On the non-existence of Feller semigroups in the non-transversal case

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In the theory of Markov processes the question arises as to whether there is a strongly continuous non-negative contraction semigroup (a Feller semigroup) of operators acting in spaces of continuous functions. Feller semigroups describe (from a probabilistic point of view) the motion of a Markovian particle in a domain. The general form of the generator of such a semigroup on an interval was studied in [1]. In the multidimensional case it was proved that the generator of a Feller semigroup is an elliptic differential operator (possibly degenerate) whose domain consists of continuous functions satisfying non-local conditions involving an integral over the closure of the domain with respect to some non-negative Borel measure [2]. However, the inverse problem remains open. Suppose that we are given an elliptic integro-differential operator whose domain is described by non-local conditions. Will the closure of this operator then be the generator of a Feller semigroup? In the transversal case the order of the non-local terms is less than the order of the local terms [3]–[7], and in the more complicated non-transversal case these orders coincide [7] (see also the references in [7]).

In [8] an example was constructed of a non-local operator (containing a transformation of the boundary into itself), whose closure is not the generator of a Feller semigroup. In this paper we give three examples to illustrate the non-existence of Feller semigroups in cases when the transformations $\Omega(y)$ (under *non-transversal* non-local conditions) map the boundary into the domain. For any y on the boundary the indicated Borel measure is the delta-function with support at the point $\Omega(y)$ in the closure of the domain. We note that the conditions 3.3 and 3.6 in [7] are violated in our first and second examples, while the conditions 3.5 and 3.9 in [7] are violated in our third example.

1. '*Jumps*' *with zero probability to the outside of a neighbourhood of the termination points of the process.* Let $G \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \mathscr{K}$, where Γ_1 and Γ_2 are C^∞-curves that are open and connected in the topology of ∂G, $\Gamma_1 \cap \Gamma_2 = \varnothing$, $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \mathscr{K}$, and the set $\mathscr{K}$ consists of two points g_1 and g_2. It is assumed that in some ε-neighbourhood $\mathscr{O}_\varepsilon(g_i)$ of g_i $(i = 1, 2)$ the domain G coincides with the flat angle with opening π.

We consider the non-local conditions

$$u(y) - b_1(y)u(\Omega_1(y)) = 0, \quad y \in \Gamma_1, \qquad u(y) = 0, \quad y \in \overline{\Gamma_2}, \tag{1}$$

where $b_1 \in C^\infty(\overline{\Gamma_1})$, $0 \leqslant b_1(y) \leqslant 1$, $b_1(y) = b_1^* > 0$ for $y \in \mathscr{O}_{\varepsilon/2}(g_1)$, $b_1(y) = 0$ for $y \notin \mathscr{O}_\varepsilon(g_1)$, Ω_1 is a smooth non-singular transformation defined in a neighbourhood of the curve $\overline{\Gamma_1}$, $\Omega_1(\Gamma_1) \subset G$, $\Omega_1(g_1) \in G$, and $\Omega_1(y)$ is the composition of a rotation about g_1 and a translation by some vector for $y \in \mathscr{O}_\varepsilon(g_1)$. From the probabilistic point of view, the Dirichlet condition means that the Markovian particle is absorbed (that is, the process terminates) once it arrives at a point $y \in \overline{\Gamma_2}$, and the non-local condition means that after some random time the particle 'jumps' from a point $y \in \Gamma_1$ to the point $\Omega_1(y) \in G$ with probability $b_1(y)$.

Let us consider the unbounded operator $\mathbf{P}_1 \colon \mathrm{D}(\mathbf{P}_1) \subset C_1(\overline{G}) \to C_1(\overline{G})$ defined by the formula

$$\mathbf{P}_1 u = \Delta u, \qquad u \in \mathrm{D}(\mathbf{P}_1) = \{u \in C_1(\overline{G}) : \Delta u \in C_1(\overline{G})\},$$

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where $C_1(\overline{G})$ is the set of functions in $C(\overline{G})$ satisfying the conditions (1), and Δ is the Laplacian acting in the sense of distributions.

2. '*Jumps*' *from conjugation points which are not termination points of the process.* We consider the non-local conditions

$$u(y) - b_1(y)u(\Omega_1(y)) = 0, \quad y \in \overline{\Gamma_1}, \qquad u(y) - b_2(y)u(\Omega_2(y)) = 0, \quad y \in \Gamma_2, \tag{2}$$

where $b_j \in C^\infty(\overline{\Gamma_j})$, $0 \leqslant b_j(y) \leqslant 1$, $b_j(y) = \text{const} > 0$ for $y \in \mathscr{O}_{\varepsilon/2}(g_1)$, $b_j(y) = 0$ for $y \notin \mathscr{O}_\varepsilon(g_1)$, Ω_j is a smooth non-singular transformation defined in a neighbourhood of the curve $\overline{\Gamma_j}$, $\Omega_j(\Gamma_j) \subset G$, $\Omega_j(g_1) \in G$, $\Omega_1(g_1) \neq \Omega_2(g_1)$, and $\Omega_j(y)$ is the composition of a rotation about the point g_1 and a translation by some vector for $y \in \mathscr{O}_\varepsilon(g_1)$.

Let $\mathbf{P}_2 \colon \mathrm{D}(\mathbf{P}_2) \subset C_2(\overline{G}) \to C_2(\overline{G})$ be the unbounded operator defined by

$$\mathbf{P}_2 u = \Delta u, \qquad u \in \mathrm{D}(\mathbf{P}_2) = \{u \in C_2(\overline{G}) : \Delta u \in C_2(\overline{G})\},$$

where $C_2(\overline{G})$ is the set of functions in $C(\overline{G})$ satisfying the non-local conditions (2).

3. '*Jumps*' *with probability 1 in a neighbourhood of the termination points of the process.* We consider the non-local conditions

$$u(y) - b_j(y)u(\Omega_j(y)) = 0, \quad y \in \Gamma_j, \quad j = 1, 2; \qquad u(y) = 0, \quad y \in \mathscr{K}, \tag{3}$$

where $b_j \in C^\infty(\overline{\Gamma_j})$, $0 \leqslant b_j(y) \leqslant 1$, $b_j(y) = 1$ for $y \in \mathscr{O}_{\varepsilon/2}(g_1)$ and $b_j(y) = 0$ for $y \notin \mathscr{O}_\varepsilon(g_1)$, Ω_j is a smooth non-singular transformation defined in a neighbourhood of the curve $\overline{\Gamma_j}$, $\Omega_j(\Gamma_j) \subset G$, $\Omega_j(g_1) = g_1$, and $\Omega_j(y)$ is a rotation by the angle $\pi/2$ into the domain G for $y \in \mathscr{O}_\varepsilon(g_1)$.

Let $\mathbf{P}_3 \colon \mathrm{D}(\mathbf{P}_3) \subset C_3(\overline{G}) \to C_3(\overline{G})$ be the unbounded operator defined by

$$\mathbf{P}_3 u = \Delta u, \qquad u \in \mathrm{D}(\mathbf{P}_3) = \{u \in C_3(\overline{G}) : \Delta u \in C_3(\overline{G})\},$$

where $C_3(\overline{G})$ is the set of functions in $C(\overline{G})$ satisfying the non-local conditions (3).

Theorem 1. *The operators* $\mathbf{P}_j$ *admit closure* $\overline{\mathbf{P}_j} \colon \mathrm{D}(\overline{\mathbf{P}_j}) \subset C_j(\overline{G}) \to C_j(\overline{G})$ $(j = 1, 2, 3)$, *and the operators* $\overline{\mathbf{P}_j}$ $(j = 1, 2, 3)$ *are not the generators of a Feller semigroup.*

Remark. It is possible to prove that $C_j(\overline{G}) \setminus \overline{\mathscr{R}(\mathbf{P}_j - q\mathbf{I})} \neq \varnothing$ for sufficiently small $q > 0$. Therefore, $C_j(\overline{G}) \setminus \mathscr{R}(\overline{\mathbf{P}}_j - q\mathbf{I}) \neq \varnothing$. By the Hille–Yosida theorem, this gives Theorem 1.

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