Solvability of Nonlocal Elliptic Problems in Sobolev Spaces, II

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Abstract. This is the second part of the paper (for the first part, see *Russ. J. Math. Phys.*, vol. 10, no. 4, pp. 436–466; the numbering of the sections continues that of part one). We study elliptic equations of order $2m$ with nonlocal boundary-value conditions in plane bounded domains for the case in which the support of nonlocal terms can nontrivially intersect the boundary. We give necessary and sufficient conditions for nonlocal problems to have the Fredholm property in Sobolev spaces and in weighted spaces with small weight exponents, respectively. We also find the asymptotic behavior of solutions of nonlocal problems near the conjugation points on the boundary, where solutions can have power-law singularities.

CONTENTS

4. NONLOCAL PROBLEMS IN BOUNDED DOMAINS FOR THE CASE IN WHICH THE LINE $\operatorname{Im}\lambda = 1 - l - 2m$ CONTAINS NO EIGENVALUES OF $\tilde{\mathcal{L}}_p(\lambda)$

In this section, using the results of Sec. 2, we construct a right regularizer for the operator

$$\mathbf{L} = \{\mathbf{P},\ \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + \mathbf{B}_{i\mu}^2\} : W^{l+2m}(G) \to \mathcal{W}^l(G,\Upsilon)$$

(see Sec. 1.1) corresponding to problem (1.7), (1.8). It follows from the existence of a right regularizer that the image of $\mathbf{L}$ is closed and of finite codimension. To prove that the kernel of $\mathbf{L}$ is of finite dimension, we will reduce $\mathbf{L}$ to an operator between weighted spaces such that the kernel of the reduced operator is finite-dimensional.

We write $\mathbf{B}^k = \{\mathbf{B}_{i\mu}^k\}_{i,\mu}$, $k = 0,\ldots,2$; $\mathbf{B} = \mathbf{B}^0 + \mathbf{B}^1 + \mathbf{B}^2$, $\mathbf{C} = \mathbf{B}^0 + \mathbf{B}^1$. Along with the nonlocal operator $\mathbf{L} = \{\mathbf{P},\ \mathbf{B}\}$, we consider the bounded operators

$$\mathbf{L}^1 = \{\mathbf{P},\ \mathbf{C}\} : W^{l+2m}(G) \to \mathcal{W}^l(G,\Upsilon) \quad \text{and} \quad \mathbf{L}^0 = \{\mathbf{P},\ \mathbf{B}^0\} : W^{l+2m}(G) \to \mathcal{W}^l(G,\Upsilon).$$

We first consider the operator $\mathbf{L}^1$ (i.e., we suppose that $\mathbf{B}_{i\mu}^2 = 0$) and then proceed with the study of the operator $\mathbf{L}$ for the general case in which $\mathbf{B}_{i\mu}^2 \neq 0$. Throughout the section, we assume that the following condition holds.

Condition 4.1. For each orbit Orb_p, $p = 1,\ldots,N_1$, the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains no eigenvalues of the corresponding operator $\tilde{\mathcal{L}}_p(\lambda)$.

4.1. *Construction of a Right Regularizer When $\mathbf{B}_{i\mu}^2 = 0$*

In this subsection we discuss the situation with $\mathbf{B}_{i\mu}^2 = 0$, i.e., the case in which the support of nonlocal terms is concentrated near the set $\mathcal{K}$.

For each curve Υ_i $(i = 1,\ldots,N_0)$, denote the endpoints of Υ_i by g_{i1} and g_{i2}. Recall that the domain G has the form of a plane angle in some neighborhood of the point g_{i1} (g_{i2}), while the curve Υ_i coincides there with a segment I_{i1} (I_{i2}). Let τ_{i1} (τ_{i2}) be the unit vector parallel to the segment I_{i1} (I_{i2}).

Let $\mathcal{S}_1^l(G,\Upsilon)$ be the set consisting of the functions $f = \{f_0, f_{i\mu}\} \in \mathcal{W}^l(G,\Upsilon)$ satisfying the following relations:

$$D^\alpha f_0(y) = 0 \ (y \in \mathcal{K}), \quad |\alpha| \leqslant l - 2, \tag{4.1}$$

$$\left.\frac{\partial^\beta f_{i\mu}}{\partial \tau_{i1}^\beta}\right|_{y=g_{i1}} = 0, \quad \left.\frac{\partial^\beta f_{i\mu}}{\partial \tau_{i2}^\beta}\right|_{y=g_{i2}} = 0, \quad \beta \leqslant l + 2m - m_{i\mu} - 2. \tag{4.2}$$

It follows from Sobolev's embedding theorem and Riesz' theorem on the general form of a continuous linear functional in a Hilbert space that $\mathcal{S}_1^l(G,\Upsilon)$ is a closed subset of the space $\mathcal{W}^l(G,\Upsilon)$ and the codimension of $\mathcal{S}_1^l(G,\Upsilon)$ in $\mathcal{W}^l(G,\Upsilon)$ is finite.

Lemma 4.1. *Let Condition* 4.1 *hold. If the number ε_0 is sufficiently small, then there exist a bounded operator* $\mathbf{R}_1 : \mathcal{S}_1^l(G,\Upsilon) \to W^{l+2m}(G)$ *and a compact operator* $\mathbf{T}_1 : \mathcal{S}_1^l(G,\Upsilon) \to \mathcal{S}_1^l(G,\Upsilon)$ *such that*

$$\mathbf{L}^1 \mathbf{R}_1 = \mathbf{I}_1 + \mathbf{T}_1, \tag{4.3}$$

where $\mathbf{I}_1$ stands for the identity operator in $\mathcal{S}_1^l(G,\Upsilon)$.

Proof. 1. By Theorem 2.1, there exist bounded operators

$$\mathbf{R}_\mathcal{K} : \{f \in \mathcal{S}_1^l(G,\Upsilon) : \operatorname{supp} f \subset \mathcal{O}_{2\varepsilon_0}(\mathcal{K})\} \to W^{l+2m}(G),$$

$$\mathbf{M}_\mathcal{K}, \mathbf{T}_\mathcal{K} : \{f \in \mathcal{S}_1^l(G,\Upsilon) : \operatorname{supp} f \subset \mathcal{O}_{2\varepsilon_0}(\mathcal{K})\} \to \mathcal{S}_1^l(G,\Upsilon)$$

such that $\|\mathbf{M}_{\mathcal{K}}f\|_{\mathcal{W}^l(G,\Upsilon)} \leqslant c\varepsilon_0\|f\|_{\mathcal{W}^l(G,\Upsilon)}$, where $c > 0$ does not depend on ε_0, the operator $\mathbf{T}_{\mathcal{K}}$ is compact, and

$$\mathbf{L}^1\mathbf{R}_{\mathcal{K}}f = f + \mathbf{M}_{\mathcal{K}}f + \mathbf{T}_{\mathcal{K}}f. \tag{4.4}$$

2. For each point $g \in \bar{G}\setminus\mathcal{O}_{2\varepsilon_0}(\mathcal{K})$, we consider its $(\varepsilon_0/2)$-neighborhood $\mathcal{O}_{\varepsilon_0/2}(g)$. The family of these neighborhoods, together with the set $\mathcal{O}_{2\varepsilon_0}(\mathcal{K})$, covers $\bar{G}$. Let us choose a finite subcovering $\mathcal{O}_{2\varepsilon_0}(\mathcal{K})$, $\mathcal{O}_{\varepsilon_0/2}(g_j)$, $j = 1, \ldots, J = J(\varepsilon_0)$. Let $\psi, \psi_j \in C_0^\infty(\mathbb{R}^2)$, $j = 1, \ldots, J$, be a partition of unity subordinated to the covering $\mathcal{O}_{2\varepsilon_0}(\mathcal{K})$, $\mathcal{O}_{\varepsilon_0/2}(g_j)$, $j = 1, \ldots, J$.

According to the general theory of elliptic boundary-value problems in smooth domains (see, e.g., [27]), there exist bounded operators

$$\mathbf{R}_{0j} : \{f \in \mathcal{W}^l(G,\Upsilon) : \text{supp}\, f \subset \mathcal{O}_{\varepsilon_0/2}(g_j)\} \to \{u \in W^{l+2m}(G) : \text{supp}\, u \subset \mathcal{O}_{\varepsilon_0}(g_j)\} \tag{4.5}$$

and compact operators

$$\mathbf{T}_{0j} : \{f \in \mathcal{W}^l(G,\Upsilon) : \text{supp}\, f \subset \mathcal{O}_{\varepsilon_0/2}(g_j)\} \to \{f \in \mathcal{W}^l(G,\Upsilon) : \text{supp}\, f \subset \mathcal{O}_{\varepsilon_0}(g_j)\}$$

such that

$$\mathbf{L}^0\mathbf{R}_{0j}f = f + \mathbf{T}_{0j}f. \tag{4.6}$$

3. For any $f \in \mathcal{S}_1^l(G,\Upsilon)$, set

$$\mathbf{R}_0 f = \sum_{j=1}^{J} \mathbf{R}_{0j}(\psi_j f)$$

and $\hat{\mathbf{R}}_1 f = \mathbf{R}_{\mathcal{K}}(\psi f) + \mathbf{R}_0 f$.

In this case,

$$\mathbf{P}\hat{\mathbf{R}}_1 f = \mathbf{P}\mathbf{R}_{\mathcal{K}}(\psi f) + \mathbf{P}\mathbf{R}_0 f. \tag{4.7}$$

Since $\text{supp}\,\mathbf{R}_0 f \subset \bar{G}\setminus\overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, it follows from the definition of the operator $\mathbf{B}^1$ that $\mathbf{B}^1\mathbf{R}_0 f = 0$. Therefore,

$$\mathbf{C}\hat{\mathbf{R}}_1 f = \mathbf{C}\mathbf{R}_{\mathcal{K}}(\psi f) + \mathbf{B}^0\mathbf{R}_0 f. \tag{4.8}$$

Relations (4.7) and (4.8), with regard to (4.4) and (4.6), imply

$$\mathbf{L}^1\hat{\mathbf{R}}_1 f = f + \mathbf{M}_{\mathcal{K}}(\psi f) + \mathbf{T}_{\mathcal{K}}(\psi f) + \mathbf{T}_0 f, \qquad \text{where} \qquad \mathbf{T}_0 f = \sum_{j=1}^{J} \mathbf{T}_{0j}(\psi_j f). \tag{4.9}$$

4. Let us estimate the norm of $\mathbf{M}_{\mathcal{K}}(\psi f)$:

$$\|\mathbf{M}_{\mathcal{K}}(\psi f)\|_{\mathcal{W}^l(G,\Upsilon)} \leqslant k_1\varepsilon_0\|\psi f\|_{\mathcal{W}^l(G,\Upsilon)}$$

$$\leqslant k_2\varepsilon_0\|f\|_{\mathcal{W}^l(G,\Upsilon)} + k_3(\varepsilon_0)\left(\|f_0\|_{W^{l-1}(G)} + \sum_{i,\mu}\|\Phi_{i\mu}\|_{W^{l+2m-m_{i\mu}-1}(G)}\right), \tag{4.10}$$

where $\Phi_{i\mu} \in W^{l+2m-m_{i\mu}}(G)$ is an extension of $f_{i\mu} \in W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ to the domain G (if $l = 0$, then the term $\|f_0\|_{W^{l-1}(G)}$ on the right-hand side of (4.10) is absent).

It follows from (4.10), from the Rellich theorem, and from Lemma 2.3 that

$$\mathbf{M}_{\mathcal{K}}(\psi f) = \hat{\mathbf{M}}_1 f + \mathbf{T}_2 f,$$

where $\hat{\mathbf{M}}_1, \mathbf{T}_2 : \mathcal{S}_1^l(G,\Upsilon) \to \mathcal{S}_1^l(G,\Upsilon)$ are such that $\|\hat{\mathbf{M}}_1\| \leqslant c\varepsilon_0$ ($c > 0$ does not depend on ε_0) and $\mathbf{T}_2$ is compact. Combining this fact with relation (4.9), we obtain

$$\mathbf{L}^1\hat{\mathbf{R}}_1 = \mathbf{I}_1 + \hat{\mathbf{M}}_1 + \hat{\mathbf{T}}_1,$$

where $\hat{\mathbf{T}}_1 f = \mathbf{T}_2 f + \mathbf{T}_{\mathcal{K}}(\psi f) + \mathbf{T}_0 f$.

The operator $\mathbf{I}_1 + \hat{\mathbf{M}}_1 : \mathcal{S}_1^l(G,\Upsilon) \to \mathcal{S}_1^l(G,\Upsilon)$ is invertible for $\varepsilon_0 \leqslant 1/(2c)$. Therefore, denoting $\mathbf{R}_1 = \hat{\mathbf{R}}_1(\mathbf{I}_1 + \hat{\mathbf{M}}_1)^{-1}$ and $\mathbf{T}_1 = \hat{\mathbf{T}}_1(\mathbf{I}_1 + \hat{\mathbf{M}}_1)^{-1}$, we obtain (4.3). This proves Lemma 4.1.

4.2. *Construction of a Right Regularizer When* $\mathbf{B}^2_{i\mu} \neq 0$

In this subsection, we assume that ε_0 is fixed. Consider the operator $\mathbf{L}$ with $\mathbf{B}^2_{i\mu} \neq 0$. In other words, we suppose that there are nonlocal terms supported both near the set $\mathcal{K}$ and outside some neighborhood of $\mathcal{K}$.

By Theorem 2.2, for any sufficiently small $\varepsilon > 0$, there exist bounded operators

$$\mathbf{R}'_{\mathcal{K}} : \{f' : \{0, f'\} \in \mathcal{S}^l_1(G, \Upsilon), \ \operatorname{supp} f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \{u \in W^{l+2m}(G) : \operatorname{supp} f' \subset \mathcal{O}_{4\varepsilon}(\mathcal{K})\},$$

$$\mathbf{M}'_{\mathcal{K}}, \mathbf{T}'_{\mathcal{K}} : \{f' : \{0, f'\} \in \mathcal{S}^l_1(G, \Upsilon), \ \operatorname{supp} f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \mathcal{S}^l_1(G, \Upsilon)$$

such that $\|\mathbf{M}'_{\mathcal{K}} f'\|_{\mathcal{W}^l(G, \Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{W}^l(G, \Upsilon)}$, where $c > 0$ does not depend on ε, the operator $\mathbf{T}'_{\mathcal{K}}$ is compact, and
$$\mathbf{L}^1 \mathbf{R}'_{\mathcal{K}} f' = \{0, f'\} + \mathbf{M}'_{\mathcal{K}} f' + \mathbf{T}'_{\mathcal{K}} f'.$$

Note that the diameter of the support of $\mathbf{R}'_{\mathcal{K}} f'$ depends on ε rather than on ε_0.

Similarly to the proof of Lemma 4.1, we can construct a covering $\mathcal{O}_{2\varepsilon}(\mathcal{K})$, $\mathcal{O}_{\varepsilon/2}(g_j)$ ($g_j \in \partial G$, $j = 1, \ldots, J$, $J = J(\varepsilon)$) of the boundary ∂G. Let $\psi', \psi'_j \in C^\infty_0(\mathbb{R}^2)$, $j = 1, \ldots, J$, be a partition of unity subordinated to this covering.

According to the general theory of elliptic boundary-value problems in smooth domains (see, e.g., [27]), there exist bounded operators

$$\mathbf{R}'_{0j} : \{f' : \{0, f'\} \in \mathcal{W}^l(G, \Upsilon), \ \operatorname{supp} f \subset \mathcal{O}_{\varepsilon/2}(g_j)\} \to \{u \in W^{l+2m}(G) : \operatorname{supp} u \subset \mathcal{O}_\varepsilon(g_j)\}$$

and compact operators

$$\mathbf{T}'_{0j} : \{f' : \{0, f'\} \in \mathcal{W}^l(G, \Upsilon) : \operatorname{supp} f \subset \mathcal{O}_{\varepsilon/2}(g_j)\} \to \{f \in \mathcal{W}^l(G, \Upsilon) : \operatorname{supp} f \subset \mathcal{O}_\varepsilon(g_j)\}$$

such that $\mathbf{L}^0 \mathbf{R}'_{0j} f' = \{0, f'\} + \mathbf{T}'_{0j} f'$. For any f' satisfying $\{0, f'\} \in \mathcal{S}^l_1(G, \Upsilon)$, write

$$\mathbf{R}'_1 f' = \mathbf{R}'_{\mathcal{K}}(\psi' f') + \sum_{j=1}^J \mathbf{R}'_{0j}(\psi'_j f'). \tag{4.11}$$

As in the proof of Lemma 4.1, one can show that

$$\mathbf{L}^1 \mathbf{R}'_1 f' = \{0, f'\} + \mathbf{M}'_1 f' + \mathbf{T}'_1 f', \tag{4.12}$$

where $\mathbf{M}'_1, \mathbf{T}'_1 : \{f' : \{0, f'\} \in \mathcal{S}^l_1(G, \Upsilon)\} \to \mathcal{S}^l_1(G, \Upsilon)$ are bounded operators such that

$$\|\mathbf{M}'_1 f'\|_{\mathcal{W}^l(G, \Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{W}^l(G, \Upsilon)},$$

where $c > 0$ does not depend on ε, and $\mathbf{T}'_1$ is a compact operator.

Using the operators $\mathbf{R}_1$ (see Lemma 4.1) and $\mathbf{R}'_1$ (see (4.11)), we shall now construct a right regularizer for the operator $\mathbf{L}$ with $\mathbf{B}^2_{i\mu} \neq 0$.

Introduce the set

$$\mathcal{S}^l(G, \Upsilon) = \left\{ f \in \mathcal{S}^l_1(G, \Upsilon) : \text{the functions } \Phi = \mathbf{B}^2 \mathbf{R}_1 f \text{ and } \mathbf{B}^2 \mathbf{R}'_1 \Phi \text{ satisfy relations (4.2)} \right\}.$$

It follows from Sobolev's embedding theorem and from Riesz' theorem on the general form of a continuous linear functional on a Hilbert space that $\mathcal{S}^l(G, \Upsilon)$ is a closed subset of finite codimension in $\mathcal{W}^l(G, \Upsilon)$. It is also clear that $\mathcal{S}^l(G, \Upsilon) \subset \mathcal{S}^l_1(G, \Upsilon)$.

Lemma 4.2. *Let Condition* 4.1 *hold. Then there exist a bounded operator*

$$\mathbf{R}\colon \mathcal{W}^l(G,\Upsilon) \to W^{l+2m}(G)$$

and a compact operator

$$\mathbf{T}\colon \mathcal{W}^l(G,\Upsilon) \to \mathcal{W}^l(G,\Upsilon)$$

such that

$$\mathbf{LR} = \mathbf{I} + \mathbf{T}, \tag{4.13}$$

where $\mathbf{I}$ *stands for the identity operator in* $\mathcal{W}^l(G,\Upsilon)$.

Proof. 1. We set $\Phi = \mathbf{B}^2\mathbf{R}_1 f$, where $f = \{f_0, f'\} \in \mathcal{S}^l(G,\Upsilon)$. Then, by the definition of the space $\mathcal{S}^l(G,\Upsilon)$, the functions Φ and $\mathbf{B}^2\mathbf{R}_1'\Phi$ belong to the domain of the operator $\mathbf{R}_1'$. Therefore, we can introduce a bounded operator $\mathbf{R}_{\mathcal{S}}\colon \mathcal{S}^l(G,\Upsilon) \to W^{l+2m}(G)$ by the formula

$$\mathbf{R}_{\mathcal{S}} f = \mathbf{R}_1 f - \mathbf{R}_1'\Phi + \mathbf{R}_1'\mathbf{B}^2\mathbf{R}_1'\Phi.$$

Let us show that the operator $\mathbf{R}_{\mathcal{S}}$ is a right inverse to $\mathbf{L}$, up to a sum of small and compact perturbations. For simplicity, we denote diverse operators (acting on the corresponding spaces) whose norms are dominated by $c\varepsilon$ by the letter M and diverse compact operators by the letter T.

By virtue of (4.3) and (4.12), we have

$$\mathbf{PR}_{\mathcal{S}} f = \mathbf{PR}_1 f - \mathbf{PR}_1'(\Phi - \mathbf{B}^2\mathbf{R}_1'\Phi)$$
$$= f_0 + Tf_0 - M(\Phi - \mathbf{B}^2\mathbf{R}_1'\Phi) - T(\Phi - \mathbf{B}^2\mathbf{R}_1'\Phi) = f_0 + Mf + Tf, \quad (4.14)$$

$$\mathbf{CR}_{\mathcal{S}} f = \mathbf{CR}_1 f - \mathbf{CR}_1'\Phi + \mathbf{CR}_1'\mathbf{B}^2\mathbf{R}_1'\Phi$$
$$= (f' + Tf') - (\Phi + M\Phi + T\Phi) + (\mathbf{B}^2\mathbf{R}_1'\Phi + M\mathbf{B}^2\mathbf{R}_1'\Phi + T\mathbf{B}^2\mathbf{R}_1'\Phi) \quad (4.15)$$
$$= f' - \Phi + \mathbf{B}^2\mathbf{R}_1'\Phi + Mf + Tf.$$

Applying the operator $\mathbf{B}^2$ to the function $\mathbf{R}_{\mathcal{S}} f$, we obtain

$$\mathbf{B}^2\mathbf{R}_{\mathcal{S}} f = \Phi - \mathbf{B}^2\mathbf{R}_1'\Phi + \mathbf{B}^2\mathbf{R}_1'\mathbf{B}^2\mathbf{R}_1'\Phi. \tag{4.16}$$

Summing relations (4.15) and (4.16), we obtain

$$\mathbf{BR}_{\mathcal{S}} f = f' + Mf + Tf + \mathbf{B}^2\mathbf{R}_1'\mathbf{B}^2\mathbf{R}_1'\Phi. \tag{4.17}$$

Let us show that

$$\mathbf{B}^2\mathbf{R}_1'\mathbf{B}^2\mathbf{R}_1'\Phi = 0 \tag{4.18}$$

for sufficiently small $\varepsilon = \varepsilon(\varkappa_1, \varkappa_2, \rho)$, where $\varkappa_1, \varkappa_2, \rho$ are the constants occurring in Condition 1.2. (Note that ε does not depend on ε_0.)

It follows from (4.11) that $\operatorname{supp}\mathbf{R}_1'\Phi \subset \bar{G} \setminus \bar{G}_{4\varepsilon}$. Take a small number ε such that $4\varepsilon < \rho$. Then the estimate (1.6) implies that $\operatorname{supp}\mathbf{B}^2\mathbf{R}_1'\Phi \subset \mathcal{O}_{\varkappa_2}(\mathcal{K})$.

Furthermore, take a small number ε such that $4\varepsilon < \varkappa_1$ and $\varkappa_2 + 3\varepsilon/2 < \varkappa_1$. Then, using (4.11) again, we see that $\operatorname{supp}\mathbf{R}_1'\mathbf{B}^2\mathbf{R}_1'\Phi \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. Combining this fact with inequality (1.5), we obtain (4.18).

It follows from relations (4.14), (4.17), and (4.18) that

$$\mathbf{LR}_{\mathcal{S}} = \mathbf{I}_{\mathcal{S}} + M + T,$$

where $\mathbf{I}_{\mathcal{S}}, M, T\colon \mathcal{S}^l(G,\Upsilon) \to \mathcal{W}^l(G,\Upsilon)$ are bounded operators for which $\mathbf{I}_{\mathcal{S}} f = f$, $\|M\| \leqslant c\varepsilon$ ($c > 0$ does not depend on ε), and T is compact.

3. Since the subspace $\mathcal{S}^l(G, \Upsilon)$ is of finite codimension in $\mathcal{W}^l(G, \Upsilon)$, the operator $\mathbf{I}_{\mathcal{S}}$ has the Fredholm property. Therefore, by Theorems 16.2 and 16.4 in [28], the operator $\mathbf{I}_{\mathcal{S}} + M + T$ also has the Fredholm property provided that ε is sufficiently small. Now it follows from Theorem 15.2 in [28] that there exists a bounded operator $\tilde{\mathbf{R}}$ and a compact operator $\mathbf{T}$ acting from $\mathcal{W}^l(G, \Upsilon)$ to $\mathcal{S}^l(G, \Upsilon)$ and to $\mathcal{W}^l(G, \Upsilon)$, respectively, which satisfy the relation $(\mathbf{I}_{\mathcal{S}} + M + T)\tilde{\mathbf{R}} = \mathbf{I} + \mathbf{T}$. Denoting $\mathbf{R} = \mathbf{R}_{\mathcal{S}}\tilde{\mathbf{R}} : \mathcal{W}^l(G, \Upsilon) \to W^{l+2m}(G)$, we obtain (4.13), which proves Lemma 4.2.

Remark 4.1. We stress that the numbers ε_0, $\varkappa_1$, $\varkappa_2$, and ρ are fixed in the course of the proof of Lemma 4.2.

Remark 4.2. The construction of the operator $\mathbf{R}$ is close to that in [18], where nonlocal problems in weighted spaces are treated for the case in which $\mathbf{B}^1 = 0$ (i.e., the support of nonlocal terms is disjoint from the set $\mathcal{K}$).

4.3. *Fredholm Solvability of Nonlocal Problems*

In this subsection, we prove the following result concerning the solvability of problem (1.7), (1.8) in a bounded domain in Sobolev spaces.

Theorem 4.1. *Let Condition* 4.1 *hold; then the operator*

$$\mathbf{L} \colon W^{l+2m}(G) \to \mathcal{W}^l(G, \Upsilon)$$

has the Fredholm property, $\operatorname{ind} \mathbf{L} = \operatorname{ind} \mathbf{L}^1$.
Conversely, let the operator

$$\mathbf{L} \colon W^{l+2m}(G) \to \mathcal{W}^l(G, \Upsilon)$$

have the Fredholm property; then Condition 4.1 *holds.*

We shall show below that, if Condition 4.1 fails, then the image of $\mathbf{L}$ is not closed (Lemma 4.5). Combining this fact with Theorem 4.1 of this paper and with Theorem 7.1 in [28] yields the following corollary.

Corollary 4.1. *Condition* 4.1 *holds if and only if the following* a priori *estimate holds*:

$$\|u\|_{W^{l+2m}(G)} \leqslant c(\|\mathbf{L}u\|_{\mathcal{W}^l(G, \Upsilon)} + \|u\|_{L_2(G)}),$$

where $c > 0$ does not depend on u.

4.3.1. Proof of Theorem 4.1. Sufficiency. Let us show that the kernel of $\mathbf{L}$ is finite-dimensional. To do this, we consider problem (1.7), (1.8) in weighted spaces. Denote by $\tilde{H}_a^k(G)$ the completion of the set $C_0^\infty(\bar{G} \setminus \mathcal{K})$ with respect to the norm

$$\|u\|_{H_a^k(G)} = \Big(\sum_{|\alpha| \leqslant k} \int_G \rho^{2(a-k+|\alpha|)} |D^\alpha u|^2 \Big)^{1/2},$$

where $k \geqslant 0$ is an integer, $a \in \mathbb{R}$, and $\rho = \rho(y) = \operatorname{dist}(y, \mathcal{K})$. For an integer $k \geqslant 1$, denote by $H_a^{k-1/2}(\Upsilon)$ the space of traces on a smooth curve $\Upsilon \subset \bar{G}$ with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Upsilon)} = \inf \|u\|_{H_a^k(G)} \quad (u \in H_a^k(G) : u|_\Upsilon = \psi).$$

Let us introduce the operator corresponding to problem (1.7), (1.8) in weighted spaces,

$$\mathbf{L}_a = \{\mathbf{P}, \ \mathbf{B}\} \colon H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon), \quad a > l + 2m - 1,$$

where

$$\mathcal{H}_a^l(G, \Upsilon) = H_a^l(G) \times \prod_{i=1}^{N_0} \prod_{\mu=1}^{m} H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i).$$

Note that, by (1.5) and by Lemma 5.2 in [18],

$$\mathbf{B}_{i\mu}^2 u \in W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i) \subset H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$$

for all $u \in H_a^{l+2m}(G) \subset W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, $a > l + 2m - 1$. Since the functions $\mathbf{B}_{i\mu}^0 u$ and $\mathbf{B}_{i\mu}^1 u$ also belong to $H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$, it follows that the operator $\mathbf{L}_a$ is well defined.

Thus, the operators $\mathbf{L}$ and $\mathbf{L}_a$ correspond to the same nonlocal problem (1.7), (1.8) regarded in Sobolev spaces and in weighted spaces, respectively.

Lemma 4.3. *The kernel of the operator $\mathbf{L}$ is finite-dimensional.*

Proof. It follows from Lemma 2.1 in [15] and from Theorem 3.2 in [16][1]that the operator $\mathbf{L}_a$ has the Fredholm property for almost all $a > l + 2m - 1$. Choose some $a > l + 2m - 1$ for which the operator $\mathbf{L}_a$ has the Fredholm property. Then $W^{l+2m}(G) \subset H_a^{l+2m}(G)$ by Lemma 5.2 in [18], and therefore $\ker \mathbf{L} \subset \ker \mathbf{L}_a$. Since $\ker \mathbf{L}_a$ is finite-dimensional for the number a chosen above, it follows that $\ker \mathbf{L}$ is also finite-dimensional. This proves Lemma 4.3.

Remark 4.3. We stress that the kernel of the operator $\mathbf{L}$ is finite-dimensional for any arrangement of the eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$.

By Theorem 15.2 in [28] and by Lemma 4.2, the image of the operator $\mathbf{L}$ is a closed subspace of finite codimension. Combining this fact with Lemma 4.3, we see that $\mathbf{L}$ has the Fredholm property.

Let us show that $\operatorname{ind} \mathbf{L} = \operatorname{ind} \mathbf{L}^1$. We introduce the operator

$$\mathbf{L}_t u = \{\mathbf{P}u, \ \mathbf{C}u + (1 - t)\mathbf{B}^2 u\}.$$

Clearly, $\mathbf{L}_0 = \mathbf{L}$ and $\mathbf{L}_1 = \mathbf{L}^1$.

It follows from what was proved above that the operators $\mathbf{L}_t$ have the Fredholm property for any t. Furthermore, the following estimate holds for any t_0 and t:

$$\|\mathbf{L}_t u - \mathbf{L}_{t_0} u\|_{\mathcal{W}^l(G, \Upsilon)} \leqslant k_{t_0} |t - t_0| \cdot \|u\|_{W^{l+2m}(G)},$$

where $k_{t_0} > 0$ does not depend on t. Therefore, it follows from Theorem 16.2 in [28] that we have $\operatorname{ind} \mathbf{L}_t = \operatorname{ind} \mathbf{L}_{t_0}$ for any t in a sufficiently small neighborhood of the point t_0. Since t_0 is arbitrary, these neighborhoods cover the segment $[0, 1]$. Choosing a finite subcovering, we obtain the relations $\operatorname{ind} \mathbf{L} = \operatorname{ind} \mathbf{L}_0 = \operatorname{ind} \mathbf{L}_1 = \operatorname{ind} \mathbf{L}^1$. This proves the sufficiency of Condition 4.1 in Theorem 4.1.

4.3.2. **Proof of Theorem** 4.1. **Necessity.** Suppose that the model problem (1.18), (1.19) in the plane angles $K_j = K_j^p$ with the sides $\gamma_{j\sigma} = \gamma_{j\sigma}^p$, $j = 1, \ldots, N = N_{1p}$, $\sigma = 1, 2$, corresponds to the orbit Orb_p.

For any $d > 0$, consider the sets $K_j^d = K_j \cap \{y \in \mathbb{R}^2 : |y| < d\}$ and $\gamma_{j\sigma}^d = \gamma_{j\sigma} \cap \{y \in \mathbb{R}^2 : |y| < d\}$ and the spaces

$$H_a^{l,N}(K^d) = \prod_{j=1}^{N} H_a^l(K_j^d), \qquad \mathcal{W}^{l,N}(K^d, \gamma^d) = \prod_{j=1}^{N} \mathcal{W}^l(K_j^d, \gamma_j^d),$$

$$W^{l,N}(K^d) = \prod_{j=1}^{N} W^l(K_j^d), \quad \mathcal{W}^l(K_j^d, \gamma_j^d) = W^l(K_j^d) \times \prod_{\sigma=1,2} \prod_{\mu=1}^{m} W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^d).$$

Set $d_1 = \min\{\chi_{j\sigma ks}, 1\}/2$, $d_2 = 2\max\{\chi_{j\sigma ks}, 1\}$ and $d = d(\varepsilon) = 2d_2\varepsilon$.

---

[1]Theorem 3.2 was stated in [16] for the case in which the operators $\mathbf{B}_{i\mu}^2$ have the same specific form as in Example 1.1. However, the proof of Theorem 3.2 in [16] is based on inequalities (1.5) and (1.6) and does not depend on the explicit form of the operators $\mathbf{B}_{i\mu}^2$.

Lemma 4.4. *Suppose that the image of the operator* $\mathbf{L}$ *is closed. Then the estimate*

$$\|U\|_{W^{l+2m,N}(K^\varepsilon)} \leqslant c\Big(\|\mathcal{L}_p U\|_{\mathcal{W}^{l,N}(K^{2\varepsilon},\gamma^{2\varepsilon})} + \sum_{j=1}^{N} \|\mathcal{P}_j(D_y)U_j\|_{W^l(K_j^d)} + \|U\|_{W^{l+2m-1,N}(K^d)}\Big) \quad (4.19)$$

holds for each orbit Orb_p, *for any sufficiently small* ε, *and for all functions* $U \in W^{l+2m,N}(K^d)$.

Proof. 1. Since the image of $\mathbf{L}$ is closed, it follows from Lemma 4.3, from the compactness of the embedding $W^{l+2m}(G) \subset W^{l+2m-1}(G)$, and from Theorem 7.1 in [28] that

$$\|u\|_{W^{l+2m}(G)} \leqslant c(\|\mathbf{L}u\|_{\mathcal{W}^l(G,\Upsilon)} + \|u\|_{W^{l+2m-1}(G)}). \quad (4.20)$$

Let us substitute functions $u \in W^{l+2m}(G)$ such that

$$\mathrm{supp}\, u \subset \bigcup_{j=1}^{N_{1p}} \mathcal{O}_{2\varepsilon}(g_j^p)$$

and $2\varepsilon < \min\{\varepsilon_0, \varkappa_1\}$ into (4.20). It follows from (1.5) that $\mathbf{B}^2 u = 0$ for these functions u. Therefore, by using Lemma 3.2 in [22, Ch. 2], we obtain the following estimate:

$$\|U\|_{W^{l+2m,N}(K)} \leqslant c(\|\mathcal{L}_p U\|_{\mathcal{W}^{l,N}(K,\gamma)} + \|U\|_{W^{l+2m-1,N}(K)}), \quad (4.21)$$

which holds for any $U \in W^{l+2m,N}(K)$ with $\mathrm{supp}\, U \subset \mathcal{O}_{2\varepsilon}(0)$ if ε is sufficiently small.

2. Let us now get rid of the assumption $\mathrm{supp}\, U \subset \mathcal{O}_{2\varepsilon}(0)$ and show that the estimate (4.19) remains valid for any $U \in W^{l+2m,N}(K^d)$.

We introduce a function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi(y) = 1$ for $|y| \leqslant \varepsilon$, $\mathrm{supp}\, \psi \subset \mathcal{O}_{2\varepsilon}(0)$, and ψ does not depend on the polar angle ω. Using inequality (4.22) and Leibniz' formula, we obtain

$$\|U\|_{W^{l+2m,N}(K^\varepsilon)} \leqslant \|\psi U\|_{W^{l+2m,N}(K)} \leqslant k_1(\|\mathcal{L}_p(\psi U)\|_{\mathcal{W}^{l,N}(K,\gamma)} + \|\psi U\|_{W^{l+2m-1,N}(K)})$$

$$\leqslant k_2(\|\psi \mathcal{L}_p U\|_{\mathcal{W}^{l,N}(K,\gamma)} + \sum_{j,\sigma,\mu} \sum_{(k,s)\neq(j,0)} \|J_{j\sigma\mu ks}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} + \|U\|_{W^{l+2m-1,N}(K^{2\varepsilon})}) \quad (4.22)$$

for any $U \in W^{l+2m,N}(K^d)$, where

$$J_{j\sigma\mu ks} = \big(\psi(\mathcal{G}_{j\sigma ks}y) - \psi(y)\big)\big(B_{j\sigma\mu ks}(D_y)U_k\big)\big(\mathcal{G}_{j\sigma ks}y\big)\big|_{\gamma_{j\sigma}}.$$

Let us estimate the norm of $J_{j\sigma\mu ks}$. Note that, for $(k,s) \neq (j,0)$, the operator $\mathcal{G}_{j\sigma ks}$ maps the ray $\gamma_{j\sigma}$ onto the ray

$$\{y \in \mathbb{R}^2 : r > 0, \ \omega = (-1)^\sigma b_j + \omega_{j\sigma ks}\},$$

and the latter is located strictly inside the angle K_k. Therefore, there exists a function

$$\xi_{j\sigma ks} \in C_0^\infty(-b_k, b_k)$$

taking the value 1 at the point $\omega = (-1)^\sigma b_j + \omega_{j\sigma ks}$.

Furthermore, the support of the function $\psi(y) - \psi(\mathcal{G}_{j\sigma ks}^{-1}y)$ is contained in the set

$$\{d_1\varepsilon < |y| < d_2\varepsilon\}.$$

Therefore, there exists a function $\psi_1 \in C_0^\infty(K_k)$ which is identically equal to 1 on the support of the function $\xi(\omega)\big(\psi(y) - \psi(\mathcal{G}_{j\sigma ks}^{-1}y)\big)$ and satisfies the condition $\mathrm{supp}\, \psi_1 \subset \{d_1\varepsilon < |y| < d_2\varepsilon\}$. In this case, similarly to (2.38), we obtain

$$\|J_{j\sigma\mu ks}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_3\|\psi_1 U_k\|_{W^{l+2m}(K_k)}.$$

Let us estimate the norm on the right-hand side of this inequality by using Theorem 5.1 in [22, Ch. 2] and Leibniz' formula. Taking into account the fact that ψ_1 is compactly supported and vanishes both near the origin and near the sides of K_k, we obtain

$$\|J_{j\sigma\mu ks}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_4(\|\mathcal{P}_k(D_y)U_k\|_{W^l(\{d_1\varepsilon/2<|y|<2d_2\varepsilon\})}$$

$$+ \|U_k\|_{W^{l+2m-1}(\{d_1\varepsilon/2<|y|<2d_2\varepsilon\})}). \quad (4.23)$$

The estimate (4.19) follows now from (4.22) and (4.23). This proves Lemma 4.4.

Lemma 4.5. *Let the line* $\operatorname{Im}\lambda = 1 - l - 2m$ *contain an eigenvalue of the operator* $\tilde{\mathcal{L}}_p(\lambda)$ *for some* p. *Then the image of the operator* **L** *is not closed.*

Proof. 1. Suppose that the image of **L** is closed. The following two cases are possible: either (a) the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains an improper eigenvalue, or (b) the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains only the eigenvalue $\lambda_0 = i(1 - l - 2m)$, which is proper (see Definitions 3.1 and 3.2).

2. We first assume that there is an improper eigenvalue $\lambda = \lambda_0$. Let us show that the estimate (4.19) does not hold in this case. Denote by $\varphi^{(0)}(\omega), \ldots, \varphi^{(\varkappa-1)}(\omega)$ an eigenvector and some associated vectors (forming a *Jordan chain* of length $\varkappa \geqslant 1$) corresponding to the eigenvalue λ_0 (see [23]). According to Remark 2.1 in [29], the vectors $\varphi^{(k)}(\omega)$ belong to $W^{l+2m,N}(-b, b)$, and it follows from Lemma 2.1 in [29] that

$$\mathcal{L}_p V^k = 0, \qquad \text{where} \qquad V^k = r^{i\lambda_0} \sum_{s=0}^{k} \frac{1}{s!}(i \log r)^k \varphi^{(k-s)}(\omega), \quad k = 0, \ldots, \varkappa - 1. \qquad (4.24)$$

Since λ_0 is not a proper eigenvalue, it follows that the function $V^k(y)$ is not a polynomial vector for some $k \geqslant 0$. For simplicity, suppose that $V^0 = r^{i\lambda_0}\varphi^{(0)}(\omega)$ is not a polynomial vector (the case in which $k > 0$ can be treated analogously).

We introduce the sequence $U^\delta = r^\delta V^0 / \|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}$. The denominator is finite for any $\delta > 0$; however,

$$\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)} \to \infty \qquad \text{as} \qquad \delta \to 0$$

because V^0 is not a polynomial vector. However, $\|r^\delta V^0\|_{W^{l+2m-1,N}(K^d)} \leqslant c$, where $c > 0$ does not depend on $\delta \geqslant 0$; therefore,

$$\|U^\delta\|_{W^{l+2m-1,N}(K^d)} \to 0 \quad \text{as} \quad \delta \to 0. \qquad (4.25)$$

Moreover, it follows from relation (4.24) that

$$\mathcal{P}_j(D_y)U^\delta = \frac{r^\delta \mathcal{P}_j(D_y)V^0 + \sum_{|\alpha|+|\beta|=2m,|\alpha|\geqslant 1} p_{j\alpha\beta} D^\alpha r^\delta \cdot D^\beta V_j^0}{\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}}$$

$$= \frac{\sum_{|\alpha|+|\beta|=2m,|\alpha|\geqslant 1} p_{j\alpha\beta} D^\alpha r^\delta \cdot D^\beta V_j^0}{\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}},$$

where $p_{j\alpha\beta}$ are some complex constants. Hence,

$$|D^\xi \mathcal{P}_j(D_y)U^\delta| \leqslant c_{j\xi}\delta r^{l-1-|\xi|+\delta}/\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)} \qquad (|\xi| \leqslant l),$$

which implies that

$$\|\mathcal{P}_j(D_y)U^\delta\|_{W^l(K_j^d)} \to 0 \quad \text{as} \quad \delta \to 0. \qquad (4.26)$$

Similarly, by using (4.24), one can prove that

$$\|\mathcal{B}_{j\sigma\mu}(D_y)U^\delta|_{\gamma_{j\sigma}}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^{2\varepsilon})} \to 0 \quad \text{as} \quad \delta \to 0. \qquad (4.27)$$

(To obtain (4.27), one must additionally estimate the expression

$$\frac{\sum_{(k,s)\neq(j,0)} \|(\chi_{j\sigma ks}^\delta - 1)r^\delta (B_{j\sigma\mu ks}(y, D_y)V^0)(\mathcal{G}_{j\sigma ks}y)|_{\gamma_{j\sigma}}\|_{W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^{2\varepsilon})}}{\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}},$$

which also tends to zero as $\delta \to 0$ by virtue of the inequality $|\chi_{j\sigma ks}^\delta - 1| \leqslant k_6 \delta$.)

However, assertions (4.25)–(4.27) contradict the estimate (4.19) because $\|U^\delta\|_{W^{l+2m,N}(K^\varepsilon)} = 1$. This completes the proof in the case under consideration.

3. It remains to consider the case in which the line $\mathrm{Im}\,\lambda = 1 - l - 2m$ contains only the eigenvalue $\lambda_0 = i(1 - l - 2m)$ of $\tilde{\mathcal{L}}_p(\lambda)$, and this eigenvalue is proper. In this case, we cannot repeat the above arguments because V^0 is a polynomial vector, and the norm $\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}$ is uniformly bounded as $\delta \to 0$.

Let us use the results in Sec. 3. By Lemma 3.2, there is a sequence $f^\delta \in \hat{\mathcal{S}}^{l,N}(K,\gamma)$, $\delta > 0$, such that $\mathrm{supp}\,f^\delta \subset \mathcal{O}_\varepsilon(0)$ and f^δ converges in $\mathcal{W}^{l,N}(K,\gamma)$ to $f^0 \notin \hat{\mathcal{S}}^{l,N}(K,\gamma)$ as $\delta \to 0$. By Lemma 3.5, for each f^δ, there exists a function $U^\delta \in W^{l+2m,N}(K^d)$ such that

$$\mathcal{L}_p U^\delta = f^\delta, \tag{4.28}$$

$$\|U^\delta\|_{W^{l+2m-1,N}(K^d)} \leqslant c\|f^\delta\|_{\mathcal{W}^{l,N}(K,\gamma)} \tag{4.29}$$

($c > 0$ does not depend on δ), and U^δ satisfies relations (3.8). It follows from inequalities (4.19) and (4.29), from relation (4.25), and from the convergence of f^δ in $\mathcal{W}^{l,N}(K,\gamma)$ that the "sequence" U^δ is a Cauchy sequence in $W^{l+2m,N}(K^\varepsilon)$. Therefore, U^δ converges in $W^{l+2m,N}(K^\varepsilon)$ to some function U as $\delta \to 0$. Moreover, the limit function U also satisfies relations (3.8), and, since the operator

$$\mathcal{L}_p \colon W^{l+2m,N}(K^\varepsilon) \to \mathcal{W}^{l,N}(K^{2d_1\varepsilon}, \gamma^{2d_1\varepsilon})$$

is bounded, the following relation holds:

$$\mathcal{L}_p U = f^0 \quad \text{for} \quad y \in \mathcal{O}_{2d_1\varepsilon}(0).$$

Consider a function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi(y) = 1$ for $|y| \leqslant d_1^2\varepsilon$ and $\mathrm{supp}\,\psi \subset \mathcal{O}_{2d_1^2\varepsilon}(0)$. Clearly, $\psi U \in W^{l+2m,N}(K)$, ψU satisfies relations (3.8), and $\mathrm{supp}\,\mathcal{L}_p(\psi U) \subset \mathcal{O}_{2d_1\varepsilon}(0)$. Therefore,

$$\mathcal{L}_p(\psi U) = \psi f^0 + \hat{f},$$

where $\hat{f} \in \mathcal{W}^{l,N}(K,\gamma)$, and the support of $\hat{f}$ is compact and does not contain the origin. Hence, the function $\psi f^0 + \hat{f}$, together with f^0, does not belong to $\hat{\mathcal{S}}^{l,N}(K,\gamma)$, which contradicts Lemma 3.1. This completes the proof of Lemma 4.5.

Now the necessity of Condition 4.1 in Theorem 4.1 follows from Lemma 4.5.

5. ASYMPTOTICS OF SOLUTIONS OF NONLOCAL PROBLEMS IN SOBOLEV SPACES

5.1. *Smoothness of Solutions Outside the Set $\mathcal{K}$*

In this subsection, we prove the following result on smoothness of solutions of problem (1.7), (1.8) inside the domain and near a smooth part of the boundary.

Lemma 5.1. *Let $u \in W^{l+2m}(G)$ be a solution of problem (1.7), (1.8). Suppose that the right-hand side $f = \{f_0, f_{i\mu}\}$ belongs to $\mathcal{W}^{l_1}(G, \Upsilon)$ with $l_1 > l$ and Condition 1.2 holds for l_1 substituted for l. Then*

$$u \in W^{l_1+2m}\big(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})}\big) \quad \text{for any } \delta > 0. \tag{5.1}$$

Proof. 1. We denote by $W_{\mathrm{loc}}^l(G)$ the space of distributions v in G such that $\psi v \in W^l(G)$ for all $\psi \in C_0^\infty(G)$. By Theorem 3.2 in [22, Ch. 2], we have

$$u \in W_{\mathrm{loc}}^{l_1+2m}(G). \tag{5.2}$$

Combining (5.2) with estimate (1.6) implies that

$$\mathbf{B}_{i\mu}^2 u \in W^{l_1+2m-m_{i\mu}-1/2}(\Upsilon_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}). \tag{5.3}$$

We fix an arbitrary point $g \in \Upsilon_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}$ and choose a $\delta > 0$ such that

$$\overline{\mathcal{O}_\delta(g) \cap \Upsilon_i} \subset \Upsilon_i \setminus \overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})}; \qquad g \in \mathcal{O}_{\varepsilon_0}(\mathcal{K}) \implies \overline{\Omega_{is}(\mathcal{O}_\delta(g) \cap \mathcal{O}_{\varepsilon_0}(\mathcal{K}))} \subset G. \tag{5.4}$$

Then, in the neighborhood $\mathcal{O}_\delta(g)$, the function u is a solution of the following problem:

$$\mathbf{P}(y, D_y)u = f_0(y) \qquad (y \in \mathcal{O}_\delta(g) \cap G), \tag{5.5}$$

$$B_{i\mu 0}(y, D_y)u = f_{i\mu}^2(y) \qquad (y \in \mathcal{O}_\delta(g) \cap \Upsilon_i; \ \mu = 1, \ldots, m), \tag{5.6}$$

where

$$f_{i\mu}^2(y) = f_{i\mu}(y) - \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)\big(\Omega_{is}(y)\big) - \mathbf{B}_{i\mu}^2 u(y), \qquad y \in \mathcal{O}_\delta(g) \cap \Upsilon_i.$$

It follows from relations (5.2), (5.3), and (5.4) that

$$f_{i\mu}^2 \in W^{l_1+2m-m_{i\mu}-1/2}(\mathcal{O}_\delta(g) \cap \Upsilon_i).$$

Applying Theorem 5.1 in [22, Ch. 2][2] to problem (5.5), (5.6), we see that

$$u \in W^{l_1+2m}(\mathcal{O}_{\delta/2}(g) \cap G). \tag{5.7}$$

By using the method of partition of unity, we derive from (5.2) and (5.7) that

$$u \in W^{l_1+2m}\big(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}\big). \tag{5.8}$$

2. It follows from the inclusion in (5.8) and from inequality (1.5) that

$$\mathbf{B}_{i\mu}^2 u \in W^{l_1+2m-m_{i\mu}-1/2}(\Upsilon_i). \tag{5.9}$$

Taking into account formula (5.9), we can repeat the arguments of part 1 of this proof for an arbitrary point $g \in \Upsilon_i$ and for any number δ, $\delta > 0$, such that

$$\overline{\mathcal{O}_\delta(g) \cap \Upsilon_i} \subset \Upsilon_i; \qquad g \in \mathcal{O}_{\varepsilon_0}(\mathcal{K}) \implies \overline{\Omega_{is}(\mathcal{O}_\delta(g) \cap \mathcal{O}_{\varepsilon_0}(\mathcal{K}))} \subset G.$$

As a result, we obtain relation (5.7), which thus holds for an arbitrary element $g \in \Upsilon_i$. Combining this fact with (5.2) and using the method of partition of unity, we obtain (5.1), which completes the proof of Lemma 5.1.

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[2]In Theorem 5.1 of [22, Ch. 2] it is also assumed in addition that the operators $B_{i\mu 0}(y, D_y)$ are normal on Υ_i and their orders do not exceed $2m-1$. However, one can readily see that Theorem 5.1 of [22, Ch. 2] remains valid without these assumptions (see [22, Ch. 2, § 8.3]).

5.2. *Asymptotics of Solutions Near the Set* $\mathcal{K}$

In this subsection, we obtain an asymptotic formula for the solution u near an arbitrary orbit $\mathrm{Orb}_p \subset \mathcal{K}$ provided that the line $\mathrm{Im}\,\lambda = 1 - l_1 - 2m$ contains no eigenvalues of the operator $\tilde{\mathcal{L}}_p(\lambda)$.

Thus, let us choose some orbit $\mathrm{Orb}_p \subset \mathcal{K}$, and let this orbit consist of the points g_j^p, $j = 1, \dots, N = N_{1p}$. Choose a number ε, $\varepsilon > 0$, such that $\mathcal{O}_\varepsilon(g_j^p) \subset \mathcal{V}(g_j^p)$. In this case, the function u is a solution of the following problem in the neighborhood

$$\bigcup_{j=1}^{N} \mathcal{O}_\varepsilon(g_j^p)$$

of the orbit Orb_p:

$$\mathbf{P}(y, D_y)u_j = f(y) \qquad (y \in \mathcal{O}_\varepsilon(g_j) \cap G), \tag{5.10}$$

$$B_{i\mu 0}(y, D_y)u_j(y)|_{\Upsilon_i} + \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u_k)\big)\big(\Omega_{is}(y)\big)|_{\Upsilon_i} = f'_{i\mu}(y) \tag{5.11}$$

$$\big(y \in \mathcal{O}_\varepsilon(g_j^p) \cap \Upsilon_i;\ i \in \{1 \leqslant i \leqslant N_0 : g_j \in \bar{\Upsilon}_i\};\ j = 1, \dots, N;\ \mu = 1, \dots, m\big).$$

Here $u_1(y), \dots, u_N(y)$ stand for the same functions as in 1.3 and $f'_{i\mu}(y) = f_{i\mu}(y) - \mathbf{B}^2_{i\mu}u(y)$ for $y \in \mathcal{O}_\varepsilon(g_j^p) \cap \Upsilon_i$. It follows from (5.9) that $f'_{i\mu} \in W^{l_1 + 2m - m_{i\mu} - 1/2}(\mathcal{O}_\varepsilon(g_j^p) \cap \Upsilon_i)$.

Let $y \mapsto y'(g_j^p)$ be the change of variables described in Sec. 1.1. As in 1.3, we introduce the function $U_j(y') = u_j(y(y'))$ and denote y' by y again. For the index p chosen above, we set $b_j = b_j^p$, $K_j = K_j^p$, and $\gamma_{j\sigma} = \{y \in \mathbb{R}^2 : r > 0,\ \omega = (-1)^\sigma b_j\}$ ($\sigma = 1, 2$). Then problem (5.10), (5.11) becomes

$$\mathbf{P}_j(y, D_y)U_j = f_j(y) \quad (y \in K_j^\varepsilon), \tag{5.12}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}^\varepsilon} \equiv \sum_{k,s}(B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y)|_{\gamma_{j\sigma}^\varepsilon} = f_{j\sigma\mu}(y) \quad (y \in \gamma_{j\sigma}^\varepsilon) \tag{5.13}$$

(cf. (1.15), (1.16)); here $f = \{f_j, f_{j\sigma\mu}\} \in \mathcal{W}^{l_1, N}(K^\varepsilon, \gamma^\varepsilon)$ and $U \in W^{l+2m, N}(K^d)$, where $d = \varepsilon \max\{\chi_{j\sigma ks}, 1\}$ (the symbols $\chi_{j\sigma ks}$ stand for the coefficients of the homothety operators corresponding to the orbit Orb_p).

To obtain the asymptotics of the solution u of problem (1.7), (1.8) near the orbit Orb_p, we preliminarily investigate the asymptotics of the solution U of problem (5.12), (5.13) near the origin.

By Lemma 4.11 in [21], the function $U_j \in W^{l+2m}(K_j^d)$ can be represented in the form

$$U_j(y) = Q_j(y) + U_j^1(y), \tag{5.14}$$

where $Q_j(y)$ is a polynomial of order $l + 2m - 2$, while $U_j^1 \in W^{l+2m}(K_j^d) \cap H_a^{l+2m}(K_j^d)$ for any $a > 0$. By setting $Q = (Q_1, \dots, Q_N)$, we see that the function $U^1 = (U_1^1, \dots, U_N^1)$ is a solution of the problem

$$\mathbf{P}_j(y, D_y)U_j^1 = f_j(y) - \mathbf{P}_j(y, D_y)Q_j(y) \equiv f_j^1(y) \qquad (y \in K_j^\varepsilon), \tag{5.15}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U^1|_{\gamma_{j\sigma}^\varepsilon} = f_{j\sigma\mu}(y) - \mathbf{B}_{j\sigma\mu}(y, D_y)Q|_{\gamma_{j\sigma}^\varepsilon} \equiv f_{j\sigma\mu}^1(y) \quad (y \in \gamma_{j\sigma}^\varepsilon), \tag{5.16}$$

where $f^1 = \{f_j^1, f_{j\sigma\mu}^1\} \in \mathcal{W}^{l_1, N}(K^\varepsilon, \gamma^\varepsilon)$.

Using Lemma 4.11 in [21], we represent the function $f_j^1 \in W^{l_1}(K_j^\varepsilon)$ as follows:

$$f_j^1(y) = P_j(y) + f_j^2(y), \tag{5.17}$$

where $P_j(y)$ is a polynomial of order $l_1 - 2$ (if $l_1 \geqslant 2$), while $f_j^2 \in W^{l_1}(K_j^\varepsilon) \cap H_a^{l_1}(K_j^\varepsilon)$ for any $a > 0$. If $l_1 \leqslant 1$, then we set $P_j(y) \equiv 0$, in which case $f_j^1 = f_j^2 \in H_a^{l_1}(K_j^\varepsilon)$ by Lemma 2.1. Note that, on one hand, the inclusion $U_j^1 \in H_a^{l+2m}(K_j^d)$ implies the inclusion $f_j^1 \in H_a^l(K_j^\varepsilon)$ and, on the other hand, $f_j^2 \in H_a^{l_1}(K_j^\varepsilon) \subset H_a^l(K_j^\varepsilon)$. Thus, $P_j \in H_a^l(K_j^\varepsilon)$, and therefore the polynomial P_j consists of monomials whose order is greater than or equal to $l - 1$.

We similarly have

$$f_{j\sigma\mu}^1(y) = P_{j\sigma\mu}(y) + f_{j\sigma\mu}^2(y), \tag{5.18}$$

where $P_{j\sigma\mu}(y)$ is a polynomial of order $l_1 + 2m - m_{j\sigma\mu} - 2$ (if $l_1 + 2m - m_{j\sigma\mu} \geqslant 2$), and $P_{j\sigma\mu}(y)$ consists of monomials whose order is greater than or equal to $l + 2m - m_{j\sigma\mu} - 1$, while

$$f_{j\sigma\mu}^2 \in W^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon) \cap H_a^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon) \quad \text{for any} \quad a > 0.$$

If $l_1 + 2m - m_{j\sigma\mu} \leqslant 1$, then $P_{j\sigma\mu}(y) \equiv 0$.

By Lemma 3.1 in [14],[3] there exist functions

$$W_j = \sum_{s=l+2m-1}^{l_1+2m-1} \sum_{q=0}^{q_j} r^s (i \log r)^q \varphi_{jsq}(\omega) \in H_a^{l+2m}(K_j^\varepsilon), \quad a > 0,$$

with $\varphi_{jsk} \in C^\infty([-b_j, b_j])$ such that the vector $W = (W_1, \ldots, W_N)$ satisfies the following relations:

$$\mathbf{P}_j(y, D_y)W_j - P_j \in H_0^{l_1}(K_j^\varepsilon), \tag{5.19}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)W - P_{j\sigma\mu} \in H_0^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon). \tag{5.20}$$

Further, since

$$f_j^2 \in W^{l_1}(K_j^\varepsilon) \cap H_a^{l_1}(K_j^\varepsilon), \quad f_{j\sigma\mu}^2 \in W^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon) \cap H_a^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon)$$

for any $a > 0$, it follows that the functions f_j^2 and $f_{j\sigma\mu}^2$ satisfy the relations

$$D^\alpha f_j^2|_{y=0} = 0, \quad |\alpha| \leqslant l_1 - 2, \tag{5.21}$$

$$\left. \frac{\partial^\beta f_{j\sigma\mu}^2}{\partial \tau_{j\sigma}^\beta} \right|_{y=0} = 0, \quad \beta \leqslant l_1 + 2m - m_{j\sigma\mu} - 2. \tag{5.22}$$

Therefore, by virtue of Lemma 2.4 and Corollary 2.1, there exist functions

$$V_j \in W^{l_1+2m}(K_j^d) \cap H_a^{l_1+2m}(K_j^d),$$

where $a > 0$ is arbitrary, such that the vector $V = (V_1, \ldots, V_N)$ satisfies the relations

$$\mathbf{P}_j(y, D_y)V_j - f_j^2 \in H_0^{l_1}(K_j^\varepsilon), \tag{5.23}$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)V - f_{j\sigma\mu}^2 \in H_0^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon). \tag{5.24}$$

It follows from (5.15)–(5.24) that the vector

$$U^2 = U^1 - V - W \in H_a^{l+2m,N}(K^d) \tag{5.25}$$

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[3]In Lemma 3.1 [14] (as well as in Lemma 3.2 [14]), it is assumed that the nonlocal terms contain rotation operators only (rather than expansion operators). However, the corresponding results remain valid in our case (see [29]).

is a solution of the problem

$$\mathbf{P}_j(y, D_y)U_j^2 = (P_j - \mathbf{P}_j(y, D_y)W_j) + (f_j^2 - \mathbf{P}_j(y, D_y)V_j) \in H_0^{l_1}(K_j^\varepsilon), \qquad (5.26)$$

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U^2|_{\gamma_{j\sigma}^\varepsilon} = (P_{j\sigma\mu} - \mathbf{B}_{j\sigma\mu}(y, D_y)W)|_{\gamma_{j\sigma}^\varepsilon}$$
$$+ (f_{j\sigma\mu}^2 - \mathbf{B}_{j\sigma\mu}(y, D_y)V|_{\gamma_{j\sigma}^\varepsilon}) \in H_0^{l_1+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^\varepsilon). \qquad (5.27)$$

Let us choose a small number a, $a > 0$, such that the strip $1 - l - 2m < \operatorname{Im}\lambda \leqslant a + 1 - l - 2m$ contains no eigenvalues of $\tilde{\mathcal{L}}_p(\lambda)$ (this is possible because the spectrum of $\tilde{\mathcal{L}}_p(\lambda)$ is discrete). In this case, equalities (5.26) and (5.27) and Lemma 3.2 in [14] imply the following asymptotic formula for $U_j^2 \in H_a^{l+2m}(K_j^d)$:

$$U_j^2 = \sum_{1-l_1-2m < \operatorname{Im}\lambda_n \leqslant 1-l-2m} \sum_{s,q} r^{i\lambda_n+s}(i\log r)^q \psi_{jnsq}(\omega) + U_j^3 \quad (y \in K_j^\varepsilon), \qquad (5.28)$$

where $U_j^3 \in H_0^{l_1+2m}(K_j^\varepsilon)$, λ_n are the eigenvalues of $\tilde{\mathcal{L}}_p(\lambda)$, $\psi_{jnsq} \in C^\infty([-b_j, b_j])$, $s = 0, \dots, s_n$, $s_n = [l_1 + 2m - 1 + \operatorname{Im}\lambda_n]$, and $q = 0, \dots, q_{jn}$, $q_{jn} \geqslant 0$.

Formula (5.28) and relations (5.14) and (5.25) imply

$$U_j = \sum_n \sum_{s,q} r^{i\lambda_n+s}(i\log r)^q \psi_{jnsq}(\omega) + \sum_{s,q} r^s(i\log r)^q \varphi_{jsk}(\omega) + U_j^4 \quad (y \in K_j^\varepsilon), \qquad (5.29)$$

where $U_j^4 = U_j^3 + V_j + Q_j \in W^{l_1+2m}(K^d)$.

Note that the function

$$J_j = \sum_{\operatorname{Im}\lambda_n = 1-l-2m} \sum_{q=0}^{q_{jn}} r^{i\lambda_n}(i\log r)^q \psi_{j,n0q}(\omega) + \sum_{q=0}^{q_j} r^{l+2m-1}(i\log r)^q \varphi_{j,l+2m-1,k}(\omega)$$

is a homogeneous polynomial of order $l + 2m - 1$ with respect to y_1, y_2 (Lemma 4.20 in [21] would otherwise imply that $J_j \notin W^{l+2m}(K_j^d)$, while the other terms in (5.29) belong to $W^{l+2m}(K_j^d)$). Thus, we finally obtain

$$U_j = \sum_{1-l_1-2m < \operatorname{Im}\lambda_n \leqslant 1-l-2m} \sum_{s,q} r^{i\lambda_n+s}(i\log r)^q \psi_{jnsq}(\omega)$$
$$+ \sum_{s=l+2m}^{l_1+2m-1} \sum_{q=0}^{q_j} r^s(i\log r)^q \varphi_{jsk}(\omega) + U_j^5 \quad (y \in K_j^\varepsilon), \qquad (5.30)$$

where $U_j^5 = U_j^4 + J_j \in W^{l_1+2m}(K^d)$ and the indices in the first interior sum range as follows: $s = 1, \dots, s_n$ if $\operatorname{Im}\lambda_n = 1 - l - 2m$, $s = 0, \dots, s_n$ if $\operatorname{Im}\lambda_n < 1 - l - 2m$, and $q = 0, \dots, q_{jn}$ for $q_{jn} \geqslant 0$.

Let us now derive the main result of this section from Lemma 5.1 and from the representation (5.30).

Theorem 5.1. *Let $u \in W^{l+2m}(G)$ be a solution of problem (1.7), (1.8), and let the conditions of Lemma 5.1 hold. Then the solution u satisfies relations (5.1). If we additionally assume that the line $\operatorname{Im}\lambda = 1 - l_1 - 2m$ contains no eigenvalues of the operator $\tilde{\mathcal{L}}_p(\lambda)$ for some $p \in \{1, \dots, N_1\}$, then the following representation holds in the neighborhood $\mathcal{O}_\varepsilon(g_j^p)$ $(j = 1, \dots, N_{1p})$:*

$$u = \sum_n \sum_{s,q} r^{i\lambda_n+s}(i\log r)^q \psi'_{jnsq}(\omega) + \sum_{s,q} r^s(i\log r)^q \varphi'_{jsk}(\omega) + u' \quad (y \in \mathcal{O}_\varepsilon(g_j^p) \cap G). \qquad (5.31)$$

Here (ω, r) are polar coordinates with origin at g_j^p, while ψ'_{jnsq} and φ'_{jsk} are infinitely differentiable functions with respect to ω which turn into the functions ψ_{jnsq} and φ_{jsk}, respectively, after the change of variables $y \mapsto y'(g_j^p)$, and, finally, $u' \in W^{l_1+2m}(\mathcal{O}_\varepsilon(g_j^p) \cap G)$, while the indices in (5.31) range as in (5.30).

In particular, Theorem 5.1 means that, if $u \in W^{l+2m}(G)$ is a solution of problem (1.7), (1.8) with a right-hand side $f = \{f_0, f_{i\mu}\}$ belonging to $\mathcal{W}^{l_1}(G, \Upsilon)$ $(l_1 > l)$, and if the closed strip $1 - l_1 - 2m \leqslant \operatorname{Im}\lambda \leqslant 1 - l - 2m$ contains no eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \dots, N_1$, then $u \in W^{l_1+2m}(G)$.

6. NONLOCAL PROBLEMS IN BOUNDED DOMAINS IN WEIGHTED SPACES WITH SMALL WEIGHT EXPONENTS

6.1. *Statement of the Main Result*

In Sec. 4.3, we have introduced the operator

$$\mathbf{L}_a = \{\mathbf{P}, \, \mathbf{B}\} : H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon), \quad a > l + 2m - 1. \tag{6.1}$$

As was mentioned in the proof of Lemma 4.3, the operator $\mathbf{L}_a$ has the Fredholm property for almost any a, $a > l + 2m - 1$, due to Lemma 2.1 in [15] and Theorem 3.2 in [16].

In this subsection, we consider problem (1.7), (1.8) in weighted spaces with weight exponents $a > 0$. In that case, we have

$$\mathbf{B}_{i\mu}^2 u \in W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i) \quad \text{for any} \quad u \in H_a^{l+2m}(G) \subset W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}),$$

as above. However, the difficulty is that the function $\mathbf{B}_{i\mu}^2 u$ can now be outside the space $H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$, in which case the operator $\mathbf{L}_a$ given by (6.1) can be not well defined.

Introduce the set

$$S_a^{l+2m}(G) = \left\{ u \in H_a^{l+2m}(G) : \text{ the functions } \mathbf{B}_{i\mu}^2 u \text{ satisfy conditions (4.2)} \right\}.$$

Using inequality (1.5), we obtain

$$\|\mathbf{B}_{i\mu}^2 u\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant k_1 \|u\|_{W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})} \leqslant k_2 \|u\|_{H_a^{l+2m}(G)}$$

for all $u \in H_a^{l+2m}(G)$. Combining this inequality with Sobolev's embedding theorem and Riesz' theorem on the general form of a continuous linear functional on a Hilbert space, we see that $S_a^{l+2m}(G)$ is a closed subspace of finite codimension in $H_a^{l+2m}(G)$.

On the other hand, it follows from Lemma 2.1 that $\mathbf{B}_{i\mu}^2 u \in H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ for any $u \in S_a^{l+2m}(G)$, $a > 0$. Since the functions $\mathbf{B}_{i\mu}^0 u$ and $\mathbf{B}_{i\mu}^1 u$ belong to $H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ for any $a \in \mathbb{R}$ and $u \in S_a^{l+2m}(G)$ (and even for any $u \in H_a^{l+2m}(G)$), it follows that

$$\{\mathbf{P}u, \, \mathbf{B}u\} \in \mathcal{H}_a^l(G, \Upsilon) \quad \text{for any} \quad u \in S_a^{l+2m}(G), \, a > 0.$$

Thus, there exists a *finite-dimensional* space $\mathcal{R}_a^l(G, \Upsilon)$ (which is naturally embedded in the product

$$\{0\} \times \prod_{i,\mu} H_{a'}^{l+2m-m_{i\mu}-1/2}(\Upsilon_i),$$

$a' > l + 2m - 1$) such that $\mathcal{H}_a^l(G, \Upsilon) \cap \mathcal{R}_a^l(G, \Upsilon) = \{0\}$ and

$$\{\mathbf{P}u, \, \mathbf{B}u\} \in \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon) \quad \text{for all} \quad u \in H_a^{l+2m}(G), \quad a > 0.$$

Therefore, we can introduce the bounded operator

$$\mathbf{L}_a = \{\mathbf{P}, \, \mathbf{B}\} : H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon), \quad a > 0.$$

Clearly, here we can set $\mathcal{R}_a^l(G, \Upsilon) = \{0\}$ if $a > l + 2m - 1$.

Theorem 6.1. *Let $a > 0$ and let the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contain no eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$. In this case, the operator $\mathbf{L}_a : H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$ has the Fredholm property.*

Conversely, let the operator

$$\mathbf{L}_a \colon H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$$

have the Fredholm property. In this case, the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contains no eigenvalues of either of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$.

Note that, if $f \in \mathcal{H}_a^l(G, \Upsilon)$, then $\|f\|_{\mathcal{H}_a^l(G,\Upsilon) \oplus \mathcal{R}_a^l(G,\Upsilon)} = \|f\|_{\mathcal{H}_a^l(G,\Upsilon)}$. Combining this fact with Theorem 6.1 and with Riesz' theorem on the general form of continuous linear functionals on Hilbert spaces, we obtain the following result.

Corollary 6.1. *Let $a > 0$ and let the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contain no eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$. Then there exist functions $f^q \in \mathcal{H}_a^l(G, \Upsilon)$, $q = 1, \ldots, q_1$, such that problem (1.7), (1.8) admits a solution $u \in H_a^{l+2m}(G)$ if the right-hand side f of problem (1.7), (1.8) belongs to $\mathcal{H}_a^l(G, \Upsilon)$ and*

$$(f, \; f^q)_{\mathcal{H}_a^l(G,\Upsilon)} = 0, \quad q = 1, \ldots, q_1.$$

Corollary 6.1 shows that, *generally, the inclusion $u \in H_a^{l+2m}(G)$ for $0 < a \leqslant l + 2m - 1$ does not imply the inclusion $\mathbf{L}_a u \in \mathcal{H}_a^l(G, \Upsilon)$; however, if we impose finitely many orthogonality conditions on the right-hand side $f \in \mathcal{H}_a^l(G, \Upsilon)$, then problem (1.7), (1.8) admits a solution $u \in H_a^{l+2m}(G)$.*

6.2. *Proof of the Main Result*

6.2.1. Proof of Theorem 6.1. Sufficiency.

Lemma 6.1. *The kernel of the operator $\mathbf{L}_a$ is finite-dimensional.*

Proof. Note that $H_a^{l+2m}(G) \subset H_{a'}^{l+2m}(G)$ for $a \leqslant a'$. Thus, the lemma can be proved in the same way as Lemma 4.3.

Let us proceed by constructing a right regularizer for the operator $\mathbf{L}_a$.

As was mentioned above, the functions $\mathbf{B}_{i\mu}^0 u$ and $\mathbf{B}_{i\mu}^1 u$ belong to $H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ for any $u \in H_a^{l+2m}(G)$ and $a \in \mathbb{R}$. Therefore, we can introduce the bounded operator

$$\mathbf{L}_a^1 = \{\mathbf{P}, \; \mathbf{C}\} : H_a^{l+2m}(G) \to \mathcal{H}_a^l(G, \Upsilon).$$

In [16, § 3] it was proved that one can find a bounded operator $\mathbf{R}_{a,1} : \mathcal{H}_a^l(G, \Upsilon) \to H_a^{l+2m}(G)$ and a compact operator $\mathbf{T}_{a,1} : \mathcal{H}_a^l(G, \Upsilon) \to \mathcal{H}_a^l(G, \Upsilon)$ such that

$$\mathbf{L}_a^1 \mathbf{R}_{a,1} = \mathbf{I}_a + \mathbf{T}_a, \tag{6.2}$$

where $\mathbf{I}_a$ stands for the identity operator in $\mathcal{H}_a^l(G, \Upsilon)$.

Further, it follows from Theorem 2.3 that, for any sufficiently small number $\varepsilon > 0$, there exist bounded operators

$$\mathbf{R}_{a,\mathcal{K}}' : \{f' : \{0, f'\} \in \mathcal{H}_a^l(G, \Upsilon), \; \operatorname{supp} f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \{u \in H_a^{l+2m}(G) : \operatorname{supp} f' \subset \mathcal{O}_{4\varepsilon}(\mathcal{K})\},$$

$$\mathbf{M}_{a,\mathcal{K}}', \mathbf{T}_{a,\mathcal{K}}' : \{f' : \{0, f'\} \in \mathcal{H}_a^l(G, \Upsilon), \; \operatorname{supp} f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \mathcal{H}_a^l(G, \Upsilon)$$

such that $\|\mathbf{M}_{a,\mathcal{K}}' f'\|_{\mathcal{H}_a^l(G,\Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{H}_a^l(G,\Upsilon)}$, where $c > 0$ does not depend on ε, the operator $\mathbf{T}_{a,\mathcal{K}}'$ is compact, and

$$\mathbf{L}_a^1 \mathbf{R}_{a,\mathcal{K}}' f' = \{0, f'\} + \mathbf{M}_{a,\mathcal{K}}' f' + \mathbf{T}_{a,\mathcal{K}}' f'.$$

For any f' such that $\{0, f'\} \in \mathcal{H}_a^l(G, \Upsilon)$, we set

$$\mathbf{R}_{a,1}' f' = \mathbf{R}_{a,\mathcal{K}}'(\psi' f') + \sum_{j=1}^J \mathbf{R}_{0j}'(\psi_j' f'), \tag{6.3}$$

where the functions ψ', ψ_j' and the operators $\mathbf{R}_{0j}'$ are the same as in 4.2.

By using Theorem 2.3, one can immediately show that

$$\mathbf{L}_a^1 \mathbf{R}_{a,1}' f' = \{0, f'\} + \mathbf{M}_{a,1}' f' + \mathbf{T}_{a,1}' f'. \tag{6.4}$$

Here $\mathbf{M}_{a,1}', \mathbf{T}_{a,1}' : \{f' : \{0, f'\} \in \mathcal{H}_a^l(G, \Upsilon)\} \to \mathcal{H}_a^l(G, \Upsilon)$ are bounded operators such that $\|\mathbf{M}_{a,1}' f'\|_{\mathcal{H}_a^l(G,\Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\mathcal{H}_a^l(G,\Upsilon)}$, where $c > 0$ does not depend on ε and the operator $\mathbf{T}_{a,1}'$ is compact.

Let us construct a right regularizer for problem (1.7), (1.8) with nonzero $\mathbf{B}_{i\mu}^2$ in weighted spaces by using the operators $\mathbf{R}_{a,1}$ and $\mathbf{R}_{a,1}'$.

For $a > 0$, we introduce the set

$$\mathcal{S}_a^l(G, \Upsilon) = \left\{ f \in \mathcal{H}_a^l(G, \Upsilon) : \text{the functions } \Phi = \mathbf{B}^2 \mathbf{R}_{a,1} f \text{ and } \mathbf{B}^2 \mathbf{R}_{a,1}' \Phi \text{ satisfy conditions (4.2)} \right\}.$$

We claim that $\mathcal{S}_a^l(G, \Upsilon)$ is a closed subspace of finite codimension in $\mathcal{H}_a^l(G, \Upsilon)$. Indeed, by using inequality (1.5), we obtain

$$\|\Phi_{i\mu}\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant k_1 \|\mathbf{R}_{a,1} f\|_{W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})} \leqslant k_2 \|\mathbf{R}_{a,1} f\|_{H_a^{l+2m}(G)} \leqslant k_3 \|f\|_{\mathcal{H}_a^l(G,\Upsilon)}. \tag{6.5}$$

Since the function $\Phi_{i\mu}$ satisfies conditions (4.2), it follows from (6.5) and from Lemma 2.1 that $\Phi_{i\mu} \in H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$ and

$$\|\Phi_{i\mu}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant k_4 \|f\|_{\mathcal{H}_a^l(G,\Upsilon)}. \tag{6.6}$$

Therefore, the expression $\mathbf{B}^2 \mathbf{R}_{a,1}' \Phi$ is well defined. Similarly, using (6.6) and (4.2), we obtain

$$\|[\mathbf{B}^2 \mathbf{R}_{a,1}' \Phi]_{i\mu}\|_{W^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant k_5 \|f\|_{\mathcal{H}_a^l(G,\Upsilon)}, \tag{6.7}$$

$$\|[\mathbf{B}^2 \mathbf{R}_{a,1}' \Phi]_{i\mu}\|_{H_a^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)} \leqslant k_6 \|f\|_{\mathcal{H}_a^l(G,\Upsilon)}, \tag{6.8}$$

where $[\,\cdot\,]_{i\mu}$ stands for the corresponding component of the vector.

It follows from (6.5) and (6.7), from Sobolev's embedding theorem, and from Riesz' theorem on the general form of continuous linear functionals on Hilbert spaces that $\mathcal{S}_a^l(G, \Upsilon)$ is a closed subspace of finite codimension in $\mathcal{H}_a^l(G, \Upsilon)$. Hence,

$$\mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon) = \mathcal{S}_a^l(G, \Upsilon) \oplus \hat{\mathcal{R}}_a^l(G, \Upsilon), \tag{6.9}$$

where $\hat{\mathcal{R}}_a^l(G, \Upsilon)$ is some *finite-dimensional* space. We can now prove the following result.

Lemma 6.2. *Let $a > 0$ and let the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contain no eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$. Then one can find a bounded operator*

$$\mathbf{R}_a : \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon) \to H_a^{l+2m}(G)$$

and a compact operator

$$\mathbf{T}_a : \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon) \to \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$$

such that

$$\mathbf{LR} = \hat{\mathbf{I}}_a + \mathbf{T}_a, \tag{6.10}$$

where $\hat{\mathbf{I}}_a$ stands for the identity operator in $\mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$.

Proof. 1. Set $\Phi = \mathbf{B}^2 \mathbf{R}_{a,1} f$, where $f \in \mathcal{S}_a^l(G, \Upsilon)$. It follows from (6.6) and (6.8) that the functions $\{0, \Phi\}$ and $\{0, \mathbf{B}^2 \mathbf{R}'_{a,1} \Phi\}$ belong to $\mathcal{H}_a^l(G, \Upsilon)$. Therefore, the functions Φ and $\mathbf{B}^2 \mathbf{R}'_{a,1} \Phi$ belong to the domain of the operator $\mathbf{R}'_{a,1}$, and we can introduce the bounded operator

$$\mathbf{R}_{a,\mathcal{S}} : \mathcal{S}_a^l(G, \Upsilon) \to H_a^{l+2m}(G) \quad \text{by setting} \quad \mathbf{R}_{a,\mathcal{S}} f = \mathbf{R}_{a,1} f - \mathbf{R}'_{a,1} \Phi + \mathbf{R}'_{a,1} \mathbf{B}^2 \mathbf{R}'_{a,1} \Phi.$$

As in the proof of Lemma 4.2, using relations (6.2) and (6.4), one can show that

$$\mathbf{L}_a \mathbf{R}_{a,\mathcal{S}} = \mathbf{I}_{a,\mathcal{S}} + M + T,$$

where $\mathbf{I}_{a,\mathcal{S}}, M, T : \mathcal{S}_a^l(G, \Upsilon) \to \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$ are bounded operators such that $\mathbf{I}_{a,\mathcal{S}} f = f$, $\|M\| \leqslant c\varepsilon$ ($c > 0$ does not depend on ε), and T is compact.

2. Due to (6.9), the subspace $\mathcal{S}_a^l(G, \Upsilon)$ is of finite codimension in $\mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$. Therefore, the operator $\mathbf{I}_{a,\mathcal{S}}$ has the Fredholm property. By Theorems 16.2 and 16.4 in [28], the operator $\mathbf{I}_{a,\mathcal{S}} + M + T$ also has the Fredholm property if ε is sufficiently small. It follows from Theorem 15.2 in [28] that one can find a bounded operator $\tilde{\mathbf{R}}_a$ and a compact operator $\mathbf{T}_a$ acting from the space $\mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$ to $\mathcal{S}_a^l(G, \Upsilon)$ and to $\mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon)$, respectively, which satisfy $(\mathbf{I}_{a,\mathcal{S}} + M + T)\tilde{\mathbf{R}}_a = \hat{\mathbf{I}}_a + \mathbf{T}_a$. Set

$$\mathbf{R}_a = \mathbf{R}_{a,\mathcal{S}} \tilde{\mathbf{R}}_a : \mathcal{H}_a^l(G, \Upsilon) \oplus \mathcal{R}_a^l(G, \Upsilon) \to H_a^{l+2m}(G);$$

this yields (6.10) and completes the proof of Lemma 6.2.

By Theorem 15.2 in [28] and by Lemma 6.2, the image of the operator $\mathbf{L}_a$, $a > 0$, is closed and of finite codimension. Combining this fact with Lemma 6.1 proves the sufficiency part of Theorem 6.1.

6.2.2. Proof of Theorem 6.1. Necessity.

Lemma 6.3. *Let $a > 0$ and let the line $\operatorname{Im} \lambda = a + 1 - l - 2m$ contain an eigenvalue of the operator $\tilde{\mathcal{L}}_p(\lambda)$ for some p. Then the image of $\mathbf{L}_a$ is not closed.*

Proof. 1. Assume that, to the orbit Orb_p, there corresponds a model problem of the form (1.18), (1.19) in the angles $K_j = K_j^p$ with the sides $\gamma_{j\sigma} = \gamma_{j\sigma}^p$, $j = 1, \ldots, N = N_{1p}$, $\sigma = 1, 2$.

For any $d > 0$, we introduce the spaces

$$\mathcal{H}_a^l(K_j^d, \gamma_j^d) = H_a^l(K_j^d) \times \prod_{\sigma=1,2} \prod_{\mu=1}^m H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^d), \qquad \mathcal{H}_a^{l,N}(K^d, \gamma^d) = \prod_{j=1}^N \mathcal{H}_a^l(K_j^d, \gamma_j^d).$$

Set $d_1 = \min\{\chi_{j\sigma ks}, 1\}/2$, $d_2 = 2\max\{\chi_{j\sigma ks}, 1\}$, and $d = d(\varepsilon) = 2d_2\varepsilon$.

Assume that the image of $\mathbf{L}_a$ is closed. Then, as in the proof of Lemma 4.4, one can apply Lemma 6.1, the compactness of the embedding $H_a^{l+2m}(G) \subset H_a^{l+2m-1}(G)$, and Theorem 7.1 in [28] to show that

$$\|U\|_{H_a^{l+2m,N}(K^\varepsilon)} \leqslant c\Big(\|\mathcal{L}_p U\|_{\mathcal{H}_a^{l,N}(K^{2\varepsilon}, \gamma^{2\varepsilon})} + \sum_{j=1}^N \|\mathcal{P}_j(D_y)U_j\|_{H_a^l(K_j^d)} + \|U\|_{H_a^{l+2m-1,N}(K^d)}\Big) \tag{6.11}$$

for any $U \in H_a^{l+2m,N}(K^d)$ and for any sufficiently small ε.

2. Let λ_0 be an eigenvalue of $\tilde{\mathcal{L}}_p(\lambda)$ lying on the line $\operatorname{Im} \lambda = a + 1 - l - 2m$, and let $\varphi^{(0)}(\omega)$ be an eigenvector corresponding to the eigenvalue λ_0. According to Remark 2.1 in [29], the vector $\varphi^{(0)}(\omega)$ belongs to the space $W^{l+2m,N}(-b,b)$, and it follows from Lemma 2.1 in [29] that

$$\mathcal{L}_p V^0 = 0, \tag{6.12}$$

where $V^0 = r^{i\lambda_0} \varphi^{(0)}(\omega)$.

Substitute the sequence $U^\delta = r^\delta V^0 / \|r^\delta V^0\|_{H_a^{l+2m,N}(K^\varepsilon)}$, $\delta > 0$, into (6.11). Let δ tend to zero. As in the proof of Lemma 4.5, one can use relation (6.12) to show that the right-hand side of inequality (6.11) tends to zero while the left-hand side remains equal to one. This contradiction proves Lemma 6.3.

The other part of Theorem 6.1 follows from Lemma 6.3.

7. NONLOCAL PROBLEMS IN BOUNDED DOMAINS WHEN THE LINE $\operatorname{Im} \lambda = 1 - l - 2m$ CONTAINS AN EIGENVALUE OF $\tilde{\mathcal{L}}_p(\lambda)$

In the previous sections, we proved the Fredholm solvability and obtained the asymptotics of solutions of problem (1.7), (1.8) for the case in which the corresponding line in the complex plane contains no eigenvalues of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$. In this section, by using the results of Sec. 3, we study the case in which the line $\operatorname{Im} \lambda = 1 - l - 2m$ contains only the proper eigenvalue $\lambda_0 = i(1 - l - 2m)$ of the operators $\tilde{\mathcal{L}}_p(\lambda)$ for some $p \in \{1, \ldots, N_1\}$. In this case, the operator $\mathbf{L} \colon W^{l+2m}(G) \to \mathcal{W}^l(G, \Upsilon)$ fails to have the Fredholm property due to Theorem 4.1 (its image is not closed). For this reason, we assign to problem (1.7), (1.8) an operator acting on another space and prove that this operator has the Fredholm property.

7.1. *Construction of a Right Regularizer When* $\mathbf{B}_{i\mu}^2 = 0$

We study the nonlocal elliptic problem (1.7), (1.8) under the following condition.

Condition 7.1. The number $\lambda_0 = i(1 - l - 2m)$ is a proper eigenvalue of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p \in \Pi$, where Π is a nonempty subset of the set $\{1, \ldots, N_1\}$. None of the eigenvalues λ, $\lambda \neq \lambda_0$, of the operators $\tilde{\mathcal{L}}_p(\lambda)$, $p = 1, \ldots, N_1$, belongs to the line $\operatorname{Im} \lambda = 1 - l - 2m$.

Introduce functions $\psi^p \in C_0^\infty(\mathbb{R}^2)$ such that $\psi^p(y) = 1$ for

$$y \in \bigcup_{j=1}^{N_{1p}} \mathcal{O}_{\varepsilon/2}(g_j^p) \quad \text{and} \quad \operatorname{supp} \psi^p \subset \bigcup_{j=1}^{N_{1p}} \mathcal{O}_\varepsilon(g_j^p).$$

Here $\varepsilon > 0$ is assumed to be so small that $\mathcal{O}_\varepsilon(g_j^p) \subset \mathcal{V}(g_j^p)$. We also write

$$\psi = 1 - \sum_{p=1}^{N_1} \psi^p.$$

Let a vector $f^p = \{f_j^p, f_{j\sigma\mu}^p\}$ of the right-hand sides in problem (1.15), (1.16) correspond to a vector $\psi^p f = \{\psi^p f_0, \psi^p f_{i\mu}\}$ of the right-hand sides in problem (1.7), (1.8). Clearly, $\operatorname{supp} f^p \subset \mathcal{O}_\varepsilon(0)$.

We introduce the space $\hat{\mathcal{S}}^l(G, \Upsilon)$ with the norm

$$\|f\|_{\hat{\mathcal{S}}^l(G, \Upsilon)} = \left(\|\psi f\|_{\mathcal{W}^l(G, \Upsilon)}^2 + \sum_{p \in \Pi} \|f^p\|_{\hat{\mathcal{S}}^l(K^p, \gamma^p)}^2 + \sum_{p \notin \Pi} \|f^p\|_{\mathcal{S}^l(K^p, \gamma^p)}^2 \right)^{1/2}. \tag{7.1}$$

According to Condition 7.1, the set of indices Π is not empty; therefore, by Lemma 3.2, the set $\hat{\mathcal{S}}^l(G, \Upsilon)$ is not closed in the topology of $\mathcal{W}^l(G, \Upsilon)$.

On the other hand, it follows from Lemma 3.1 that, if $u \in W^{l+2m}(G)$ satisfies the relations

$$D^\alpha u|_{y=g_j^p} = 0, \quad |\alpha| \leqslant l + 2m - 2; \ p = 1, \dots, N_1; \ j = 1, \dots, N_{1p}, \tag{7.2}$$

then $\{\mathbf{P}u, \ \mathbf{C}u\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$ (the operator $\mathbf{C} = \mathbf{B}^0 + \mathbf{B}^1$ was defined in Sec. 4). Introduce the space

$$S^{l+2m}(G) = \big\{ u \in W^{l+2m}(G) : u \text{ satisfies relations (7.2)} \big\}$$

and consider the operator

$$\hat{\mathbf{L}}^1 = \{\mathbf{P}, \ \mathbf{C}\} : S^{l+2m}(G) \to \hat{\mathcal{S}}^l(G, \Upsilon).$$

By Lemma 3.1, the operator $\hat{\mathbf{L}}^1$ is bounded.

Lemma 7.1. *Let Condition* 7.1 *hold. Then there exist a bounded operator*

$$\hat{\mathbf{R}}_1 : \hat{\mathcal{S}}^l(G, \Upsilon) \to S^{l+2m}(G)$$

and a compact operator

$$\hat{\mathbf{T}}_1 : \hat{\mathcal{S}}^l(G, \Upsilon) \to \hat{\mathcal{S}}^l(G, \Upsilon)$$

such that

$$\hat{\mathbf{L}}^1 \hat{\mathbf{R}}_1 = \hat{\mathbf{I}} + \hat{\mathbf{T}}_1, \tag{7.3}$$

where $\hat{\mathbf{I}}$ stands for the identity operator on the space $\hat{\mathcal{S}}^l(G, \Upsilon)$.

Proof. The proof is similar to that of Lemma 4.1 with the following modifications: (I) Theorem 2.1 (which is now applied to the orbits Orb_p, $p \notin \Pi$) must be completed with Theorem 3.1 (applied to the orbits Orb_p, $p \in \Pi$) and (II) Remark 2.1 must be taken into account.

7.2. *Construction of Right Regularizer when* $\mathbf{B}_{i\mu}^2 \neq 0$

Theorem 2.2, Remark 2.1, and Theorem 3.2 imply that, for any sufficiently small $\varepsilon > 0$, there exist bounded operators

$$\hat{\mathbf{R}}_{\mathcal{K}}' : \{f' : \{0, f'\} \in \hat{\mathcal{S}}^l(G, \Upsilon), \ \mathrm{supp}\, f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \{u \in S^{l+2m}(G) : \mathrm{supp}\, f' \subset \mathcal{O}_{4\varepsilon}(\mathcal{K})\},$$

$$\hat{\mathbf{M}}_{\mathcal{K}}', \hat{\mathbf{T}}_{\mathcal{K}}' : \{f' : \{0, f'\} \in \hat{\mathcal{S}}^l(G, \Upsilon), \ \mathrm{supp}\, f' \subset \mathcal{O}_{2\varepsilon}(\mathcal{K})\} \to \hat{\mathcal{S}}^l(G, \Upsilon)$$

such that $\|\hat{\mathbf{M}}_{\mathcal{K}}' f'\|_{\hat{\mathcal{S}}^l(G, \Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\hat{\mathcal{S}}^l(G, \Upsilon)}$, where $c > 0$ does not depend on ε, the operator $\hat{\mathbf{T}}_{\mathcal{K}}'$ is compact, and

$$\hat{\mathbf{L}}^1 \hat{\mathbf{R}}_{\mathcal{K}}' f' = \{0, f'\} + \hat{\mathbf{M}}_{\mathcal{K}}' f' + \hat{\mathbf{T}}_{\mathcal{K}}' f'.$$

For any f' such that $\{0, f'\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$, we set

$$\hat{\mathbf{R}}_1' f' = \hat{\mathbf{R}}_{\mathcal{K}}'(\psi' f') + \sum_{j=1}^{J} \mathbf{R}_{0j}'(\psi_j' f'),$$

where the functions ψ' and ψ_j' and the operators $\mathbf{R}_{0j}'$ are the same as in 4.2.

By using Theorems 2.2 and 3.2, one can directly show that

$$\hat{\mathbf{L}}^1 \hat{\mathbf{R}}_1' f' = \{0, f'\} + \hat{\mathbf{M}}_1' f' + \hat{\mathbf{T}}_1' f'. \tag{7.4}$$

Here $\hat{\mathbf{M}}_1', \hat{\mathbf{T}}_1' : \{f' : \{0, f'\} \in \hat{\mathcal{S}}^l(G, \Upsilon)\} \to \hat{\mathcal{S}}^l(G, \Upsilon)$ are bounded operators satisfying the inequality $\|\hat{\mathbf{M}}_1' f'\|_{\hat{\mathcal{S}}^l(G, \Upsilon)} \leqslant c\varepsilon \|\{0, f'\}\|_{\hat{\mathcal{S}}^l(G, \Upsilon)}$, where $c > 0$ does not depend on ε, and the operator $\hat{\mathbf{T}}_1'$ is compact.

Let us construct a right regularizer for problem (1.7), (1.8) with nonzero operator $\mathbf{B}_{i\mu}^2$ by using the operators $\hat{\mathbf{R}}_1$ and $\hat{\mathbf{R}}_1'$. To this end, we need the following consistency condition.

Condition 7.2. For any $u \in S^{l+2m}(G)$, we have $\{0, \mathbf{B}^2 u\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$ and

$$\|\{0, \mathbf{B}^2 u\}\|_{\hat{\mathcal{S}}^l(G, \Upsilon)} \leqslant c\|u\|_{W^{l+2m}(G)}.$$

Remark 7.1. According to (1.5), the operator $\mathbf{B}^2$ corresponds to nonlocal terms whose support is outside the set $\mathcal{K}$. Therefore, if Condition 7.2 holds for the functions $u \in S^{l+2m}(G)$, then it also holds for the functions $u \in W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$.

Remark 7.2. Example 1.1 shows how to achieve the validity of Condition 7.2.

Consider problem (1.9), (1.10) and assume in addition that the transformations Ω_{is} in this problem satisfy condition (1.2) (which is a condition on the geometric structure of the transformations Ω_{is}). In this case, it follows from the continuity of Ω_{is} that $\Omega_{is}(\mathcal{O}_\delta(g)) \subset \mathcal{O}_{\varepsilon_0/2}(\mathcal{K})$ for any $g \in \bar{\Upsilon}_i \cap \mathcal{K}$ if the number δ, $\delta > 0$, is sufficiently small. Therefore,

$$\mathbf{B}^2_{i\mu} u(y) = 0 \quad \text{for} \quad y \in \mathcal{O}_\delta(\mathcal{K}) \tag{7.5}$$

for any $u \in W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$ because $1 - \zeta(\Omega_{is}(y)) = 0$ for $y \in \mathcal{O}_\delta(\mathcal{K})$. In this case, Condition 7.2 obviously holds.

One can replace condition (1.2) by the following condition: if $\Omega_{is}(g) \notin \mathcal{K}$ (where $g \in \bar{\Upsilon}_i \cap \mathcal{K}$), then the coefficients of $B_{i\mu s}(y, D_y)$ have zeros of certain orders at the points $\Omega_{is}(g)$. This also ensures that $\{0, \mathbf{B}^2 u\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$ for any $u \in W^{l+2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. However, we do not study this issue in detail in this paper.

By Lemma 3.1 and Condition 7.2, we have

$$\{\mathbf{P}u, \ \mathbf{B}u\} \in \hat{\mathcal{S}}^l(G, \Upsilon) \quad \text{for all } u \in S^{l+2m}(G).$$

Therefore, the operator $\hat{\mathbf{L}}_S = \{\mathbf{P}, \ \mathbf{B}\} : S^{l+2m}(G) \to \hat{\mathcal{S}}^l(G, \Upsilon)$ is well defined and bounded, by Lemma 3.1 and Condition 7.2 again.

Lemma 7.2. *Assume that Conditions* 7.1 *and* 7.2 *hold. Then there exist a bounded operator* $\hat{\mathbf{R}} : \hat{\mathcal{S}}^l(G, \Upsilon) \to S^{l+2m}(G)$ *and a compact operator* $\hat{\mathbf{T}} : \hat{\mathcal{S}}^l(G, \Upsilon) \to \hat{\mathcal{S}}^l(G, \Upsilon)$ *such that*

$$\hat{\mathbf{L}}_S \hat{\mathbf{R}} = \hat{\mathbf{I}} + \hat{\mathbf{T}}. \tag{7.6}$$

Proof. We set $\Phi = \mathbf{B}^2 \hat{\mathbf{R}}_1 f$, where $f = \{f_0, f'\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$ and $\hat{\mathbf{R}}_1$ enters (7.3). According to Condition 7.2, the functions Φ and $\mathbf{B}^2 \hat{\mathbf{R}}'_1 \Phi$ belong to the domain of the operator $\hat{\mathbf{R}}'_1$. Therefore, we can define the bounded operator $\hat{\mathbf{R}}_{\mathcal{S}} : \hat{\mathcal{S}}^l(G, \Upsilon) \to S^{l+2m}(G)$ by the formula

$$\hat{\mathbf{R}}_{\mathcal{S}} f = \hat{\mathbf{R}}_1 f - \hat{\mathbf{R}}'_1 \Phi + \hat{\mathbf{R}}'_1 \mathbf{B}^2 \hat{\mathbf{R}}'_1 \Phi.$$

As in the proof of Lemma 4.2, one can use relations (7.3) and (7.4) to show that

$$\hat{\mathbf{L}}_S \hat{\mathbf{R}}_{\mathcal{S}} = \hat{\mathbf{I}} + M + T,$$

where $M, T : \hat{\mathcal{S}}^l(G, \Upsilon) \to \hat{\mathcal{S}}^l(G, \Upsilon)$ are bounded operators such that $\|M\| \leqslant c\varepsilon$ (where $c > 0$ does not depend on ε) and the operator T is compact.

The operator $\hat{\mathbf{I}} + M : \hat{\mathcal{S}}^l(G, \Upsilon) \to \hat{\mathcal{S}}^l(G, \Upsilon)$ is invertible for $\varepsilon \leqslant 1/(2c)$. Therefore, writing $\hat{\mathbf{R}} = \hat{\mathbf{R}}_{\mathcal{S}}(\hat{\mathbf{I}} + M)^{-1}$ and $\mathbf{T} = T(\hat{\mathbf{I}} + M)^{-1}$, one obtains (7.6) and completes the proof of Lemma 7.2.

7.3. *Fredholm Solvability of Nonlocal Problems*

Since the subspace $S^{l+2m}(G)$ is of finite codimension in $W^{l+2m}(G)$, there exists a *finite-dimensional* subspace $\mathcal{R}^l(G, \Upsilon)$ in $\mathcal{W}^l(G, \Upsilon)$ such that

$$\{\mathbf{P}u,\ \mathbf{B}u\} \in \hat{\mathcal{S}}^l(G, \Upsilon) \oplus \mathcal{R}^l(G, \Upsilon) \quad \text{for all } u \in W^{l+2m}(G).$$

Therefore, we can introduce the bounded operator

$$\hat{\mathbf{L}} = \{\mathbf{P},\ \mathbf{B}\} : W^{l+2m}(G) \to \hat{\mathcal{S}}^l(G, \Upsilon) \oplus \mathcal{R}^l(G, \Upsilon).$$

Theorem 7.1. *Let Conditions* 7.1 *and* 7.2 *hold. Then the operator* $\hat{\mathbf{L}}$ *has the Fredholm property.*

Proof. Lemmas 4.3 and 7.2 of this paper and Theorem 15.2 in [28] imply that the operator

$$\hat{\mathbf{L}}_S : S^{l+2m}(G) \to \hat{\mathcal{S}}^l(G, \Upsilon)$$

has the Fredholm property. Since the domain $W^{l+2m}(G)$ of the operator $\hat{\mathbf{L}}$ is an extension of the domain $S^{l+2m}(G)$ of the operator $\hat{\mathbf{L}}_S$ by a finite-dimensional subspace and the operator $\hat{\mathbf{L}}$ coincides with the operator $\hat{\mathbf{L}}_S$ on $S^{l+2m}(G)$, it follows that $\hat{\mathbf{L}}$ also has the Fredholm property. This proves Theorem 7.1.

8. ELLIPTIC PROBLEMS WITH HOMOGENEOUS NONLOCAL CONDITIONS

In this section, we study the operator corresponding to problem (1.7), (1.8) with homogeneous nonlocal conditions. By using the results of Sec. 7, we show that, if the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains a proper eigenvalue only, then the operator under consideration can have the Fredholm property, in contrast to the operator $\mathbf{L}$. This turns out to depend on whether or not some algebraic relations among the operators $\mathbf{P}$, $\mathbf{B}^0$, and $\mathbf{B}^1$ hold at the points of the set $\mathcal{K}$.

8.1. *Case in Which the Line* $\operatorname{Im}\lambda = 1 - l - 2m$ *Contains No Eigenvalues of* $\tilde{\mathcal{L}}_p(\lambda)$

Introduce the space
$$W_B^{l+2m}(G) = \{u \in W^{l+2m}(G) : \mathbf{B}u = 0\}.$$

Clearly, the space $W_B^{l+2m}(G)$ is a closed subspace of $W^{l+2m}(G)$. Consider the bounded operator $\mathbf{L}_B : W_B^{l+2m}(G) \to W^l(G)$ given by

$$\mathbf{L}_B u = \mathbf{P}u, \qquad u \in W_B^{l+2m}(G).$$

To study problem (1.7), (1.8) with homogeneous nonlocal conditions, we impose the following assumptions on the operators $B_{i\mu s}(y, D_y)$ (see, e.g., [22, Ch. 2, § 1]).

Condition 8.1. The system $\{B_{i\mu 0}(y, D_y)\}_{\mu=1}^m$ is normal on $\bar{\Upsilon}_i$ for any i, $i = 1, \ldots, N_0$, and the orders of the operators $B_{i\mu s}(y, D_y)$ $(s = 0, \ldots, S_i)$ are less than of equal to $2m - 1$.

In this subsection, we prove the following result.

Theorem 8.1. *Let Condition* 4.1 *hold. Then the operator* $\mathbf{L}_B$ *has the Fredholm property.*

Let the line $\operatorname{Im}\lambda = 1 - l - 2m$ *contain an improper eigenvalue* λ_0 *of the operator* $\tilde{\mathcal{L}}_p(\lambda)$ *for some* p, *and let Condition* 8.1 *hold. Then the image of the operator* $\mathbf{L}_B$ *is not closed* (*and therefore* $\mathbf{L}_B$ *fails to have the Fredholm property*).

Let a model problem (1.18), (1.19) in the angles $K_j = K_j^p$ with the sides $\gamma_{j\sigma} = \gamma_{j\sigma}^p$, $j = 1, \ldots, N = N_{1p}$, $\sigma = 1, 2$, correspond to the orbit Orb_p.

The following lemma enables one to reduce nonlocal problems with nonhomogeneous nonlocal conditions to the corresponding problems with homogeneous conditions.

Lemma 8.1. *Let Condition* 8.1 *hold. Then, for any* $f_{j\sigma\mu} \in H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ *such that* $\operatorname{supp} f_{j\sigma\mu} \subset \mathcal{O}_{\varepsilon_1}(0)$ ($\varepsilon_1 > 0$ *is fixed), there exists a function* $V \in H_a^{l+2m,N}(K)$ *satisfying the conditions* $\operatorname{supp} V \subset \mathcal{O}_{2\varepsilon_1}(0)$ *and*

$$\mathbf{B}_{j\sigma\mu}(y, D_y)V|_{\gamma_{j\sigma}} = f_{j\sigma\mu}, \tag{8.1}$$

$$\|V\|_{H_a^{l+2m,N}(K)} \leqslant c_{\varepsilon_1} \sum_{j,\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}, \tag{8.2}$$

where $c_{\varepsilon_1} > 0$ *does not depend on* $f_{j\sigma\mu}$.

Proof. 1. As in the proof of Lemma 3.1 in [30] (which deals with differential operators with constant coefficients), one can construct functions $V_{j\sigma} \in H_a^{l+2m}(K_j)$ such that

$$B_{j\sigma\mu j0}(y, D_y)V_{j\sigma}|_{\gamma_{j\sigma}} = f_{j\sigma\mu}, \tag{8.3}$$

$$\|V_{j\sigma}\|_{H_a^{l+2m}(K_j)} \leqslant k_2 \sum_{\mu=1}^{m} \|f_{j\sigma\mu}\|_{H_a^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}. \tag{8.4}$$

Since $\operatorname{supp} f_{j\sigma\mu} \subset \mathcal{O}_{\varepsilon_1}(0)$, one can assume that $\operatorname{supp} V_{j\sigma} \subset \mathcal{O}_{2\varepsilon_1}(0)$.

2. Write $\delta = \min |(-1)^\sigma b_j + \omega_{j\sigma ks} \pm b_k|/2$ ($j, k = 1, \dots, N$; $\sigma = 1, 2$; $s = 1, \dots, S_{j\sigma k}$) and introduce functions $\zeta_{j\sigma} \in C_0^\infty(\mathbb{R}^2)$ such that $\zeta_{j\sigma}(\omega) = 1$ for $|(-1)^\sigma b_j - \omega| < \delta/2$ and $\zeta_{j\sigma}(\omega) = 0$ for $|(-1)^\sigma b_j - \omega| > \delta$. Since the functions $\zeta_{j\sigma}$ are multipliers on the space $H_a^{l+2m}(K_j)$, it follows from (8.3) and (8.4) that the function $V = (\zeta_{11}V_{11} + \zeta_{12}V_{12}, \dots, \zeta_{N1}V_{N1} + \zeta_{N2}V_{N2})$ satisfies conditions (8.1) and (8.2). This completes the proof of Lemma 8.1.

Remark 8.1. One cannot use similar arguments for Sobolev spaces because the functions $\zeta_{j\sigma}$ are not multipliers on the spaces $W^{l+2m}(K_j)$. Moreover, it is possible to construct functions $f_{j\sigma\mu} \in W^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ ($j = 1, \dots, N$; $\sigma = 1, 2$; $\mu = 1, \dots, m$) such that none of the functions $V \in W^{l+2m,N}(K)$ satisfy conditions (8.1). This explains why the problem with homogeneous nonlocal conditions is not equivalent to the problem with nonhomogeneous conditions (i.e., the former can have the Fredholm property in contrast to the latter, see examples in Sec. 9).

As above, for any chosen orbit Orb_p, we write $d_1 = \min\{\chi_{j\sigma ks}, 1\}/2$, $d_2 = 2\max\{\chi_{j\sigma ks}, 1\}$, and $d = d(\varepsilon) = 2d_2\varepsilon$. The following result will be used below when studying the image of the operator $\mathbf{L}_B$ (cf. Lemma 4.4).

Lemma 8.2. *Let Condition* 8.1 *hold, and let the image of* $\mathbf{L}_B$ *be closed. For each orbit* Orb_p, *for any sufficiently small* ε, *and for any* $U \in W^{l+2m,N}(K^d)$ *satisfying relations* (3.8) *and the conditions*

$$\mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}^{2\varepsilon}} = 0 \qquad (j = 1, \dots, N; \ \sigma = 1, 2; \ \mu = 1, \dots, m), \tag{8.5}$$

the following estimate holds:[4]

$$\|U\|_{W^{l+2m,N}(K^\varepsilon)} \leqslant c \sum_{j=1}^{N} \left(\|\mathcal{P}_j(D_y)U_j\|_{W^l(K_j^d)} + \|U_j\|_{H_0^{l+2m-1}(K_j^d)} \right). \tag{8.6}$$

Proof. 1. Since the image of $\mathbf{L}_B$ is closed, it follows from Lemma 4.3, from the fact that the embedding $W^{l+2m}(G) \subset W^{l+2m-1}(G)$ is compact, and from Theorem 7.1 in [28] that

$$\|u\|_{W^{l+2m}(G)} \leqslant c(\|\mathbf{P}(y, D_y)u\|_{W^l(G)} + \|u\|_{W^{l+2m-1}(G)}) \tag{8.7}$$

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[4]Under the assumptions of this lemma, it follows from Lemma 2.1 that $U_j \in H_a^{l+2m}(K_j^d)$ for any $a > 0$. Therefore, $U_j \in H_0^{l+2m-1}(K_j^d)$ and the estimate (8.6) is correct.

for all $u \in W_B^{l+2m}(G)$. Let us substitute a function $u \in W_B^{l+2m}(G)$ such that

$$\operatorname{supp} u \subset \bigcup_{j=1}^{N_{1p}} \mathcal{O}_{2\varepsilon_1}(g_j^p), \qquad 2\varepsilon_1 < \min\{\varepsilon_0, \varkappa_1\},$$

into (8.7). By (1.5), we have $\mathbf{B}^2 u = 0$ for any function of this kind. Therefore, using Lemma 3.2 in [22, Ch. 2], we see that, if ε_1 is sufficiently small, then the estimate

$$\|U\|_{W^{l+2m,N}(K)} \leqslant k_1 \sum_{j=1}^{N} \left(\|\mathcal{P}_j(D_y)U_j\|_{W^l(K_j)} + \|U_j\|_{W^{l+2m-1}(K_j)} \right) \tag{8.8}$$

holds for any $U \in W^{l+2m,N}(K)$ such that $\operatorname{supp} U \subset \mathcal{O}_{2\varepsilon_1}(0)$ and

$$\mathbf{B}_{j\sigma\mu}(y, D_y)U|_{\gamma_{j\sigma}} = 0 \quad (j = 1, \dots, N; \ \sigma = 1, 2; \ \mu = 1, \dots, m). \tag{8.9}$$

2. Let us show that, if $\varepsilon_2 < \varepsilon_1 d_1$ is sufficiently small, then the estimate (8.8) holds for all $U \in W^{l+2m,N}(K)$ satisfying relations (3.8) and the conditions $\operatorname{supp} U \subset \mathcal{O}_{2\varepsilon_2}(0)$ and

$$\mathcal{B}_{j\sigma\mu}(D_y)U|_{\gamma_{j\sigma}} = 0 \quad (j = 1, \ \dots, \ N; \ \sigma = 1, \ 2; \ \mu = 1, \ \dots, \ m). \tag{8.10}$$

We set $\Phi_{j\sigma\mu} = \mathbf{B}_{j\sigma\mu}(y, \ D_y)U|_{\gamma_{j\sigma}}$; clearly,

$$\operatorname{supp} \Phi \subset \mathcal{O}_{\varepsilon_2/d_1}(0) \subset \mathcal{O}_{\varepsilon_1}(0). \tag{8.11}$$

Let us choose some a, $0 < a < 1$, and prove that

$$\|\Phi_{j\sigma\mu}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_2 \varepsilon_2^{1-a} \|U\|_{W^{l+2m,N}(K)}. \tag{8.12}$$

It follows from (8.10), with regard to the fact that the trace operator in the weighted spaces in question is bounded, that it suffices to estimate the terms of the following type:

$$\left(a_\alpha(y) - a_\alpha(0)\right)D^\alpha U_j \quad (|\alpha| = m_{j\sigma\mu}), \qquad a_\beta(y)D^\beta U_j \quad (|\beta| \leqslant m_{j\sigma\mu} - 1),$$

where a_α and a_β are infinitely differentiable functions. Using the assumptions concerning the support of U_j and taking account of Lemma 3.3$'$ in [21] and Lemma 2.1, we obtain

$$\|\left(a_\alpha(y) - a_\alpha(0)\right)D^\alpha U_j\|_{H_0^{l+2m-m_{j\sigma\mu}}(K_j)} \leqslant k_3 \varepsilon_2^{1-a} \|\left(a_\alpha(y) - a_\alpha(0)\right)D^\alpha U_j\|_{H_{a-1}^{l+2m-m_{j\sigma\mu}}(K_j)}$$

$$\leqslant k_4 \varepsilon_2^{1-a} \|D^\alpha U_j\|_{H_a^{l+2m-m_{j\sigma\mu}}(K_j)} \leqslant k_5 \varepsilon_2^{1-a} \|U_j\|_{W^{l+2m}(K_j)}.$$

Similarly, using Lemma 2.1, we obtain

$$\|a_\beta(y)D^\beta U_j\|_{H_0^{l+2m-m_{j\sigma\mu}}(K_j)} \leqslant k_6 \varepsilon_2^{1-a} \|U_j\|_{H_{a-1}^{l+2m-1}(K_j)} \leqslant k_7 \varepsilon_2^{1-a} \|U_j\|_{W^{l+2m}(K_j)}.$$

Thus, the estimate (8.12) is proved.

Further, by virtue of (8.11) and Lemma 8.1, there exists a function $V = (V_1, \dots, V_N) \in H_0^{l+2m,N}(K)$ such that $\operatorname{supp} V \subset \mathcal{O}_{2\varepsilon_1}(0)$ and

$$\mathbf{B}_{j\sigma\mu}(y, \ D_y)V|_{\gamma_{j\sigma}} = \Phi_{j\sigma\mu}, \tag{8.13}$$

$$\|V\|_{H_0^{l+2m,N}(K)} \leqslant c_{\varepsilon_1} \sum_{j,\sigma,\mu} \|\Phi_{j\sigma\mu}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}, \tag{8.14}$$

where c_{ε_1} does not depend on ε_2.

Estimating $U - V$ with the help of (8.8) and using inequalities (8.14) and (8.12), we obtain

$$\|U\|_{W^{l+2m,N}(K)} \leqslant \|U - V\|_{W^{l+2m,N}(K)} + \|V\|_{W^{l+2m,N}(K)}$$

$$\leqslant k_8 \sum_{j=1}^{N} \left(\|\mathcal{P}_j(D_y)U_j\|_{W^l(K_j)} + \|U_j\|_{W^{l+2m-1}(K_j)} + \varepsilon_2^{1-a}\|U_j\|_{W^{l+2m}(K_j)} \right).$$

Now, choosing a sufficiently small number ε_2, we obtain the estimate (8.8) for any $U \in W^{l+2m,N}(K)$ such that relations (3.8) and (8.10) hold and $\operatorname{supp} U \subset \mathcal{O}_{2\varepsilon_2}(0)$.

3. Let us omit the assumption $\operatorname{supp} U \subset \mathcal{O}_{2\varepsilon_2}(0)$ and prove that the estimate (8.6) holds for $\varepsilon < \varepsilon_2 d_1$ and for any $U \in W^{l+2m,N}(K^d)$ satisfying (3.8) and (8.5).

Introduce a function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi(y) = 1$ for $|y| \leqslant \varepsilon$, $\operatorname{supp}\psi \subset \mathcal{O}_{2\varepsilon}(0)$, and ψ does not depend on the polar angle ω.

Set $\Psi_{j\sigma\mu} = \mathcal{B}_{j\sigma\mu}(D_y)(\psi U)|_{\gamma_{j\sigma}}$; clearly,

$$\operatorname{supp}\Psi_{j\sigma\mu} \subset \mathcal{O}_{\varepsilon/d_1}(0) \subset \mathcal{O}_{\varepsilon_2}(0). \tag{8.15}$$

Let us show that

$$\|\Psi_{j\sigma\mu}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_9 \sum_{k=1}^{N} \left(\|\mathcal{P}_k(D_y)U_k\|_{W^l(K_k^d)} + \|U_k\|_{H_0^{l+2m-1}(K_j^d)} \right). \tag{8.16}$$

Taking into account relations (8.5), we can represent the function $\Psi_{j\sigma\mu}$ as follows:

$$\Psi_{j\sigma\mu} = \sum_{k,s} \Psi_{j\sigma\mu ks} + \sum_{(k,s)\neq(j,0)} J_{j\sigma\mu ks}, \tag{8.17}$$

where

$$\Psi_{j\sigma\mu ks} = \left([B_{j\sigma\mu ks}(D_y), \psi]U_k \right)\left(\mathcal{G}_{j\sigma ks}y \right)\big|_{\gamma_{j\sigma}},$$

$$J_{j\sigma\mu ks} = \left(\psi(\mathcal{G}_{j\sigma ks}y) - \psi(y) \right)\left(B_{j\sigma\mu ks}(D_y)U_k \right)\left(\mathcal{G}_{j\sigma ks}y \right)\big|_{\gamma_{j\sigma}}$$

$([\cdot,\cdot]$ stands for the commutator).

Since the expression for $\Psi_{j\sigma\mu ks}$ contains derivatives of U_k whose order is less than or equal to $m_{j\sigma\mu} - 1$, it follows that

$$\|\Psi_{j\sigma\mu ks}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_{10}\|U_k\|_{H_0^{l+2m-1}(K_k^d)}. \tag{8.18}$$

Further, repeating the arguments in part 1 of the proof of Lemma 4.5, we obtain

$$\|J_{j\sigma\mu ks}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})} \leqslant k_{11}(\|\mathcal{P}_k(D_y)U_k\|_{W^l(\{d_1\varepsilon/2<|y|<2d_2\varepsilon\})}$$

$$+ \|U_k\|_{W^{l+2m-1}(\{d_1\varepsilon/2<|y|<2d_2\varepsilon\})}). \tag{8.19}$$

The estimate (8.16) follows now from (8.17), (8.18), and (8.19).

4. By virtue of (8.15) and Lemma 8.1 (applied to the operators $\mathcal{B}_{j\sigma\mu}(D_y)$), there exists a function $V = (V_1,\dots,V_N) \in H_0^{l+2m,N}(K)$ such that $\operatorname{supp} V \subset \mathcal{O}_{2\varepsilon_2}(0)$ and

$$\mathcal{B}_{j\sigma\mu}(D_y)V|_{\gamma_{j\sigma}} = \Psi_{j\sigma\mu}, \tag{8.20}$$

$$\|V\|_{H_0^{l+2m,N}(K)} \leqslant k_{12} \sum_{j,\sigma,\mu} \|\Psi_{j\sigma\mu}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}. \tag{8.21}$$

Estimating $\psi U - V$ with the help of (8.8), using Leibniz' formula and inequalities (8.21), (8.16), we obtain

$$\|U\|_{W^{l+2m,N}(K^\varepsilon)} \leqslant \|\psi U\|_{W^{l+2m,N}(K)} \leqslant \|\psi U - V\|_{W^{l+2m,N}(K)} + \|V\|_{W^{l+2m,N}(K)}$$

$$\leqslant k_{11} \sum_{j=1}^N \big(\|\mathcal{P}_j(D_y)U_j\|_{W^l(K_j^d)} + \|U_j\|_{H_0^{l+2m-1}(K_j^d)} \big).$$

This proves Lemma 8.2.

Lemma 8.2 enables us to prove that, if the line $\operatorname{Im}\lambda = 1-l-2m$ contains an improper eigenvalue, then the operator $\mathbf{L}_B$ fails to have the Fredholm property, like $\mathbf{L}$.

Lemma 8.3. *Let the line* $\operatorname{Im}\lambda = 1 - l - 2m$ *contain an improper eigenvalue* λ_0 *of the operator* $\tilde{\mathcal{L}}_p(\lambda)$ *for some* p, *and let Condition* 8.1 *hold. Then the image of* $\mathbf{L}_B$ *is not closed.*

Proof. 1. Assume that the image of $\mathbf{L}_B$ is closed. Denote by $\varphi^{(0)}(\omega), \ldots, \varphi^{(\varkappa-1)}(\omega)$ an eigenvector and associated vectors corresponding to the eigenvalue λ_0 (see [23]). By Remark 2.1 in [29], the vectors $\varphi^{(k)}(\omega)$ belong to $W^{l+2m,N}(-b, b)$ and satisfy the relations

$$\mathcal{P}_j(D_y)V_j^k = 0, \qquad \mathcal{B}_{j\sigma\mu}(D_y)V^k = 0, \tag{8.22}$$

where

$$V^k = r^{i\lambda_0} \sum_{s=0}^k \frac{1}{s!}(i\log r)^k \varphi^{(k-s)}(\omega), \qquad k = 0, \ldots, \varkappa - 1.$$

Since λ_0 is not a proper eigenvalue, it follows that the function $V^k(y)$ is not a polynomial vector for some $k \geqslant 0$. For simplicity, we assume that $V^0 = r^{i\lambda_0}\varphi^{(0)}(\omega)$ is not a polynomial vector (the case in which $k > 0$ can be treated in the similar way).

Let ε and $d = d(\varepsilon)$ be the same constants as in Lemma 8.2. Consider the sequence

$$U^\delta = r^\delta V^0 / \|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)}.$$

The denominator is finite for any $\delta > 0$; however, $\|r^\delta V^0\|_{W^{l+2m,N}(K^\varepsilon)} \to \infty$ as $\delta \to 0$ because V^0 is not a polynomial vector. Note that

$$\|r^\delta V^0\|_{H_0^{l+2m-1,N}(K^d)} \leqslant c,$$

where $c > 0$ does not depend on $\delta \geqslant 0$, and therefore

$$\|U^\delta\|_{H_0^{l+2m-1,N}(K^d)} \to 0 \quad \text{as} \quad \delta \to 0. \tag{8.23}$$

By using (8.22), as in the proof of Lemma 4.5, one can show that

$$\|\mathcal{P}_j(D_y)U_j^\delta\|_{W^l(K_j^d)} \to 0 \quad \text{as} \quad \delta \to 0, \tag{8.24}$$

$$\|\mathcal{B}_{j\sigma\mu}(D_y)U^\delta\big|_{\gamma_{j\sigma}^{3\varepsilon}}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^{3\varepsilon})} \to 0 \quad \text{as } \delta \to 0. \tag{8.25}$$

2. Introduce a function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi(y) = 1$ for $y \in \mathcal{O}_{2\varepsilon}(0)$ and $\operatorname{supp}\psi \subset \mathcal{O}_{3\varepsilon}(0)$.

Applying Lemma 8.1 to the operators $\mathcal{B}_{j\sigma\mu}(D_y)$ and to the functions $f_{j\sigma\mu} = \psi\mathcal{B}_{j\sigma\mu}(D_y)U^\delta|_{\gamma_{j\sigma}}$ (note that $\operatorname{supp} f_{j\sigma\mu} \subset \mathcal{O}_{3\varepsilon}(0)$), we obtain a function $W^\delta \in H_0^{l+2m,N}(K)$ $(\delta > 0)$ such that $\operatorname{supp} W^\delta \subset \mathcal{O}_{6\varepsilon}(0)$ and

$$\mathcal{B}_{j\sigma\mu}(D_y)W^\delta|_{\gamma_{j\sigma}^{2\varepsilon}} = \mathcal{B}_{j\sigma\mu}(D_y)U^\delta|_{\gamma_{j\sigma}^{2\varepsilon}}, \tag{8.26}$$

$$\|W^\delta\|_{H_0^{l+2m,N}(K^{6\varepsilon})} \leqslant k_1 \sum_{j,\sigma,\mu} \|\mathcal{B}_{j\sigma\mu}(D_y)U^\delta|_{\gamma_{j\sigma}^{3\varepsilon}}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^{3\varepsilon})}. \tag{8.27}$$

Moreover, the function $U^\delta - W^\delta$ satisfies relations (3.8); therefore, we can apply Lemma 8.2 to the function $U^\delta - W^\delta$. The estimate (8.6) and inequality (8.27), together with the fact that the embedding $H_0^{l+2m}(K_j^{6\varepsilon}) \subset W^{l+2m}(K_j^{6\varepsilon})$ is bounded, implies

$$\|U^\delta\|_{W^{l+2m,N}(K^\varepsilon)} \leqslant \|U^\delta - W^\delta\|_{W^{l+2m,N}(K^\varepsilon)} + \|W^\delta\|_{W^{l+2m,N}(K^\varepsilon)}$$

$$\leqslant k_2 \sum_{j=1}^N \left(\|\mathcal{P}_j(D_y)U_j^\delta\|_{W^l(K_j^d)} + \sum_{\sigma,\mu} \|\mathcal{B}_{j\sigma\mu}(D_y)U^\delta|_{\gamma_{j\sigma}^{3\varepsilon}}\|_{H_0^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}^{3\varepsilon})} \right.$$

$$\left. + \|U_j^\delta\|_{H_0^{l+2m-1}(K_j^d)} \right). \tag{8.28}$$

However, relations (8.23)–(8.25) contradict the estimate (8.28) because $\|U^\delta\|_{W^{l+2m,N}(K^\varepsilon)} = 1$. This proves Lemma 8.3.

Proof of Theorem 8.1. The first part of Theorem 8.1 follows from Theorem 4.1. The other part follows from Lemma 8.3.

8.2. *Case in Which the Line* $\operatorname{Im}\lambda = 1 - l - 2m$ *Contains the Proper Eigenvalue of* $\tilde{\mathcal{L}}_p(\lambda)$

It remains to study the case in which the line $\operatorname{Im}\lambda = 1 - l - 2m$ contains the proper eigenvalue only. Let Condition 7.1 hold. We claim that the Fredholm property of the operator $\mathbf{L}_B$, for a chosen $l \geqslant 1$, is a consequence of the following condition.

Condition 8.2. For $l \geqslant 1$ and for any $p \in \Pi$, system (3.4) corresponding to the orbit Orb_p contains the operators $D^\xi \mathcal{P}_j(D_y)$ $(|\xi| = l - 1, j = 1, \ldots, N = N_{1p})$.

Theorem 8.2. *Suppose Conditions* 7.1 *and* 7.2 *hold. Then*
   1. *The operator*
$$\mathbf{L}_B \colon W_B^{2m}(G) \to L_2(G)$$

   has the Fredholm property;
   2. *If* $l \geqslant 1$ *and Condition* 8.2 *holds, then the operator*

$$\mathbf{L}_B \colon W_B^{l+2m}(G) \to W^l(G)$$

   has the Fredholm property;
   2'. *If* $l \geqslant 1$ *and if Condition* 8.2 *fails and Condition* 8.1 *holds, then the image of the operator*

$$\mathbf{L}_B \colon W_B^{l+2m}(G) \to W^l(G)$$

   is not closed (*and, therefore,* $\mathbf{L}_B$ *fails to have the Fredholm property*).

Proof. 1. By Lemma 4.3, the kernel of $\mathbf{L}_B$ is finite-dimensional. Let us study the image $\mathcal{R}(\mathbf{L}_B)$ of the operator $\mathbf{L}_B$.
   2. Assume first that $l \geqslant 1$ and Condition 8.2 holds. We claim that the set

$$\{f_0 \in W^l(G) : \{f_0, 0\} \in \hat{\mathcal{S}}^l(G, \Upsilon)\} \tag{8.29}$$

is a closed subset of finite codimension in $W^l(G)$. Indeed, let ψ^p be the functions occurring in the definition of the space $\hat{\mathcal{S}}^l(G, \Upsilon)$ (see 7.1). Then some vector $\{f_j^p, 0\}$ of the right-hand sides in problem (1.15), (1.16) corresponds to the vector $\{\psi^p f_0, 0\}$ of the right-hand sides in problem (1.7), (1.8). Let $p \in \Pi$; clearly, $\mathcal{T}_{j\sigma\mu}\{f_j^p, 0\} = 0$. Moreover, by Condition 8.2, relations (3.6) are absent. Thus, due to (7.1), the norm of the function $\{f_0, 0\} \in \hat{\mathcal{S}}^l(G, \Upsilon)$ in $\hat{\mathcal{S}}^l(G, \Upsilon)$ is equivalent to the norm of f_0 in $W^l(G)$, while the set (8.29) is the subspace of $W^l(G)$ consisting of the functions satisfying relations (4.1).

Further, since $\hat{\mathcal{S}}^l(G, \Upsilon) \subset \hat{\mathcal{S}}^l(G, \Upsilon) \oplus \mathcal{R}^l(G, \Upsilon)$, it follows that the set

$$\left\{ f_0 \in W^l(G) : \{f_0, 0\} \in \hat{\mathcal{S}}^l(G, \Upsilon) \oplus \mathcal{R}^l(G, \Upsilon) \right\} \tag{8.30}$$

(which contains the set (8.29)) is also a close subset in $W^l(G)$ of finite codimension. On the other hand, $f_0 \in \mathcal{R}(\mathbf{L}_B)$ if and only if $\{f_0, 0\} \in \mathcal{R}(\hat{\mathbf{L}})$, where $\hat{\mathbf{L}}$ is the operator defined in Sec. 7.3. Combining this with the fact that the operator $\hat{\mathbf{L}}$ has the Fredholm property, we see that the image of $\mathbf{L}_B$ is closed and of finite codimension.

3. Assume now that $l \geqslant 1$ and Condition 8.2 fails. Let us prove, using the results of Sec. 3, that the image of $\mathbf{L}_B$ is not closed. Suppose the contrary. Let $\mathcal{R}(\mathbf{L}_B)$ be closed.

Since Condition 8.2 fails, the set of conditions (3.6) is not empty. For some j, ξ, the norm (3.7) contains the corresponding term $\|\mathcal{T}_{j\xi} f\|_{H_0^1(\mathbb{R}^2)}$. Therefore, as follows from the proof of Lemma 3.2, there is a sequence $f^\delta = \{f_j^\delta, 0\} \in \hat{\mathcal{S}}^{l,N}(K, \gamma)$, $\delta > 0$, such that $\operatorname{supp} f^\delta \subset \mathcal{O}_\varepsilon(0)$ and f^δ converges in $\mathcal{W}^{l,N}(K, \gamma)$ to $f^0 \notin \hat{\mathcal{S}}^{l,N}(K, \gamma)$ as $\delta \to 0$.

By Lemma 3.5, for each f^δ, there exists a function $U^\delta \in W^{l+2m,N}(K^d)$ such that

$$\mathcal{P}_j(D_y)U_j^\delta = f_j^\delta, \qquad \mathcal{B}_{j\sigma\mu}(D_y)U^\delta = 0, \tag{8.31}$$

$$\|U^\delta\|_{H_0^{l+2m-1,N}(K^d)} \leqslant c\|f^\delta\|_{\mathcal{W}^{l,N}(K,\gamma)} \tag{8.32}$$

(where $c > 0$ does not depend on δ) and U^δ satisfies relations (3.8). By the second relation in (8.31) and relations (3.8), we can apply Lemma 8.2 to the function U^δ. By using the estimate (8.6), the convergence of f^δ to $f^0 \notin \hat{\mathcal{S}}^{l,N}(K, \gamma)$, and inequality (8.32), we arrive at a contradiction (cf. the proof of Lemma 4.5).

4. If $l = 0$, then the set of conditions (3.6) is empty because these conditions occur for $l \geqslant 1$ only. As in part 2 of the proof, this implies the assertion of the theorem.

9. EXAMPLES OF NONLOCAL ELLIPTIC PROBLEMS IN SOBOLEV SPACES

In this section, we consider two examples illustrating the results of the research.

9.1. *Example* 1

9.1.1. A Problem with Nonhomogeneous Nonlocal Conditions. Let

$$\partial G \setminus \mathcal{K} = \bigcup_{i=1}^{2} \Upsilon_i,$$

where Υ_i are open (in the topology of ∂G) smooth curves and $\mathcal{K} = \bar{\Upsilon}_1 \cap \bar{\Upsilon}_2 = \{g_1, g_2\}$, where g_1 and g_2 are the ends of the curves $\bar{\Upsilon}_1$ and $\bar{\Upsilon}_2$. We assume that, in some neighborhoods of the points g_1, g_2, the domain G coincides with the plane angles of the same aperture $2\omega_0$, $0 < 2\omega_0 < 2\pi$. We consider the following nonlocal problem in the domain G:

$$\Delta u = f_0(y) \quad (y \in G), \tag{9.1}$$

$$u|_{\Upsilon_i} + b_i u\big(\Omega_i(y)\big)\big|_{\Upsilon_i} = f_i(y) \quad (y \in \Upsilon_i; \ i = 1, 2). \tag{9.2}$$

Here $b_1, b_2 \in \mathbb{R}$, and Ω_i is an infinitely differentiable nondegenerate transformation taking a neighborhood $\mathcal{O}_i$ of the curve Υ_i onto $\Omega(\mathcal{O}_i)$ in such a way that $\Omega(\Upsilon_i) \subset G$, $\Omega_i(g_j) = g_j$, $j = 1, 2$, and the transformation Ω_i is the rotation of Υ_i through an angle of ω_0 inwards G (into the domain G) near the points g_1 and g_2 (see Fig. 9.1).

According to Remark 7.2, Condition 7.2 holds. Clearly, Condition 8.1 also holds.

Figure 9.1: Domain G with the boundary $\partial G = \bar{\Upsilon}_1 \cup \bar{\Upsilon}_2$.

One and the same model problem in the plane angle corresponds to each of the points g_1 and g_2:

$$\Delta U = f_0(y) \qquad (y \in K), \tag{9.3}$$

$$U|_{\gamma_j} + b_j U(\mathcal{G}_j y)|_{\gamma_j} = f_j(y) \quad (y \in \gamma_j;\ j = 1, 2), \tag{9.4}$$

where $K = \{y \in \mathbb{R}^2 : r > 0,\ |\omega| < \omega_0\}$, $\gamma_j = \{y \in \mathbb{R}^2 : r > 0,\ \omega = (-1)^j \omega_0\}$, and

$$\mathcal{G}_j = \begin{pmatrix} \cos \omega_0 & (-1)^j \sin \omega_0 \\ (-1)^{j+1} \sin \omega_0 & \cos \omega_0 \end{pmatrix}$$

is the operator of rotation through an angle of $(-1)^{j+1} \omega_0$ about the origin, $j = 1, 2$.

The eigenvalue problem corresponding to problem (9.3), (9.4) is

$$\frac{d^2 \varphi(\omega)}{d\omega^2} - \lambda^2 \varphi(\omega) = 0 \quad (|\omega| < \omega_0), \tag{9.5}$$

$$\varphi(-\omega_0) + b_1 \varphi(0) = 0, \quad \varphi(\omega_0) + b_2 \varphi(0) = 0. \tag{9.6}$$

Let us find the eigenvalues of problem (9.5), (9.6).

I. First, consider the case in which $\lambda \neq 0$. Substituting the general solution $\varphi(\omega) = c_1 e^{\lambda \omega} + c_2 e^{-\lambda \omega}$ of Eq. (9.5) into the nonlocal condition (9.6), we obtain the following system of equations for c_1, c_2:

$$\begin{pmatrix} e^{-\lambda \omega_0} + b_1 & e^{\lambda \omega_0} + b_1 \\ e^{\lambda \omega_0} + b_2 & e^{-\lambda \omega_0} + b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.7}$$

Equating the determinant of system (9.7) with zero, we obtain

$$(e^{-\lambda \omega_0} - e^{\lambda \omega_0})(e^{\lambda \omega_0} + e^{-\lambda \omega_0} + b_1 + b_2) = 0.$$

1. It follows from the equation $e^{-\lambda\omega_0} - e^{\lambda\omega_0} = 0$ that

$$\lambda = \frac{\pi k}{\omega_0}i, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{9.8}$$

2. Consider the equation $e^{\lambda\omega_0} + e^{-\lambda\omega_0} + b_1 + b_2 = 0$. If $b_1 + b_2 = 0$, then

$$\lambda = \frac{\pi/2 + \pi k}{\omega_0}i, \qquad k \in \mathbb{Z}. \tag{9.9}$$

If $b_1 + b_2 \neq 0$, then

$$\lambda^{\pm} = \begin{cases} \dfrac{\log\left(-\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2}\right)}{\omega_0} + \dfrac{2\pi n}{\omega_0}i & \text{for} \quad b_1 + b_2 < -2, \\[4ex] \dfrac{\pm\arctan\dfrac{\sqrt{4 - (b_1 + b_2)^2}}{b_1 + b_2} + 2\pi n}{\omega_0}i & \text{for} \quad -2 < b_1 + b_2 < 0, \\[4ex] \dfrac{\pm\arctan\dfrac{\sqrt{4 - (b_1 + b_2)^2}}{b_1 + b_2} + (2n + 1)\pi}{\omega_0}i & \text{for} \quad 0 < b_1 + b_2 < 2, \\[4ex] \dfrac{\log\left(\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2}\right)}{\omega_0} + \dfrac{(2n + 1)\pi}{\omega_0}i & \text{for} \quad b_1 + b_2 > 2, \end{cases} \tag{9.10}$$

where $n \in \mathbb{Z}$. If $|b_1 + b_2| = 2$, we obtain the eigenvalues from the series (9.8).

II. Similarly, one can consider the case in which $\lambda = 0$ and show that $\lambda = 0$ is an eigenvalue of problem (9.5), (9.6) if and only if $b_1 + b_2 = -2$.

Let us study the special case in which $\omega_0 = \pi/2$ (this implies that $\partial G \in C^{\infty}$).

I. Let $\lambda \neq 0$.

1. Relation (9.8) implies the following purely imaginary eigenvalues with integral imaginary parts:

$$\lambda_{2k} = 2ki, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{9.11}$$

2. If $b_1 + b_2 = 0$, then we obtain the following purely imaginary eigenvalues with integral imaginary parts from (9.9):

$$\lambda_{2k+1} = (2k + 1)i, \qquad k \in \mathbb{Z}. \tag{9.12}$$

If $b_1 + b_2 \neq 0$, then we obtain the following eigenvalues from (9.10):

$$\lambda_n^{\pm} = \begin{cases} \dfrac{2\log\left(-\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2}\right)}{\pi} + 4ni & \text{for} \quad b_1 + b_2 < -2, \\[4ex] \dfrac{\pm 2\arctan\dfrac{\sqrt{4 - (b_1 + b_2)^2}}{b_1 + b_2}}{\pi}i + 4ni & \text{for} \quad -2 < b_1 + b_2 < 0, \\[4ex] \dfrac{\pm 2\arctan\dfrac{\sqrt{4 - (b_1 + b_2)^2}}{b_1 + b_2}}{\pi}i + (4n + 2)i & \text{for} \quad 0 < b_1 + b_2 < 2, \\[4ex] \dfrac{2\log\left(\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2}\right)}{\pi} + (4n + 2)i & \text{for} \quad b_1 + b_2 > 2, \end{cases} \tag{9.13}$$

where $n \in \mathbb{Z}$. If $|b_1 + b_2| = 2$, then we obtain the eigenvalues from the series (9.11).

II. The number $\lambda_0 = 0$ is an eigenvalue of problem (9.5), (9.6) if and only if $b_1 + b_2 = -2$.

Let us consider the operator $\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$ corresponding to problem (9.1), (9.2) with $\omega_0 = \pi/2$. It follows from (9.11)–(9.13) and from Theorem 4.1 that the following assertion holds.

Theorem 9.1. *Suppose that $\omega_0 = \pi/2$. Let l be even. Then the operator*

$$\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$$

has the Fredholm property if and only if $b_1 + b_2 \neq 0$.

Let l be odd. Then the operator

$$\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$$

fails to have the Fredholm property for any $b_1, b_2 \in \mathbb{R}$.

Note that, if l is even and $b_1 = b_2 = 0$, then the operator $\mathbf{L}$ corresponding to the "local" boundary-value problem fails to have the Fredholm property (its image is not closed). However, if we add nonlocal terms with *arbitrarily small* coefficients b_1 and b_2 (such that $b_1 + b_2 \neq 0$) to the boundary-value conditions, then the problem obtains the Fredholm property.

9.1.2. A Problem with Homogeneous Nonlocal Conditions. Let us study problem (9.1), (9.2) with homogeneous nonlocal conditions for the case in which $\omega_0 = \pi/2$. Write

$$W_B^{l+2}(G) = \left\{ u \in W^{l+2}(G) : u|_{\Upsilon_i} + b_i u\big(\Omega_i(y)\big)\big|_{\Upsilon_i} = 0, \ i = 1, 2 \right\}$$

and introduce the corresponding operator $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ given by

$$\mathbf{L}_B u = \Delta u, \qquad u \in W_B^{l+2}(G).$$

The Fredholm property of the operator $\mathbf{L}_B$ depends only on the eigenvalues of problem (9.5), (9.6) lying on the line $\operatorname{Im} \lambda = -(l+1)$, $l \geqslant 0$. Thus, we must consider only the eigenvalues (9.11), (9.12) for $k \leqslant -1$ and the eigenvalues (9.13) for $|b_1 + b_2| > 2$ and $n \leqslant -1$. Clearly, the eigenvalues (9.13) for $|b_1 + b_2| > 2$ are improper because they are not purely imaginary. Let us find the values of the coefficients b_1 and b_2 for which the eigenvalues (9.11) and (9.12) are proper.

1. Consider the numbers $\lambda_{2k} = 2ki$, $k = -1, -2, \ldots$, which are eigenvalues of problem (9.5), (9.6) for any b_1 and b_2. Let us show that λ_{2k} is a proper eigenvalue if and only if $b_1 + b_2 \neq 2(-1)^{k+1}$.

The eigenvector

$$\varphi_{2k}^{(0)}(\omega) = e^{i2k\omega} - e^{-i2k\omega} = 2i \sin(2k\omega)$$

corresponds to the eigenvalue λ_{2k} (for $b_1 = b_2 = (-1)^{k+1}$, there is another eigenvector $\psi_{2k}^{(0)}(\omega) = e^{i2k\omega} + e^{-i2k\omega} = 2\cos(2k\omega)$). If an associate vector $\varphi_{2k}^{(1)}$ exists, then it satisfies the equation

$$\frac{d^2 \varphi_{2k}^{(1)}(\omega)}{d\omega^2} + 4k^2 \varphi_{2k}^{(1)}(\omega) = 4ik\varphi_{2k}^{(0)}(\omega) \qquad (|\omega| < \pi/2) \tag{9.14}$$

and the nonlocal conditions (9.6). Substituting the general solution

$$\varphi_{2k}^{(1)}(\omega) = c_1 e^{i2k\omega} + c_2 e^{-i2k\omega} + \omega(e^{i2k\omega} + e^{-i2k\omega})$$

of Eq. (9.14) into nonlocal conditions (9.6), we obtain the following system of equations for c_1 and c_2:

$$\begin{pmatrix} (-1)^k + b_1 & (-1)^k + b_1 \\ (-1)^k + b_2 & (-1)^k + b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \pi(-1)^k \\ -\pi(-1)^k \end{pmatrix}.$$

Clearly, this system is incompatible if and only if $b_1 + b_2 \neq 2(-1)^{k+1}$. Combining this observation with the fact that $r^{-2k}\varphi_{2k}^{(0)}(\omega)$ is a polynomial in y_1 and y_2 for $k = -1, -2, \ldots$, we see that λ_{2k} is a proper eigenvalue if and only if $b_1 + b_2 \neq 2(-1)^{k+1}$.

2. Consider the numbers $\lambda_{2k+1} = (2k+1)i$, $k = -1, -2, \ldots$, which are eigenvalues of problem (9.5), (9.6) if and only if $b_1 + b_2 = 0$. Let us prove that the eigenvalues λ_{2k+1} are proper for $b_1 + b_2 = 0$.

A unique (up to factor) eigenvector

$$\varphi_{2k+1}^{(0)}(\omega) = e^{i(2k+1)\omega} + e^{-i(2k+1)\omega} = 2i\sin((2k+1)\omega)$$

corresponds to the eigenvalue λ_{2k+1}. If an associate eigenvector $\varphi_{2k+1}^{(1)}$ exists, then it satisfies the equation

$$\frac{d^2\varphi_{2k+1}^{(1)}(\omega)}{d\omega^2} + (2k+1)^2\varphi_{2k+1}^{(1)}(\omega) = 2i(2k+1)\varphi_{2k+1}^{(0)}(\omega) \quad (|\omega| < \pi/2) \tag{9.15}$$

and the nonlocal conditions (9.6). Substituting the general solution

$$\varphi_{2k+1}^{(1)}(\omega) = c_1 e^{i(2k+1)\omega} + c_2 e^{-i(2k+1)\omega} + \omega(e^{i(2k+1)\omega} - e^{-i(2k+1)\omega})$$

of (9.15) into nonlocal conditions (9.6), we obtain the following system of equations for c_1 and c_2:

$$\begin{pmatrix} -i(-1)^k + b_1 & i(-1)^k + b_1 \\ i(-1)^k + b_2 & -i(-1)^k + b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -i\pi(-1)^k \\ -i\pi(-1)^k \end{pmatrix}.$$

This system is clearly incompatible for $b_1 + b_2 = 0$. This observation, together with the fact that $r^{-(2k+1)}\varphi_{2k+1}^{(0)}(\omega)$ is a polynomial in y_1 and y_2 for $k = -1, -2, \ldots$, implies that the eigenvalues λ_{2k+1} are proper for $b_1 + b_2 = 0$.

Remark 9.1. When finding out whether or not an eigenvalue is proper, we used first associate vectors only. Obviously, we can continue this procedure and find an entire Jordan chain (see, e.g., Example 2.1 in [29]); however, we avoid this procedure here because already the existence of a first associate vector implies that the corresponding eigenvalue is improper.

I. Consider the operator
$$\mathbf{L}_B \colon W_B^2(G) \to L_2(G).$$

The line $\operatorname{Im}\lambda = -1$ contains either no eigenvalues of problem (9.5), (9.6) (for $b_1 + b_2 \neq 0$) or only the proper eigenvalue $\lambda_{-1} = -i$ (for $b_1 + b_2 = 0$). Applying either Theorem 8.1 (for $b_1 + b_2 \neq 0$) or Theorem 8.2 (for $b_1 + b_2 = 0$), we see that the *operator*

$$\mathbf{L}_B \colon W_B^2(G) \to L_2(G)$$

has the Fredholm property for any b_1 and b_2.

II. Consider the operator
$$\mathbf{L}_B \colon W_B^3(G) \to W^1(G).$$

(a) Let $b_1 + b_2 > 2$. Then the line $\operatorname{Im}\lambda = -2$ contains the proper eigenvalue $\lambda_{-2} = -2i$ and the two improper eigenvalues

$$\lambda_{-2}^{\pm} = \frac{2\log\left(\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2}\right)}{\pi} - 2i.$$

Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ fails to have the Fredholm property.

(b) Let $b_1 + b_2 = 2$. Then the line $\operatorname{Im} \lambda = -2$ contains only the improper eigenvalue $\lambda_{-2} = -2i$. Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ fails to have the Fredholm property.

(c) Let $b_1 + b_2 < 2$. Then the line $\operatorname{Im} \lambda = -2$ contains only the proper eigenvalue $\lambda_{-2} = -2i$. We must verify Condition 8.2. Differentiating the expression $U(y) + b_j U(\mathcal{G}_j y)$ with respect to y_2 twice and replacing the values of the corresponding function at the point $\mathcal{G}_j y$ by the values at y, we see that system (2.11) acquires the following form:

$$\hat{\mathcal{B}}_1(D_y)U = \frac{\partial^2 U}{\partial y_2^2} + b_1 \frac{\partial^2 U}{\partial y_1^2}, \quad \hat{\mathcal{B}}_2(D_y)U = \frac{\partial^2 U}{\partial y_2^2} + b_2 \frac{\partial^2 U}{\partial y_1^2}.$$

(c$_1$) Let $b_1 \neq b_2$. Then the operators $\hat{\mathcal{B}}_1(D_y)U$ and $\hat{\mathcal{B}}_2(D_y)U$ are linearly independent, and therefore both of them enter system (3.4). Clearly, the operator ΔU does not enter this system because the system

$$\mathcal{B}_1(D_y)U, \qquad \hat{\mathcal{B}}_2(D_y)U, \qquad \Delta U$$

is linearly dependent. Hence, Condition 8.2 fails, and Theorem 8.2 implies that the operator

$$\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$$

cannot have the Fredholm property.

(c$_2$) Let $b_1 = b_2$ (and therefore $b_1 = b_2 < 1$). Then the operators $\hat{\mathcal{B}}_1(D_y)U$ and $\hat{\mathcal{B}}_2(D_y)U$ coincide. Since $b_1 < 1$, it follows that the system

$$\hat{\mathcal{B}}_1(D_y)U, \qquad \Delta U$$

is linearly independent and forms system (3.4). Hence, Condition 8.2 holds, and, by Theorem 8.2, the operator $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ has the Fredholm property.

Thus, we have proved that the *operator* $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ *has the Fredholm property if and only if* $b_1 = b_2 < 1$.

III. Consider the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G) \quad \text{with even} \quad l, \ l \geqslant 2.$$

(a) Let $b_1 + b_2 \neq 0$. Then the line $\operatorname{Im} \lambda = -(l+1)$ contains no eigenvalues of problem (9.5), (9.6). Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ has the Fredholm property.

(b) Let $b_1 + b_2 = 0$. Then the line $\operatorname{Im} \lambda = -(l+1)$ contains only the proper eigenvalue $\lambda_{-(l+1)} = -(l+1)i$. In contrast to the case in which $l = 0$, we must now verify Condition 8.2. Differentiating the expression $U(y) + b_j U(\mathcal{G}_j y)$ with respect to y_2 ($l+1$ times) and replacing the values of the corresponding function at the point $\mathcal{G}_j y$ by the values at y, we see that system (2.11) acquires the form

$$\hat{\mathcal{B}}_1(D_y)U = \frac{\partial^{l+1} U}{\partial y_2^{l+1}} - b_1 \frac{\partial^{l+1} U}{\partial y_1^{l+1}}, \quad \hat{\mathcal{B}}_2(D_y)U = \frac{\partial^{l+1} U}{\partial y_2^{l+1}} + b_2 \frac{\partial^{l+1} U}{\partial y_1^{l+1}}.$$

Since $b_2 = -b_1$, only the operator $\hat{\mathcal{B}}_1(D_y)U$ enters system (3.4).

Let us show that the system consisting of the operator $\hat{\mathcal{B}}_1(D_y)U$ and the operators

$$\frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U \equiv \frac{\partial^{l+1} U}{\partial y_1^{\xi_1+2} \partial y_2^{\xi_2}} + \frac{\partial^{l+1} U}{\partial y_1^{\xi_1} \partial y_2^{\xi_2+2}}, \qquad \xi_1 + \xi_2 = l - 1,$$

is linearly independent. To do this, to each derivative

$$\frac{\partial^{l+1} U}{\partial y_1^s \partial y_2^{l+1-s}}, \qquad s = 0, \dots, l+1,$$

we assign the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length $l + 2$ such that its $(s+1)$st component is equal to one and the other components are equal to zero. In this case, the operator $\hat{\mathcal{B}}_1(D_y)U$ is assigned to the vector

$$(1, 0, \ldots, 0, -b_1), \tag{9.16}$$

and the operators $\dfrac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}}\Delta$, $\xi_1 = 0, \ldots, l-1$, are assigned to the vectors

$$(0, \ldots, 1, 0, 1, \ldots, 0) \tag{9.17}$$

such that their $(\xi_1 + 1)$st and $(\xi_1 + 3)$rd components are equal to one and the other components are equal to zero. Thus, we must show that the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & -b_1 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \end{pmatrix}$$

(of order $(l+1) \times (l+2)$) formed by the rows (9.16), (9.17) is equal to $l+1$. Denote by A' the matrix obtained from A by deleting the last column of A. Expanding the determinant of A' with respect to the first row, we see that $\det A' = \det A_l$, where

$$A_l = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$$

is a tridiagonal matrix of order $l \times l$. One can directly show by induction that

$$\det A_l = \begin{cases} 0 & \text{for} \quad l = 2n-1, \\ 1 & \text{for} \quad l = 4n, \\ -1 & \text{for} \quad l = 4n-2, \end{cases} \tag{9.18}$$

where $n \geqslant 1$. It follows from (9.18) that $|\det A'| = |\det A_l| = 1$. Therefore, the system

$$\hat{\mathcal{B}}_1(D_y)U, \quad \frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}}\Delta U, \ \xi_1 + \xi_2 = l - 1,$$

is linearly independent, and Theorem 8.2 implies that the operator $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ has the Fredholm property.

Thus, we have proved that the *operator* $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ *with even* l, $l \geqslant 2$, *has the Fredholm property for any* b_1 *and* b_2.

IV. Finally, consider the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G) \qquad \text{with odd } l, \qquad l \geqslant 3.$$

Assume first that $l + 1 = 4n$ for some $n \geqslant 1$.

(a) Let $b_1 + b_2 < -2$. Then the line $\operatorname{Im} \lambda = -(l+1) = -4n$ contains the proper eigenvalue $\lambda_{-4n} = -4ni$ and the two improper eigenvalues

$$\lambda^{\pm}_{-4n} = \frac{2 \log \left(\dfrac{b_1 + b_2}{2} \pm \dfrac{\sqrt{(b_1 + b_2)^2 - 4}}{2} \right)}{\pi} - 4ni.$$

Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W^{l+2}_B(G) \to W^l(G)$ cannot have the Fredholm property.

(b) Let $b_1 + b_2 = -2$. Then the line $\operatorname{Im} \lambda = -(l+1) = -4n$ contains only the improper eigenvalue $\lambda_{-4n} = -4ni$. Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W^{l+2}_B(G) \to W^l(G)$ cannot have the Fredholm property.

(c) Let $b_1 + b_2 > -2$. Then the line $\operatorname{Im} \lambda = -(l+1) = -4n$ contains only the proper eigenvalue $\lambda_{-2} = -4ni$. We must verify Condition 8.2. Differentiating the expression $U(y) + b_j U(\mathcal{G}_j y)$ with respect to y_2 ($l+1$ times) and replacing the values of the corresponding function at the point $\mathcal{G}_j y$ by the values at y, we see that system (2.11) has the form

$$\hat{\mathcal{B}}_1(D_y)U = \frac{\partial^{l+1} U}{\partial y_2^{l+1}} + b_1 \frac{\partial^{l+1} U}{\partial y_1^{l+1}}, \qquad \hat{\mathcal{B}}_2(D_y)U = \frac{\partial^{l+1} U}{\partial y_2^{l+1}} + b_2 \frac{\partial^{l+1} U}{\partial y_1^{l+1}}.$$

(c_1) Let $b_1 \neq b_2$. Then the operators $\hat{\mathcal{B}}_1(D_y)U$ and $\hat{\mathcal{B}}_2(D_y)U$ are linearly independent, and therefore both of them are included in system (3.4). Let us show that the system

$$\hat{\mathcal{B}}_1(D_y)U, \qquad \hat{\mathcal{B}}_2(D_y)U, \qquad \frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U, \qquad \xi_1 + \xi_2 = l - 1,$$

is linearly dependent. (Note that, unlike the case in which $l = 1$, this system now contains all the derivatives of U of order $l + 1$.) Since $\hat{\mathcal{B}}_1(D_y)U$ and $\hat{\mathcal{B}}_2(D_y)U$ are linearly independent, it suffices to show that the system

$$\frac{\partial^{l+1} U}{\partial y_2^{l+1}}, \qquad \frac{\partial^{l+1} U}{\partial y_1^{l+1}}, \qquad \frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U, \qquad \xi_1 + \xi_2 = l - 1,$$

is linearly dependent. Let us consider the corresponding matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \end{pmatrix}$$

of order $(l+2) \times (l+2)$. Decomposing the determinant of A with respect to the first row and then decomposing the determinant of the resulting matrix with respect to the first row again, we see that $\det A = \det A_l$. Since l is odd, it follows from (9.18) that $\det A = 0$. Therefore, Condition 8.2 fails, and Theorem 8.2 implies that the operator $\mathbf{L}_B \colon W^{l+2}_B(G) \to W^l(G)$ is not Fredholm.

(c_2) Let $b_1 = b_2$ (and therefore $b_1 = b_2 > -1$). Then system (3.4) contains only the operator $\hat{\mathcal{B}}_1(D_y)U$. Let us show that the system

$$\hat{\mathcal{B}}_1(D_y)U, \qquad \frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U, \qquad \xi_1 + \xi_2 = l - 1,$$

is linearly independent. Consider the corresponding matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & b_1 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \end{pmatrix}$$

of order $(l+1) \times (l+2)$. Deleting the second column from A, decomposing the determinant of the matrix thus obtained with respect to the first row, and using relation (9.18), we obtain

$$\begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & b_1 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \end{vmatrix} = 1 - b_1 \begin{vmatrix} 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{vmatrix} = 1 - b_1 \det A_{l-1} = 1 + b_1 \neq 0$$

because $b_1 > -1$. Therefore, Condition 8.2 holds, and Theorem 8.2 implies that the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property.

Thus, we have proved that the *operator* $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ *with* $l+1 = 4n$, $n \geqslant 1$, *has the Fredholm property if and only if* $b_1 = b_2 > -1$.

Similary, by using (9.18) and Theorem 8.2, one can show that the *operator*

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

with $l+1 = 4n+2$, $n \geqslant 1$, *has the Fredholm property if and only if* $b_1 = b_2 < 1$.

The following theorem summarizes the results thus obtained.

Theorem 9.2. *Let l be even. Then the operator*

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property for any $b_1, b_2 \in \mathbb{R}$.

Let l be odd and let $l = 4n+1$, where $n = 0, 1, 2, \ldots$. Then the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property if and only if $b_1 = b_2 < 1$.

Let l be odd and $l = 4n+3$, where $n = 0, 1, 2, \ldots$. Then the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property if and only if $b_1 = b_2 > -1$.

Note that, for $\omega_0 = \pi/2$ and $b_1 = b_2 = 0$, we obtain the "local" Dirichlet problem in a smooth domain with homogeneous boundary conditions. In this case, as is well known, the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

corresponding to problem (9.1), (9.2) with homogeneous boundary conditions is invertible for any $l \geqslant 0$ rather than simply an operator with the Fredholm property.

9.2. *Example 2*

9.2.1. A Problem with Nonhomogeneous Nonlocal Conditions. Let $G \subset \mathbb{R}^2$ be a domain such that its boundary $\partial G \in C^\infty$ coincides outside the disks $B_{1/8}((i4/3, j4/3))$ $(i, j = 0, 1)$ with the boundary of the square $(0, 4/3) \times (0, 4/3)$. We write $\Upsilon_1 = \{y \in \partial G : y_1 < 1/3, \ y_2 < 1/3\}$, $\Upsilon_2 = \{y \in \partial G : y_1 > 1, \ y_2 > 1\}$, $\Upsilon_3 = \partial G \setminus (\bar{\Upsilon}_1 \cup \bar{\Upsilon}_2)$. Thus, $\mathcal{K} = \{g_1, \ldots, g_4\}$, where $g_1 = (1/3, 0)$, $g_2 = (0, 1/3)$, $g_3 = (4/3, 1)$, $g_4 = (1, 4/3)$ (see Fig. 9.2).

Figure 9.2: Domain G with smooth boundary $\partial G = \bar{\Upsilon}_1 \cup \bar{\Upsilon}_2 \cup \bar{\Upsilon}_3$.

We consider the following nonlocal elliptic problem in the domain G:

$$\Delta u = f_0(y) \qquad (y \in G), \tag{9.19}$$

$$u(y)|_{\Upsilon_i} + b_i u(y + h_i)|_{\Upsilon_i} = f_i(y) \quad (y \in \Upsilon_i; \ i = 1, 2), \qquad u(y)|_{\Upsilon_3} = f_3(y) \quad (y \in \Upsilon_3), \tag{9.20}$$

where $h_1 = (1, 1)$, $h_2 = (-1, -1)$, and $b_1, b_2 \in \mathbb{R}$. Clearly, $\mathcal{K} = \mathrm{Orb}_1 \cup \mathrm{Orb}_2$, where the orbit Orb_1 consists of the points g_1 and $g_3 = g_1 + h_1$ and the orbit Orb_2 consists of the points g_2 and $g_4 = g_2 + h_2$.

According to Remark 7.2, Condition 7.2 holds. Clearly, Condition 8.1 also holds.

Assume first that $b_1^2 + b_2^2 \neq 0$ (to be definite, we suppose that $b_1 \neq 0$).

One and the same model problem in the plane angles corresponds to each of the orbits Orb_1 and Orb_2,

$$\Delta U_j = f_j(y) \quad (y \in K), \tag{9.21}$$

$$\begin{array}{ll} U_1|_{\gamma_1} = f_{11}(y) \ (y \in \gamma_1), & U_1|_{\gamma_2} + b_1 U_2(\mathcal{G}y)|_{\gamma_2} = f_{12}(y) \ (y \in \gamma_2), \\ U_2|_{\gamma_1} = f_{21}(y) \ (y \in \gamma_1), & U_2|_{\gamma_2} + b_2 U_1(\mathcal{G}y)|_{\gamma_2} = f_{22}(y) \ (y \in \gamma_2). \end{array} \tag{9.22}$$

Here $K = \{y \in \mathbb{R}^2 : r > 0, \ |\omega| < \pi/2\}$, $\gamma_j = \{y \in \mathbb{R}^2 : r > 0, \ \omega = (-1)^j \pi/2\}$, and

$$\mathcal{G} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the operator of rotation through the angle of $-\pi/2$.

The eigenvalue problem corresponding to problem (9.21), (9.22) has the following form:

$$\frac{d^2\varphi_j(\omega)}{d\omega^2} - \lambda^2\varphi_j(\omega) = 0 \qquad (|\omega| < \pi/2; \ j = 1, 2), \tag{9.23}$$

$$\begin{array}{ll} \varphi_1(-\pi/2) = 0, & \varphi_1(\pi/2) + b_1\varphi_2(0) = 0, \\ \varphi_2(-\pi/2) = 0, & \varphi_2(\pi/2) + b_2\varphi_1(0) = 0. \end{array} \tag{9.24}$$

One can find the eigenvalues of problem (9.23), (9.24) by straightforward computations (see [19]). They are as follows:

$$\lambda_{2k} = 2ki, \qquad k \in \mathbb{Z} \setminus \{0\} \qquad (\text{for any } b_1, b_2, \ b_1^2 + b_2^2 \neq 0), \tag{9.25}$$

$$\lambda_{2k+1} = (2k+1)i, \qquad k \in \mathbb{Z} \qquad (\text{for } b_2 = 0, \ b_1 \neq 0), \tag{9.26}$$

and

$$\lambda_n^{\pm} = \begin{cases} \dfrac{2}{\pi}\log\left|\dfrac{\sqrt{-b_1b_2}}{2} \pm \dfrac{\sqrt{4 - b_1b_2}}{2}\right| + (2n+1)i & \text{for} \quad b_1b_2 < 0, \\[3mm] \left(\pm\dfrac{2}{\pi}\arctan\sqrt{4(b_1b_2)^{-1} - 1} + 2n\right)i & \text{for} \quad 0 < b_1b_2 < 4, \\[3mm] \dfrac{2}{\pi}\log\left(\dfrac{\sqrt{b_1b_2}}{2} \pm \dfrac{\sqrt{b_1b_2 - 4}}{2}\right) + 2ni & \text{for} \quad b_1b_2 \geqslant 4, \end{cases} \tag{9.27}$$

where $n \in \mathbb{Z}$. If $b_1b_2 = 4$, then there is another eigenvalue, namely, $\lambda_0 = 0$.

Remark 9.2. If $b_2 = 0$, then we can consider another setting of a nonlocal problem which differs from problem (9.19), (9.20), namely,

$$\Delta u = f(y) \qquad (y \in G), \qquad u(y)|_{\Upsilon_1} + b_1 u(y + h_1)|_{\Upsilon_1} = f_1(y) \qquad (y \in \Upsilon_1), \tag{9.28}$$

$$u(y)|_{\bar{\Upsilon}_2 \cup \Upsilon_3} = f_2(y) \qquad (y \in \bar{\Upsilon}_2 \cup \Upsilon_3). \tag{9.29}$$

In this case, $\mathcal{K} = \{g_1, g_2\}$ (note that Condition 7.2 fails here). Solutions of problem (9.28), (9.29) can have singularities only near the points g_1 and g_2, while solutions of problem (9.19), (9.20) can have singularities near the points $g_1, \ldots, g_4$.

To each of the points g_1 and g_2, the same model "*local*" problem corresponds, namely,

$$\Delta U_1 = f_1(y) \qquad (y \in K), \tag{9.30}$$

$$U_1|_{\gamma_1} = f_1(y) \ (y \in \gamma_1), \qquad U_1|_{\gamma_2} = f_2(y) \ (y \in \gamma_2). \tag{9.31}$$

The eigenvalues problem for problem (9.30), (9.31) has the following form:

$$\frac{d^2\varphi_1(\omega)}{d\omega^2} - \lambda^2\varphi_1(\omega) = 0 \quad (|\omega| < \pi/2), \tag{9.32}$$

$$\varphi_1(-\pi/2) = \varphi_1(\pi/2) = 0. \tag{9.33}$$

The eigenvalues of problem (9.32), (9.33) are as follows:

$$\lambda_k = ki, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{9.34}$$

They coincide with the eigenvalues of problem (9.23), (9.24) for $b_2 = 0$. Therefore, according to Theorem 4.1, problem (9.28), (9.29) has the Fredholm property if and only if problem (9.19), (9.20) has the Fredholm property.

Let us consider the operator

$$\mathbf{L}\colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$$

corresponding to problem (9.19), (9.20). The following theorem results from (9.25)–(9.27) and from Theorem 4.1.

Theorem 9.3. *Let l be even. The operator $\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$ has the Fredholm property if and only if $b_1 b_2 > 0$.*

Let l be odd. The operator

$$\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$$

fails to have the Fredholm property for any $b_1, b_2 \in \mathbb{R}$.

Note that Theorem 9.3 is proved under the assumption that $b_1^2 + b_2^2 \neq 0$; however, the operator $\mathbf{L} \colon W^{l+2}(G) \to \mathcal{W}^l(G, \Upsilon)$ (with $b_1 = b_2 = 0$) corresponding to problem (9.19), (9.20) cannot have the Fredholm property either. This follows from the fact that, to each of the points $g_1, \ldots, g_4 \in \mathcal{K}$, one assigns the model problem (9.32), (9.33) with the eigenvalues (9.34) lying on the lines $-(l+1)$, $l \geqslant 0$.

9.2.2. **A Problem with Homogeneous Nonlocal Conditions.** Let us study problem (9.19), (9.20) with homogeneous nonlocal conditions. Write

$$W_B^{l+2}(G) = \big\{ u \in W^{l+2}(G) : u|_{\Upsilon_i} + b_i u(y + h_i)|_{\Upsilon_i} = 0, \ i = 1, 2; \ u|_{\Upsilon_3} = 0 \big\}$$

and introduce the corresponding operator $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ by

$$\mathbf{L}_B u = \Delta u, \qquad u \in W_B^{l+2}(G).$$

Assume first that $b_1^2 + b_2^2 \neq 0$ (to be definite, we again suppose that $b_1 \neq 0$).

Remark 9.3. Problem (9.28), (9.29) with homogeneous nonlocal conditions is equivalent to problem (9.19), (9.20) with $b_2 = 0$. Hence, one need not study problem (9.28), (9.29) independently.

The Fredholm property of the operator $\mathbf{L}_B$ is related only to the eigenvalues of problem (9.23), (9.24) lying on the line $\operatorname{Im} \lambda = -(l+1)$, $l \geqslant 0$. Thus, we must consider only the eigenvalues λ_{2k} and λ_{2k+1} (for $b_2 = 0$) and $\lambda_n^{\pm}$ (for $b_1 b_2 < 0$ or $b_1 b_2 \geqslant 4$) for $k, n \leqslant -1$. Clearly, the eigenvalues $\lambda_n^{\pm}$ (for $b_1 b_2 < 0$ or $b_1 b_2 \geqslant 4$) are improper because they are not purely imaginary. Let us find the values of the coefficients b_1 and b_2 for which the eigenvalues λ_{2k} and λ_{2k+1} (if $b_2 = 0$) are proper.

1. Consider the numbers $\lambda_{2k} = 2ki$, $k = -1, -2, \ldots$, which are eigenvalues of problem (9.23), (9.24) for any b_1 and b_2. Let us show that λ_{2k} is a proper eigenvalue.

Two linearly independent eigenvectors correspond to the eigenvalue λ_{2k},

$$\big(\varphi_{1,2k}^{(0)}(\omega), \ \varphi_{2,2k}^{(0)}(\omega)\big) = \big(e^{i2k\omega} - e^{-i2k\omega}, \ 0\big) = \big(2i\sin(2k\omega), \ 0\big),$$

$$\big(\psi_{1,2k}^{(0)}(\omega), \ \psi_{2,2k}^{(0)}(\omega)\big) = \big(0, \ e^{i2k\omega} - e^{-i2k\omega}\big) = \big(0, \ 2i\sin(2k\omega)\big).$$

If an associate vector $(\varphi_{1,2k}^{(1)}, \ \varphi_{2,2k}^{(1)})$ corresponding to the first of the eigenvectors exists, then it satisfies the equations

$$\frac{d^2 \varphi_{1,2k}^{(1)}(\omega)}{d\omega^2} + 4k^2 \varphi_{1,2k}^{(1)}(\omega) = 4ik\big(e^{i2k\omega} - e^{-i2k\omega}\big) \qquad (|\omega| < \pi/2),$$

$$\frac{d^2 \varphi_{2,2k}^{(1)}(\omega)}{d\omega^2} + 4k^2 \varphi_{2,2k}^{(1)}(\omega) = 0 \qquad (|\omega| < \pi/2) \tag{9.35}$$

and the nonlocal conditions (9.24). Substituting the general solution

$$\varphi_{1,2k}^{(1)}(\omega) = c_1 e^{i2k\omega} + c_2 e^{-i2k\omega} + \omega\big(e^{i2k\omega} + e^{-i2k\omega}\big), \qquad \varphi_{2,2k}^{(1)}(\omega) = c_3 e^{i2k\omega} + c_4 e^{-i2k\omega},$$

of Eqs. (9.35) into the nonlocal conditions (9.24), we obtain the following system of equations for the indeterminates $c_1, \ldots, c_4$:

$$\begin{pmatrix} (-1)^k & (-1)^k & 0 & 0 \\ (-1)^k & (-1)^k & b_1 & b_1 \\ 0 & 0 & (-1)^k & (-1)^k \\ b_2 & b_2 & (-1)^k & (-1)^k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \pi(-1)^k \\ -\pi(-1)^k \\ 0 \\ 0 \end{pmatrix}.$$

One can readily see that this system is incompatible; therefore, the first eigenvector has no associate vectors. One can similarly see that the second eigenvector has no associate vectors either. Combining this observation with the fact that $r^{-2k}\varphi_{j,2k}^{(0)}(\omega)$ and $r^{-2k}\psi_{j,2k}^{(0)}(\omega)$ $(j=1,2)$ are polynomials in y_1, y_2 for $k = -1, -2, \ldots$, we see that λ_{2k} is a proper eigenvalue.

2. Consider the numbers $\lambda_{2k+1} = (2k+1)i$, $k = -1, -2, \ldots$, which are eigenvalues of problem (9.23), (9.24) with $b_2 = 0$ (recall that $b_1 \neq 0$). Let us show that λ_{2k+1} is an improper eigenvalue.

The only eigenvector (up to factor) corresponding to the eigenvalue λ_{2k+1} is

$$\left(\varphi_{1,2k+1}^{(0)}(\omega),\ \varphi_{2,2k+1}^{(0)}(\omega)\right) = (e^{i(2k+1)\omega} + e^{-i(2k+1)\omega},\ 0) = (2\cos((2k+1)\omega),\ 0).$$

If an associate eigenvector $(\varphi_{1,2k+1}^{(1)},\ \varphi_{2,2k+1}^{(1)})$ exists, then it satisfies the equations

$$
\begin{aligned}
\frac{d^2\varphi_{1,2k+1}^{(1)}(\omega)}{d\omega^2} + (2k+1)^2\varphi_{1,2k+1}^{(1)}(\omega) &= 2(2k+1)i(e^{i(2k+1)\omega} + e^{-i(2k+1)\omega}) \quad (|\omega| < \pi/2), \\
\frac{d^2\varphi_{2,2k+1}^{(1)}(\omega)}{d\omega^2} + (2k+1)^2\varphi_{2,2k+1}^{(1)}(\omega) &= 0 \qquad (|\omega| < \pi/2)
\end{aligned}
\tag{9.36}
$$

and the nonlocal conditions (9.24). Substituting the general solution

$$
\begin{aligned}
\varphi_{1,2k}^{(1)}(\omega) &= c_1 e^{i(2k+1)\omega} + c_2 e^{-i(2k+1)\omega} + \omega(e^{i(2k+1)\omega} - e^{-i(2k+1)\omega}), \\
\varphi_{2,2k}^{(1)}(\omega) &= c_3 e^{i(2k+1)\omega} + c_4 e^{-i(2k+1)\omega},
\end{aligned}
$$

of Eqs. (9.36) into the nonlocal conditions (9.24), we obtain the following system of equations for the indeterminates $c_1, \ldots, c_4$:

$$
\begin{pmatrix}
i(-1)^{k+1} & i(-1)^k & 0 & 0 \\
i(-1)^k & i(-1)^{k+1} & b_1 & b_1 \\
0 & 0 & i(-1)^{k+1} & i(-1)^k \\
0 & 0 & i(-1)^k & i(-1)^{k+1}
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4
\end{pmatrix}
=
\begin{pmatrix}
\pi i(-1)^{k+1} \\ \pi i(-1)^{k+1} \\ 0 \\ 0
\end{pmatrix}.
$$

One can readily see that this system is compatible; therefore, λ_{2k+1} is an improper eigenvalue.

I. Consider the operator $\mathbf{L}_B \colon W_B^2(G) \to L_2(G)$. The line $\operatorname{Im}\lambda = -1$ either has no eigenvalues of problem (9.23), (9.24) (for $b_1 b_2 > 0$) or contains an improper eigenvalue λ_{-1} (for $b_2 = 0$) or $\lambda_{-1}^{\pm}$ (for $b_1 b_2 < 0$). Therefore, by Theorem 8.1, the *operator*

$$\mathbf{L}_B \colon W_B^2(G) \to L_2(G)$$

has the Fredholm property if and only if $b_1 b_2 > 0$.

II. Consider the operator

$$\mathbf{L}_B \colon W_B^3(G) \to W^1(G).$$

(a) Let $b_1 b_2 \geqslant 4$. Then the line $\operatorname{Im}\lambda = -2$ contains a proper eigenvalue λ_{-2} and two improper eigenvalues $\lambda_{-1}^{\pm}$. Therefore, by Theorem 8.1, the operator $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ is not Fredholm.

(b) Let $b_1 b_2 < 4$. Then the only eigenvalue on the line $\operatorname{Im}\lambda = -2$ is the proper eigenvalue $\lambda_{-2} = -2i$. Let us show that Condition 8.2 fails.

Differentiating the expressions $U_1(y) + b_1 U_2(\mathcal{G}y)$ and $U_2(y) + b_2 U_1(\mathcal{G}y)$ with respect to y_2 twice and replacing the values of the corresponding functions at the point $\mathcal{G}y$ by the values at y, we see that system (2.11) has the following form:

$$
\begin{aligned}
\hat{\mathcal{B}}_{11}(D_y)U &= \frac{\partial^2 U_1}{\partial y_2^2}, \quad \hat{\mathcal{B}}_{12}(D_y)U = \frac{\partial^2 U_1}{\partial y_2^2} + b_1 \frac{\partial^2 U_2}{\partial y_1^2}, \\
\hat{\mathcal{B}}_{21}(D_y)U &= \frac{\partial^2 U_2}{\partial y_2^2}, \quad \hat{\mathcal{B}}_{22}(D_y)U = \frac{\partial^2 U_2}{\partial y_2^2} + b_2 \frac{\partial^2 U_1}{\partial y_1^2}.
\end{aligned}
$$

Since $b_1 \neq 0$, the operators $\hat{\mathcal{B}}_{11}(D_y)U$, $\hat{\mathcal{B}}_{12}(D_y)U$, and $\hat{\mathcal{B}}_{21}(D_y)U$ are linearly independent, and are therefore included in system (3.4). However, the system consisting of these three operators and of the operators ΔU_1 and ΔU_2 is linearly dependent. Therefore, Condition 8.2 fails, and it follows from Theorem 8.2 that the operator

$$\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$$

cannot have the Fredholm property.

Thus, we have proved that the *operator* $\mathbf{L}_B \colon W_B^3(G) \to W^1(G)$ *cannot have the Fredholm property for any* b_1, b_2 $(b_1^2 + b_2^2 \neq 0)$.

III. Consider the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

with even l, $l \geqslant 2$. The line $\operatorname{Im}\lambda = -(l+1)$ either has no eigenvalues of problem (9.23), (9.24) (for $b_1 b_2 > 0$) or contains an improper eigenvalue $\lambda_{-(l+1)}$ (for $b_2 = 0$) or $\lambda_{-1-l/2}^{\pm}$ (for $b_1 b_2 < 0$). Therefore, by Theorem 8.1, the *operator* $\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$ *with even* l, $l \geqslant 2$, *has the Fredholm property if and only if* $b_1 b_2 > 0$.

IV. Consider the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

with odd l, $l \geqslant 3$.

(a) Let $b_1 b_2 \geqslant 4$. Then the line $\operatorname{Im}\lambda = -(l+1)$ contains a proper eigenvalue $\lambda_{-(l+1)}$ and the two improper eigenvalues $\lambda_{-1/2-l/2}^{\pm}$. Therefore, by Theorem 8.1, the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

cannot have the Fredholm property.

(b) Let $b_1 b_2 < 4$. Then the line $\operatorname{Im}\lambda = -(l+1)$ contains the proper eigenvalue $\lambda_{-(l+1)} = -(l+1)i$ only. Let us show that Condition 8.2 fails. Differentiating the expressions $U_1(y) + b_1 U_2(\mathcal{G}y)$ and $U_2(y) + b_2 U_1(\mathcal{G}y)$ with respect to y_2 ($l+1$ times) and replacing the values of the corresponding functions at the point $\mathcal{G}y$ by the values at y, we see that system (2.11) has the following form:

$$\hat{\mathcal{B}}_{11}(D_y)U = \frac{\partial^{l+1}U_1}{\partial y_2^{l+1}}, \quad \hat{\mathcal{B}}_{12}(D_y)U = \frac{\partial^{l+1}U_1}{\partial y_2^{l+1}} + b_1 \frac{\partial^{l+1}U_2}{\partial y_1^{l+1}},$$

$$\hat{\mathcal{B}}_{21}(D_y)U = \frac{\partial^{l+1}U_2}{\partial y_2^{l+1}}, \quad \hat{\mathcal{B}}_{22}(D_y)U = \frac{\partial^{l+1}U_2}{\partial y_2^{l+1}} + b_2 \frac{\partial^{l+1}U_1}{\partial y_1^{l+1}}.$$

Since $b_1 \neq 0$, it follows that the operators $\hat{\mathcal{B}}_{11}(D_y)U$, $\hat{\mathcal{B}}_{12}(D_y)U$, and $\hat{\mathcal{B}}_{21}(D_y)U$ are linearly independent, and therefore they are included in system (3.4). Let us show that the system consisting of these three operators and the operators

$$\frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U_1 \equiv \frac{\partial^{l+1}U_1}{\partial y_1^{\xi_1+2} \partial y_2^{\xi_2}} + \frac{\partial^{l+1}U_1}{\partial y_1^{\xi_1} \partial y_2^{\xi_2+2}}, \quad \xi_1 + \xi_2 = l - 1,$$

$$\frac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U_2 \equiv \frac{\partial^{l+1}U_2}{\partial y_1^{\xi_1+2} \partial y_2^{\xi_2}} + \frac{\partial^{l+1}U_2}{\partial y_1^{\xi_1} \partial y_2^{\xi_2+2}}, \quad \xi_1 + \xi_2 = l - 1,$$

is linearly dependent. To do this, to each derivative

$$\frac{\partial^{l+1}U_1}{\partial y_1^s \partial y_2^{l+1-s}}, \qquad s = 0, \dots, l+1,$$

we assign the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length $2l + 4$ such that its $(s+1)$st component is equal to one and the other components are equal to zero. Further, to each derivative

$$\frac{\partial^{l+1} U_2}{\partial y_1^s \partial y_2^{l+1-s}}, \qquad s = 0, \ldots, l+1,$$

we assign the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ of length $2l + 4$ such that its $(l + 2 + s + 1)$st component is equal to one and the other components are equal to zero. Thus, it suffices to show that the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & b_1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 \\ \hline 1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

(of order $(2l + 3) \times (2l + 4)$) is less than $2l + 3$. (In the matrix A, the first three rows correspond to the operators $\hat{\mathcal{B}}_{11}(D_y)U$, $\hat{\mathcal{B}}_{12}(D_y)U$, and $\hat{\mathcal{B}}_{21}(D_y)U$, respectively, the next $l + 2$ rows correspond to the operators $\dfrac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U_1$, and the last $l + 2$ rows correspond to the operators $\dfrac{\partial^{l-1}}{\partial y_1^{\xi_1} \partial y_2^{\xi_2}} \Delta U_2$.)

Delete the 1st column, the $(l + 3)$rd column, or the $(2l + 4)$th column from the matrix A. Then the 1st row, the 3rd row, or the difference between the 1st and 2nd rows in the resulting matrix vanishes. Denote by $\hat{A}$ the matrix obtained from A by deleting any other column. Then, consecutively decomposing the determinant of $\hat{A}$ with respect to the first three rows, we see that $|\det \hat{A}| = |b_1 \det A'|$, where A' is the matrix of order $2l \times 2l$ obtained by deleting the corresponding column from the matrix

$$A'' = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\ \hline 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

of order $2l \times (2l + 1)$. Note that the last l rows of A'' form the matrix $(0 \ A_l)$, and thus are linearly dependent by virtue of (9.18). Therefore, after deleting any column from A'', we obtain a degenerate matrix A'. Hence, $\det \hat{A} = 0$, and the rank of the matrix A is less than $2l + 3$. Thus, Condition 8.2 fails, and Theorem 8.2 implies that the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

cannot have the Fredholm property.

We have thus proved that *the operator*

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

with odd l, $l \geqslant 3$, cannot have the Fredholm property for any b_1 and b_2.

We have considered the case in which $b_1^2 + b_2^2 \neq 0$. If $b_1 = b_2 = 0$, then one can similarly show that the corresponding operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property for any $l \geqslant 0$. However, we omit the proof of this fact because, for $b_1 = b_2 = 0$, we obtain the "local" Dirichlet problem in a smooth domain. As is well known, this problem is uniquely solvable for any $l \geqslant 0$ rather than simply have the Fredholm property.

The following theorem summarizes the results obtained in this direction.

Theorem 9.4. *Let l be even. Then the operator*

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property if and only if either $b_1 b_2 > 0$ or $b_1 = b_2 = 0$.

Let l be odd. Then the operator

$$\mathbf{L}_B \colon W_B^{l+2}(G) \to W^l(G)$$

has the Fredholm property if and only if $b_1 = b_2 = 0$.

REFERENCES

1. Feller, W., Diffusion Processes in One Dimension, *Trans. Amer. Math. Soc.*, 1954, vol. 77, pp. 1–31.
2. Samarskii, A. A., On Some Problems of the Theory of Differential Equations, *Differentsial'nye Uravneniya*, 1980, vol. 16, no. 11, pp. 1925–1935; English transl. in *Different. Equat.*, 1980, vol. 16.
3. Onanov, G. G., Skubachevskii, A. L., Differential Equations with Deviating Arguments in Stationary Problems in the Mechanics of Deformable Bodies, *Prikladnaya Mekhanika*, 1979, vol. 15, pp. 39–47; English transl. in *Soviet Applied Mech.*, 1979, vol. 15.
4. Sommerfeld, A., Ein Beitrag zur hydrodinamischen Erklärung der turbulenten Flussigkeitsbewegungen, *Proc. Intern. Congr. Math. (Rome, 1908), Reale Accad. Lincei. Roma.*, 1909, vol. 3, pp. 116–124.
5. Tamarkin, J. D., *Some General Problems of the Theory of Ordinary Linear Differential Equations and Series Expansion of Arbitrary Functions*, Petrograd, 1917; Abridged English translation in vol. 27, 1927, pp. 1–54.
6. Picone, M., Equazione Integrale Traducente il Più Generale Problema Lineare per le Equazioni Differenziali Lineari Ordinarie di Qualsivoglia Ordine, *Academia nazionale dei Lincei. Atti dei convegni.*, 1932, vol. 15, pp. 942–948.
7. Carleman, T., Sur la théorie des equations intégrales et ses applications, *Verhandlungen des Internat. Math. Kongr. Zürich.*, 1932, vol. 1, pp. 138–151.
8. Bitsadze, A. V. and Samarskii, A. A., Some Elementary Generalizations of Linear Elliptic Boundary Value Problems, *Dokl. Akad. Nauk SSSR*, 1969, vol. 185, pp. 739–740; English transl. in *Soviet Math. Dokl.*, 1969, vol. 10.
9. Eidelman, S. D. and Zhitarashu, N. V., Nonlocal Boundary Value Problems for Elliptic Equations, *Mat. Issled.*, 1971, vol. 6, no. 2 (20), pp. 63–73.
10. Roitberg, Ya. A. and Sheftel', Z. G., Nonlocal Problems for Elliptic Equations and Systems, *Sib. Mat. Zh.*, 1972, vol. 13, pp. 165–181; English transl. in: *Siberian Math. J.*, 1972, vol. 13.
11. Kishkis, K. Yu., The Index of the Bitsadze–Samarskii Problem for Harmonic Functions, *Differentsial'nye Uravneniya*, 1988, vol. 24, no. 1, pp. 105–110; English transl. in: *Different. Equat.*, 1988, vol. 24.
12. Gushchin, A. K. and Mikhailov, V. P., On the Solvability of Nonlocal Problems for a Second-Order Elliptic Equation, *Mat. Sb.*, 1994, vol. 185, pp. 121–160; English transl. in: *Math. Sb.*, 1994, vol. 185.
13. Skubachevskii, A. L., Nonlocal Elliptic Problems with a Parameter, *Mat. Sb.*, 1983, vol. 121 (163), pp. 201–210; English transl. in: *Math. USSR Sb.*, 1984, vol. 49.
14. Skubachevskii, A. L., Elliptic Problems with Nonlocal Conditions Near the Boundary, *Mat. Sb.*, 1986, vol. 129 (171), pp. 279–302; English transl. in: *Math. USSR Sb.*, 1987, vol. 57.
15. Skubachevskii, A. L., Model Nonlocal Problems for Elliptic Equations in Dihedral Angles, *Differentsial'nye Uravneniya*, 1990, vol. 26, pp. 120–131; English transl. in: *Different. Equat.*, 1990, vol. 26.
16. Skubachevskii, A. L., The Method of Cut-Off Functions in the Theory of Nonlocal Problems, *Differentsial'nye Uravneniya*, 1991, vol. 27, pp. 128–139; English transl. in: *Different. Equat.*, 1991, vol. 27.

17. Skubachevskii, A. L., *Elliptic Functional-Differential Equations and Applications*, Basel: Birkhäuser, 1997.
18. Kovaleva, O. A. and Skubachevskii, A. L., Solvability of Nonlocal Elliptic Problems in Weighted Spaces, *Mat. Zametki.*, 2000, vol. 67, pp. 882–898; English transl. in: *Math. Notes.*, 2000, vol. 67.
19. Skubachevskii, A. L., Regularity of Solutions of a Nonlocal Elliptic Problem, *Russ. J. Math. Phys.*, 2001, vol. 8, no. 3, pp. 365–374.
20. Gurevich, P. L., Nonlocal Problems for Elliptic Equations in Dihedral Angles and the Green Formula, *Mitt. Math. Sem. Giessen*, 2001, vol. 247, pp. 1–74.
21. Kondrat'ev, V. A., Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points, *Trudy Moskov. Mat. Obshch.*, 1967, vol. 16, pp. 209–292; English transl. in: *Trans. Moscow Math. Soc.*, 1967, vol. 16.
22. Lions, J. L. and Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications,* vol. 1, New York–Heidelberg: Springer–Verlag, 1973.
23. Gohberg, I. C. and Sigal, E. I., An Operator Generalization of the Logarithmic Residue Theorem and Rouché's Theorem, *Mat. Sb.*, 1971, vol. 84 (126), pp. 607–629; English transl. in: *Math. USSR Sb.*, 1971, vol. 13.
24. Riesz, F. and Sz.-Nagy, *Leçons d'Analyse Fonctionnelle*, Deuxième édition, Budapest: Akademiai Kiado, 1953.
25. Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton: Princeton Univ. Press, 1970.
26. Agranovich, M. S. and Vishik, M. I., Elliptic Problems with a Parameter and Parabolic Problems of General Type, *Uspekhi Mat. Nauk*, 1964, vol. 19, pp. 53–161; English transl. in: *Russian Math. Surveys*, 1964, vol. 19.
27. Volevich, L. R., Solvability of Boundary Value Problems for General Elliptic Systems, *Mat. sb.*, 1965, vol. 68, no. 3, pp. 373–416.
28. Krein, S. G., *Linear Equations in Banach Spaces*, Nauka, Moscow, 1971; English transl. in: Birkhäuser, Boston, 1982.
29. Gurevich, P. L., Asymptotics of Solutions for Nonlocal Elliptic Problems in Plane Angles, *Trudy Sem. Petrovsk.*, 2003, vol. 23; English transl. in:, *J. Math. Sci.*, vol. 120, no. 3, pp. 1295–1312.
30. Maz'ya [Maz'ja], V. G. and Plamenevskii, B. A., L_p-Estimates of Solutions of Elliptic Boundary Value Problems in Domains with Ribs, *Trudy Moskov. Mat. Obshch.*, 1978, vol. 37, pp. 49–97; English transl. in: *Trans. Moscow Math. Soc.*, 1980, vol. 1.