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On the Stability of the Index of Unbounded Nonlocal Operators in Sobolev Spaces

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Abstract—Unbounded operators corresponding to nonlocal elliptic problems on a bounded region $G \subset \mathbb{R}^2$ are considered. The domain of these operators consists of functions in the Sobolev space $W_2^m(G)$ that are generalized solutions of the corresponding elliptic equation of order $2m$ with the right-hand side in $L_2(G)$ and satisfy homogeneous nonlocal boundary conditions. It is known that such unbounded operators have the Fredholm property. It is proved that lower order terms in the differential equation do not affect the index of the operator. Conditions under which nonlocal perturbations on the boundary do not change the index are also formulated.

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INTRODUCTION

In the one-dimensional case, nonlocal problems were studied by A. Sommerfeld [20], Ya.D. Tamarkin [13], and M. Picone [17]. T. Carleman [14] considered the problem of finding a function harmonic on a two-dimensional bounded domain and subject to a nonlocal condition connecting the values of this function at different points of the boundary. A.V. Bitsadze and A.A. Samarskii [1] suggested another nonlocal problem arising in plasma theory: find a function harmonic on a bounded domain and satisfying nonlocal conditions on shifts of the boundary that can take points of the boundary inside the domain. Different generalizations of the above nonlocal problems were investigated by many authors (see [19] and references therein).

It turns out that the most difficult situation occurs if the support of nonlocal terms intersects the boundary. In this case, solutions of nonlocal problems may have power-law singularities near some points even if the boundary and the right-hand sides are infinitely smooth [10]. For this reason, such problems are naturally studied in weighted spaces (introduced by V.A. Kondrat'ev for boundary-value problems in nonsmooth domains [7]). The most complete theory of nonlocal problems in weighted spaces is developed by A.L. Skubachevskii [10–12, 18, 19] and his students.

Note that the study of nonlocal problems is motivated both by significant theoretical progress in this field and important applications arising in biophysics, theory of diffusion processes, plasma theory, and so on.

In this paper, we investigate the influence of lower order terms in an elliptic equation and the influence of nonlocal perturbations in boundary conditions upon the index of the unbounded nonlocal operator in $L_2(G)$. This issue was earlier studied by A.L. Skubachevskii [18] for bounded operators in weighted spaces. It is proved in [18] that nonlocal perturbations supported outside the points of conjugation of the boundary conditions do not change the index of the corresponding bounded operator. A similar assertion was later established in Sobolev spaces in the two-dimensional case [16]. In both cases, one can either use the method of continuation with respect to a parameter or reduce the original problem to that where nonlocal perturbations have compact square. As for lower order terms in the elliptic equation, they are simply compact perturbations.

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The situation is quite different in the case of unbounded operators. The difficulty is that the lower order terms in elliptic equations are not compact or relatively compact (see Definition A.2 in the Appendix); moreover, if the order of the elliptic equation is greater than two, they are not even relatively bounded, and therefore, they may change the domain of definition of the operator. As for nonlocal perturbations in the boundary conditions, they explicitly change the domain of definition, and therefore, they cannot be regarded as compact perturbations (in any sense).

To overcome the above difficulties, we consider an auxiliary operator (whose index equals the index of the original operator) acting on weighted spaces. In Section 2, we prove that lower order terms in elliptic equations are relatively compact perturbations of the auxiliary operator and, therefore, do not affect the index. In Section 3, we consider nonlocal perturbations in boundary conditions, which explicitly change the domain of definition. We make use of the notion of a *gap between unbounded operators* (see Definition A.3). We show that if the nonlocal perturbations in boundary conditions satisfy some regularity conditions at the conjugation points, then multiplying the perturbations by a small parameter leads to a small gap between the corresponding operators. Combining this fact with the method of continuation with respect to a parameter, we prove the index stability theorem.

Finally, we note that the Fredholm property of unbounded nonlocal operators on $L_2(G)$ was earlier studied either in the case when nonlocal conditions were set on shifts of the boundary [19] or in the case of a nonlocal perturbation of the Dirichlet problem for a second-order elliptic equation [5, 4]. For elliptic equations of order $2m$ with general nonlocal conditions, this question is being investigated for the first time.

1. SETTING OF NONLOCAL PROBLEMS IN BOUNDED DOMAINS

1.1. Setting of nonlocal problems. Let $G \subset \mathbb{R}^2$ be a bounded domain with boundary ∂G. We introduce a set $\mathcal{K} \subset \partial G$ consisting of finitely many points and assume that $\partial G \setminus \mathcal{K} = \bigcup_{i=1}^{N} \Gamma_i$, where Γ_i are open (in the topology of ∂G) C^∞ curves. In a neighborhood of each point $g \in \mathcal{K}$, the domain G is supposed to coincide with some plane angle.

For any domain Q and for integer $k \geq 0$, we denote by $W^k(Q) = W_2^k(Q)$ the Sobolev space with the norm

$$\|u\|_{W^k(Q)} = \left(\sum_{|\alpha| \leq k} \int\limits_Q |D^\alpha u|^2 \, dy \right)^{1/2}$$

(we set $W^0(Q) = L_2(Q)$ for $k = 0$). For an integer $k \geq 1$, we introduce the space $W^{k-1/2}(\Gamma)$ of traces on a smooth curve $\Gamma \subset \overline{Q}$, with the norm

$$\|\psi\|_{W^{k-1/2}(\Gamma)} = \inf \|u\|_{W^k(Q)}, \qquad u \in W^k(Q)\colon\ u|_\Gamma = \psi. \tag{1.1}$$

For any set $X \subset \mathbb{R}^2$ with nonempty interior, we denote by $C_0^\infty(X)$ the set of functions that are infinitely differentiable on $\mathbb{R}^2$ and compactly supported on X.

Now we introduce different weighted spaces for different domains Q. Consider the following cases:

(1) $Q = G$; denote $\mathcal{M} = \mathcal{K}$;

(2) Q is a plane angle, $Q = \{y \in \mathbb{R}^2\colon |\omega| < \omega_0\}$, where $0 < \omega_0 < \pi$; denote $\mathcal{M} = \{0\}$;

(3) $Q = \{y \in \mathbb{R}^2\colon |\omega| < \omega_0,\ 0 < r < \varepsilon\}$ for some $\varepsilon > 0$; denote $\mathcal{M} = \{0\}$;

here (ω, r) are polar coordinates of the point y.

Introduce the weighted Kondrat'ev space $H_a^k(Q)=H_a^k(Q,\mathcal{M})$ as the completion of the set $C_0^\infty(\overline{Q}\setminus\mathcal{M})$ with respect to the norm

$$\|u\|_{H_a^k(Q)}=\left(\sum_{|\alpha|\le k}\int\limits_Q \rho^{2(a+|\alpha|-k)}|D^\alpha u|^2\,dx\right)^{1/2},$$

where $k\ge 0$, $a\in\mathbb{R}$, and $\rho(y)=\operatorname{dist}(y,\mathcal{M})$; clearly, $\rho(y)=r$ in cases (2) and (3).

Denote by $H_a^{k-1/2}(\Gamma)$ ($k\ge 1$ is an integer) the space of traces on a smooth curve $\Gamma\subset\overline{Q}$, with the norm

$$\|\psi\|_{H_a^{k-1/2}(\Gamma)}=\inf\|v\|_{H_a^k(Q)},\qquad v\in H_a^k(Q)\colon\ v|_\Gamma=\psi.$$

We denote by $\mathbf{A}(y,D_y)$ and $B_{i\mu s}(y,D_y)$ linear differential operators of orders $2m$ and $m_{i\mu}$ (with $m_{i\mu}\le m-1$), respectively, with complex-valued C^∞ coefficients ($i=1,\dots,N$; $\mu=1,\dots,m$; $s=0,\dots,S_i$). Set $\mathbf{B}_{i\mu}^0u=B_{i\mu 0}(y,D_y)u|_{\Gamma_i}$.

Condition 1.1 (see, e.g., [9])**.** The operator $\mathbf{A}(y,D_y)$ is properly elliptic for all $y\in\overline{G}$, and the system of operators $\{\mathbf{B}_{i\mu}^0\}_{\mu=1}^m$ covers $\mathbf{A}(y,D_y)$ for all $i=1,\dots,N$ and $y\in\overline{\Gamma}_i$.

The operators $\mathbf{A}(y,D_y)$ and $\mathbf{B}_{i\mu}^0$ will *correspond to a "local" boundary-value problem.*

Now we define operators corresponding to nonlocal conditions near the set $\mathcal{K}$. For $\varepsilon>0$ and any closed set $\mathcal{N}$, denote by $\mathcal{O}_\varepsilon(\mathcal{N})=\{y\in\mathbb{R}^2\colon \operatorname{dist}(y,\mathcal{N})<\varepsilon\}$ its ε-neighborhood.

Let Ω_{is} ($i=1,\dots,N$; $s=1,\dots,S_i$) be C^∞ diffeomorphisms taking some neighborhood $\mathcal{O}_i$ of the curve $\overline{\Gamma_i\cap\mathcal{O}_\varepsilon(\mathcal{K})}$ to the set $\Omega_{is}(\mathcal{O}_i)$ in such a way that $\Omega_{is}(\Gamma_i\cap\mathcal{O}_\varepsilon(\mathcal{K}))\subset G$ and $\Omega_{is}(g)\in\mathcal{K}$ for $g\in\overline{\Gamma}_i\cap\mathcal{K}$. Thus, under the transformations Ω_{is}, the curves $\Gamma_i\cap\mathcal{O}_\varepsilon(\mathcal{K})$ are mapped strictly inside the domain G, whereas the set of end points $\overline{\Gamma}_i\cap\mathcal{K}$ is mapped to itself.

Let us specify the structure of the transformations Ω_{is} near the set $\mathcal{K}$. Denote by the symbol Ω_{is}^{+1} the transformation $\Omega_{is}\colon\mathcal{O}_i\to\Omega_{is}(\mathcal{O}_i)$ and by Ω_{is}^{-1} the inverse transformation. The set of all points $\Omega_{i_qs_q}^{\pm1}(\dots\Omega_{i_1s_1}^{\pm1}(g))\in\mathcal{K}$ ($1\le s_j\le S_{i_j}$, $j=1,\dots,q$), i.e., the set of all points that can be obtained by consecutively applying the transformations $\Omega_{i_js_j}^{+1}$ or $\Omega_{i_js_j}^{-1}$ (taking the points of $\mathcal{K}$ to $\mathcal{K}$) to the point $g\in\mathcal{K}$, is called the *orbit* of the point g and is denoted by $\operatorname{Orb}(g)$.

Clearly, for any $g,g'\in\mathcal{K}$, either $\operatorname{Orb}(g)=\operatorname{Orb}(g')$ or $\operatorname{Orb}(g)\cap\operatorname{Orb}(g')=\varnothing$. In what follows, we assume that the set $\mathcal{K}$ consists of one orbit and the number of points in the orbit is equal to the number N of the curves Γ_i. Denote the points of the set (orbit) $\mathcal{K}$ by g_j, $j=1,\dots,N$.

Take a small number $\varepsilon>0$ such that there exist neighborhoods $\mathcal{O}_{\varepsilon_1}(g_j)$ of the points $g_j\in\mathcal{K}$ satisfying the following conditions:

(1) $\mathcal{O}_{\varepsilon_1}(g_j)\supset\mathcal{O}_\varepsilon(g_j)$;

(2) the boundary ∂G coincides with some plane angle in the neighborhood $\mathcal{O}_{\varepsilon_1}(g_j)$;

(3) $\overline{\mathcal{O}_{\varepsilon_1}(g_j)}\cap\overline{\mathcal{O}_{\varepsilon_1}(g_k)}=\varnothing$ for any $g_j,g_k\in\mathcal{K}$, $j\ne k$;

(4) if $g_j\in\overline{\Gamma}_i$ and $\Omega_{is}(g_j)=g_k$, then $\mathcal{O}_\varepsilon(g_j)\subset\mathcal{O}_i$ and $\Omega_{is}(\mathcal{O}_\varepsilon(g_j))\subset\mathcal{O}_{\varepsilon_1}(g_k)$.

For each point $g_j\in\overline{\Gamma}_i\cap\mathcal{K}$, we fix a transformation $Y_j\colon y\mapsto y'(g_j)$ that is the composition of the shift by the vector $-\overrightarrow{Og_j}$ and the rotation through some angle so that

$$Y_j(\mathcal{O}_{\varepsilon_1}(g_j))=\mathcal{O}_{\varepsilon_1}(0),\qquad Y_j(G\cap\mathcal{O}_{\varepsilon_1}(g_j))=K_j\cap\mathcal{O}_{\varepsilon_1}(0),$$

$$Y_j(\Gamma_i\cap\mathcal{O}_{\varepsilon_1}(g_j))=\gamma_{j\sigma}\cap\mathcal{O}_{\varepsilon_1}(0),\qquad \sigma=1\text{ or }2,$$

where $K_j=\{y\in\mathbb{R}^2\colon r>0,\ |\omega|<\omega_j\}$, $\gamma_{j\sigma}=\{y\in\mathbb{R}^2\colon r>0,\ \omega=(-1)^\sigma\omega_j\}$, (ω,r) are polar coordinates, and $0<\omega_j<\pi$.

Condition 1.2. Let $g_j \in \overline{\Gamma}_i \cap \mathcal{K}$ and $\Omega_{is}(g_j) = g_k \in \mathcal{K}$. Then the transformation $Y_k \circ \Omega_{is} \circ Y_j^{-1} \colon \mathcal{O}_\varepsilon(0) \to \mathcal{O}_{\varepsilon_1}(0)$ is the composition of a rotation and a homothety.

Remark 1.1. Condition 1.2, being combined with the assumption $\Omega_{is}(\Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K})) \subset G$, means, in particular, that if $g \in \Omega_{is}(\overline{\Gamma}_i \cap \mathcal{K}) \cap \overline{\Gamma}_j \cap \mathcal{K} \neq \varnothing$, then the curves $\Omega_{is}(\overline{\Gamma}_i)$ and $\overline{\Gamma}_j$ are not tangent to each other at the point g.

Consider a number ε_0, $0 < \varepsilon_0 \le \varepsilon$, satisfying the following condition: if $g_j \in \overline{\Gamma}_i$ and $\Omega_{is}(g_j) = g_k$, then $\mathcal{O}_{\varepsilon_0}(g_k) \subset \Omega_{is}(\mathcal{O}_\varepsilon(g_j))$. Introduce a function $\zeta \in C^\infty(\mathbb{R}^2)$ such that $\zeta(y) = 1$ for $y \in \mathcal{O}_{\varepsilon_0/2}(\mathcal{K})$ and $\operatorname{supp}\zeta \subset \mathcal{O}_{\varepsilon_0}(\mathcal{K})$.

Now we define nonlocal operators $\mathbf{B}^1_{i\mu}$ by the formula

$$\mathbf{B}^1_{i\mu}u = \sum_{s=1}^{S_i} \big(B_{i\mu s}(y, D_y)(\zeta u)\big)(\Omega_{is}(y)), \quad y \in \Gamma_i \cap \mathcal{O}_\varepsilon(\mathcal{K}), \qquad \mathbf{B}^1_{i\mu}u = 0, \quad y \in \Gamma_i \setminus \mathcal{O}_\varepsilon(\mathcal{K}),$$

where $(B_{i\mu s}(y, D_y)u)(\Omega_{is}(y)) = B_{i\mu s}(x, D_x)u(x)|_{x=\Omega_{is}(y)}$. Since $\mathbf{B}^1_{i\mu}u = 0$ if $\operatorname{supp} u \subset \overline{G} \setminus \overline{\mathcal{O}_{\varepsilon_0}(\mathcal{K})}$, we say that the operators $\mathbf{B}^1_{i\mu}$ *correspond to nonlocal terms supported near the set* $\mathcal{K}$.

For any $\rho > 0$, we denote $G_\rho = \{y \in G \colon \operatorname{dist}(y, \partial G) > \rho\}$. Consider linear operators $\mathbf{B}^2_{i\mu}$ satisfying the following condition (cf. [10, 18, 15]).

Condition 1.3. There exist numbers $\varkappa_1 > \varkappa_2 > 0$ and $\rho > 0$ such that the following inequalities hold:

$$\|\mathbf{B}^2_{i\mu}u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le c_1 \|u\|_{W^{2m}(G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})}, \tag{1.2}$$

$$\|\mathbf{B}^2_{i\mu}u\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i\setminus\overline{\mathcal{O}_{\varkappa_2}(\mathcal{K})})} \le c_2 \|u\|_{W^{2m}(G_\rho)}. \tag{1.3}$$

Remark 1.2. In (1.2), (1.3) and throughout the paper, we denote by $c, c_1, c_2, \dots$ and $k_1, k_2, \dots$ positive constants that do not depend on the functions involved in the corresponding inequality.

We assume that Conditions 1.1–1.3 hold throughout, including the formulation of lemmas.

It follows from (1.2) that $\mathbf{B}^2_{i\mu}u = 0$ whenever $\operatorname{supp} u \subset \mathcal{O}_{\varkappa_1}(\mathcal{K})$. For this reason, we say that the operators $\mathbf{B}^2_{i\mu}$ *correspond to nonlocal terms supported outside the set* $\mathcal{K}$.

We study the following nonlocal elliptic problem:

$$\mathbf{A}(y, D_y)u = f(y), \qquad y \in G, \tag{1.4}$$

$$\mathbf{B}_{i\mu}u \equiv \mathbf{B}^0_{i\mu}u + \mathbf{B}^1_{i\mu}u + \mathbf{B}^2_{i\mu}u = 0, \qquad y \in \Gamma_i, \quad i = 1, \dots, N, \quad \mu = 1, \dots, m, \tag{1.5}$$

where $f \in L_2(G)$. Introduce a space $W^m(G, \mathbf{B})$ consisting of functions $u \in W^m(G)$ that satisfy the homogeneous nonlocal conditions (1.5). Consider the unbounded operator $\mathbf{P} \colon \mathrm{D}(\mathbf{P}) \subset L_2(G) \to L_2(G)$ given by

$$\mathbf{P}u = \mathbf{A}(y, D_y)u, \qquad u \in \mathrm{D}(\mathbf{P}) = \big\{u \in W^m(G, \mathbf{B}) \colon \mathbf{A}(y, D_y)u \in L_2(G)\big\}.$$

Definition 1.1. A function u is called a *generalized solution* of problem (1.4), (1.5) with right-hand side $f \in L_2(G)$ if $u \in \mathrm{D}(\mathbf{P})$ and $\mathbf{P}u = f$.

An equivalent definition of a generalized solution can be given in terms of an integral identity [3].

Note that generalized solutions a priori belong to the space $W^m(G)$, whereas Condition 1.3 is formulated for functions that belong to the space W^{2m} outside the set $\mathcal{K}$. Such a formulation can be justified by the following result (see Lemma 2.1 in [3] and Lemma 5.1 in [16]).

Lemma 1.1. *Let $u \in W^m(G)$ be a generalized solution of problem* (1.4), (1.5) *with right-hand side $f \in W^k(G)$. Then*

$$\|u\|_{W^{k+2m}(G\setminus\overline{\mathcal{O}_\delta(\mathcal{K})})} \le c_\delta\big(\|f\|_{W^k(G\setminus\overline{\mathcal{O}_{\delta_1}(\mathcal{K})})} + \|u\|_{L_2(G)}\big) \qquad \forall\delta>0,$$

where $\delta_1=\delta_1(\delta)>0$ and $c_\delta>0$ do not depend on u.

Theorem 1.1 (see Theorem 2.1 in [3])**.** *Let Conditions* 1.1–1.3 *hold. Then the operator* $\mathbf{P}$ *has the Fredholm property.*[1]

The aim of this paper is to investigate the influence of lower order terms in (1.4) and nonlocal operators $\mathbf{B}^1_{i\mu}$ and $\mathbf{B}^2_{i\mu}$ in (1.5) upon the index of the operator $\mathbf{P}$.

1.2. Nonlocal problems near the set $\mathcal{K}$. When studying problem (1.4), (1.5), one must pay particular attention to the behavior of solutions near the set $\mathcal{K}$ of conjugation points. Let us consider the corresponding model problems. Denote by $u_j(y)$ the function $u(y)$ for $y\in\mathcal{O}_{\varepsilon_1}(g_j)$. If $g_j\in\overline{\Gamma}_i$, $y\in\mathcal{O}_\varepsilon(g_j)$, and $\Omega_{is}(y)\in\mathcal{O}_{\varepsilon_1}(g_k)$, then we denote the function $u(\Omega_{is}(y))$ by $u_k(\Omega_{is}(y))$. Using this notation, we rewrite the nonlocal problem (1.4), (1.5) in the ε-neighborhood of the set (orbit) $\mathcal{K}$ as follows:

$$\mathbf{A}(y,D_y)u_j=f(y), \qquad y\in\mathcal{O}_\varepsilon(g_j)\cap G,$$

$$B_{i\mu0}(y,D_y)u_j(y)\big|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i}+\sum_{s=1}^{S_i}\big(B_{i\mu s}(y,D_y)(\zeta u_k)\big)(\Omega_{is}(y))\big|_{\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i}=f_{i\mu}(y), \qquad y\in\mathcal{O}_\varepsilon(g_j)\cap\Gamma_i,$$

$$i\in\{1\le i\le N\colon g_j\in\overline{\Gamma}_i\}, \qquad j=1,\dots,N, \qquad \mu=1,\dots,m,$$

where $f_{i\mu}=-\mathbf{B}^2_{i\mu}u$.

Let $y\mapsto y'(g_j)$ be the change of variables described in Subsection 1.1. Denote $K_j^\varepsilon=K_j\cap\mathcal{O}_\varepsilon(0)$ and $\gamma^\varepsilon_{j\sigma}=\gamma_{j\sigma}\cap\mathcal{O}_\varepsilon(0)$. Introduce the functions

$$U_j(y')=u_j(y(y')), \qquad f_j(y')=f(y(y')), \qquad y'\in K_j^\varepsilon,$$

and

$$f_{j\sigma\mu}(y')=f_{i\mu}(y(y')), \qquad y'\in\gamma^\varepsilon_{j\sigma},$$

where $\sigma=1$ ($\sigma=2$) if, under the transformation $y\mapsto y'(g_j)$, the curve Γ_i is mapped to the side γ_{j1} (γ_{j2}) of the angle K_j. Denote y' by y again. Then, by virtue of Condition 1.2, problem (1.4), (1.5) acquires the form

$$\mathbf{A}_j(y,D_y)U_j=f_j(y), \qquad y\in K_j^\varepsilon, \tag{1.6}$$

$$\sum_{k,s}\big(B_{j\sigma\mu ks}(y,D_y)U_k\big)(\mathcal{G}_{j\sigma ks}y)=f_{j\sigma\mu}(y), \qquad y\in\gamma^\varepsilon_{j\sigma}; \tag{1.7}$$

here $j,k=1,\dots,N$; $\sigma=1,2$; $\mu=1,\dots,m$; $s=0,\dots,S_{j\sigma k}$; $\mathbf{A}_j(y,D_y)$ and $B_{j\sigma\mu ks}(y,D_y)$ are differential operators of orders $2m$ and $m_{j\sigma\mu}$ ($m_{j\sigma\mu}\le m-1$), respectively, with C^∞ complex-valued coefficients; and $\mathcal{G}_{j\sigma ks}$ is the operator of rotation through an angle $\omega_{j\sigma ks}$ and the homothety with a coefficient $\chi_{j\sigma ks}$ ($\chi_{j\sigma ks}>0$). Moreover, $|(-1)^\sigma b_j+\omega_{j\sigma ks}|<b_k$ for $(k,s)\ne(j,0)$ (cf. Remark 1.1), $\omega_{j\sigma j0}=0$, and $\chi_{j\sigma j0}=1$ (i.e., $\mathcal{G}_{j\sigma j0}y\equiv y$).

Set $D_\chi=2\max\{\chi_{j\sigma ks}\}$. The following lemma establishes the regularity property for the solutions of nonlocal problems near the set $\mathcal{K}$.

[1] See Definition A.1.

Lemma 1.2 (see[2] Lemma 2.3 in [3]). *Let* $(U_1, \dots, U_N)$ *be a solution of problem* (1.6), (1.7) *such that*

$$U_j \in W^{2m}\big(K_j^{D_\chi \varepsilon} \cap \{|y| > \delta\}\big) \quad \forall \delta > 0, \qquad U_j \in H^0_{a-2m}\big(K_j^{D_\chi \varepsilon}\big),$$

where $a \in \mathbb{R}$. *Suppose that* $f_j \in H^0_a(K_j^\varepsilon)$ *and* $f_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma^\varepsilon_{j\sigma})$. *Then*

$$\sum_j \|U_j\|_{H_a^{2m}(K_j^{\varepsilon/D_\chi^3})} \le c \sum_j \left(\|f_j\|_{H^0_a(K_j^\varepsilon)} + \sum_{\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma^\varepsilon_{j\sigma})} + \|U_j\|_{H^0_{a-2m}(K_j^\varepsilon)} \right).$$

We write the principal homogeneous parts of the operators $\mathbf{A}_j(0, D_y)$ and $B_{j\sigma\mu ks}(0, D_y)$ in the polar coordinates as $r^{-2m}\widetilde{\mathcal{A}}_j(\omega, D_\omega, rD_r)$ and $r^{-m_{j\sigma\mu}}\widetilde{B}_{j\sigma\mu ks}(\omega, D_\omega, rD_r)$, respectively, and consider the analytic operator-valued function

$$\widetilde{\mathcal{L}}(\lambda)\colon \prod_{j=1}^N W^{l+2m}(-\omega_j, \omega_j) \to \prod_{j=1}^N \big(W^l(-\omega_j, \omega_j) \times \mathbb{C}^{2m}\big),$$

$$\widetilde{\mathcal{L}}(\lambda)\varphi = \left\{ \widetilde{\mathcal{A}}_j(\omega, D_\omega, \lambda)\varphi_j,\ \sum_{k,s} (\chi_{j\sigma ks})^{i\lambda - m_{j\sigma\mu}} \widetilde{B}_{j\sigma\mu ks}(\omega, D_\omega, \lambda)\varphi_k(\omega + \omega_{j\sigma ks})\big|_{\omega = (-1)^\sigma \omega_j} \right\}.$$

The basic definitions and facts concerning eigenvalues, eigenvectors, and associate vectors of analytic operator-valued functions can be found in [2]. In the sequel, it is essential that the spectrum of the operator $\widetilde{\mathcal{L}}(\lambda)$ is discrete (see Lemma 2.1 in [11]).

2. PERTURBATIONS BY LOWER ORDER TERMS

2.1. Reduction to weighted spaces. Introduce the lower order terms operator

$$A'(y, D_y) = \sum_{|\alpha| \le 2m-1} a_\alpha(y) D^\alpha, \tag{2.1}$$

where $a_\alpha \in C^\infty(\mathbb{R}^2)$. Consider the perturbed operator $\mathbf{P}'\colon \mathrm{D}(\mathbf{P}') \subset L_2(G) \to L_2(G)$ given by

$$\mathbf{P}'u = \mathbf{A}(y, D_y)u + A'(y, D_y)u,$$
$$u \in \mathrm{D}(\mathbf{P}') = \big\{u \in W^m(G, \mathbf{B})\colon\ \mathbf{A}(y, D_y)u + A'(y, D_y)u \in L_2(G)\big\}.$$

By Theorem 1.1, the unbounded operator $\mathbf{P}'$ has the Fredholm property (just as $\mathbf{P}$ does). The main result of this section (to be proved in Subsection 2.2) is as follows.

Theorem 2.1. *Let Conditions* 1.1–1.3 *hold. Then* $\operatorname{ind}\mathbf{P}' = \operatorname{ind}\mathbf{P}$.

This theorem shows that the lower order terms in (1.4) do not affect the index of the unbounded operator $\mathbf{P}$. The difficulty is that the above perturbations are, in general, neither compact nor $\mathbf{P}$-compact in the sense of Definition A.2. If $m = 1$, then $u \in \mathrm{D}(\mathbf{P})$ implies only $u \in W^1(G)$, which ensures the $\mathbf{P}$-boundedness of the perturbation but not its $\mathbf{P}$-compactness. However, if $m \ge 2$, then $u \in \mathrm{D}(\mathbf{P})$ does not imply $u \in W^{2m-1}(G)$, and the perturbation is not even $\mathbf{P}$-bounded. Moreover, $\mathrm{D}(\mathbf{P}') \ne \mathrm{D}(\mathbf{P})$ in the latter case.

To overcome this difficulty, we introduce the operator $\mathbf{Q}\colon \mathrm{D}(\mathbf{Q}) \subset L_2(G) \to H^0_a(G)$ given by

$$\mathbf{Q}u = \mathbf{A}(y, D_y)u, \qquad u \in \mathrm{D}(\mathbf{Q}) = \big\{u \in W^m(G, \mathbf{B})\colon\ \mathbf{A}(y, D_y)u \in H^0_a(G)\big\}. \tag{2.2}$$

[2] Lemma 2.3 in [3] was formulated for $a > 2m - 1$. However, its proof remains true for any $a \in \mathbb{R}$.

In this definition and further (unless otherwise stated), we assume that

$$m - 1 < a < m.$$

We will prove that $\operatorname{ind}\mathbf{Q} = \operatorname{ind}\mathbf{P}$. On the other hand, we will show that the operator $A'(y, D_y)$ is a $\mathbf{Q}$-compact perturbation and, therefore, does not change the index of $\mathbf{Q}$ and hence of $\mathbf{P}$.

Lemma 2.1. *Let the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contain no eigenvalues of the operator* $\widetilde{\mathcal{L}}(\lambda)$. *Then the operator* $\mathbf{Q}$ *has the Fredholm property and* $\operatorname{ind}\mathbf{Q} = \operatorname{ind}\mathbf{P}$.

Proof. 1. It is shown in [16, Section 6] that $\mathbf{B}_{i\mu}u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i) \dotplus R_a^{i\mu}(\Gamma_i)$ for $u \in H_a^{2m}(G)$, where $R_a^{i\mu}(\Gamma_i)$ is a finite-dimensional subspace in $H_{a'}^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $a' > 2m - 1$. Set

$$\mathcal{H}_a^0(G,\Gamma) = H_a^0(G) \times \prod_{i=1}^{N}\prod_{\mu=1}^{m} H_a^{2m-m_{i\mu}-1/2}(\Gamma_i), \qquad \mathcal{R}_a^0(G,\Gamma) = \{0\} \times \prod_{i=1}^{N}\prod_{\mu=1}^{m} R_a^{i\mu}(\Gamma_i).$$

By Theorem 6.1 in [16], the bounded operator

$$\mathbf{L} = \{\mathbf{A}(y, D_y), \mathbf{B}_{i\mu}\}\colon\ H_a^{2m}(G) \to \mathcal{H}_a^0(G,\Gamma) \dotplus \mathcal{R}_a^0(G,\Gamma) \tag{2.3}$$

has the Fredholm property. Therefore, by virtue of the compactness of the embedding $H_a^{2m}(G) \subset L_2(G)$ (see Lemma A.1) and by Theorem A.1, we have

$$\|u\|_{H_a^{2m}(G)} \le k_1\big(\|\mathbf{L}u\|_{\mathcal{H}_a^0(G,\Gamma)\dotplus\mathcal{R}_a^0(G,\Gamma)} + \|u\|_{L_2(G)}\big). \tag{2.4}$$

2. Introduce an unbounded operator $\dot{\mathbf{Q}}\colon \mathrm{D}(\dot{\mathbf{Q}}) \subset L_2(G) \to H_a^0(G)$ given by

$$\dot{\mathbf{Q}}u = \mathbf{A}(y, D_y)u, \qquad u \in \mathrm{D}(\dot{\mathbf{Q}}) = \{u \in H_a^{2m}(G)\colon\ \mathbf{B}_{i\mu}u = 0\}. \tag{2.5}$$

Since $H_a^{2m}(G) \subset W^m(G)$, it follows that $\dot{\mathbf{Q}}$ is a restriction of $\mathbf{Q}$, i.e., $\dot{\mathbf{Q}} \subset \mathbf{Q}$.

First, we prove that $\dot{\mathbf{Q}}$ has the Fredholm property. Let $u \in \mathrm{D}(\dot{\mathbf{Q}})$; then $u \in \mathrm{D}(\mathbf{L}) = H_a^{2m}(G)$ and $\mathbf{A}(y, D_y)u \in H_a^0(G)$, $\mathbf{B}_{i\mu}u = 0$. Therefore, estimate (2.4) acquires the form

$$\|u\|_{H_a^{2m}(G)} \le k_1\big(\|\dot{\mathbf{Q}}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big) \qquad \forall u \in \mathrm{D}(\dot{\mathbf{Q}}). \tag{2.6}$$

It follows from (2.6) that the operator $\dot{\mathbf{Q}}$ is closed, $\dim\ker\dot{\mathbf{Q}} < \infty$, and $\mathcal{R}(\dot{\mathbf{Q}}) = \overline{\mathcal{R}(\dot{\mathbf{Q}})}$ (to obtain the latter two properties, one must apply Theorem A.1).

Let us prove that $\operatorname{codim}\mathcal{R}(\dot{\mathbf{Q}}) < \infty$. Since $\mathbf{L}$ has the Fredholm property, there exist finitely many linearly independent functions $F_1, \dots, F_d \in H_a^0(G)$ such that a function $f \in H_a^0(G)$ belongs to the image of $\dot{\mathbf{Q}}$ if and only if $(f, F_j)_{H_a^0(G)} = 0$, $j = 1, \dots, d$. Thus, $\dot{\mathbf{Q}}$ has the Fredholm property.

3. Now we prove that $\mathbf{Q}$ has the Fredholm property. Since $\ker\mathbf{Q} = \ker\mathbf{P}$ and $\mathbf{P}$ has the Fredholm property, it follows that

$$\dim\ker\mathbf{Q} = \dim\ker\mathbf{P} < \infty. \tag{2.7}$$

On the other hand, $\mathbf{Q}$ is an extension of the Fredholm operator $\dot{\mathbf{Q}}$; therefore,

$$\mathcal{R}(\mathbf{Q}) = \overline{\mathcal{R}(\mathbf{Q})}, \qquad \operatorname{codim}\mathcal{R}(\mathbf{Q}) < \infty. \tag{2.8}$$

Thus, $\mathbf{Q}$ is an extension of the Fredholm operator $\dot{\mathbf{Q}}$ and possesses properties (2.7) and (2.8). Applying Theorem A.2, we see that $\mathbf{Q}$ has the Fredholm property.

4. By virtue of (2.7), it remains to be proved that $\operatorname{codim}\mathcal{R}(\mathbf{Q}) = \operatorname{codim}\mathcal{R}(\mathbf{P})$.

Let $\operatorname{codim}\mathcal{R}(\mathbf{Q}) = d_1$, where $d_1 \le d$. Take an arbitrary function $f \in L_2(G)$. Then $f \in \mathcal{R}(\mathbf{P})$ if and only if $f \in \mathcal{R}(\mathbf{Q})$ because $L_2(G) \subset H_a^0(G)$. However, the inclusion $f \in \mathcal{R}(\mathbf{Q})$ is equivalent to the relations $(f, F_j)_{H_a^0(G)} = 0$, $j = 1, \dots, d_1$, where $F_1, \dots, F_{d_1} \in H_a^0(G)$ are linearly independent functions. Using Schwarz' inequality, the boundedness of the embedding $L_2(G) \subset H_a^0(G)$, and Riesz' theorem, we see that these relations are equivalent to the following ones: $(f, f_j)_{L_2(G)} = 0$, $j = 1, \dots, d_1$, where $f_j \in L_2(G)$. Moreover, the functions $f_1, \dots, f_{d_1}$ are linearly independent. (Otherwise, some linear combination of the functions $F_1, \dots, F_{d_1}$ would be orthogonal in $H_a^0(G)$ to any function that lies in $L_2(G)$. This is impossible because $F_1, \dots, F_{d_1}$ are linearly independent, while $L_2(G)$ is dense in $H_a^0(G)$.) Thus, we have proved that $\operatorname{codim}\mathcal{R}(\mathbf{P}) = d_1$. $\square$

Introduce a perturbed operator $\mathbf{Q}' \colon \mathrm{D}(\mathbf{Q}') \subset L_2(G) \to H_a^0(G)$ given by

$$\mathbf{Q}'u = \mathbf{A}(y, D_y)u + A'(y, D_y)u,$$

$$u \in \mathrm{D}(\mathbf{Q}') = \bigl\{u \in W^m(G, \mathbf{B})\colon\ \mathbf{A}(y, D_y)u + A'(y, D_y)u \in H_a^0(G)\bigr\}.$$

In the following subsection, we prove that $\operatorname{ind}\mathbf{Q}' = \operatorname{ind}\mathbf{Q}$, provided that the line $\operatorname{Im}\lambda = a + 1 - 2m$ contains no eigenvalues of the operator $\widetilde{\mathcal{L}}(\lambda)$. Then, using the discreteness of the spectrum of $\widetilde{\mathcal{L}}(\lambda)$ and Lemma 2.1, we will prove Theorem 2.1.

2.2. Compactness of lower order terms in weighted spaces.

Lemma 2.2. *Let the line* $\operatorname{Im}\lambda = a + 1 - 2m$ *contain no eigenvalues of the operator* $\widetilde{\mathcal{L}}(\lambda)$. *Then*

$$\|u\|_{W^m(G)} \le c\bigl(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\bigr) \qquad \forall u \in \mathrm{D}(\mathbf{Q}).$$

Proof. Consider the unbounded operator $\widehat{\mathbf{Q}} \colon \mathrm{D}(\widehat{\mathbf{Q}}) \subset W^m(G) \to H_a^0(G)$ given by $\widehat{\mathbf{Q}}u = \mathbf{A}(y, D_y)u$, $u \in \mathrm{D}(\widehat{\mathbf{Q}}) = \mathrm{D}(\mathbf{Q})$. Since $\mathbf{Q}$ has the Fredholm property, the same is true for $\widehat{\mathbf{Q}}$. Therefore, the desired estimate follows from the compactness of the embedding $W^m(G) \subset L_2(G)$ and from Theorem A.1. $\square$

Take a number b such that

$$m - 1 < b < a < m. \tag{2.9}$$

Consider a function $\psi_j \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a small neighborhood of the point $g_j \in \mathcal{K}$ and vanishing outside a larger neighborhood of g_j. The following lemma describes the behavior of $u \in \mathrm{D}(\mathbf{Q})$ near the set $\mathcal{K}$.

Lemma 2.3. *For any* $u \in \mathrm{D}(\mathbf{Q})$, *we have*

$$u(y) = \sum_{j=1}^{N} P_j(y) + v(y), \tag{2.10}$$

where

$$P_j(y) = \psi_j(y) \sum_{|\alpha| \le m-2} p_{j\alpha}(y - g_j)^\alpha, \qquad p_{j\alpha} \in \mathbb{C}, \tag{2.11}$$

and $v \in H_{b+1}^{2m}(G)$ (*if* $m = 1$, *we set* $P_j(y) \equiv 0$); *moreover,*

$$\sum_{j,\alpha} |p_{j\alpha}| + \|v\|_{H_{b+1}^{2m}(G)} \le c\bigl(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\bigr). \tag{2.12}$$

Proof. 1. It follows from Lemma 1.1 that $u \in W^{2m}(G \setminus \overline{\mathcal{O}_\delta(\mathcal{K})})$ for any $\delta > 0$ and

$$\|u\|_{W^{2m}(G\setminus\overline{\mathcal{O}_\delta(\mathcal{K})})} \le k_{1\delta}\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big), \tag{2.13}$$

where $k_{1\delta}$ does not depend on u. Therefore, it suffices to consider the behavior of u near the set $\mathcal{K}$.

By Lemma A.2, $u \in W^m(G)$ can be represented in the form (2.10), where $P_j(y)$ is given by (2.11), $v \in H_{b-m+1}^m(G)$, and

$$\sum_{j,\alpha} |p_{j\alpha}| + \|v\|_{H_{b-m+1}^m(G)} \le k_2\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big) \tag{2.14}$$

(to obtain (2.14), we have also applied Lemma 2.2).

Moreover, relations (2.10), (2.13), and (2.14) imply that

$$\|v\|_{W^{2m}(G\setminus\overline{\mathcal{O}_\delta(\mathcal{K})})} \le k_{2\delta}\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big) \qquad \forall \delta > 0, \tag{2.15}$$

where $k_{2\delta}$ does not depend on u. It remains to be proved that $v \in H_{b+1}^{2m}(G)$.

2. By using (1.4) and (1.5), we see that v is a solution of the problem

$$\mathbf{A}(y, D_y)v = f - \mathbf{A}(y, D_y)P \equiv f', \qquad \mathbf{B}_{i\mu}^0 v + \mathbf{B}_{i\mu}^1 v = -\mathbf{B}_{i\mu}P - \mathbf{B}_{i\mu}^2 v \equiv f'_{i\mu}, \tag{2.16}$$

where $P(y) = \sum_{j=1}^N P_j(y)$ and $f = \mathbf{Q}u \in H_a^0(G)$. It follows from the boundedness of the embedding $H_a^0(G) \subset H_{b+1}^0(G)$ (see (2.9)) and from the estimate of the coefficients $p_{j\alpha}$ (see (2.14)) that

$$\|f'\|_{H_{b+1}^0(G)} \le k_3\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big). \tag{2.17}$$

Similarly, using additionally inequalities (1.2) and (2.15), we obtain $f'_{i\mu} = -\mathbf{B}_{i\mu}P - \mathbf{B}_{i\mu}^2 v \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ and

$$\|f'_{i\mu}\|_{W^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le k_4\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big). \tag{2.18}$$

On the other hand, $v \in H_{b-m+1}^m(G)$; hence, $f'_{i\mu} = \mathbf{B}_{i\mu}^0 v + \mathbf{B}_{i\mu}^1 v \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma_i)$. We claim that

$$\|f'_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le k_5\big(\|\mathbf{Q}u\|_{H_a^0(G)} + \|u\|_{L_2(G)}\big). \tag{2.19}$$

To prove this assertion, we fix i and μ and set $\Gamma = \Gamma_i$. Let $g \in \overline{\Gamma} \setminus \Gamma$. Assume, without loss of generality, that $g = 0$ and Γ coincides with the axis Oy_1 in a sufficiently small neighborhood $\mathcal{O}_\varepsilon(0)$ of the origin. Denote

$$G^\varepsilon = G \cap \mathcal{O}_\varepsilon(0), \qquad \Gamma^\varepsilon = \Gamma \cap \mathcal{O}_\varepsilon(0),$$

in which case $H_a^k(G^\varepsilon) = H_a^k(G^\varepsilon, \{0\})$.

Using Lemma A.3(1), we represent $f'_{i\mu} \in W^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$ near the origin as follows:

$$f'_{i\mu}(r) = P_1(r) + f''_{i\mu}(r), \qquad 0 < r < \varepsilon,$$

where $P_1(r)$ is a polynomial of order $2m - m_{i\mu} - 2$, whereas $f''_{i\mu} \in H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$ (in fact, we can replace $b+1$ by any positive number in the last relation). Now we have $f'_{i\mu}, f''_{i\mu} \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$; therefore, $P_1 \in H_{b-m+1}^{m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$; i.e., P_1 consists of monomials of order greater than or equal to $m - m_{i\mu} - 1$. This implies that $P_1 \in H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)$. Using Lemma A.3(3), we obtain

$$\|f'_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} \le \|P_1\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} + \|f''_{i\mu}\|_{H_{b+1}^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)} \le k_6\|f'_{i\mu}\|_{W^{2m-m_{i\mu}-1/2}(\Gamma^\varepsilon)}.$$

Combining this estimate with (2.18) yields (2.19).

3. Applying Lemma 1.2 to problem (2.16) and taking into account (2.15), (2.17), (2.19), and (2.14), we obtain

$$\|v\|_{H^{2m}_{b+1}(G)} \le k_7\biggl(\|f'\|_{H^0_{b+1}(G)} + \sum_{i,\mu}\|f'_{i\mu}\|_{H^{2m-m_{i\mu}-1/2}_{b+1}(\Gamma_i)} + \|v\|_{H^0_{b-2m+1}(G)}\biggr)$$
$$\le k_8\bigl(\|\mathbf{Q}u\|_{H^0_a(G)} + \|u\|_{L_2(G)}\bigr).$$

Combining this inequality with (2.14) yields (2.12). $\square$

The following corollary results from Lemma 2.3.

Corollary 2.1. *Let $A'(y,D_y)$ be the differential operator of order $2m-1$ given by* (2.1). *Then*

$$\|A'(y,D_y)u\|_{H^1_{b+1}(G)} \le c\bigl(\|\mathbf{Q}u\|_{H^0_a(G)} + \|u\|_{L_2(G)}\bigr) \qquad \forall u\in \mathrm{D}(\mathbf{Q}). \tag{2.20}$$

Now we can prove that lower order perturbations in (1.4) do not change the index of $\mathbf{Q}$.

Lemma 2.4. *Let the line $\operatorname{Im}\lambda = a+1-2m$ contain no eigenvalues of the operator $\widetilde{\mathcal{L}}(\lambda)$. Then the operators $\mathbf{Q}$ and $\mathbf{Q}'$ have the Fredholm property and* $\operatorname{ind}\mathbf{Q}' = \operatorname{ind}\mathbf{Q}$.

Proof. By Lemma 2.1, $\mathbf{Q}$ and $\mathbf{Q}'$ have the Fredholm property.

Introduce an operator $\mathbf{A}'\colon \mathrm{D}(\mathbf{A}')\subset L_2(G)\to H^0_a(G)$ given by $\mathbf{A}'u = \mathbf{A}'(y,D_y)u$, $u\in\mathrm{D}(\mathbf{A}') = \mathrm{D}(\mathbf{Q})$. It follows from Corollary 2.1 and from the compactness of the embedding $H^1_{b+1}(G)\subset H^0_a(G)$ (see (2.9) and Lemma A.1) that $\mathbf{Q}' = \mathbf{Q}+\mathbf{A}'$ and $\mathbf{A}'$ is a $\mathbf{Q}$-compact operator. Therefore, by Theorem A.4, we have $\operatorname{ind}\mathbf{Q}' = \operatorname{ind}\mathbf{Q}$. $\square$

Proof of Theorem 2.1. It follows from Lemma 2.1 in [11] that the spectrum of $\widetilde{\mathcal{L}}(\lambda)$ is discrete. Therefore, one can find a number a such that $m-1<a<m$ and the line $\operatorname{Im}\lambda = a+1-2m$ contains no eigenvalues of $\widetilde{\mathcal{L}}(\lambda)$. In this case, Lemmas 2.1 and 2.4 imply $\operatorname{ind}\mathbf{P}' = \operatorname{ind}\mathbf{Q}' = \operatorname{ind}\mathbf{Q} = \operatorname{ind}\mathbf{P}$. $\square$

3. PERTURBATIONS IN NONLOCAL CONDITIONS

3.1. Formulation of the main result. In this section, we study the stability of the index for nonlocal operators under the perturbation of nonlocal conditions by operators of the same form as $\mathbf{B}^1_{i\mu}$ and $\mathbf{B}^2_{i\mu}$. This situation is more difficult than that in Section 2 because the above perturbations explicitly change the domain of the corresponding unbounded operators. Therefore, these perturbations cannot be treated as relatively compact ones, and we make use of another approach based on the notion of a *gap between closed operators*.

We consider differential operators $C_{i\mu s}(y,D_y)$, $i=1,\dots,N$, $\mu=1,\dots,m$, $s=1,\dots,S'_i$, of the same order $m_{i\mu}$ as $B_{i\mu s}$ in Subsection 1.1, given by

$$C_{i\mu s}(y,D_y)u = \sum_{|\alpha|\le m_{i\mu}} c_{i\mu s\alpha}(y)D^\alpha u,$$

where $c_{i\mu s\alpha}\in C^\infty(\mathbb{R}^2)$. Introduce an operator $\mathbf{C}^1_{i\mu}$ by the formula

$$\mathbf{C}^1_{i\mu}u = \sum_{s=1}^{S'_i}\bigl(C_{i\mu s}(y,D_y)(\zeta u)\bigr)(\Omega'_{is}(y)),\quad y\in\Gamma_i\cap\mathcal{O}_\varepsilon(\mathcal{K}),\qquad \mathbf{C}^1_{i\mu}u = 0,\quad y\in\Gamma_i\setminus\mathcal{O}_\varepsilon(\mathcal{K}),$$

where ζ and ε are the same as in the definition of $\mathbf{B}^1_{i\mu}$, whereas Ω'_{is} are C^∞ diffeomorphisms possessing the same properties as Ω_{is} (in particular, they satisfy Condition 1.2 with S_i and Ω_{is} replaced by S'_i and Ω'_{is}).

We also consider operators $\mathbf{C}^2_{i\mu}$ satisfying Condition 1.3 with $\mathbf{B}^2_{i\mu}$ replaced by $\mathbf{C}^2_{i\mu}$. Set

$$\mathbf{C}_{i\mu} = \mathbf{C}^1_{i\mu} + \mathbf{C}^2_{i\mu}.$$

We prove an index stability theorem under the following conditions (which are assumed to hold along with Conditions 1.1–1.3 throughout this section, including the formulation of lemmas).

Condition 3.1 (see, e.g., [9]). The system $\{\mathbf{B}^0_{i\mu}\}_{\mu=1}^m$ is normal on $\overline{\Gamma}_i$, $i = 1, \dots, N$.

Condition 3.2. $D^\sigma c_{i\mu s\alpha}(g_{i1}) = D^\sigma c_{i\mu s\alpha}(g_{i2}) = 0$ for $|\sigma| = 0, \dots, (m-1) - (m_{i\mu} - |\alpha|)$.

Denote by g_{i1} and g_{i2} the end points of $\overline{\Gamma}_i$. Let τ_{i1} (τ_{i2}) be a unit vector tangent to $\overline{\Gamma}_i$ at the point g_{i1} (g_{i2}).

Condition 3.3.

$$\frac{\partial^\beta \mathbf{C}^2_{i\mu} u}{\partial \tau_{i1}^\beta}\bigg|_{y=g_{i1}} = \frac{\partial^\beta \mathbf{C}^2_{i\mu} u}{\partial \tau_{i2}^\beta}\bigg|_{y=g_{i2}} = 0, \qquad \beta = 0, \dots, m-1-m_{i\mu}, \quad \forall u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})}).$$

The following lemma is a consequence of Conditions 3.2 and 3.3 (recall that $m-1 < a < m$ throughout).

Lemma 3.1. *The following inequalities hold*:

$$\|\mathbf{C}^1_{i\mu} u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le c_1 \|u\|_{H^{2m}_{a+m}(G)}, \tag{3.1}$$

$$\|\mathbf{C}^2_{i\mu} u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le c_2 \|u\|_{W^{2m}(G\setminus\overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})}. \tag{3.2}$$

Proof. 1. For any $u \in H^{2m}_{a+m}(G)$, we have

$$(D^\alpha u)(\Omega'_{is}(y))\big|_{\Gamma_i} \in H^{2m-|\alpha|-1/2}_{a+m}(\Gamma_i) \subset H^{2m-m_{i\mu}-1/2}_{a+m-(m_{i\mu}-|\alpha|)}(\Gamma_i).$$

Therefore, by Condition 3.2 and Lemma A.5, we have $(c_{i\mu s\alpha} D^\alpha u)(\Omega'_{is}(y))|_{\Gamma_i} \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Estimate (3.1) follows from the boundedness of the above embedding and from inequality (A.5).

2. It follows from Condition 1.3 (applied to $\mathbf{C}^2_{i\mu}$) that $\mathbf{C}^2_{i\mu} u \in W^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $u \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$. Now, by Condition 3.3 and Lemma A.4, $\mathbf{C}^2_{i\mu} u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Estimate (3.2) follows from inequality (1.2) (applied to $\mathbf{C}^2_{i\mu}$) and from (A.2). $\square$

In this section, we write $\mathbf{A} = \mathbf{A}(y, D_y)$. Consider the operators $\mathbf{P}_t \colon \mathrm{D}(\mathbf{P}_t) \subset L_2(G) \to L_2(G)$, $t \in \mathbb{C}$, given by

$$\mathbf{P}_t u = \mathbf{A}u, \qquad u \in \mathrm{D}(\mathbf{P}_t) = \big\{u \in W^m(G, \mathbf{B} + t\mathbf{C}) \colon \mathbf{A}u \in L_2(G)\big\},$$

where $W^m(G, \mathbf{B} + t\mathbf{C})$ is the space of functions $u \in W^m(G)$ that satisfy the nonlocal conditions $(\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + t\mathbf{C}_{i\mu})u = 0$. The main result of this section (to be proved in Subsection 3.2) is as follows.

Theorem 3.1. *Let Conditions* 1.1–1.3 *and* 3.1–3.3 *hold. Then* $\operatorname{ind} \mathbf{P}_t = \mathrm{const}\ \forall t \in \mathbb{C}$.

3.2. The gap between nonlocal operators in weighted spaces. As in Section 2, we preliminarily study the operators $\mathbf{Q}_t \colon \mathrm{D}(\mathbf{Q}_t) \subset L_2(G) \to H^0_a(G)$ given by

$$\mathbf{Q}_t u = \mathbf{A}u, \qquad u \in \mathrm{D}(\mathbf{Q}_t) = \big\{u \in W^m(G, \mathbf{B} + t\mathbf{C}) \colon \mathbf{A}u \in H^0_a(G)\big\},$$

where $t \in \mathbb{C}$ and $W^m(G, \mathbf{B} + t\mathbf{C})$ is the same as in the definition of the operator $\mathbf{P}_t$. The operators $\mathbf{P}_t$ and $\mathbf{Q}_t$ correspond to the problem

$$\mathbf{A}u = f(y), \qquad y \in G, \tag{3.3}$$

$$(\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + t\mathbf{C}_{i\mu})u = 0, \qquad y \in \Gamma_i, \quad i = 1, \dots, N, \quad \mu = 1, \dots, m. \tag{3.4}$$

Remark 3.1. The operator $\widetilde{\mathcal{L}}(\lambda)$ was constructed in Subsection 1.2 by means of the principal homogeneous parts of the operators $\mathbf{A}$ and $B_{i\mu s}(y, D_y)$ at the points of the set $\mathcal{K}$. Due to Condition 3.2, the principal homogeneous parts of the operators $C_{i\mu s}(y, D_y)$ are equal to zero at these points. Therefore, the same operator $\widetilde{\mathcal{L}}(\lambda)$ corresponds to problem (3.3), (3.4) for any t.

Fix a number a such that $m - 1 < a < m$ and the line $\operatorname{Im}\lambda = a + 1 - 2m$ contains no eigenvalues of $\widetilde{\mathcal{L}}(\lambda)$ (which is possible due to the discreteness of the spectrum of $\widetilde{\mathcal{L}}(\lambda)$). It follows from Remark 3.1 and Lemma 2.1 that $\mathbf{Q}_t$ has the Fredholm property. Therefore, its graph $\operatorname{Gr}\mathbf{Q}_t$ is a closed subspace in the Hilbert space $L_2(G) \times H^0_a(G)$; this space is endowed with the norm

$$\|(u, f)\| = \big(\|u\|^2_{L_2(G)} + \|f\|^2_{H^0_a(G)}\big)^{1/2} \qquad \forall (u, f) \in L_2(G) \times H^0_a(G).$$

Denote

$$\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}) = \sup_{u \in \mathrm{D}(\mathbf{Q}_t)\colon \|(u, \mathbf{Q}_t u)\| = 1} \operatorname{dist}\big((u, \mathbf{Q}_t u), \operatorname{Gr}\mathbf{Q}_{t+s}\big). \tag{3.5}$$

By Definition A.3, the number $\widehat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s}) = \max\{\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}), \delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)\}$ is the *gap between the operators* $\mathbf{Q}_t$ *and* $\mathbf{Q}_{t+s}$.

The main tool that enables us to prove the index stability theorem is Theorem A.5 and the following result (to be proved later on).

Theorem 3.2. *Let Conditions* 1.1–1.3 *and* 3.1–3.3 *hold. Suppose that the lines* $\operatorname{Im}\lambda = a + 1 - 2m$ *and* $\operatorname{Im}\lambda = a + 1 - m$ *contain no eigenvalues of* $\widetilde{\mathcal{L}}(\lambda)$. *Then*

$$\widehat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s}) \le c_t s, \qquad |s| \le s_t, \tag{3.6}$$

where $s_t > 0$ *is sufficiently small, while* $c_t > 0$ *does not depend on* s.

First, we prove several auxiliary results.

Lemma 3.2. *Let the line* $\operatorname{Im}\lambda = a + 1 - m$ *contain no eigenvalues of* $\widetilde{\mathcal{L}}(\lambda)$. *Then, for all sufficiently small* $|s|$, *we have*

$$\|u\|_{H^{2m}_{a+m}(G)} \le c_t \|(u, \mathbf{A}u)\| \qquad \forall u \in \mathrm{D}(\mathbf{Q}_{t+s}), \tag{3.7}$$

where $c_t > 0$ *does not depend on* s *and* u.

Proof. 1. Consider the bounded operator

$$\mathbf{M}_t = \{\mathbf{A}, \mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + t\mathbf{C}_{i\mu}\}\colon\ H^{2m}_{a+m}(G) \to \mathcal{H}^0_{a+m}(G, \Gamma). \tag{3.8}$$

If $v \in H^{2m}_{a+m}(G)$, then $(\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + t\mathbf{C}^1_{i\mu})v \in H^{2m-m_{i\mu}-1/2}_{a+m}(\Gamma_i)$ and $\mathbf{C}^2_{i\mu}v \in W^{2m-m_{i\mu}-1/2}(\Gamma_i) \subset H^{2m-m_{i\mu}-1/2}_{a+m}(\Gamma_i)$ (the latter relations are due to Condition 1.3 and Lemma A.3(1)); thus, the operator $\mathbf{M}_t$ is well defined.

By Theorem 6.1 in [16] and by Remark 3.1, the operator $\mathbf{M}_t$ has the Fredholm property for any $t \in \mathbb{C}$. Therefore, applying Theorem A.1 and noting that the embedding $H^{2m}_{a+m} \subset L_2(G)$ is compact for $a < m$ (see Lemma A.1), we obtain

$$\|u\|_{H^{2m}_{a+m}(G)} \le k_1\big(\|\mathbf{M}_t u\|_{\mathcal{H}^0_{a+m}(G,\Gamma)} + \|u\|_{L_2(G)}\big) \qquad \forall u \in H^{2m}_{a+m}(G), \tag{3.9}$$

where $k_1 > 0$ may depend on t but does not depend on s and u.

2. Now take a function $u \in \mathrm{D}(\mathbf{Q}_{t+s})$. By Lemma 2.3, $u \in H^{2m}_{a+m}(G)$. Inequality (3.9), estimate (1.2) (for $\mathbf{C}^2_{i\mu}$), and the boundedness of the embedding $W^{2m-m_{i\mu}-1/2}(\Gamma_i) \subset H^{2m-m_{i\mu}-1/2}_{a+m}(\Gamma_i)$ (see Lemma A.3(1)) yield

$$\|u\|_{H^{2m}_{a+m}(G)} \le k_1\big(\|\mathbf{A}u\|_{H^0_{a+m}(G)} + \|u\|_{L_2(G)}\big) + k_2|s| \cdot \|u\|_{H^{2m}_{a+m}(G)} \qquad \forall u \in \mathrm{D}(\mathbf{Q}_{t+s}),$$

where $k_2 > 0$ may depend on t but does not depend on s and u. Choosing $|s| \le 1/(2k_2)$ and noting that the embedding $H^0_a(G) \subset H^0_{a+m}(G)$ is bounded, we obtain (3.7). $\square$

Lemmas 3.1 and 3.2 imply

Corollary 3.1. *Let the line* $\operatorname{Im}\lambda = a + 1 - m$ *contain no eigenvalues of* $\widetilde{\mathcal{L}}(\lambda)$. *Then*

$$\|\mathbf{C}_{i\mu}u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le c_t\|(u, \mathbf{A}u)\| \qquad \forall u \in \mathrm{D}(\mathbf{Q}_{t+s}), \tag{3.10}$$

where $c_t > 0$ *does not depend on* s *and* u, *provided that* $|s|$ *is sufficiently small.*

The following two lemmas enable us to reduce nonlocal problems with nonhomogeneous nonlocal conditions to nonlocal problems with homogeneous ones. This is the place where Condition 3.1 is needed.

Lemma 3.3 (see Lemma 8.1 in [16]). *Let* $a \in \mathbb{R}$. *Then, for any right-hand sides* $f_{j\sigma\mu} \in H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})$ *in* (1.7) *such that* $\operatorname{supp} f_{j\sigma\mu} \subset \gamma^{\varepsilon/2}_{j\sigma}$, *there exist functions* $U_j \in H^{2m}_a(K_j)$ *such that* $\operatorname{supp} U_j \subset \overline{K^\varepsilon_j}$,

$$B_{j\sigma\mu j0}(y, D_y)U_j(y) = f_{j\sigma\mu}(y), \qquad (B_{j\sigma\mu ks}(y, D_y)U_k)(\mathcal{G}_{j\sigma ks}y) = 0, \qquad y \in \gamma_{j\sigma}, \quad (k,s) \ne (j,0),$$

$$\sum_j \|U_j\|_{H^{2m}_a(K_j)} \le c \sum_{j,\sigma,\mu} \|f_{j\sigma\mu}\|_{H_a^{2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma})}.$$

Lemma 3.4. *Let* $f_{i\mu} \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. *Then, for* $t \in \mathbb{C}$ *and* $|s| \le 1$, *there is a function* $u \in H^{2m}_a(G)$ *such that*

$$(\mathbf{B}^0_{i\mu} + \mathbf{B}^1_{i\mu} + (t+s)\mathbf{C}_{i\mu})u = f_{i\mu}, \tag{3.11}$$

$$\|u\|_{H^{2m}_a(G)} \le c_t \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \tag{3.12}$$

where $c_t > 0$ *does not depend on* $f_{i\mu}$ *and* s.

Proof. Using Lemma 3.3 and a partition of unity, we construct a function $v \in H^{2m}_a(G)$ such that

$$\operatorname{supp} v \subset \overline{G} \setminus \overline{G}_\rho, \tag{3.13}$$

$$\mathbf{B}^0_{i\mu}v = f_{i\mu}, \qquad \mathbf{B}^1_{i\mu}v = 0, \qquad \mathbf{C}^1_{i\mu}v = 0, \tag{3.14}$$

$$\|v\|_{H^{2m}_a(G)} \le k_1 \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \tag{3.15}$$

where $k_1 > 0$ does not depend on $f_{i\mu}$, t, and s.

By (3.13) and (1.3), we have $\operatorname{supp} \mathbf{C}^2_{i\mu}v \subset \mathcal{O}_{\varkappa_2}(\mathcal{K})$. Moreover, by virtue of Lemma 3.1, $\mathbf{C}^2_{i\mu}v \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$. Therefore, using Lemma 3.3 and a partition of unity again, we construct a function $w \in H^{2m}_a(G)$ such that

$$\operatorname{supp} w \subset \mathcal{O}_{\varkappa_1}(\mathcal{K}), \tag{3.16}$$

$$\mathbf{B}_{i\mu}^0 w = -(t+s)\mathbf{C}_{i\mu}^2 v, \qquad \mathbf{B}_{i\mu}^1 w = 0, \qquad \mathbf{C}_{i\mu}^1 w = 0, \tag{3.17}$$

$$\|w\|_{H_a^{2m}(G)} \le k_1 \sum_{i,\mu} \|(t+s)\mathbf{C}_{i\mu}^2 v\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}.$$

Using the relation $|s| \le 1$ and inequalities (3.2) and (3.15), from the last inequality we infer

$$\|w\|_{H_a^{2m}(G)} \le k_1 \sum_{i,\mu}(|t|+1)\|\mathbf{C}_{i\mu}^2 v\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \le k_2\|v\|_{H_a^{2m}(G)} \le k_2 k_1 \sum_{i,\mu} \|f_{i\mu}\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}, \tag{3.18}$$

where $k_2 > 0$ may depend on t but does not depend on $f_{i\mu}$ and s.

By (3.16) and (3.2), we have $\mathbf{C}_{i\mu}^2 w = 0$. It follows from this relation, (3.14), and (3.17) that $u = v + w$ satisfies (3.11). Inequality (3.12) follows from inequalities (3.15) and (3.18). □

Remark 3.2. One can easily see that if $(\mathbf{C}_{i\mu}^2 v)(y) = 0$ in $\mathcal{O}_\varkappa(\mathcal{K})$ for some $\varkappa > 0$ and for any $v \in W^{2m}(G \setminus \overline{\mathcal{O}_{\varkappa_1}(\mathcal{K})})$, then Lemma 3.4 is true for any $a \in \mathbb{R}$.

Proof of Theorem 3.2. 1. We must prove inequality (3.6) for the quantity $\widehat{\delta}(\mathbf{Q}_t, \mathbf{Q}_{t+s})$ replaced by $\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s})$ and $\delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)$. Let us prove the inequality

$$\delta(\mathbf{Q}_t, \mathbf{Q}_{t+s}) \le c_t|s|, \qquad |s| \le s_t. \tag{3.19}$$

(The corresponding inequality for $\delta(\mathbf{Q}_{t+s}, \mathbf{Q}_t)$ is proved in a similar way.)

Fix an arbitrary number t and take a function $u \in \mathrm{D}(\mathbf{Q}_t)$. According to the definition (3.5), it suffices to find a function $v_s \in \mathrm{D}(\mathbf{Q}_{t+s})$ (which depends on u) such that

$$\|u - v_s\|_{L_2(G)} + \|\mathbf{A}u - \mathbf{A}v_s\|_{H_a^0(G)} \le k_1|s| \cdot \|(u, \mathbf{A}u)\|, \tag{3.20}$$

where $|s|$ is sufficiently small and $k_1, k_2, \ldots > 0$ may depend on t but do not depend on u and s.

Let us seek $v_s \in \mathrm{D}(\mathbf{Q}_{t+s})$ in the form

$$v_s = u + w_s, \tag{3.21}$$

where $w_s \in H_a^{2m}(G)$ is a solution of the problem

$$\mathbf{A}w_s = \sum_{j=1}^{J_s} \beta_j^s f_j^s, \qquad (\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})w_s = -s\mathbf{C}_{i\mu}u; \tag{3.22}$$

the numbers J_s and β_j^s, as well as the functions $f_j^s \in H_a^0(G)$, will be defined later in such a way that the solution $w_s \in H_a^{2m}(G)$ exists.

2. To solve problem (3.22), we first note that $\mathbf{C}_{i\mu}u \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ due to Corollary 3.1. Hence, we can apply Lemma 3.4 and construct a function $W_s \in H_a^{2m}(G)$ such that

$$(\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})W_s = -s\mathbf{C}_{i\mu}u, \tag{3.23}$$

$$\|W_s\|_{H_a^{2m}(G)} \le k_2|s| \sum_{i,\mu} \|\mathbf{C}_{i\mu}u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)}. \tag{3.24}$$

Combining (3.24) with (3.10), we obtain

$$\|W_s\|_{H_a^{2m}(G)} \le k_3|s| \cdot \|(u, \mathbf{A}u)\|. \tag{3.25}$$

Clearly, problem (3.22) is equivalent to the following one:

$$\mathbf{A}Y_s = -\mathbf{A}W_s + \sum_{j=1}^{J_s} \beta_j^s f_j^s, \qquad (\mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu})Y_s = 0, \tag{3.26}$$

where

$$Y_s = w_s - W_s \in H_a^{2m}(G). \tag{3.27}$$

3. To solve problem (3.26), we consider the bounded operator

$$\mathbf{L}_t = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}\}\colon\ H_a^{2m}(G) \to \mathcal{H}_a^0(G,\Gamma). \tag{3.28}$$

Note that $\mathbf{C}_{i\mu}^2 v \in H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)$ for any $v \in H_a^{2m}(G)$ due to Lemma 3.1; for this reason, we can write $\mathcal{H}_a^0(G,\Gamma)$ instead of $\mathcal{H}_a^0(G,\Gamma) \dotplus \mathcal{R}_a^0(G,\Gamma)$ in the definition of the operator $\mathbf{L}_t$ (cf. (2.3)). It follows from Theorem 6.1 in [16] and from Remark 3.1 that the operator $\mathbf{L}_t$ has the Fredholm property for any $t \in \mathbb{C}$.

Let us expand the space $H_a^{2m}(G)$ in the orthogonal sum $H_a^{2m}(G) = \ker \mathbf{L}_t \oplus E_t$, where E_t is a closed subspace in $H_a^{2m}(G)$. Clearly, the operator

$$\mathbf{L}_t' = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + t\mathbf{C}_{i\mu}\}\colon\ E_t \to \mathcal{H}_a^0(G,\Gamma) \tag{3.29}$$

has the Fredholm property and its kernel is trivial. In particular, this means that

$$\|u\|_{H_a^{2m}(G)} \le k_4 \|\mathbf{L}_t' u\|_{\mathcal{H}_a^0(G,\Gamma)} \qquad \forall u \in E_t. \tag{3.30}$$

Let $J = \operatorname{codim} \mathcal{R}(\mathbf{L}_t')$. It follows from Lemma 3.1 and Theorem A.3 that the operator

$$\mathbf{L}_{ts}' = \{\mathbf{A}, \mathbf{B}_{i\mu}^0 + \mathbf{B}_{i\mu}^1 + (t+s)\mathbf{C}_{i\mu}\}\colon\ E_t \to \mathcal{H}_a^0(G,\Gamma)$$

also has the Fredholm property, its kernel is trivial, and $\operatorname{codim} \mathcal{R}(\mathbf{L}_{ts}') = J$, provided that $|s| \le s_t$, where $s_t > 0$ is sufficiently small. Moreover, using estimates (3.30), (3.1), and (3.2), we obtain

$$\|u\|_{H_a^{2m}(G)} \le k_4 \Bigg(\|\mathbf{L}_{ts}' u\|_{\mathcal{H}_a^0(G,\Gamma)} + s_t \sum_{i,\mu} \|\mathbf{C}_{i\mu} u\|_{H_a^{2m-m_{i\mu}-1/2}(\Gamma_i)} \Bigg) \le k_5 \Big(\|\mathbf{L}_{ts}' u\|_{\mathcal{H}_a^0(G,\Gamma)} + s_t \|u\|_{H_a^{2m}(G)} \Big)$$

for all $u \in E_t$. Taking $s_t \le 1/(2k_6)$, we arrive at

$$\|u\|_{H_a^{2m}(G)} \le k_6 \|\mathbf{L}_{ts}' u\|_{\mathcal{H}_a^0(G,\Gamma)} \qquad \forall u \in E_t. \tag{3.31}$$

Since $\mathbf{L}_{ts}'$ has the Fredholm property, the set $\{f \in H_a^0(G)\colon (f,0) \in \mathcal{R}(\mathbf{L}_{ts}')\}$ is closed and is of finite codimension J_s in $H_a^0(G)$. It is easy to see that $J_s \le J$.

Let $f_1^s, \dots, f_{J_s}^s$ be an orthogonal normalized basis for the space

$$H_a^0(G) \ominus \{f \in H_a^0(G)\colon\ (f,0) \in \mathcal{R}(\mathbf{L}_{ts}')\}.$$

Set $\beta_j^s = (\mathbf{A}W_s, f_j^s)_{H_a^0(G)}$. In this case, problem (3.26) admits a unique solution $Y_s \in E_t$, and, by virtue of (3.31) and (3.25), we have

$$\|Y_s\|_{H_a^{2m}(G)} \le k_6 \Bigg(\|\mathbf{A}W_s\|_{H_a^0(G)} + \sum_{j=1}^{J_s} |\beta_j^s| \Bigg) \le k_7 |s| \cdot \|(u, \mathbf{A}u)\| + k_6 J \max\{\beta_1^s, \dots, \beta_{J_s}^s\}. \tag{3.32}$$

Estimating $\beta_j^s = (\mathbf{A}W_s, f_j^s)_{H_a^0(G)}$ by Schwarz' inequality and using (3.25), we obtain

$$|\beta_j^s| \le \|\mathbf{A}W_s\|_{H_a^0(G)} \le k_8|s| \cdot \|(u, \mathbf{A}u)\|.$$

Combining this inequality with (3.32) yields

$$\|Y_s\|_{H_a^{2m}(G)} \le k_9|s| \cdot \|(u, \mathbf{A}u)\|. \tag{3.33}$$

4. Taking into account equality (3.27), from estimates (3.25) and (3.33) we deduce

$$\|w_s\|_{L_2(G)} \le k_{10}\|w_s\|_{H_a^{2m}(G)} \le k_{11}|s| \cdot \|(u, \mathbf{A}u)\|, \tag{3.34}$$

$$\|\mathbf{A}w_s\|_{H_a^0(G)} \le k_{12}\|w_s\|_{H_a^{2m}(G)} \le k_{12}k_{11}|s| \cdot \|(u, \mathbf{A}u)\|, \tag{3.35}$$

where $w_s = Y_s + W_s$ is a solution of problem (3.22).

It follows from the boundedness of the embedding $H_a^{2m}(G) \subset W^m(G)$ that the function v_s defined by (3.21) belongs to $W^m(G)$, and $v_s \in \mathrm{D}(\mathbf{Q}_{t+s})$ due to the second relation in (3.22). The desired inequality (3.20) follows from (3.21), (3.34), and (3.35). $\square$

Proof of Theorem 3.1. It follows from Lemma 2.1 in [11] that the spectrum of $\widetilde{\mathcal{L}}(\lambda)$ is discrete. Therefore, one can find a number a such that $m-1 < a < m$ and the lines $\operatorname{Im}\lambda = a+1-2m$ and $\operatorname{Im}\lambda = a+1-m$ contain no eigenvalues of $\widetilde{\mathcal{L}}(\lambda)$. Fix two arbitrary numbers $t_1, t_2 \in \mathbb{C}$. By Lemma 2.1 and Remark 3.1, the operators $\mathbf{Q}_t$ have the Fredholm property for all t in the interval $I_{t_1t_2} \subset \mathbb{C}$ with the end points t_1 and t_2. Covering each point of the interval $I_{t_1t_2}$ by a disk of sufficiently small radius, choosing a finite subcovering of $I_{t_1t_2}$, and applying Theorems 3.2 and A.5, we see that $\operatorname{ind}\mathbf{Q}_{t_1} = \operatorname{ind}\mathbf{Q}_{t_2}$. It follows from this fact and from Lemma 2.1 that $\operatorname{ind}\mathbf{P}_{t_1} = \operatorname{ind}\mathbf{P}_{t_2}$. $\square$

Remark 3.3. Theorems 2.1 and 3.1 remain true in the case where the set $\mathcal{K}$ consists of finitely many disjoint orbits. The proofs need evident modifications.

APPENDIX

A.1. Some properties of Sobolev and weighted spaces. Let G and Γ_i be the same as in Section 1.

Lemma A.1 (see Lemma 3.5 in [7]). *Let $k_2 > k_1$ and $k_2 - a_2 > k_1 - a_1$. Then the space $H_{a_2}^{k_2}(G)$ is compactly embedded in $H_{a_1}^{k_1}(G)$.*

Fix an arbitrary index i and set $\Gamma = \Gamma_i$. Let $g \in \overline{\Gamma} \setminus \Gamma$. Throughout this section, we assume without loss of generality that $g = 0$ and Γ coincides with the axis Oy_1 in a sufficiently small neighborhood $\mathcal{O}_\varepsilon(0)$ of the origin. In this appendix, we use the notation $G^\varepsilon = G \cap \mathcal{O}_\varepsilon(0)$ and $\Gamma^\varepsilon = \Gamma \cap \mathcal{O}_\varepsilon(0)$, in which case $H_a^k(G^\varepsilon) = H_a^k(G^\varepsilon, \{0\})$.

Lemma A.2. *If $u \in W^k(G^\varepsilon)$, $k \ge 1$, then the following assertions are true*:

(1) *$u(y) = P(y) + v(y)$ for $y \in G^\varepsilon$, where $P(y) = \sum_{|\alpha| \le k-2} p_\alpha y^\alpha$, $v \in W^k(G^\varepsilon) \cap H_\delta^k(G^\varepsilon)$ $\forall \delta > 0$ (if $k = 1$, we set $P(y) \equiv 0$); in particular, $u \in H_{k-1+\delta}^k(G^\varepsilon)$*;

(2) *$D^\alpha u|_{y=0} = D^\alpha P|_{y=0}$ for $|\alpha| \le k-2$*;

(3) *$\sum_{|\alpha| \le k-2} |p_\alpha| + \|v\|_{H_\delta^k(G^\varepsilon)} \le c_\delta \|u\|_{W^k(G^\varepsilon)}$, where $c_\delta > 0$ does not depend on u.*

Proof. The proof follows from Lemma 4.9 in [7] for $k = 1$ and from Lemma 4.11 in [7] for $k \ge 2$.

Lemma A.3. *If $\psi \in W^{k-1/2}(\Gamma^\varepsilon)$, $k \ge 1$, then the following assertions are true*:

(1) *$\psi(r) = P_1(r) + \varphi(r)$ for $0 < r < \varepsilon$, where $P_1(r) = \sum_{\beta=0}^{k-2} p_\beta r^\beta$ and $\varphi \in W^{k-1/2}(\Gamma^\varepsilon) \cap H_\delta^{k-1/2}(\Gamma^\varepsilon)$ $\forall \delta > 0$ (if $k = 1$, we set $P_1(r) \equiv 0$); in particular, $\psi \in H_{k-1+\delta}^{k-1/2}(\Gamma^\varepsilon)$*;

(2) $(d^\beta\psi/dr^\beta)|_{r=0} = (d^\beta P_1/dr^\beta)|_{r=0}$ *for* $\beta = 0, \dots, k-2$;

(3) $\sum_{\beta=0}^{k-2} |p_\beta| + \|\varphi\|_{H_\delta^{k-1/2}(\Gamma^\varepsilon)} \le c_\delta \|\psi\|_{W^{k-1/2}(\Gamma^\varepsilon)}$, *where* $c_\delta > 0$ *does not depend on* ψ.

Proof. Consider a function $u \in W^k(G^\varepsilon)$ such that $u|_{\Gamma^\varepsilon} = \psi$ and $\|u\|_{W^k(G^\varepsilon)} \le 2\|\psi\|_{W^{k-1/2}(\Gamma^\varepsilon)}$. Now it remains to apply Lemma A.2. $\square$

Lemma A.4. *Let* $\psi \in W^{k-1/2}(\Gamma)$, $k \ge 2$, *and let*

$$\frac{d^s\psi}{dr^s}\bigg|_{y=0} = 0, \qquad s = 0, \dots, l, \tag{A.1}$$

for a fixed $l \le k-2$. *Then* $\psi \in H_{k-2-l+\delta}^{k-1/2}(\Gamma)$ $\forall \delta > 0$ *and*

$$\|\psi\|_{H_{k-2-l+\delta}^{k-1/2}(\Gamma)} \le c_\delta \|\psi\|_{W^{k-1/2}(\Gamma)}, \tag{A.2}$$

where $c_\delta > 0$ *does not depend on* ψ.

Proof. It follows from relations (A.1) and Lemma A.3 (assertions (1) and (2)) that

$$\psi(r) = \sum_{\beta=l+1}^{k-2} p_\beta r^\beta + \varphi(r), \qquad 0 < r < \varepsilon, \tag{A.3}$$

where

$$\varphi \in H_\delta^{k-1/2}(\Gamma^\varepsilon) \subset H_{k-2-l+\delta}^{k-1/2}(\Gamma^\varepsilon), \qquad \delta > 0. \tag{A.4}$$

If $l = k-2$, then the sum in (A.3) is absent and the lemma follows from (A.4) and Lemma A.3(3).

If $l \le k-3$, then the sum comprises the terms r^β for $\beta \ge l+1$. One can directly verify that $r^\beta \in H_{k-2-l+\delta}^{k-1/2}(\Gamma^\varepsilon)$ for the above β and for all $\delta > 0$. Therefore, combining (A.3) with (A.4) and with Lemma A.3(3), we complete the proof. $\square$

Lemma A.5. *Let* $\psi \in H_{a+l}^{k-1/2}(\Gamma)$, $l, k \in \mathbb{N}$, $a \in \mathbb{R}$, *and let* $b \in C^\infty(\overline{\Gamma})$ *be a compactly supported function satisfying the relations* $\frac{\partial^s b}{\partial r^s}\big|_{r=0} = 0$, $s = 0, \dots, l-1$. *Then*

$$\|b\psi\|_{H_a^{k-1/2}(\Gamma)} \le c\|\psi\|_{H_{a+l}^{k-1/2}(\Gamma)}. \tag{A.5}$$

Proof. Clearly, it suffices to carry out the proof for compactly supported functions ψ and for G and Γ replaced by $K = \{y \in \mathbb{R}^2 \colon 0 < \omega < \omega_0\}$ and $\gamma = \{y \in \mathbb{R}^2 \colon \omega = 0\}$, respectively.

Denote by $\widehat{b} \in C^\infty(\mathbb{R})$ an extension of $b(y_1)$ to $\mathbb{R}$ and introduce the function $B(y_1, y_2) = \widehat{b}(y_1)$ for $(y_1, y_2) \in \mathbb{R}^2$. Clearly, we have

$$B \in C^\infty(\overline{K}), \qquad D^\sigma B|_{y=0} = 0, \quad |\sigma| \le l-1. \tag{A.6}$$

Let $u \in H_{a+l}^k(K)$ be a compactly supported extension of ψ to the angle K such that

$$\|u\|_{H_{a+l}^k(K)} \le c_1 \|\psi\|_{H_{a+l}^{k-1/2}(\gamma)}. \tag{A.7}$$

It follows from Teylor's formula and (A.6) that $|D^\sigma B| = O\big(r^{l-|\sigma|}\big)$ for any σ; therefore,

$$\|Bu\|_{H_a^k(K)}^2 = \sum_{|\alpha| \le k} \int\limits_K r^{2(a+|\alpha|-k)} |D^\alpha(Bu)|^2\,dy \le c_2 \sum_{|\sigma|+|\zeta| \le k} \int\limits_K r^{2(a+|\sigma|+|\zeta|-k)} |D^\sigma B|^2 |D^\zeta u|^2\,dy$$

$$\le c_3 \sum_{|\zeta| \le k} \int\limits_K r^{2(a+l+|\zeta|-k)} |D^\zeta u|^2\,dy = c_3 \|u\|_{H_{a+l}^k(K)}^2$$

(recall that u is compactly supported). Combining this estimate with (A.7), we finally obtain

$$\|b\psi\|_{H_a^{k-1/2}(\gamma)} \le \|Bu\|_{H_a^k(K)} \le c_3^{1/2}\|u\|_{H_{a+l}^k(K)} \le c_3^{1/2}c_1\|\psi\|_{H_{a+l}^{k-1/2}(\gamma)}. \qquad \square$$

A.2. Some properties of Fredholm operators. Let H_1 and H_2 be Hilbert spaces, and let $P\colon \mathrm{D}(P) \subset H_1 \to H_2$ be a linear (in general, unbounded) operator.

Definition A.1. The operator P is said *to have the Fredholm property* if it is closed, its image is closed, and the dimension of its kernel $\ker P$ and the codimension of its image $\mathcal{R}(P)$ are finite. The number $\operatorname{ind} P = \dim\ker P - \operatorname{codim}\mathcal{R}(P)$ is called the *index* of the Fredholm operator P.

Theorem A.1 (see Theorem 7.1 in [8])**.** *Let H be a Hilbert space such that H_1 is compactly embedded in H, and let the operator P be closed. Then* $\dim\ker P < \infty$ *and* $\mathcal{R}(P) = \overline{\mathcal{R}(P)}$ *if and only if*

$$\|u\|_{H_1} \le c(\|Pu\|_{H_2} + \|u\|_H) \qquad \forall u \in \mathrm{D}(P).$$

The proof of the following result is contained in part 2 of the proof of Lemma 2.5 in [3].

Theorem A.2. *Let $\dot P\colon \mathrm{D}(\dot P) \subset H_1 \to H_2$ be a Fredholm operator such that P is an extension of $\dot P$, i.e., $\dot P \subset P$. Suppose that* $\dim\ker P < \infty$, $\mathcal{R}(P) = \overline{\mathcal{R}(P)}$, *and* $\operatorname{codim}\mathcal{R}(P) < \infty$. *Then the operator P is closed* (*hence, it has the Fredholm property*)*.*

Let $A\colon \mathrm{D}(A) \subset H_1 \to H_2$ be a linear operator.

Theorem A.3 (see Section 16 in [8])**.** *Let the operator P have the Fredholm property, A be bounded, and* $\mathrm{D}(A) = H_1$. *Then the operator $P+A$ has the Fredholm property,* $\operatorname{ind}(P+A) = \operatorname{ind} P$, $\dim\ker(P+A) \le \dim\ker P$, *and* $\operatorname{codim}\mathcal{R}(P+A) \le \operatorname{codim}\mathcal{R}(P)$, *provided that $\|A\|$ is sufficiently small.*

Definition A.2 (see, e.g., [8, 6])**.** The operator A is said to be *relatively compact with respect to P* or simply *P-compact* if $\mathrm{D}(P) \subset \mathrm{D}(A)$ and, for any sequence $u_n \in \mathrm{D}(P)$ with both $\{u_n\}$ and $\{Pu_n\}$ bounded, $\{Au_n\}$ contains a convergent subsequence.

Theorem A.4 (see Theorem 5.26 in Chapter 4 of [6])**.** *Suppose that the operator P has the Fredholm property and the operator A is P-compact. Then the operator $P+A$ also has the Fredholm property and* $\operatorname{ind}(P+A) = \operatorname{ind} P$.

Finally, we introduce a concept of a gap between closed operators. Let $S\colon \mathrm{D}(S) \subset H_1 \to H_2$ be a linear operator. In the space $H_1 \times H_2$, we introduce the norm

$$\|(u,f)\| = \bigl(\|u\|_{H_1}^2 + \|f\|_{H_2}^2\bigr)^{1/2} \qquad \forall (u,f) \in H_1 \times H_2.$$

Set $\delta(P,S) = \sup_{u\in \mathrm{D}(P)\colon \|(u,Pu)\|=1} \operatorname{dist}\bigl((u,Pu), \operatorname{Gr} S\bigr)$, where $\operatorname{Gr} S$ is the graph of the operator S.

Definition A.3. The number $\widehat{\delta}(P,S) = \max\{\delta(P,S), \delta(S,P)\}$ is called a *gap between the operators P and S.*

Theorem A.5 (see Theorem 5.17 in Chapter 4 of [6])**.** *Let the operator P have the Fredholm property and S be closed. Then the operator S has the Fredholm property,* $\operatorname{ind} S = \operatorname{ind} P$, $\dim\ker S \le \dim\ker P$, *and* $\operatorname{codim}\mathcal{R}(S) \le \operatorname{codim}\mathcal{R}(P)$ *provided that the gap $\widehat{\delta}(P,S)$ is sufficiently small.*

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