\, y\in\Gamma_j,\ j=1,\dots,N, \\ \dfrac{1}{c_1(1+c_1)} & \text{otherwise,} \end{cases}$$

where c_1 is the constant from Theorem 2.1.

Remark 4.1. The operators $\widehat{\mathbf{B}}^1_{\beta i}, \widetilde{\mathbf{B}}^1_{\beta i}\colon C_{\mathscr{N}}(\overline{G}) \to C_{\mathscr{N}}(\overline{\Gamma}_i)$ are bounded if and only if the operator $\widehat{\mathbf{B}}^1_{\beta i}+\widetilde{\mathbf{B}}^1_{\beta i}\colon C_{\mathscr{N}}(\overline{G}) \to C_{\mathscr{N}}(\overline{\Gamma}_i)$ is bounded. This it follows from the relations

$$\widehat{\mathbf{B}}^1_{\beta i}u = (\widehat{\mathbf{B}}^1_{\beta i}+\widetilde{\mathbf{B}}^1_{\beta i})(\widehat{\zeta}_p u) \qquad \text{and} \qquad \widetilde{\mathbf{B}}^1_{\beta i}u = (\widehat{\mathbf{B}}^1_{\beta i}+\widetilde{\mathbf{B}}^1_{\beta i})(\widetilde{\zeta}_p u)$$

and the continuity of the functions $\widehat{\zeta}_p$ and $\widetilde{\zeta}_p$.

Condition 4.3. The operators

$$\mathbf{B}^2_{\beta i}\colon C_{\mathscr{N}}(\overline{G}) \to C_{\mathscr{N}}(\overline{\Gamma}_i), \qquad i=1,\dots,N,$$

are compact.

It follows from (3.4) and (4.3) that the measures $\mu_i(y,\cdot)$ can be expressed as

$$\mu_i(y,\cdot) = \sum_{s=1}^{S_i} \delta_{is}(y,\cdot) + \alpha_i(y,\cdot) + \beta_i^1(y,\cdot) + \beta_i^2(y,\cdot), \qquad y\in\Gamma_i.$$

The measures $\delta_{is}(y,\cdot)$ correspond to nonlocal terms with support near the set $\mathscr{K}$ of points of conjugation. The measures $\alpha_i(y,\cdot)$ correspond to nonlocal terms with support outside the set $\mathscr{K}$. The measures $\beta_i^1(y,\cdot)$ and $\beta_i^2(y,\cdot)$ correspond to nonlocal terms whose support has an arbitrary geometric structure (in particular, it can intersect with the set $\mathscr{K}$); however, for small p, the measure $\beta_i^1(y,\mathscr{M}_p)$ of the set $\mathscr{M}_p$ must be small (condition 4.2) and the measure $\beta_i^2(y,\cdot)$ must generate a compact operator (condition 4.3).

Lemma 4.2. *Let conditions* 2.1–2.3*,* 3.1–3.5*, and* 4.1–4.3 *hold. Then there exists a bounded operator*

$$(\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1}\colon \mathscr{C}_{\mathscr{N}}(\partial G) \to \mathscr{C}_{\mathscr{N}}(\partial G), q \ge q_1,$$

where $q_1>0$ is sufficiently large.

[5] Part 2 of condition 4.2 can be replaced by the stronger assumption: "$\|\widehat{\mathbf{B}}^1_{\beta i}\| \to 0$ as $p\to 0$," which is simpler to verify in specific examples.

Proof. 1. Consider the bounded operators

$$\widehat{\mathbf{B}}^1_\beta = \{\widehat{\mathbf{B}}^1_{\beta i}\}, \qquad \widetilde{\mathbf{B}}^1_\beta = \{\widetilde{\mathbf{B}}^1_{\beta i}\}, \qquad \mathbf{B}^2_\beta = \{\mathbf{B}^2_{\beta i}\}, \qquad \text{and} \qquad \mathbf{B}_\alpha = \{\mathbf{B}_{\alpha i}\},$$

acting from $C_\mathscr{N}(\overline{G})$ to $\mathscr{C}_\mathscr{N}(\partial G)$ (see (4.1)).

Let us prove that the operator

$$\mathbf{I} - \mathbf{B}_\alpha \mathbf{S}_q \colon \mathscr{C}_\mathscr{N}(\partial G) \to \mathscr{C}_\mathscr{N}(\partial G)$$

has a bounded inverse.

We introduce a function $\zeta \in C^\infty(\overline{G})$ such that $0 \le \zeta(y) \le 1$, $\zeta(y) = 1$ for $y \in \overline{G_\sigma}$ and $\zeta(y) = 0$ for $y \notin G_{\sigma/2}$, where $\sigma > 0$ is a number from condition 3.4.

We have

$$\mathbf{I} - \mathbf{B}_\alpha \mathbf{S}_q = \mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q - \mathbf{B}_\alpha \zeta \mathbf{S}_q. \tag{4.4}$$

1a. First, let us show that the operator $\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q$ has a bounded inverse. By Lemma 4.1 and Theorem 2.1, we have

$$\|\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\| \le c_1. \tag{4.5}$$

Further, $(1-\zeta)\mathbf{S}_q\psi = 0$ in $\overline{G_\sigma}$ for any $\psi \in \mathscr{C}_\mathscr{N}(\partial G)$. Therefore, by condition 3.4, we have

$$\operatorname{supp} \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\psi \subset \partial G \cap \overline{\mathscr{O}_{\varkappa_2}(\mathscr{K})}. \tag{4.6}$$

Let us show that

$$\|[\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q]^2\| \le \frac{c}{q}, \qquad q \ge q_1, \tag{4.7}$$

where $q_1 > 0$ is sufficiently large and $c > 0$ is independent of q. Successively applying

(I) Lemma 4.1;

(II) Lemma 2.2 and relation (4.6);

(III) Lemma 4.1 and Theorem 2.1,

we obtain

$$\begin{aligned} \|\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\psi\|_{\mathscr{C}_\mathscr{N}(\partial G)} &\le \|\mathbf{S}_q\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\psi\|_{C_\mathscr{N}(\overline{G}\setminus\mathscr{O}_{\varkappa_1}(\mathscr{K}))} \\ &\le \frac{c_3}{q}\|\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q\psi\|_{C_\mathscr{N}(\partial G\cap\overline{\mathscr{O}_{\varkappa_2}(\mathscr{K})})} \\ &\le \frac{c_3c_1}{q}\|\psi\|_{\mathscr{C}_\mathscr{N}(\partial G)}. \end{aligned}$$

This implies (4.7) with $c = c_3c_1$.

If $q \ge 2c$, then the operator $\mathbf{I} - [\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q]^2$ has a bounded inverse. In that case, the operator $\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q$ also has a bounded inverse and

$$[\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q]^{-1} = [\mathbf{I} + \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q][\mathbf{I} - (\mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q)^2]^{-1}. \tag{4.8}$$

Relation (4.8), Lemma 4.1, Theorem 2.1, and relations (4.5) and (4.7) imply

$$\|[\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q]^{-1}\| = 1 + c_1 + O(q^{-1}), \qquad q \to +\infty. \tag{4.9}$$

1b. Let us now estimate the norm of the operator $\mathbf{B}_\alpha\zeta\mathbf{S}_q$. From Lemma 4.1 and 2.2, we obtain

$$\|\mathbf{B}_\alpha\zeta\mathbf{S}_q\psi\|_{\mathscr{C}_\mathscr{N}(\partial G)} \le \|\mathbf{S}_q\psi\|_{C(\overline{G}_{\sigma/2})} \le \frac{c_2}{q}\|\psi\|_{\mathscr{C}_\mathscr{N}(\partial G)}. \tag{4.10}$$

Therefore, using expression (4.4), we see that the operator $\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q$ has a bounded inverse for all sufficiently large q and

$$(\mathbf{I} - \mathbf{B}_\alpha\mathbf{S}_q)^{-1} = [\mathbf{I} - (\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q)^{-1}\mathbf{B}_\alpha\zeta\mathbf{S}_q]^{-1}[\mathbf{I} - \mathbf{B}_\alpha(1-\zeta)\mathbf{S}_q]^{-1}. \tag{4.11}$$

From (4.9)–(4.11), we obtain

$$\|(\mathbf{I}-\mathbf{B}_\alpha\mathbf{S}_q)^{-1}\| = 1 + c_1 + O(q^{-1}), \qquad q \to +\infty. \tag{4.12}$$

2. Let us prove that the operator $\mathbf{I}-(\mathbf{B}_\alpha+\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\colon \mathscr{C}_{\mathscr{N}}(\partial G)\to\mathscr{C}_{\mathscr{N}}(\partial G)$ has a bounded inverse.

2a. It follows from the definition of the operator $\widetilde{\mathbf{B}}_\beta^1$ and Lemma 2.1 (with $Q_1=\overline{\mathscr{M}}$ and $Q_2=\overline{G}\setminus\mathscr{M}_{p/2}$) that

$$\|\widetilde{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \|\mathbf{S}_q\psi\|_{C(\overline{G}\setminus\mathscr{M}_{p/2})} \le \frac{c_2}{q}\|\psi\|_{\mathscr{C}_{\mathscr{N}}(\partial G)}, \tag{4.13}$$

because $(\overline{G}\setminus\mathscr{M}_{p/2})\cap\overline{\mathscr{M}}=\varnothing$ and $\operatorname{supp}(\mathbf{S}_q\psi)|_{\partial G}\subset\overline{\mathscr{M}}$ for $\psi\in\mathscr{C}_{\mathscr{N}}(\partial G)$.

2b. Suppose that $\alpha_j(y,\overline{G})\ne 0$ for some j and $y\in\Gamma_j$. By condition 4.2 (part 2) and Theorem 2.1, there exists a number d such that $0<2d<1/(1+c_1)$ and

$$\|\widehat{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \biggl(\frac{1}{c_1(1+c_1)}-\frac{2d}{c_1}\biggr)\|\mathbf{S}_q\psi\|_{C_{\mathscr{N}}(\overline{G})} \le \biggl(\frac{1}{1+c_1}-2d\biggr)\|\psi\|_{\mathscr{C}_{\mathscr{N}}(\partial G)}. \tag{4.14}$$

From inequalities (4.13) and (4.14), we obtain

$$\|(\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| \le \frac{1}{1+c_1}-d \tag{4.15}$$

for all sufficiently large q. From (4.12) and (4.15), we obtain

$$\|(\mathbf{I}-\mathbf{B}_\alpha\mathbf{S}_q)^{-1}(\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| < 1$$

for sufficiently large q. Therefore, the bounded inverse operator

$$[\mathbf{I}-(\mathbf{B}_\alpha+\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q]^{-1} = [\mathbf{I}-(\mathbf{I}-\mathbf{B}_\alpha\mathbf{S}_q)^{-1}(\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q]^{-1}[\mathbf{I}-\mathbf{B}_\alpha\mathbf{S}_q]^{-1} \tag{4.16}$$

exists.

2c. If $\alpha_j(y,\overline{G})=0$ for $y\in\Gamma_j$, $j=1,\dots,N$, then, in view of condition 4.2 (part 1), inequality (4.14) takes the form

$$\|\widehat{\mathbf{B}}_{\beta i}^1\mathbf{S}_q\psi\|_{C_{\mathscr{N}}(\overline{\Gamma}_i)} \le \biggl(\frac{1}{c_1}-\frac{2d}{c_1}\biggr)\|\mathbf{S}_q\psi\|_{C_{\mathscr{N}}(\overline{G})} \le (1-2d)\|\psi\|_{\mathscr{C}_{\mathscr{N}}(\partial G)}.$$

Therefore, inequality (4.15) can be written as

$$\|(\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q\| \le 1-d. \tag{4.17}$$

Since, in this case, $\mathbf{B}_\alpha=0$, it follows from (4.17) that the operator

$$\mathbf{I}-(\mathbf{B}_\alpha+\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q = \mathbf{I}-(\widehat{\mathbf{B}}_\beta^1+\widetilde{\mathbf{B}}_\beta^1)\mathbf{S}_q$$

has a bounded inverse.

3. It remains to to prove that the operator $\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q$ also has a bounded inverse. By condition 4.3, the operator $\mathbf{B}_\beta^2$ is compact. Therefore, the operator $\mathbf{B}_\beta^2\mathbf{S}_q$ is also compact. Since the index of a Fredholm operator is stable with respect to compact perturbations it follows that the operator $\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q$ is Fredholm and $\operatorname{ind}(\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q)=0$. To prove that $\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q$ has a bounded inverse, it suffices to show that

$$\dim\ker(\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q)=0.$$

Suppose that $\psi\in\mathscr{C}_{\mathscr{N}}(\partial G)$ and $(\mathbf{I}-\mathbf{B}_{\alpha\beta}\mathbf{S}_q)\psi=0$. Then the function $u=\mathbf{S}_q\psi\in C^\infty(G)\cap C_{\mathscr{N}}(\overline{G})$ is a solution of the problem

$$P_0u-qu=0, \qquad y\in G,$$
$$u(y)-\mathbf{B}_iu(y)-\mathbf{B}_{\alpha i}u(y)-\mathbf{B}_{\beta i}u(y)=0, \quad y\in\Gamma_i, \qquad u(y)=0, \quad y\in\mathscr{K}.$$

By Corollary 3.1, we have $u=0$. Hence

$$\psi=\mathbf{B}_{\alpha\beta}\mathbf{S}_q\psi=\mathbf{B}_{\alpha\beta}u=0. \quad \square$$

5. EXISTENCE OF FELLER SEMIGROUPS

In this section, we prove that bounded perturbations of elliptic operators with nonlocal terms satisfying conditions of Secs. 2–4, are the generators of Feller semigroups.

Reducing nonlocal problems to the boundary and using Lemma 4.2, we can prove that nonlocal problems are solvable in spaces of continuous functions.

Lemma 5.1. *Let conditions* 2.1–2.3*,* 3.2–3.5*, and* 4.1–4.3 *hold, and let* $q_1 > 0$ *be sufficiently large. Then, for all* $q \geq q_1$ *and* $f_0 \in C(\overline{G})$*, the problem*

$$qu(y) - P_0 u(y) = f_0(y), \qquad y \in G, \tag{5.1}$$

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i, \qquad u(y) = 0, \quad y \in \mathscr{K}, \tag{5.2}$$

has a unique solution $u \in C_B(\overline{G}) \cap W^2_{2,\mathrm{loc}}(G)$*.*

Proof. 1. Consider the auxiliary problem

$$qv(y) - P_0 v(y) = f_0(y), \quad y \in G, \qquad v(y) - \mathbf{B}_i v(y) = 0, \quad y \in \Gamma_i, \quad i = 1, \dots, N. \tag{5.3}$$

Since $f_0 \in C(\overline{G})$, it follows from Theorem 2.1 that there exists a unique solution $v \in C_{\mathscr{K}}(\overline{G})$ of problem (5.3). Therefore, $v \in C_{\mathscr{N}}(\overline{G})$.

2. Set $w = u - v$. The unknown function w belongs to $C_{\mathscr{N}}(\overline{G})$ and, by (5.1)–(5.3), satisfies the relations

$$\begin{gathered} qw(y) - P_0 w(y) = 0, \qquad y \in G, \\ w(y) - \mathbf{B}_i w(y) - \mathbf{B}_{\alpha i} w(y) - \mathbf{B}_{\beta i} w(y) = \mathbf{B}_{\alpha i} v(y) + \mathbf{B}_{\beta i} v(y), \qquad y \in \Gamma_i, \quad i = 1, \dots, N, \\ w(y) = 0, \qquad y \in \mathscr{K}. \end{gathered} \tag{5.4}$$

In view of condition 4.1, problem (5.4) is equivalent to the operator equation

$$\psi - \mathbf{B}_{\alpha\beta} \mathbf{S}_q \psi = \mathbf{B}_{\alpha\beta} v$$

with respect to the unknown function $\psi \in \mathscr{C}_{\mathscr{N}}(\partial G)$. By Lemma 4.2, this equation has a unique solution $\psi \in \mathscr{C}_{\mathscr{N}}(\partial G)$. Then problem (5.1), (5.2) also has a unique solution, namely,

$$u = v + w = v + \mathbf{S}_q \psi = v + \mathbf{S}_q (\mathbf{I} - \mathbf{B}_{\alpha\beta} \mathbf{S}_q)^{-1} \mathbf{B}_{\alpha\beta} v \in C_B(\overline{G}).$$

Moreover, $u \in W^2_{2,\mathrm{loc}}(G)$ by the theorem on the inner smoothness of solutions of elliptic equations. □

Using Lemma 5.1 and condition 3.1, we can prove that problems with bounded perturbations are also solvable in spaces of continuous functions.

Lemma 5.2. *Let conditions* 2.1–2.3*,* 3.1–3.5*, and* 4.1–4.3 *hold, and let* $q_1 > 0$ *be sufficiently large. Then, for all* $q \geq q_1$ *and* $f_0 \in C(\overline{G})$*, the problem*

$$qu - (P_0 + P_1)u = f_0(y), \qquad y \in G, \tag{5.5}$$

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \quad y \in \Gamma_i, \qquad u(y) = 0, \quad y \in \mathscr{K}, \tag{5.6}$$

has a unique solution $u \in C_B(\overline{G}) \cap W^2_{2,\mathrm{loc}}(G)$*.*

Proof. Consider the operator $qI - P_0$ as an operator acting from $C(\overline{G})$ to $C(\overline{G})$ whose domain of definition is

$$\mathrm{D}(qI - P_0) = \{u \in C_B(\overline{G}) \cap W^2_{2,\mathrm{loc}}(G) : P_0 u \in C(\overline{G})\}.$$

By Lemma 5.1 and Corollary 3.1, there exists a bounded operator

$$(qI - P_0)^{-1} \colon C(\overline{G}) \to C(\overline{G})$$

and, moreover,

$$\|(qI - P_0)^{-1}\| \le \frac{1}{q}.$$

We introduce the operator

$$qI - P_0 - P_1 \colon C(\overline{G}) \to C(\overline{G})$$

whose domain of definition is

$$\mathrm{D}(qI - P_0 - P_1) = \mathrm{D}(qI - P_0).$$

Since

$$qI - P_0 - P_1 = (I - P_1(qI - P_0)^{-1})(qI - P_0),$$

it follows that, for $q \ge q_1$, the operator

$$qI - P_0 - P_1 \colon C(\overline{G}) \to C(\overline{G})$$

has a bounded inverse; here q_1 is so large that

$$\|P_1\| \cdot \|(qI - P_0)^{-1}\| \le 1/2, \qquad q \ge q_1.$$

□

Consider the unbounded operator

$$\mathbf{P}_B \colon \mathrm{D}(\mathbf{P}_B) \subset C_B(\overline{G}) \to C_B(\overline{G})$$

defined by the formula

$$\mathbf{P}_B u = P_0 u + P_1 u, \quad u \in \mathrm{D}(\mathbf{P}_B) = \{u \in C_B(\overline{G}) \cap W^2_{2,\mathrm{loc}}(G) \colon P_0 u + P_1 u \in C_B(\overline{G})\}. \tag{5.7}$$

Lemma 5.3. *Let conditions* 2.1–2.3, 3.1–3.5, *and* 4.1–4.3 *hold. Then* $\mathrm{D}(\mathbf{P}_B)$ *is dense in* $C_B(\overline{G})$.

Proof. We shall carry out the proof using the scheme proposed in [9].

1. Suppose that $u \in C_B(\overline{G})$. Since $C_B(\overline{G}) \subset C_{\mathscr{N}}(\overline{G})$ by (3.6), it follows that, for all $\varepsilon > 0$ and $q \ge q_1$, there exists a function

$$u_1 \in C^\infty(\overline{G}) \cap C_{\mathscr{N}}(\overline{G})$$

such that

$$\|u - u_1\|_{C(\overline{G})} \le \min\left(\varepsilon, \frac{\varepsilon}{2c_1 k_q}\right), \tag{5.8}$$

where $k_q = \|(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1}\|$.

Set

$$\begin{gathered} f_0(y) \equiv q u_1 - P_0 u_1, \qquad y \in G, \\ \psi_i(y) \equiv u_1(y) - \mathbf{B}_i u_1(y) - \mathbf{B}_{\alpha i} u_1(y) - \mathbf{B}_{\beta i} u_1(y), \qquad y \in \Gamma_i, \quad i = 1, \dots, N. \end{gathered} \tag{5.9}$$

Since $u_1 \in C_{\mathscr{N}}(\overline{G})$, by condition 4.1, we have $\{\psi_i\} \in \mathscr{C}_{\mathscr{N}}(\partial G)$. Using the relation

$$u(y) - \mathbf{B}_i u(y) - \mathbf{B}_{\alpha i} u(y) - \mathbf{B}_{\beta i} u(y) = 0, \qquad y \in \Gamma_i,$$

inequality (5.8), and Lemma 4.1, we obtain

$$\|\{\psi_i\}\|_{\mathscr{C}_{\mathscr{N}}(\partial G)} \le \|u - u_1\|_{C(\overline{G})} + \|(\mathbf{B} + \mathbf{B}_{\alpha\beta})(u - u_1)\|_{\mathscr{C}_{\mathscr{N}}(\partial G)} \le \frac{\varepsilon}{c_1 k_q}. \tag{5.10}$$

Consider the auxiliary nonlocal problem

$$\begin{gathered} q u_2 - P_0 u_2 = f_0(y), \qquad y \in G, \\ u_2(y) - \mathbf{B}_i u_2(y) - \mathbf{B}_{\alpha i} u_2(y) - \mathbf{B}_{\beta i} u_2(y) = 0, \quad y \in \Gamma_i, \qquad u_2(y) = 0, \quad y \in \mathscr{K}. \end{gathered} \tag{5.11}$$

Since $f_0 \in C^\infty(\overline{G})$, by Lemma 5.1, problem (5.11) has the unique solution

$$u_2 \in C_B(\overline{G}) \subset C_{\mathcal{N}}(\overline{G}).$$

Using (5.9), (5.11), and the relations $u_1(y) = u_2(y) = 0$, $y \in \mathcal{K}$, we see that the function $w_1 = u_1 - u_2$ satisfies the relations

$$\begin{gathered} qw_1 - P_0 w_1 = 0, \qquad y \in G, \\ w_1(y) - \mathbf{B}_i w_1(y) - \mathbf{B}_{\alpha i} w_1(y) - \mathbf{B}_{\beta i} w_1(y) = \psi_i(y), \quad y \in \Gamma_i, \qquad w_1(y) = 0, \quad y \in \mathcal{K}. \end{gathered} \tag{5.12}$$

It follows from condition 4.1 that problem (5.12) is equivalent the operator equation

$$\phi - \mathbf{B}_{\alpha\beta}\mathbf{S}_q \phi = \psi$$

in $\mathscr{C}_{\mathcal{N}}(\partial G)$, where $w_1 = \mathbf{S}_q \phi$. By Lemma 4.2, this equation has a unique solution $\phi \in \mathscr{C}_{\mathcal{N}}(\partial G)$. Thus, using Theorem 2.1 and inequality (5.10), we obtain

$$\|w_1\|_{C(\overline{G})} \le c_1 \|(\mathbf{I} - \mathbf{B}_{\alpha\beta}\mathbf{S}_q)^{-1}\| \cdot \|\{\psi_i\}\|_{\mathscr{C}_{\mathcal{N}}(\partial G)} \le c_1 \frac{k_q \varepsilon}{c_1 k_q} = \varepsilon. \tag{5.13}$$

2. Finally, consider the problem

$$\begin{gathered} \lambda u_3 - P_0 u_3 - P_1 u_3 = \lambda u_2, \qquad y \in G, \\ u_3(y) - \mathbf{B}_i u_3(y) - \mathbf{B}_{\alpha i} u_3(y) - \mathbf{B}_{\beta i} u_3(y) = 0, \quad y \in \Gamma_i, \qquad u_3(y) = 0, \quad y \in \mathcal{K}. \end{gathered} \tag{5.14}$$

Since $u_2 \in C_B(\overline{G})$, by Lemma 5.2, problem (5.14) has a unique solution $u_3 \in \mathrm{D}(\mathbf{P}_B)$ for all sufficiently large λ.

Denote $w_2 = u_2 - u_3$. It follows from (5.2) that

$$\lambda w_2 - P_0 w_2 - P_1 w_2 = -P_0 u_2 - P_1 u_2 = f_0 - q u_2 - P_1 u_2.$$

Using Corollary 3.1, we find

$$\|w_2\|_{C(\overline{G})} \le \frac{1}{\lambda} \|f_0 - q u_2 - P_1 u_2\|_{C(\overline{G})}.$$

Choosing a sufficiently large λ, we obtain

$$\|w_2\|_{C(\overline{G})} \le \varepsilon. \tag{5.15}$$

It follows from inequalities (5.8), (5.13), and (5.15) that

$$\|u - u_3\|_{C(\overline{G})} \le \|u - u_1\|_{C(\overline{G})} + \|u_1 - u_2\|_{C(\overline{G})} + \|u_2 - u_3\|_{C(\overline{G})} \le 3\varepsilon.$$

□

Let us now prove the main result of this paper.

Theorem 5.1. *Let conditions* 2.1–2.3*,* 3.1–3.5*, and* 4.1–4.3 *hold. Then the operator*

$$\mathbf{P}_B\colon\ \mathrm{D}(\mathbf{P}_B) \subset C_B(\overline{G}) \ \to C_B(\overline{G})$$

is the generator of a Feller semigroup.

Proof. 1. By Lemma 5.2 and Corollary 3.1, for all sufficiently large $q > 0$, there exists a bounded operator

$$(qI - \mathbf{P}_B)^{-1}\colon C_B(\overline{G}) \to C_B(\overline{G})$$

such that

$$\|(qI - \mathbf{P}_B)^{-1}\| \le \frac{1}{q}.$$

2. Since the operator $(qI - \mathbf{P}_B)^{-1}$ is bounded and defined on the whole space $C_B(\overline{G})$, it is closed. Therefore, the operator

$$qI - \mathbf{P}_B \colon \mathrm{D}(\mathbf{P}_B) \subset C_B(\overline{G}) \to C_B(\overline{G})$$

is closed. Hence the operator

$$\mathbf{P}_B \colon \mathrm{D}(\mathbf{P}_B) \subset C_B(\overline{G}) \to C_B(\overline{G})$$

is also closed.

3. Let us prove that the operator $(qI - \mathbf{P}_B)^{-1}$ is nonnegative. Assume the converse; then there exists a function $f_0 \ge 0$ such that the solution $u \in \mathrm{D}(\mathbf{P}_B)$ of the equation $qu - \mathbf{P}_B u = f_0$ attains a negative minimum at some point $y^0 \in \overline{G}$. In this case, the function $v = -u$ attains a positive maximum at the point y^0. By Lemma 3.2, there exists a point $y^1 \in G$ such that $v(y^1) = v(y^0)$ and $\mathbf{P}_B v(y^1) \le 0$. Therefore,

$$0 < v(y^0) = v(y^1) = (\mathbf{P}_B v(y^1) - f_0(y^1))/q \le 0.$$

The resulting contradiction proves that $u \ge 0$.

Thus, all the assumptions of the Hille–Yosida theorem (Theorem 1.1) hold, and the operator

$$\mathbf{P}_B \colon \mathrm{D}(\mathbf{P}_B) \subset C_B(\overline{G}) \to C_B(\overline{G})$$

is the generator of a Feller semigroup. □

In conclusion, let us present an example of nonlocal operators satisfying the assumptions of the present paper.

Suppose that $G \subset \mathbb{R}^2$ is a bounded domain with boundary

$$\partial G = \Gamma_1 \cup \Gamma_2 \cup \mathcal{K},$$

where Γ_1 and Γ_2 are open connected (in the topology of ∂G) curves of the class C^∞; moreover, $\Gamma_1 \cap \Gamma_2 = \varnothing$ and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \mathcal{K}$; the set $\mathcal{K}$ consists of two points: g_1 and g_2. Suppose that the domain G coincides with a plane angle in an ε-neighborhood of the points g_i, $i = 1, 2$. Let Ω_j, $j = 1, \dots, 4$, be continuous transformations defined on $\overline{\Gamma}_1$ and satisfying the following conditions (see the figure):

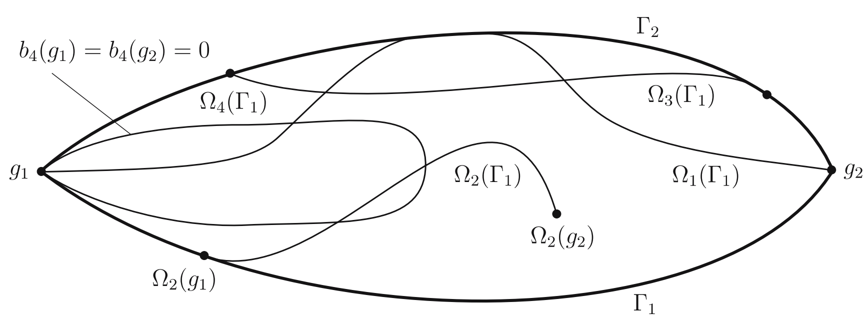

Figure.

1) $\Omega_1(\mathcal{K}) \subset \mathcal{K}$, $\Omega_1(\Gamma_1 \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$, $\Omega_1(\Gamma_1 \setminus \mathcal{O}_\varepsilon(\mathcal{K})) \subset G \cup \Gamma_2$, and $\Omega_1(y)$ is the composition of operators of translation of the argument, of rotation, and of homothety for $y \in \overline{\Gamma}_1 \cap \mathcal{O}_\varepsilon(\mathcal{K})$;

2) there exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\sigma > 0$ such that

$$\Omega_2(\overline{\Gamma}_1) \subset \overline{G} \setminus \mathcal{O}_{\varkappa_1}(\mathcal{K}) \qquad \text{and} \qquad \Omega_2(\overline{\Gamma}_1 \setminus \mathcal{O}_{\varkappa_2}(\mathcal{K})) \subset \overline{G}_\sigma;$$

besides, $\Omega_2(g_1) \in \Gamma_1$ and $\Omega_2(g_2) \in G$;

3) $\Omega_3(\overline{\Gamma}_1) \subset G \cup \Gamma_2$ and $\Omega_3(\mathcal{K}) \subset \Gamma_2$;

4) $\Omega_4(\overline{\Gamma}_1) \subset G \cup \overline{\Gamma}_2$ and $\Omega_4(\mathscr{K}) \subset \mathscr{K}$.

Suppose that $b_1 \in C(\overline{\Gamma}_1) \cap C^\infty(\overline{\Gamma}_1 \cap \mathscr{O}_\varepsilon(\mathscr{K}))$, $b_2, b_3, b_4 \in C(\overline{\Gamma}_1)$, and $b_j \geq 0$, $j = 1, \dots, 4$.

Suppose that G_1 is a bounded domain, $G_1 \subset G$, and $\Gamma \subset \overline{G}$ is a curve of class C^1. We introduce continuous nonnegative functions $c(y, \eta)$, $y \in \overline{\Gamma}_1$, $\eta \in \overline{G}_1$, and $d(y, \eta)$, $y \in \overline{\Gamma}_1$, $\eta \in \overline{\Gamma}$.

Consider the following nonlocal conditions:

$$
\begin{gathered}
u(y) - \sum_{j=1}^{4} b_j(y) u(\Omega_j(y)) - \int_{G_1} c(y, \eta) u(\eta)\, d\eta - \int_{\Gamma} d(y, \eta) u(\eta)\, d\Gamma_\eta = 0, \qquad y \in \Gamma_1, \\
u(y) = 0, \qquad y \in \overline{\Gamma}_2.
\end{gathered}
\tag{5.16}
$$

Suppose that $Q \subset \overline{G}$ is an an arbitrary Borel set; let us introduce the measures $\mu(y, \cdot\,)$, $y \in \partial G$:

$$
\begin{gathered}
\mu(y, Q) = \sum_{j=1}^{4} b_j(y) \chi_Q(\Omega_j(y)) + \int_{G_1 \cap Q} c(y, \eta)\, d\eta + \int_{\Gamma \cap Q} d(y, \eta) u(\eta)\, d\Gamma_\eta, \qquad y \in \Gamma_1, \\
\mu(y, Q) = 0, \qquad y \in \overline{\Gamma}_2.
\end{gathered}
$$

Suppose that $\mathscr{N}$ and $\mathscr{M}$ are defined as above. Suppose that

$$
\mu(y, \overline{G}) = \sum_{j=1}^{4} b_j(y) + \int_{G_1} c(y, \eta)\, d\eta + \int_{\Gamma} d(y, \eta)\, d\Gamma_\eta \leq 1, \qquad y \in \partial G,
$$

$$
\int_{\Gamma \cap \mathscr{M}} d(y, \eta)\, d\Gamma_\eta < 1, \qquad y \in \mathscr{M};
$$

$$
\begin{gathered}
b_2(g_1) = 0 \qquad \text{or} \qquad \mu(\Omega_2(g_1), \overline{G}) = 0, \quad b_2(g_2) = 0, \quad b_4(g_j) = 0; \\
c(g_j, \cdot\,) = 0, \qquad d(g_j, \cdot\,) = 0.
\end{gathered}
$$

Setting $b(y) = 1 - \mu(y, \overline{G})$, we can rewrite (5.16) as (see (3.1))

$$
b(y) u(y) + \int_{\overline{G}} [u(y) - u(\eta)]\, \mu(y, d\eta) = 0, \qquad y \in \partial G.
$$

We introduce a cut-off function $\zeta \in C^\infty(\mathbb{R}^2)$ with support in $\mathscr{O}_\varepsilon(\mathscr{K})$ equal to 1 on $\mathscr{O}_{\varepsilon/2}(\mathscr{K})$ and such that $0 \leq \zeta(y) \leq 1$ for $y \in \mathbb{R}^2$. Suppose that $y \in \overline{\Gamma}_1$ and $Q \subset \overline{G}$ is an an arbitrary Borel set; denote

$$
\begin{gathered}
\delta(y, Q) = \zeta(y) b_1(y) \chi_Q(\Omega_1(y)), \qquad \alpha(y, Q) = b_2(y) \chi_Q(\Omega_2(y)), \\
\beta^1(y, Q) = (1 - \zeta(y)) b_1(y) \chi_Q(\Omega_1(y)) + \sum_{j=3,4} b_j(y) \chi_Q(\Omega_j(y)), \\
\beta^2(y, Q) = \int_{G_1 \cap Q} c(y, \eta)\, d\eta + \int_{\Gamma \cap Q} d(y, \eta) u(\eta)\, d\Gamma_\eta
\end{gathered}
$$

(for simplicity, we omit the subscript "1" in the notation of the measures). It is readily verified that these measures satisfy conditions 2.2, 2.3, 3.2–3.5, and 4.1–4.3.

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REFERENCES

1. W. Feller, "The parabolic differential equations and the associated semi-groups of transformations," Ann. of Math. (2) **55** (3), 468–519 (1952).
2. W. Feller, "Diffusion processes in one dimension," Trans. Amer. Math. Soc. **77** (1), 1–31 (1954).
3. A. D. Venttsel′, "On boundary conditions for multidimensional diffusion processes," Teor. Veroyatnost. i Primenen. **4** (2), 172–185 (1959) [Theory Probab. Appl. **4** (2), 164–177 (1960)].
4. K. Taira, *Semigroups, Boundary-Value Problems, and Markov Processes*, in *Springer Monographs in Mathematics* (Springer-Verlag, Berlin, 2004).
5. J. M. Bony, P. Courrége, and P. Priouret, "Semi-groups de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum," Ann. Inst. Fourier (Grenoble) **18** (2), 369–521 (1968).
6. E. I. Galakhov and A. L. Skubachevskii, "On Feller semigroups generated by elliptic operators with integro-differential boundary conditions," J. Differential Equations **176** (2), 315–355 (2001).
7. Y. Ishikawa, "A remark on the existence of a diffusion process with non-local boundary conditions," J. Math. Soc. Japan **42** (1), 171–184 (1990).
8. K. Sato and T. Ueno, "Multi-dimensional diffusion and the Markov process on the boundary," J. Math. Kyoto Univ. **4**, 529–605 (1965).
9. K. Taira, *Diffusion Processes and Partial Differential Equations* (Academic Press, Boston, MA, 1988).
10. E. I. Galakhov and A. L. Skubachevskii, "On contraction nonnegative semigroups with nonlocal conditions," Mat. Sb. **189** (1), 45–78 (1998) [Russian Acad. Sci. Sb. Math. **189** (1), 43–74 (1998)].
11. A. L. Skubachevskii, "On some problems for multidimensional diffusion processes," Dokl. Akad. Nauk SSSR **307** (2), 287–291 (1989) [Soviet Math. Dokl. **40** (1), 75–79 (1990)].
12. A. L. Skubachevskii, "Nonlocal elliptic problems and multidimensional diffusion processes," Russian J. Math. Phys. **3** (3), 327–360 (1995).
13. P. L. Gurevich, "Elliptic equations with nonlocal conditions near points of conjugation in spaces of continuous functions," in *Trudy Mat. Inst. Steklov* [*Proc. Steklov Inst. Math.*] (in press).
14. D. Gilbarg and M. Trudinger, *Elliptic Differential Equations of Second Order* (Nauka, Moscow, 1989; Springer-Verlag, Berlin, 1983).