# Well-posedness of parabolic equations containing hysteresis with diffusive thresholds

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July 9, 2013

UDK: 517.9
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## 1 Introduction

In the paper, we develop a theory of reaction-diffusion equations containing discontinuous hysteresis operator — the so-called non-ideal relay. The non-ideal relay (or a bi-stable relay, or lazy switch) is the most basic, yet nontrivial *hysteresis operator*. The state (output) of the non-ideal relay switches from $-1$ to $1$ when the input exceeds a threshold value $x \in \mathbb{R}$ and switches back to state $-1$ when the input drops below a smaller threshold value $y \in \mathbb{R}$. Hence, variation of the input causes switching between two available states in such a way that the current state depends on the input history as long as the current value of the input falls within the input's bi-stability range $(y, x)$. This behavior is illustrated in Fig. 1.1.

In particular, reaction-diffusion equations with non-ideal relay arise in population dynamics, developmental biology (cell differentiation processes), bacterial metabolic processes, etc. The general feature of all those models is a hysteretic interaction between several "substances". Depending on the phys-

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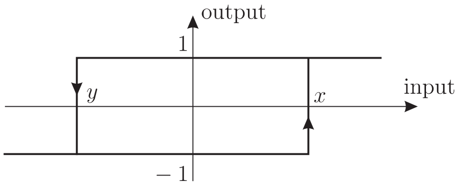

Figure 1.1: Non-ideal relay.

ical background, the substances are the amount of individuals, concentration of proteins, density of nutrients, etc.

In most existing models, the diffusion term in the reaction-diffusion equation appears because of a spatial motion of the substances. In this paper, we suggest a principally new mechanism of diffusion based on sporadical changes of the thresholds of hysteresis operators. As we will explain, this mechanism is not only natural, but also leads to a new dynamical phenomena in the systems under consideration, e.g., appearance of sign changing patterns for the states of non-ideal relays.

Let us illustrate the above mentioned mechanism with the following prototype example. Suppose we have a population of bacteria with the environment being a mixture of two types of nutrients — lactose and glucose. Each bacterium, at a given moment, can consume either lactose or glucose. In the first case, we say that it is in the state 1 and in the second case in the state $-1$. For each bacterium at a given moment, there are two (symmetric for simplicity) thresholds: $-x$ and $x$, where $x > 0$. If the deviation of the relative concentration of lactose from the value $1/2$ in the mixture of the two nutrients is greater than $x$, then the bacterium consumes lactose. If the deviation is less than $-x$, it consumes glucose. If the deviation is between $-x$ and $x$, then the state of the bacterium is defined according to the discontinuous hysteresis law (see Fig. 1.1, where $y = -x$ and the above deviation plays a role of input, while the state of the bacterium plays a role of output).

Now the key (and quite natural) assumption is that each bacterium can *sporadically change* its switching threshold or produce offsprings with different thresholds. Assume that these changes occur according to the Gaussian distribution centered at $x$. Then the density $u(x,t)$ of the biomass of bacteria with given switching thresholds $\pm x$ at a moment $t$ will satisfy the diffusion equation. We stress that the diffusion term appears not due to a spatial motion of bacterium, but rather due to the *diffusion of switching thresholds.*

Note that the collective impact of the whole population of bacteria upon the environment (concentration of lactose and glucose in our example) is mathematically described as an integral of all discontinuous hysteresis operators weighted with the density $u(x,t)$. This integral can be interpreted as the Preisach operator [18] with a time dependent density $u(x,t)$. In the classical Preisach operator, the density is usually time independent and is assumed to be given. The typical difficulty in this case is how to identify it in particular applications. Our approach allows one to overcome this difficulty (at least in some cases) because we treat the density as a component of the solution itself. This gives rise to the question of the limiting behavior of the density $u(x,t)$. For example, if one can describe a global attractor, there is no need to (precisely) identify the initial density $u(x,0)$, since, after some transition period of time, it will be in a vicinity of the known global attractor.

In the present paper, we introduce, seemingly for the first time, a notion of hysteresis (discontinuous non-ideal relay) with diffusive thresholds. This leads us to an initial boundary-value problem for a reaction-diffusion system including, as reaction terms, discontinuous hysteresis relay operators and the integral of those. We prove the well-posedness of this problem and partially address an issue of the large time behavior of its solutions.

The paper is organized as follows. In Sec. 2, we discuss various natural mechanisms leading to multi-stability in biological systems, which can be mathematically described in terms of hysteresis operators[1].

In Sec. 3, based on the above prototype example of two-phenotype bacteria, we give a rigorous model formulation — the reaction-diffusion system with discontinuous hysteresis operators. The unknown functions are the density $u(x,t)$ of the biomass of bacteria with given switching thresholds $\pm x$ at a moment $t$, and the amounts $f_1(t)$ and $f_{-1}(t)$ of the two types of nutrients. In the same section, we rewrite the system in terms of the new unknown functions: the density $u$, the total amount $v = f_1 + f_{-1}$ of nutrients, and the deviation $w = f_1/(f_1 + f_{-1}) - 1/2$ of the relative concentration of one type of nutrients from the value $1/2$ in the mixture of the two nutrients. Though our model is based on a particular biological system, it accounts for a number of quite general phenomena and can be adapted for other systems (cf. Sec. 2).

Sections 4–6 are devoted to the analysis of the model. In Sec. 4, we prove

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[1] We include this section for the convenience of a reader interested in biological background of problems with hysteresis. All the rest sections in the paper can be read independently of Sec. 2.

continuity properties of hysteresis operators provided that the input function is continuous in time and has bounded variation. Although each non-ideal relay (which we call $\mathcal{R}^x(\cdot)$ for fixed thresholds $\pm x$) is a discontinuous operator, it turns out that an infinite collection of the relays with different thresholds $\pm x$ (which we call $\mathcal{R}(\cdot)$) can be viewed as a continuous operator with values in suitable $L_q$ (with respect to $x$) spaces. An important issue, however, is that such an operator is Lipschitz continuous for $q = 1$ only, and it is Hölder continuous for $q > 1$. Similarly, the Preisach operator (which we call $\mathcal{P}(\cdot)$), i.e., the integral of the discontinuous relay operators (weighted with the time dependent density function) over all available thresholds $\pm x$ turns out to be a continuous operator.

In Sec. 5, we prove the well-posedness of the model. The existence of solutions is proved via the Schauder fixed-point theorem, where the continuity of the "collective" hysteresis $\mathcal{R}$ in $L_q$ spaces with $q > 1$ is exploited. As we said, the collective hysteresis $\mathcal{R}$ is not Lipschitz continuous for $q > 1$. Therefore, the contraction mapping principle does not apply, and the uniqueness should be proved separately. This is done via the semigroup approach and additional estimates in the $L_1$ space (where $\mathcal{R}$ is Lipschitz continuous).

In Sec. 6, we study the behavior of the density $u(x, t)$ and the total amount of nutrients $v(t)$ as $t \to \infty$. In particular, we prove that $v(t)$ monotonically decreases and tends to 0, while $u(x, t)$ converges (uniformly in $x$) to a spatially homogeneous equilibrium. The large time behavior of $w(t)$ as well as the limiting distribution of two phenotypes $\lim\limits_{t\to\infty} \mathcal{R}^x(w)(t)$ remains an open question.

The paper ends with Sec. 7 containing some discussion of the results and open questions as well as possible modifications and extensions of the model.

## 2 Motivation and biological background

The idea that epigenetic differences such as those arising in the process of cell differentiation can be attributed to multi-stability or multi-stationarity of living forms seems to have been first articulated by Max Delbrük [10]. A classical example of multi-stability in biology is multi-stable behavior of *lac-operon* in *E-Coli*. *Lac-operon* is a collection of genes associated with transport and metabolism of lactose in the bacterium. Expression of these genes can be turned on by certain small molecules that have been called inducers. Novick and Weiner [23] as well as Cohn and Horibata [8, 9], relying on prior work of others [5, 22, 28], effectively demonstrated that two phenotypes each associated with "on" and "off" state of *lac-operon* expression can be obtained from the same culture depending on the history of exposure to the inducer. Moreover, both phenotypes remain stable through multiple generations of the bacterial culture after the extracellular concentration of the inducer is reduced to lower levels, but not removed completely. This behavior resembles the definition of the *non-ideal relay* illustrated in Fig. 1.1.

The above mentioned early findings on the hysteresis of the *lac-operon* enzymes were consistent with earlier findings on regulation of other enzymes in yeast [31]. Recent experiments using molecular biology methods (such as those incorporating green fluorescent protein expression under the *lac-operon* promoter) permitted to confirm and further study the region of bi-stability of the *lac-operon* when multiple input variables (TMG that acts as the inducer and glucose) are used to switch the *lac-operon* genes on and off. Multi-stable gene expression and hysteresis has been well-documented in a number of natural as well as artificially constructed systems [4, 11, 13, 24, 25].

Several important issues require further discussion when it comes to hysteresis and multi-stability in biological systems. First is the fact that reproductive rates in different phenotype states are frequently different in a given environment. For example, the growth rate of one phenotype is high in a lactose rich environment, while the other phenotype is favoured by a glucose rich environment. This was noted in the experiments reported by Novick and Weiner, as well as others.

Second issue relates to the very essence of hysteresis — *rate-independence*. The rate-independence of the state-input relationship in non-ideal relay and other hysteresis operators means that the state does not depend on the rate at which the input may have varied, but rather on the past values of the input extrema [18]. Although rate-independence is an idealization, it is a useful one because it describes an important form of memory that can not be attained by linear dynamical systems whose memory is typically associated with certain characteristic times, rather than input features such as extrema.

The third point relates specifically to modeling switching of phenotype in bacteria using the non-ideal relay operator and, in particular, to the role and values of switching thresholds $x$ and $y$, which quantify the response of bacteria to varying environmental conditions and determine the bi-stability range. In the hybrid linear differential model of M. Thattai and A. van Oudenaarden [29], the maximal fitness (measured by the net population growth rate) was achieved by the responsive switching strategy, whereby all the bacteria switch to the currently most favoured phenotype, provided that the rate of transitions between phenotypes is higher than the rate of environment variations. This strategy is described by the memoryless relay with coinciding thresholds $x = y$ and no bi-stability region (a shifted Heaviside step function). When the rate of transitions between phenotypes is comparable to, or lower than, the rate of variations of the environment, the maximal fitness in this model can be achieved by a heterogeneous population implementing another memoryless strategy, where bacteria anticipate fluctuations of the environment by having a subpopulation ready in an appropriate phenotype before the environment changes to a state favouring this phenotype. The optimal heterogeneous distribution between phenotypes is obtained dynamically by allowing some positive rate of transitions from the currently most favoured phenotype to an unfavoured one. E. Kussell and S. Lieber argued that there is a cost of maintaining sensory machinery to respond to changes of the environment and introduced a cost of sensing in the form of an explicit reduction in the growth rate [19]. In their linear differential model, random switching of phenotype, which is not correlated to slowly varying environmental conditions, can confer more fitness to the population than the responsive switching when the penalty for sensing is large. A modification of these models, which incorporates a natural switching cost in the form of a temporary inhibition of the reproductive activity in bacteria undergoing a transition to a different phenotype (a lag phase), shows that hysteresis characterized by two different switching thresholds $x > y$ can also confer fitness to bacteria [12][2]. In the *adiabatic limit* of slowly varying environmental conditions, the switching pattern of bacteria in the model is described by the hysteretic non-ideal relay shown in Figure 1.1. In faster uncertain environments, the maximal growth was shown to be the one that results from a form of stochastic resonance where the internal characteristic time associated with the growth delay is about equal to the characteristic time between subsequent phenotype switching events (the latter time is defined by the length of the bi-stability interval $(y, x)$ and the characteristic time of the environment variations). More frequent switching in faster environments causes the

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[2] *In vitro* experimental studies give evidence that the process of changing phenotype is stressful for bacteria. In particular, bacteria may not reproduce within a period of time preceding, during, or following this process. In these experiments, a colony of bacteria grown in a Petri dish with one nutrient is swapped to a Petri dish with another nutrient. After a period of inactivity, or a shock, following the swap, bacteria start a transition to the other phenotype which is better fit for consuming the new type of food.

organisms to keep delaying their reproduction; slower environments causes the bacteria to wait too long in a sub-optimally reproducing phenotype.

An advantage of using strategies with memory has been also shown in a different framework, namely, the game theory [3] applied to models where bacteria are considered as players in an evolutionary game [19, 32, 33].

The idea of diversification or bet-hedging has been discussed in different biological contexts in many publications, often without reference to any specific mechanism by which it can be implemented. For example, the view that diversity (heterogeneity) can help improve fitness in varying conditions is very well established in ecology. The main idea behind diversification in epigenetics is that genetically identical organisms can grow their numbers faster by judiciously choosing a certain proportion of their population to be in a currently less favored phenotype when the environmental varies in time. In particular, it has been shown that most appropriate random phenotype choice strategies are based on selecting phenotype switching probabilities that are tuned in some sense to the environmental uncertainties. The main finding of [29] was that bet-hedging in the form of the anticipating switching strategy described above can lead to the maximal growth rate. If a population of bi-stable organisms includes subpopulations with different bi-stability ranges $(y, x)$, then bet-hedging can be realized by allowing permanently ongoing transitions between subpopulations. Due to the difference in the bi-stability range, each subpopulation is tuned to a specific pattern of variation of the environment, which maximizes the growth rate of this subpopulation. An exchange process between the subpopulations can bet-hedge against changes of this pattern, at least in principle.

# 3 Model description

## 3.1 Objective and modeling assumptions

In this paper, we attempt to formulate a class of models, which account for a number of phenomena listed above, namely (a) switching of bacteria between two phenotypes in response to variations of environmental conditions; (b) hysteretic switching strategy (switching rules) associated with bi-stability of phenotype states; (c) heterogeneity of the population in the form of a distribution of switching thresholds; (d) bet-hedging in the form of diffusion between subpopulations characterized by different bi-stability ranges; and,

(e) competition for nutrients. The resulting model is a reaction-diffusion system including, as reaction terms, discontinuous hysteresis relay operators and the integral of those. This integral can be interpreted as the Preisach operator [18] with a time dependent density (the density is a component of the solution describing the varying distribution of bacteria). The main objective of this paper is to prove well-posedness of the model. In the last part of the paper we will also present a preliminary discussion of a few dynamic scenarios obtained numerically, where fitness, competition and diffusion act together to select a certain distribution of switching thresholds in the population. In particular, the model seems to demonstrate interesting dynamics such as pattern formation and oscillations. However, more detailed and rigorous analysis of dynamics is beyond the scope of this paper and remains the subject of future work.

We assume that each of the two phenotypes, denoted by 1 and $-1$, consumes a different type of nutrient (for example, one consumes lactose, the other consumes glucose). The amount of nutrient available for phenotype $i$ at the moment $t$ is denoted by $f_i(t)$ where $i = \pm 1$. We base our model on the following assumptions.

- Each bacterium changes phenotype in response to the variations of the variable $w = f_1/(f_1 + f_{-1}) - 1/2$.

This variable measures the deviation of the relative concentration of the first nutrient from the value 1/2 in the mixture of the two nutrients. Bacteria sense changes of the nutrient concentrations and, by changing to the phenotype for which more food is available, can potentially increase the growth rate of the population.

- The input $w = w(t)$ is mapped to the binary time trace $r = r(t)$ of the phenotype (state) of a bacterium by the non-ideal relay operator $\mathcal{R}^x$ (see Figure 1.1) with symmetric switching thresholds $x$ and $y = -x$, where $x > 0$. This binary function of time will be denoted by $r = \mathcal{R}^x(w)$.

As transitions between the states of a relay are instantaneous, this assumption relates to the situation where the rate of transitions between phenotypes is much higher than the rate of the input variations. The non-ideal relay operator $\mathcal{R}^x$ is rigorously defined in the next subsection.

- The population includes bacteria with different bi-stability ranges $(-x, x)$; the threshold value $x$ varies over an interval $[\underline{x}, \overline{x}]$, where $0 < \underline{x} < \overline{x}$.

We will denote by $u(x,t)$ the density of the biomass of bacteria with given switching thresholds $\pm x$ at a moment $t$.

- There is a diffusion process acting on the density $u$.

The diffusion process models sporadic changes of the switching threshold $x$ in bacteria. Another source of diffusion can be the birth process if we assume that a bacterium with a threshold $x$ produces offsprings with different thresholds, for example, according to the Gaussian distribution centered at $x$. The diffusion can be viewed as a bet-hedging mechanism in the population.

Finally, we make the following important assumption.

- At any particular time moment $t$, for any given $x$, all the bacteria with the switching threshold values $\pm x$ are in the same state (phenotype).

That is, $u(x,t)$ is the total density of bacteria with the threshold $x$ at the moment $t$ and they are all in the same state. It means that when a bacterium with a threshold $x'$ sporadically changes its threshold to a different value $x$, it simultaneously copies the state from other bacteria which have the threshold $x$. (Or, if a bacterium with a threshold $x'$ produces an offspring with a threshold $x$, the offspring copies the state of other bacteria with the threshold $x$). In particular, this may require a bacterium to change the state when its threshold changes. Models where the state of a bacterium remains unchanged after a change of the threshold will be considered in a different work.

With these assumptions, we obtain the following model of the evolution of bacteria and nutrients:

$$\begin{cases} u_t = u_{xx} + \dfrac{1}{2}(1 + \mathcal{R}^x(w))\, u f_1 + \dfrac{1}{2}(1 - \mathcal{R}^x(w))\, u f_{-1}, \\ \dot{f}_1 = -\dfrac{1}{2} f_1 \displaystyle\int_{\underline{x}}^{\overline{x}} (1 + \mathcal{R}^x(w))\, u\, dx, \\ \dot{f}_{-1} = -\dfrac{1}{2} f_{-1} \displaystyle\int_{\underline{x}}^{\overline{x}} (1 - \mathcal{R}^x(w))\, u\, dx, \end{cases} \tag{3.1}$$

where $u_t$ and $u_{xx}$ are the derivatives of the population density $u$, dot denotes the derivative with respect to time, and all the non-ideal relays $\mathcal{R}^x$, $x \in [\underline{x}, \overline{x}]$, have the same input $w = f_1/(f_1 + f_{-1}) - 1/2$. Here we additionally assume the growth rate $\frac{1}{2}(1 + i\mathcal{R}^x(w))\, uf_i$ based on the mass action law for bacteria in the phenotype $i = \pm 1$. This growth rate is proportional to the product of the population density $u$ and the concentration $f_i$ of the nutrient preferred by the phenotype $i$ with the coefficient of proportionality scaled to unity. The diffusion coefficient is set to unity. The rate of the consumption of nutrient in the equation for $f_i = f_i(t)$ is proportional to the total biomass of bacteria in the phenotype $i$ (i.e., the biomass of all bacteria eating this type of nutrient), hence the integral (the coefficient of proportionality is also set to unity for simplicity); $\underline{x}$ and $\overline{x}$ are the lower and upper bounds on available threshold values, respectively.

We assume that a certain amount of nutrients is available at the initial moment; the nutrients are not supplied after that moment. Bacteria do not die but stop growing when all the nutrient has been consumed. We assume the Neumann boundary conditions for $u$, i.e., no flux of the population density $u$ through the lower and upper bounds of available threshold values.

## 3.2 Rigorous model setting

Throughout the paper, we assume that $x \in [\underline{x}, \overline{x}]$, where $0 < \underline{x} < \overline{x}$.

We begin with a rigorous definition of the hysteresis operator (non-ideal relay) with fixed thresholds $\pm x$. The *non-ideal relay* is the operator $\mathcal{R}^x$ which takes continuous functions $w = w(t)$ defined on an interval $[0, T)$ to the binary functions $r = \mathcal{R}^x(w)$ of time defined on the same interval. The operator $\mathcal{R}^x$ is given by

$$\mathcal{R}^x(w)(t) = \begin{cases} -1 & \text{if} \quad w(\tau) < -x \text{ for some } \tau \in [0,t] \\ & \qquad \text{and } w(s) < x \text{ for all } s \in [\tau, t], \\ 1 & \text{if} \quad w(\tau) \geq x \text{ for some } \tau \in [0,t] \\ & \qquad \text{and } w(s) > -x \text{ for all } s \in [\tau, t], \\ r_0 & \text{if} \quad -x \leq w(\tau) < x \text{ for all } \tau \in [0,t], \end{cases} \tag{3.2}$$

where $r_0$ is either 1 or $-1$ (*initial configuration* of the non-ideal relay $\mathcal{R}^x$). Since $r_0$ may take different values for different $x$, we write $r_0 = r_0(x)$, where $r_0(x)$ is a given function taking values $\pm 1$. In what follows, we do not explicitly indicate the dependence of the operator $\mathcal{R}^x$ on $r_0(x)$. Some additional assumptions on the structure of the initial configuration $r_0(x)$ will be made in Sec. 4 (in particular, see relations (4.1)).

The *distributed relay operator* $\mathcal{R}(w)$ taking functions $w = w(t)$ to functions $r = r(x,t)$ is defined by $\mathcal{R}(w)(x,t) = \mathcal{R}^x(w)(t)$.

We also denote

$$\mathcal{U}(u)(t) = \int\limits_{\underline{x}}^{\overline{x}} u(x,t)\,dx, \quad \mathcal{P}(u,w)(t) = \int\limits_{\underline{x}}^{\overline{x}} u(x,t)\mathcal{R}^x(w)(t)\,dx \tag{3.3}$$

and call $\mathcal{P}$ the *Preisach operator* (see more details in Sec. 4).

Now let us replace the unknown function $f_1$ and $f_{-1}$ in system (3.1) with $v = f_1 + f_{-1}$ and $w = f_1/(f_1 + f_{-1}) - 1/2$. The resulting system has the form

$$\begin{cases} u_t = u_{xx} + \mathcal{A}(v,w)u, \\ \dot{v} = \mathcal{V}(u,v,w), \\ \dot{w} = \mathcal{W}(u,w), \end{cases} \tag{3.4}$$

where we assume the Neumann boundary conditions

$$u_x|_{x=\underline{x}} = u_x|_{x=\overline{x}} = 0 \tag{3.5}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad v(0) = v_0, \quad w(0) = w_0. \tag{3.6}$$

Here

$$\begin{aligned} \mathcal{A}(v,w) &= \left(\frac{1}{2} + w\mathcal{R}(w)\right) v, \\ \mathcal{V}(u,v,w) &= -\left(\frac{1}{2}\mathcal{U}(u) + w\mathcal{P}(u,w)\right) v, \\ \mathcal{W}(u,w) &= -\left(\frac{1}{2} + w\right)\left(\frac{1}{2} - w\right)\mathcal{P}(u,w), \end{aligned}$$

and the initial configuration $r_0(x)$ of the distributed relay operator $\mathcal{R}$ is a part of initial data.

Let $T > 0$, and let $Q_T = (\underline{x}, \overline{x}) \times (0,T)$. We will use the following spaces:

1. The standard Lebesgue spaces $L_q(Q_T)$ and $L_q = L_q(\underline{x}, \overline{x})$ with $1 \le q \le \infty$.
2. The Sobolev spaces $W_q^k = W_q^k(\underline{x}, \overline{x})$, $k \in \mathbb{N}$.

3. The anisotropic Sobolev space $W_2^{2,1}(Q_T)$ with the norm

$$\|u\|_{W_2^{2,1}(Q_T)} = \left( \int\limits_0^T \|u(\cdot,t)\|_{W_2^2}^2 \, dt + \int\limits_0^T \|u_t(\cdot,t)\|_{L_2}^2 \, dt \right)^{1/2}.$$

4. The space $C([0,T];L_q)$ of $L_q$-valued functions continuous in $t \in [0,T]$.

# 4 Preisach model

## 4.1 States of the Preisach model

In this section, we establish some continuity properties of the operators $\mathcal{R}(w)$ and $\mathcal{P}(u,w)$ (the latter defined by (3.3)). In particular, we will show that, for any $w \in C[0,T]$ with bounded variation and $u \in (C[0,T];L_1) \cap L_\infty(Q_T)$, the function $\mathcal{R}(w)$ belongs to $C([0,T];L_q)$ with any $q \geq 1$ and the function $\mathcal{P}(u,w)$ belongs to $C[0,T]$.

We note that the operator $\mathcal{P}(u,w)$ is a specific case of the *Presiach operator with density* $u(x,t)$ *and input* $w(t)$ [18]. However, in the literature, the Preisach operator is usually considered for time independent density.

We begin with some definitions. Fix $T > 0$.

**Definition 4.1.** For a given input $w \in C[0,T]$, the *state* of the Preisach model at the moment $t \in [0,T]$ is defined as a subset $A(t)$ ($t \in [0,T]$) of the interval $[\underline{x},\overline{x}]$ given by

$$A(t) = \{x \in [\underline{x},\overline{x}] : \mathcal{R}^x(w)(t) = 1\}.$$

**Definition 4.2.** For a given input $w \in C[0,T]$, we say that the state $A(t)$ is *simple* for some $t \in [0,T]$ if it is a union of finitely many disjoint intervals $[x_k^-, x_k^+]$, $k = 1,\dots,N$.

In what follows, we assume that $A(0)$ is simple and satisfies the compatibility condition

$$[\underline{x}, w(0)] \subset A(0) \quad \text{if } w(0) \geq \underline{x}; \qquad [\underline{x}, -w(0)) \cap A(0) = \varnothing \quad \text{if } w(0) < -\underline{x}. \tag{4.1}$$

We will show that $A(t)$ remains simple and satisfies the compatibility condition

$$[\underline{x}, w(t)] \subset A(t) \quad \text{if } w(t) \geq \underline{x}; \qquad [\underline{x}, -w(t)) \cap A(t) = \varnothing \quad \text{if } w(t) < -\underline{x} \tag{4.2}$$

for all $t \in [0, T]$.

We remind that we have agreed that each relay $R_x(w)$ switches to the state 1 at the moment when the input reaches the value $x$ and remains in the same state 1 when the input reaches the value $-x$, switching back to the state $-1$ only when the input becomes smaller than $-x$. However, this is not essential as including/not including the end points of $[x_k^-, x_k^+]$ in $A(t)$ is not important (one cares about the classes of sets $A$ which coincide almost everywhere).

Suppose, at some moment $t_1 \in [0, T)$, the state is $A(t_1)$. Fix some moment $t_2 \in (t_1, T]$ and consider $A(t)$ for $t \in [t_1, t_2]$. From the definition of dynamics of individual relays, it follows that the state $A(t)$ will evolve in response to the changes of a continuous input $w(t)$ after the moment $t_1$ according to the following rules.

Consider the running maximum and minimum of $w$:

$$M(t, t_1) = \max_{s \in [t_1, t]} w(s), \qquad m(t, t_1) = \min_{s \in [t_1, t]} w(s).$$

1. If the input satisfies $-\underline{x} \leq w(t)$ on the time interval $[t_1, t_2]$, then on this time interval
$$A(t) = A(t_1) \cup J(t), \tag{4.3}$$
where
$$J(t) = \varnothing \quad \text{if} \quad M(t, t_1) < \underline{x}; \qquad J(t) = [\underline{x}, M(t, t_1)] \quad \text{if} \quad M(t, t_1) \geq \underline{x}.$$
In particular, $A(t)$ is expanding (non-strictly). We also note that, during any time interval within which $-\underline{x} \leq w(t) < \underline{x}$, the set $A(t)$ does not change.

2. If the input satisfies $w(t) < \underline{x}$ on the time interval $[t_1, t_2]$, then on this time interval
$$A(t) = A(t_1) \setminus I(t) \tag{4.4}$$
where
$$I(t) = \varnothing \quad \text{if } m(t, t_1) \geq -\underline{x}; \quad I(t) = (\underline{x}, -m(t, t_1)) \quad \text{if } m(t, t_1) < -\underline{x}.$$

In this case, $A(t)$ is contracting (non-strictly). Again, $A(t)$ does not change during any time interval within which $-\underline{x} \le w(t) < \underline{x}$.

Now we consider the evolution of the state $A(t)$ for all $t \in [0,T]$, provided that the input $w(t)$ is continuous and has a bounded variation. Specifically, we assume that

$$\mathrm{Var}_0^T[w] \le K$$

for some $K = K(T) > 0$. Such inputs form a closed convex subset of $C[0,T]$ for any given $T > 0$. For every such input, there is a finite sequence of moments $0 < t_1 < t_2 < \cdots < T$ such that, on any interval $[t_k, t_{k+1}]$, either $-\underline{x} \le m(t,t_k)$, or $M(t,t_k) < \underline{x}$, or both these relations hold simultaneously. Hence, on each $[t_k, t_{k+1}]$, at least one of the above two rules of evolution of the state $A(t)$ applies, thus defining the state $A(t)$ at any moment $t \in [0,T]$ by a finite number of applications of formulas (4.3), (4.4) on successive time intervals $[t_k, t_{k+1}]$. We note that the sequence $t_k$ is not unique; we call such sequences *admissible partition* sequences. The dynamics of $A(t)$, according to the above rules, is independent of the choice of an admissible partition sequence.

Note that, according to the definition of the relay, the states $A(t)$ are simple and satisfy the compatibility condition (4.2) for any $t \in [0,T]$.

## 4.2 Continuity

For any measurable sets $B_1, B_2 \subset [\underline{x}, \overline{x}]$, denote

$$\rho(B_1,B_2) = \mathrm{meas}\,\{(B_1 \setminus B_2) \cup (B_2 \setminus B_1)\}.$$

Consider the evolution of the states $A_1(t)$ and $A_2(t)$ in response to the inputs $w_1(t)$ and $w_2(t)$, respectively, for $t \in [0,T]$.

**Lemma 4.1.** *Let $w_1, w_2 \in C[0,T]$ for some $T > 0$, and let*

$$\mathrm{Var}_0^T[w_j] \le K, \quad j = 1,2,$$

*with some $K = K(T)$. Let $A_j(t)$ be the state corresponding to the input $w_j(t)$. If $A_j(0)$, $j = 1,2$, are simple and satisfy the compatibility condition* (4.1), *then*

$$\max_{t\in[0,T]} \rho(A_1(t),A_2(t)) \le \rho(A_1(0),A_2(0)) + L\|w_1 - w_2\|_{C[0,T]}, \tag{4.5}$$

*where $L = 2 + K/(2\underline{x})$.*

*Proof.* First, we choose a finite sequence $0 < t_1 < t_2 < \dots < T$ which is an admissible partition sequence for the evolution of both states $A_1(t)$ and $A_2(t)$ simultaneously.

To do so, we fix an arbitrary number $x_0 \in (0, \underline{x})$ and consider the auxiliary relay $\mathcal{R}^{x_0}$. Define the sequence of all moments $t_1 < t_2 < \dots$ on the interval $[0, T]$ as the moments when the output $\mathcal{R}^{x_0}(w_1)(t)$ switches between the states $\pm 1$ in response to the input $w_1(t)$. This sequence is finite since $w_1$ is of bounded variation. Then $0, t_1, t_2, \dots, T$ is an admissible partition sequence for the evolution of the state $A_1(t)$. Assume that, on the time interval $[0, T]$,

$$\|w_1 - w_2\|_{C[0,T]} \le \underline{x} - x_0.$$

Then, the same sequence $0, t_1, t_2, \dots, T$ is an admissible partition sequence for the evolution of the state $A_2(t)$ on the interval $[0, T]$. Moreover, the first updating rule applies to both evolutions on each time interval $[t_k, t_{k+1}]$ where $\mathcal{R}^{x_0}(w_1)(t) = 1$, while the second updating rule applies to both evolutions on each time interval $[t_k, t_{k+1}]$ where $\mathcal{R}^{x_0}(w_1)(t) = -1$. Applying formulas (4.3) and (4.4) to both evolutions, we see that

$$\rho(A_1(t), A_2(t)) \le \rho(A_1(t_k), A_2(t_k)) + \|w_1 - w_2\|_{C[0,T]}, \qquad t \in [t_k, t_{k+1}].$$

As the number of switching points $t_k$ in the interval $[0, T]$ does not exceed the value $1 + \mathrm{Var}_0^T[w_1]/(2x_0) \le 1 + K/(2x_0)$, we obtain the Lipschitz estimate

$$\max_{t \in [0,T]} \rho(A_1(t), A_2(t)) \le \rho(A_1(0), A_2(0)) + L\|w_1 - w_2\|_{C[0,T]}$$

with $L = 2 + K/(2x_0)$. Since $x_0 \in (0, \underline{x})$ is arbitrary, we have (4.5). □

Using Lemma 4.1, we obtain the main results of this section about the continuity of the operators $\mathcal{R}(w)$ and $\mathcal{P}(u, w)$ (the latter defined by (3.3).

**Lemma 4.2.** *1. Let $w \in C[0, T]$ and $\mathrm{Var}_0^T[w]$ be finite. Then $\mathcal{R}(w) \in C([0, T]; L_q)$ with any $q \ge 1$.*

*2. If, additionally, $u \in C([0, T]; L_1) \cap L_\infty(Q_T)$, then $\mathcal{P}(u, w) \in C[0, T]$.*

*Proof.* 1. Denote $r(x, t) = \mathcal{R}^x(w)(t)$. Then, for any fixed $t_0 \in [0, T]$,

$$\int_{\underline{x}}^{\overline{x}} |r(x, t) - r(x, t_0)|^q dx = 2^q \rho(A(t), A(t_0)) \to 0, \quad t \to t_0,$$

due to the updating rules (4.3) and (4.4). This proves assertion 1.

2. To prove the continuity of the function $\mathcal{P}(u,w)(t)$, we estimate

$$\begin{aligned}
&\int_{\underline{x}}^{\overline{x}} |u(x,t)r(x,t) - u(x,t_0)r(x,t_0)|dx \\
&\le \int_{\underline{x}}^{\overline{x}} |u(x,t) - u(x,t_0)|\,|r(x,t)|dx + \int_{\underline{x}}^{\overline{x}} |u(x,t_0)|\,|r(x,t) - r(x,t_0)|dx \\
&\le \|u(\cdot,t) - u(\cdot,t_0)\|_{L_1} + 2\|u(\cdot,t_0)\|_{L_\infty}\,\rho(A(t),A(t_0)) \to 0, \quad t \to t_0,
\end{aligned}$$

due to the assumptions of the lemma and the updating rules (4.3) and (4.4). □

**Lemma 4.3.** *1. Let*

$$w_j \in C[0,T], \quad \mathrm{Var}_0^T[w_j] \le K, \quad A_1(0) = A_2(0),$$

*where* $K > 0$ $(j = 1,2)$*. Then, for any* $q \ge 1$,

$$\|\mathcal{R}(w_1) - \mathcal{R}_2(w)\|_{C([0,T];L_q)} \le L_{\mathcal{R}}\|w_1 - w_2\|_{C[0,T]}^{1/q},$$

*where* $L_{\mathcal{R}} = L_{\mathcal{R}}(K,q) > 0$.

*2. If, additionally,*

$$u_j \in C([0,T];L_1), \quad \|u_j\|_{L_\infty(Q_T)} \le c$$

*for some* $c > 0$ $(j = 1,2)$*, then*

$$\|\mathcal{P}(u_1,w_1) - \mathcal{P}(u_2,w_2)\|_{C([0,T])} \le L_{\mathcal{P}}(\|u_1 - u_2\|_{C([0,T];L_1)} + \|w_1 - w_2\|_{C[0,T]}),$$

*where* $L_{\mathcal{P}} = L_{\mathcal{P}}(c,K) > 0$.

*Proof.* 1. Denote $r_j(x,t) = \mathcal{R}^x(w_j)(t)$, $j = 1,2$. Then, using Lemma 4.1, we have

$$\int_{\underline{x}}^{\overline{x}} |r_1(x,t) - r_2(x,t)|^q dx = 2^q \rho(A_1(t),A_2(t)) \le 2^q L\|w_1 - w_2\|_{C[0,T]}.$$

2. To prove the continuity of the operator $\mathcal{P}$, we estimate $|\mathcal{P}(u_1, w_1)(t) - \mathcal{P}(u_2, w_2)(t)|$ as follows (omitting the arguments of the integrands):

$$\begin{aligned}\int_{\underline{x}}^{\overline{x}} |u_1 r_1 - u_2 r_2| dx &\le \int_{\underline{x}}^{\overline{x}} |u_1|\, |r_1 - r_2| dx + \int_{\underline{x}}^{\overline{x}} |u_1 - u_2|\, |r_2| dx \\ &\le 2c\rho(A_1(t), A_2(t)) + \|u_1(\cdot, t) - u_2(\cdot, t)\|_{L_1} \\ &\le 2cL\|w_1 - w_2\|_{C[0,T]} + \|u_1 - u_2\|_{C([0,T];L_1)},\end{aligned}$$

where Lemma 4.1 was used to estimate $\rho(A_1(t), A_2(t))$. □

**Remark 4.1.** We underline (see the proof of Lemma 4.3) that the operators

$$\mathcal{R}^x : C[0,T] \to L_q, \quad \mathcal{R} : C[0,T] \to C([0,T]; L_q)$$

are continuous for any $q \ge 1$. However, they are Lipschitz continuous *only* for $q = 1$.

# 5 Well-posedness

In this section, we establish existence (first locally and then globally) and uniqueness for problem (3.4)–(3.6).

We will often write $U$ or $U(t)$ instead of $\mathcal{U}(u)(t)$ (see (3.3)). This should lead to no confusion. We will also denote

$$U_0 = \mathcal{U}(u)(0) = \int_{\underline{x}}^{\overline{x}} u_0(x)\, dx.$$

Since the right-hand sides in (3.4) contain the relays $\mathcal{R}^x$, one has to fix the initial state of those relays. We assume throughout that the initial state $A(0)$ is simple and satisfies the compatibility condition (4.1).

## 5.1 Linear parabolic problem

In this subsection, we formulate some auxiliary results on the following linear parabolic problem:

$$\begin{cases} u_t = u_{xx} + a(x,t)u + f(x,t), & x \in (\underline{x}, \overline{x}),\ t > 0, \\ u(x,0) = u_0(x) \end{cases} \tag{5.1}$$

with the homogeneous Neumann boundary conditions. In what follows, we will use the functional spaces defined in Sec. 3.2.

The first lemma follows from [20, Chap. 4].

**Lemma 5.1.** *Let $T \le T_0$ for some $T_0 > 0$, and let $\|a\|_{L_\infty(Q_T)} \le \overline{a}$ for some $\overline{a} \ge 0$. Let $f \in L_2(Q_T)$ and $u_0 \in W_2^1$. Then problem* (5.1) *has a unique solution $u \in W_2^{2,1}(Q_T)$ and*

$$\|u\|_{W_2^{2,1}(Q_T)} \le c(\|u_0\|_{W_2^1} + \|f\|_{L_2(Q_T)}),$$

*where $c = c(\overline{a}, T_0) > 0$ does not depend on $u$, $a(x,t)$ and $T$.*

The second lemma deals with continuous dependence of solutions on the coefficient $a(x,t)$. Consider a sequence $a_j \in L_\infty(Q_T)$, $j = 1, 2, \dots$. Denote by $u_j$ the solution of problem (5.1) with $a_j$ instead of $a$.

**Lemma 5.2.** *Let $\|a_j\|_{L_\infty(Q_T)} \le \overline{a}$ and $\|a - a_j\|_{L_2(Q_T)} \to 0$ as $j \to \infty$. Then*

$$\|u - u_j\|_{W_2^{2,1}(Q_T)} \to 0, \quad j \to \infty.$$

*Proof.* The function $m_j = u - u_j$ is a solution of the problem

$$\begin{cases} m_{jt} = m_{jxx} + a_j(x,t)m_j + (a - a_j)u, \\ u(x,0) = 0. \end{cases}$$

Therefore, by Lemma 5.1 and by the boundedness of the embedding $W_2^{2,1}(Q_T) \subset L_\infty(Q_T)$, we have

$$\|m_j\|_{W_2^{2,1}(Q_T)} \le k_1 \|(a - a_j)u\|_{L_2(Q_T)} \le k_2 \|(a - a_j)\|_{L_2(Q_T)} \|u\|_{W_2^{2,1}(Q_T)},$$

where $k_1, k_2 > 0$ depend only on $\overline{a}$ and $T$ and do not depend on $j$. Hence, $\|m_j\|_{W_2^{2,1}(Q_T)} \to 0$ as $j \to \infty$. □

## 5.2 Local existence of solutions

We introduce the space

$$\mathcal{W}(Q_T) = W_2^{2,1}(Q_T) \times C^1[0,T] \times C^1[0,T].$$

**Definition 5.1.** We say that $\mathbf{u} = (u, v, w) \in \mathcal{W}(Q_T)$ is a (*strong*) *solution* of problem (3.4)–(3.6) (on the interval $(0,T)$) with initial data $\mathbf{u}_0 = (u_0, v_0, w_0) \in W_2^1 \times \mathbb{R}^2$ if

1. $u$ satisfies the first equation in (3.4) a.e. in $Q_T$ and the boundary conditions (3.5) and the first initial conditions in (3.6) in the sense of traces.
2. $v$ and $w$ satisfy the second and the third equations in (3.4) and the second and the third initial conditions in (3.6), respectively, in the classical sense.

In what follows, we will often say "solution", meaning "strong solution".

In this subsection, we will prove the following result on the local existence of solutions.

**Theorem 5.1.** *Let* $\mathbf{u}_0 = (u_0, v_0, w_0) \in W_2^1 \times \mathbb{R}^2$ *and*

$$\|u_0\|_{W_2^1} \le \overline{u}, \quad |v_0| \le \overline{v}, \quad |w_0| \le \overline{w}$$

*for some* $\overline{u}, \overline{v}, \overline{w} > 0$*. Then there is* $t_0 \in (0,1]$ *such that problem* (3.4)–(3.6) *has a solution* $\mathbf{u} = (u, v, w) \in \mathcal{W}(Q_{t_0})$*. The number* $t_0$ *depends on* $\overline{u}, \overline{v}, \overline{w}$ *but does not depend on* $u_0, v_0, w_0$*.*

The idea of the proof is to construct a mapping $(v, w) \mapsto (\tilde{v}, \tilde{w})$ as follows.

Step 1. Given the functions $v$ and $w$, we solve the linear parabolic problem

$$\begin{cases} u_t = u_{xx} + \mathcal{A}(v, w)u, & x \in (\underline{x}, \overline{x}),\ t > 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{5.2}$$

with the homogeneous Neumann boundary conditions.

Step 2. After finding $u$, we find $(\tilde{v}, \tilde{w})$ by solving the equations

$$\begin{cases} \dfrac{d\tilde{v}}{dt} = \mathcal{V}(u, v, w), & \tilde{v}(0) = v_0, \\ \dfrac{d\tilde{w}}{dt} = \mathcal{W}(u, w), & \tilde{w}(0) = w_0. \end{cases} \tag{5.3}$$

Then, using the Schauder fixed point theorem, we show that the mapping $(v, w) \mapsto (\tilde{v}, \tilde{w})$ has a fixed point, which yields the solution of the original problem (3.4)–(3.6).

**Remark 5.1.** We note that the parabolic problem (5.2) is well posed in $L_q$-spaces with $q > 1$, while the hysteresis operator $\mathcal{R}$ is not Lipschitz continuous in these spaces (see Remark 4.1). Therefore, the constructed mapping $(v, w) \mapsto (\tilde{v}, \tilde{w})$ will be continuous, but not Lipschitz continuous. This is the reason why we apply the Schauder fixed point theorem and not the contraction mapping principle for the proof of Theorem 5.1. As a result, only the existence of solutions is proved. The uniqueness will be proved in Sec. 5.4 by using semigroups and additional estimates in $L_1$.

Let us formalize the above scheme.

We introduce the set

$$B[0,t_0] = \{(v,w) \in C[0,t_0] \times C[0,t_0] : \\ \|v\|_{C[0,t_0]} \le 2\overline{v},\ \|w\|_{C[0,t_0]} \le 2\overline{w},\ \mathrm{Var}_0^{t_0}[w] \le V t_0\},$$

where $\overline{v}$ and $\overline{w}$ are the constants from the assumption in Theorem 5.1, while $V > 0$ and $t_0 \le 1$ will be chosen later on. For $(v, w) \in B[0, t_0]$, we will denote

$$\|(v,w)\|_{B[0,t_0]} = \|v\|_{C[0,t_0]} + \|w\|_{C[0,t_0]}.$$

Note that $B[0, t_0]$ is a closed convex set in $C[0, t_0] \times C[0, t_0]$.

**Lemma 5.3.** *1. $\|\mathcal{A}(v,w)\|_{L_\infty(Q_{t_0})} \le \overline{a}$ for any $(v, w) \in B[0, t_0]$, where*

$$\overline{a} = 2|\overline{v}|\left(\frac{1}{2} + 2|\overline{w}|\right).$$

*2. $\|\mathcal{A}(v,w) - \mathcal{A}(v_j,w_j)\|_{L_2(Q_{t_0})} \to 0$ whenever $\|(v,w) - (v_j,w_j)\|_{B[0,t_0]} \to 0$.*

*Proof.* Part 1 is trivial. Part 2 follows from Lemma 4.2 (part 1) and Lemma 4.3 (part 1) with $q = 2$. ☐

Combining Lemmas 5.1, 5.2, and 5.3, we obtain the following result justifying Step 1 in the above scheme (recall that $t_0 \le 1$).

**Lemma 5.4.** *Let $(v, w) \in B[0, t_0]$. Then problem (5.2) has a unique solution $u \in W_2^{2,1}(Q_{t_0})$. Moreover,*

1. *the estimate*

$$\|u\|_{W_2^{2,1}(Q_{t_0})} \le c$$

*holds with $c = c(\overline{u}, \overline{v}, \overline{w}) > 0$ which does not depend on $t_0 \le 1$ and $V > 0$;*

2. *the mapping $B[0,t_0] \ni (v,w) \mapsto u \in W_2^{2,1}(Q_{t_0})$ is continuous.*

Now we justify Step 2.

**Lemma 5.5.** *There exist numbers $t_0 \in (0,1]$ and $V > 0$ such that the following hold.*

1. *For any $(v,w) \in B[0,t_0]$ and for $u \in W_2^{2,1}(Q_{t_0})$ defined by Lemma 5.4, the solution $(\tilde v, \tilde w)$ of problem (5.3) belongs to $B[0,t_0]$. Moreover,*

$$\|\tilde v\|_{C^1[0,t_0]} + \|\tilde w\|_{C^1[0,t_0]} \le c_1, \tag{5.4}$$

*where $c_1 > 0$ depends on $\overline{u}, \overline{v}, \overline{w}$, but does not depend on $(v,w) \in B[0,t_0]$.*

2. *The mapping*

$$W_2^{2,1}(Q_{t_0}) \times B[0,t_0] \ni (u,v,w) \mapsto (\tilde v, \tilde w) \in B[0,t_0]$$

*is continuous.*

*Proof.* 1. First, we note that

$$\|u\|_{C([0,t_0];L_1)} \le k_1, \quad \|u\|_{L_\infty(Q_{t_0})} \le k_2, \tag{5.5}$$

where $k_1, k_2 > 0$ depend on $\overline{u}, \overline{v}, \overline{w}$, but do not depend on $t_0 \le 1$ and $V > 0$. Indeed, for any $t_0 \le 1$, we can extend $v(t)$ and $w(t)$ to $[0,1]$ as continuous functions without increasing their norms and without changing the variation of $w$. By Lemma 5.4, we obtain a unique solution $u \in W_2^{2,1}(Q_1)$ of problem (5.2) on the time interval $(0,1)$ such that

$$\|u\|_{W_2^{2,1}(Q_1)} \le c(\overline{u}, \overline{v}, \overline{w}).$$

This estimate and the boundedness of the embeddings $W_2^{2,1}(Q_1) \subset C([0,1]; L_1)$ and $W_2^{2,1}(Q_1) \subset L_\infty(Q_1)$ imply (5.5) with $k_1, k_2 > 0$ not depending on $t_0 \le 1$.

Using Lemma 4.2 and estimates (5.5), we see that, for any $(v,w) \in B[0,t_0]$,

$$\|\mathcal{V}(u,v,w)\|_{C[0,t_0]} \le k_3, \quad \|\mathcal{W}(u,w)\|_{C[0,t_0]} \le k_4, \tag{5.6}$$

where $k_3, k_4 > 0$ depend on $\overline{u}, \overline{v}, \overline{w}$, but do not depend on $t_0 \le 1$ and $V > 0$.

Now we choose

$$t_0 = \min\left(\frac{\overline{v}}{k_3}, \frac{\overline{w}}{k_4}, 1\right), \quad V = k_4. \tag{5.7}$$

Then the solution $(\tilde{v}, \tilde{w})$ of problem (5.3) belongs to $B[0, t_0]$.

Estimate (5.4) follows from (5.6).

2. Part 2 of the lemma follows from part 2 of Lemma 4.3. ☐

*Proof of Theorem* 5.1. Combining Lemmas 5.4 and 5.5 with the compactness of the embedding $C^1[0, t_0] \subset C[0, t_0]$ and using the Schauder fixed-point theorem, we conclude that the mapping $(v, w) \mapsto (\tilde{v}, \tilde{w})$ has a fixed point, which yields the solution of the original problem (3.4)–(3.6). ☐

## 5.3 Global existence of solutions

Our next goal is to prove that the local solution of problem (3.4)–(3.6) given by Theorem 5.1 can be extended to an arbitrarily large time interval. Here we will concentrate on the physically relevant case where the initial data $\mathbf{u}_0 = (u_0, v_0, w_0)$ satisfies

$$u_0(x) \geq 0, \quad v_0 \geq 0, \quad |w_0| \leq 1/2. \tag{5.8}$$

First, we prove some a priori estimates of solutions.

**Lemma 5.6.** *Let* $\mathbf{u} = (u, v, w) \in \mathcal{W}(Q_T)$ *be a solution of problem* (3.4)–(3.6) *on some time interval* $(0, T)$ *with initial data* $\mathbf{u}_0 = (u_0, v_0, w_0)$ *satisfying* (5.8)*. Then the following hold.*

1. $u(x, t) \geq 0$ *for all* $(x, t) \in Q_T$.
2. $v(t) \geq 0$ *for all* $t \in [0, T]$.
3. $|w(t)| \leq 1/2$ *for all* $t \in [0, T]$.

*Proof.* 1. Denote $a(x, t) = \mathcal{A}(v, w)$. Then $a \in L_\infty(Q_T)$, while the first equation in (3.4) takes the form

$$u_t = u_{xx} + a(x, t)u \tag{5.9}$$

Along with (5.9), we consider the equations

$$u_t^\varepsilon = u_{xx}^\varepsilon + a(x, t)u^\varepsilon + \varepsilon, \tag{5.10}$$

where $\varepsilon > 0$. Due to Lemma 5.1, equation (5.10) with the Neumann boundary conditions and the initial condition

$$u^\varepsilon(x, 0) = u_0(x) + \varepsilon$$

has a unique solution $u^\varepsilon \in W_2^{2,1}(Q_T)$ and

$$\|u^\varepsilon - u\|_{W_2^{2,1}(Q_T)} \to 0, \quad \varepsilon \to 0. \tag{5.11}$$

We fix $\varepsilon > 0$. Denote

$$b(x,t,u^\varepsilon) = a(x,t)u^\varepsilon + \varepsilon.$$

Then we have the following:

(a) $b(x,t,0) = \varepsilon > 0$ for $(x,t) \in \overline{Q}_T$,

(b) $b(x,t,\cdot)$ is continuous near the origin (i.e., near $u^\varepsilon = 0$) uniformly with respect to $(x,t) \in \overline{Q}_T$,

(c) $u^\varepsilon(x,0) \geq \varepsilon > 0$.

Regularizing the right-hand side $b(x,t,u_\varepsilon(x,t))$ and applying the method of invariant regions (see [27] for classical solutions), we obtain that $u^\varepsilon(x,t) \geq 0$ for $(x,t) \in Q_T$. Hence, using (5.11) and the continuity of the embedding $W_2^{2,1}(Q_T) \subset C(\overline{Q}_T)$ yields $u(x,t) \geq 0$ for $(x,t) \in Q_T$.

2. The second equation in (3.4) can be written as

$$\dot{v} = c(t)v$$

with appropriate continuous function $c(t)$. Obviously, $v(t) \geq 0$ if $v_0 \geq 0$.

3. The third equation can be treated similarly to the second one. □

In the next lemma, we estimate the solutions from above. This will allow us to prove the existence of solutions on arbitrarily large time interval. The estimates will involve the total population $\mathcal{U}(u)(t)$ (see (3.3)). Whenever it appears, we keep writing $U$ or $U(t)$, as before.

**Lemma 5.7.** *Let* $\mathbf{u} = (u,v,w) \in \mathcal{W}(Q_T)$ *be a solution of problem* (3.4)–(3.6) *on some time interval* $(0,T)$ *with initial data* $\mathbf{u}_0 = (u_0,v_0,w_0)$ *satisfying* (5.8)*. Then the following hold.*

1. $v(t) \leq v_0$ *for all* $t \in [0,T]$.
2. $U(t) + v(t) = U_0 + v_0$.

3. *There is a function $\bar{u}(t)$, $t \geq 0$, depending on $\mathbf{u}_0$, but not on $T$, bounded on bounded sets and such that*

$$\|u(\cdot, t)\|_{W_2^1} \leq \bar{u}(t).$$

*Proof.* 1. By Lemma 5.6, $u(x,t) \geq 0$ and $|w(t)| \leq 1/2$. Therefore,

$$\frac{U(t)}{2} + w(t)\mathcal{P}(u,w)(t) \geq 0.$$

Hence, the second equation in (3.4) yields $\dot{v} \leq 0$, which implies the first assertion of the lemma.

2. Integrating the first equation in (3.4) with respect to $x$ and adding the second equation yields $\dot{U} + \dot{v} = 0$, which proves assertion 2.

3.1 Multiplying the first equation in (3.4) by $u$, integrating with respect to $x$, and using the fact that $|w| \leq 1/2$ and $|v| \leq v_0$ yields

$$\frac{d}{dt}\|u(\cdot,t)\|_{L_2}^2 \leq 2v_0\|u(\cdot,t)\|_{L_2}^2. \tag{5.12}$$

Therefore, by Gronwall's lemma,

$$\|u(\cdot,t)\|_{L_2}^2 \leq \|u_0\|_{L_2}^2 \exp(2v_0 t) =: \bar{u}_0(t). \tag{5.13}$$

3.2. Now, using the fact that $|w| \leq 1/2$ and $|v| \leq v_0$, we see that

$$\|\mathcal{A}(v,w)u\|_{L_2(Q_\tau)}^2 \leq v_0^2 \int_0^\tau \bar{u}_0(t)\,dt =: \bar{u}_1(\tau).$$

Since $\bar{u}_1(\tau)$ is bounded on bounded intervals, Theorems 3.2 and 3.7 in [2] imply assertion 3. □

Now we formulate the main theorem on the well-posedness in terms of strong solutions.

**Theorem 5.2.** *Let $\mathbf{u}_0 = (u_0, v_0, w_0) \in W_2^1 \times \mathbb{R}^2$ satisfies* (5.8). *Then, for any $T > 0$, problem* (3.4)–(3.6) *has a solution $\mathbf{u} = (u,v,w) \in \mathcal{W}(Q_T)$ and the estimates in Lemmas* 5.6 *and* 5.7 *hold.*

*Proof.* By Theorem 5.1, problem (3.4)–(3.6) has a solution $\mathbf{u} = (u, v, w) \in \mathcal{W}(Q_{t_0})$ for some $t_0$. We have to prove that this solution can be extended to an arbitrarily large time interval. Assume the opposite. Then there is a number $t_{max} < \infty$ and a sequence $t_j \to t_{max}$, $t_j < t_{max}$, such that the solution $\mathbf{u}$ can be extended to $[0, t_j]$ for any $j$, but cannot be extended to $[0, t_{max}]$.

By Lemmas 5.6 and 5.7, the values $\|u(\cdot, t)\|_{W_2^1}$, $|v(t)|$, and $|w(t)|$ are bounded uniformly in $t \in [0, t_{\max})$. Thus, Theorem 5.1 implies that, for any $t_j$, the solution $\mathbf{u}$ can be extended from the interval $[0, t_j]$ to the interval $[0, t_j + t_0]$, where $t_0$ does not depend on $j$.

Now, choosing $t_j \geq t_{max} - t_0/2$, we can extend the solution from the interval $[0, t_{max} - t_0/2]$ to the interval $[0, t_{max} + t_0/2]$, which contradicts the definition of the number $t_{max}$. ☐

## 5.4 Uniqueness of solutions

In this subsection, we prove the following uniqueness result.

**Theorem 5.3.** *Let* $\mathbf{u}_0 = (u_0, v_0, w_0) \in W_2^1 \times \mathbb{R}^2$. *Then, for any* $T > 0$, *problem* (3.4)–(3.6) *has no more than one solution.*

*Proof.* 1. We assume that $\mathbf{u}_j = (u_j, v_j, w_j)$, $j = 1, 2$, are two solutions on the interval $[0, T]$ for problem (3.4)–(3.6) with the same initial data $\mathbf{u}_0 = (u_0, v_0, w_0) \in W_2^1 \times \mathbb{R}^2$. Then the difference $\mathbf{v} = (u, v, w) = (u_1 - u_2, v_1 - v_2, w_1 - w_2)$ satisfies the problem

$$\begin{cases} u_t = u_{xx} + f(x,t), \\ \dot{v} = g(t), \\ \dot{w} = h(t), \end{cases} \tag{5.14}$$

where

$$\begin{aligned} f(x,t) &= \mathcal{A}(v_1, w_1)u_1 - \mathcal{A}(v_2, w_2)u_2, \\ g(t) &= \mathcal{V}(u_1, v_1, w_1) - \mathcal{V}(u_2, v_2, w_2) \\ h(t) &= \mathcal{W}(u_1, w_1) - \mathcal{W}(u_2, w_2), \end{aligned}$$

with the zero Neumann boundary conditions and zero initial condition.

Denote

$$F(x,t) = (f(x,t), g(t), h(t)).$$

2. We will prove that $\mathbf{v} = 0$ by using the semigroup theory. We introduce the operator $\mathbf{P} : D(\mathbf{P}) \subset L_q \to L_q$, $q > 1$, by the formula

$$\mathbf{P}\psi = \psi_{xx}, \quad D(\mathbf{P}) = \left\{ \psi \in W_q^2 : \psi_x|_{x=\underline{x}} = \psi_x|_{x=\overline{x}} = 0 \right\}.$$

It is well known that the operator $\mathbf{P}$ is the infinitesimal generator of an analytic semigroup of linear bounded operators $\mathbf{S}_t : L_q \to L_q$, $t \geq 0$.

Clearly, the operator

$$(\mathbf{P}, 0, 0) : L_q \times \mathbb{R}^2 \to L_q \times \mathbb{R}^2$$

generates the analytic semigroup

$$\mathcal{S}_t = (\mathbf{S}_t, 1, 1) : L_q \times \mathbb{R}^2 \to L_q \times \mathbb{R}^2, \quad t \geq 0.$$

Since $f \in L_\infty(Q_T)$ and $g, h \in C[0, T]$, while $u \in W_2^{2,1}(Q_T)$, it follows from [2, Chap. 1, Sec. 3] that the solution $\mathbf{v}$ of problem (5.14) can be represented in the form

$$\mathbf{v}(\cdot, t) = \int_0^t \mathcal{S}_{t-s} F(\cdot, s)\, ds, \tag{5.15}$$

where the equality holds in $L_2$ for a.e. $t \in [0, T]$. Since $L_2$ is continuously embedded into $L_1$, equality (5.15) also holds in $L_1$.

Due to Lemma 2 in [26, p. 19],

$$\sup_{t \in [0,T]} \|\mathbf{S}_t \psi\|_{L_1} \leq K(T) \|\psi\|_{L_1} \quad \forall \psi \in L_2,\ T \in [0, \infty),$$

where $K(T) > 0$ does not depend on $\psi$. Combining this relation with equality (5.15), we obtain for $t \in [0, T]$

$$\|\mathbf{v}(\cdot, t)\|_{L_1 \times \mathbb{R}^2} \leq K(T) \int_0^t \|F(\cdot, s)\|_{L_1 \times \mathbb{R}^2}\, ds.$$

Now using the fact that $u_j \in W_2^{2,1}(Q_T) \subset L_\infty(Q_T)$ and applying Lemma 4.3 with $q = 1$, we have

$$\|\mathbf{v}(\cdot, t)\|_{L_1 \times \mathbb{R}^2} \leq K_1(T) \int_0^t \|\mathbf{v}(\cdot, s)\|_{L_1 \times \mathbb{R}^2}\, ds,$$

where $K_1(T)$ depends on $\mathbf{u}_1, \mathbf{u}_2$, but does not depend on $t \in [0, T]$. Therefore, by Gronwall's lemma, $\mathbf{v} = 0$ in $Q_T$. $\square$

# 6 Large time behavior

Due to Theorem 5.2, problem (3.4)–(3.6) has a unique solution $\mathbf{u} = (u, v, w) \in \mathcal{W}(Q_T)$ for any $T > 0$. In this section, we still assume that $0 < \underline{x} < \overline{x}$ and additionally assume that $\overline{x} < 1/2$. Concentrating on the physically relevant case (5.8), we study the large time behavior of the solution.

We will use throughout the following basic facts, which follow from Lemmas 5.6 and 5.7:

1. $|w| \le 1/2$, while $U(t)$ and $v(t)$ are nonnegative and bounded from above uniformly in $t$.
2. $v(t)$ is nonincreasing, while $U(t)$ is nondecreasing. Indeed, the right-hand side of the second equation in (3.4) is nonpositive, while $U(t)+v(t)$ is constant.

We also remind that $U_0 = \int\limits_{\underline{x}}^{\overline{x}} u_0(x)\,dx$.

The next lemma shows that the point $w = 1/2$ is repelling in the following sense.

**Lemma 6.1.** *Let $\overline{x} < 1/2$. If $|w_0| < 1/2$, then there is a number $\delta = \delta(\mathbf{u}_0) > 0$ such that $|w(t)| \le 1/2 - \delta$ for all $t \ge 0$.*

*If $w_0 = \pm 1/2$, then $w(t) = \pm 1/2$ for all $t \ge 0$.*

*Proof.* 1. Note that if $U_0 = 0$, then $u_0(x) \equiv 0$ and problem (3.4)–(3.6) has the unique solution $u(x,t) \equiv 0$, $v(t) \equiv v_0$, $w(t) \equiv w_0$.

Thus, we assume that $U_0 > 0$. Let us prove the first part of the lemma. Suppose that $|w_0| < 1/2$. Then, $\mathcal{R}^x(w)(t) = 1$ for all $x \in (\underline{x}, \overline{x})$ whenever $w(t) \in (\overline{x}, 1/2)$. In this case, the third equation in (3.4) implies that

$$\dot w = -\left(\frac{1}{2} + w\right)\left(\frac{1}{2} - w\right)\mathcal{P}(u,w) = -\left(\frac{1}{2} + w\right)\left(\frac{1}{2} - w\right)U < 0.$$

Therefore, $w(t) \le \max(\overline{x}, |w_0|)$ for all $t \ge 0$.

Similarly, $w(t) \ge \min(-\overline{x}, |w_0|)$ for all $t \ge 0$.

2. If $w_0 = 1/2$, then we set $w(t) \equiv 1/2$. Since $\overline{x} < 1/2$, we have $\mathcal{R}(w) \equiv 1$ and $\mathcal{P}(u,w)(t) \equiv U(t)$. Therefore, $u$ and $v$ should satisfy

$$\begin{cases} u_t = u_{xx} + vu, \\ \dot v = -vU. \end{cases} \tag{6.1}$$

These equations are reaction-diffusion equations without hysteresis. Therefore, they admit a unique solution $(u, v) \in W_2^{2,1}(Q_T) \times C^1[0, T]$ for any $T > 0$. This can be proved analogously to the general Theorem 5.2.

Therefore, the whole vector $\mathbf{u} = (u, v, 1/2)$ is a solution of problem (3.4)–(3.6). It is unique due to Theorem 5.2. $\square$

In the remaining part of this section, we will prove the following result on large-time behavior of $u$ and $v$.

**Theorem 6.1.** *Let* $\overline{x} < 1/2$, $u_0(x) \not\equiv 0$, *and condition* (5.8) *hold. Then, as* $t \to \infty$, *we have*

$$v(t) \to 0, \qquad u(\cdot, t) \to \frac{U_0 + v_0}{\overline{x} - \underline{x}} \quad \text{in } C[\underline{x}, \overline{x}]$$

First we prove the following lemma.

**Lemma 6.2.** *Under the assumptions of Theorem* 6.1, *there is a constant* $C > 0$ *such that*

$$\|u(\cdot, t)\|_{L_2} \le C, \quad t \ge 0.$$

*Moreover, as* $t \to 0$, *we have*

$$v(t) \to 0, \qquad U(t) \to U_0 + v_0.$$

*Proof.* 1. If $w_0 = 1/2$, then the vector $(u, v)$ satisfies equations (6.1). Therefore, $\dot{v} \le -vU_0$ because $U$ is nondecreasing. Since $u_0(x) \not\equiv 0$ and $u_0(x) \ge 0$, it follows that $U_0 > 0$. Therefore, $v(t) \to 0$ as $t \to 0$.

Similarly, $v(t) \to 0$ as $t \to 0$ if $w_0 = -1/2$.

If $|w_0| < 1/2$, then Lemma 6.1 implies that $|w(t)| \le 1/2 - \delta$, $t \ge 0$, with some $\delta \in (0, 1/2)$. On the other hand $|\mathcal{P}(u, w)(t)| \le U(t)$ for any $u$ and $w$. Hence, the second equation in (3.4) yields

$$\dot{v} \le -\delta U v \le -\delta U_0 v.$$

Therefore, $v(t) \to 0$ as $t \to 0$ again. Note that, in both case, $v(t)$ goes to zero at least exponentially fast:

$$v(t) \le v_0 e^{-\delta U_0 t}, \quad t \ge 0. \tag{6.2}$$

Since $v(t) \to 0$ and $U(t) + v(t) = U_0 + v_0$, it follows that $U(t) \to U_0 + v_0$.

2. Multiplying the first equation in (3.4) by $u$ and integrating with respect to $x$ yields

$$\frac{d}{dt}\|u(\cdot,t)\|_{L_2}^2 \le 2v(t)\|u(\cdot,t)\|_{L_2}^2. \tag{6.3}$$

It follows from (6.2) that

$$\int\limits_0^\infty v(t)\,dt < \infty \tag{6.4}$$

Now, using (6.3), (6.4), and Gronwall's lemma, we obtain

$$\|u(\cdot,t)\|_{L_2} \le C, \quad t \ge 0,$$

where $C > 0$ does not depend on $t$. □

To prove Theorem 6.1, it remains to show that

$$u(\cdot,t) \to \frac{U_0 + v_0}{\overline{x} - \underline{x}} \quad \text{in } C[\underline{x},\overline{x}] \tag{6.5}$$

as $t \to \infty$.

We denote

$$b(x,t) = v(t)\left(\frac{1}{2} + w(t)\mathcal{R}^x(w)(t)\right)u(x,t)$$

and write the first equation in (3.4) as follows:

$$u_t = u_{xx} + b(x,t). \tag{6.6}$$

Using Lemma 4.2 and the fact the $u \in W_2^{2,1}(Q_T) \subset C([0,T];L_2)$, we see that $b(\cdot,t)$ is continuous in $t$ as an $L_2$-valued function of variable $t$. By Lemma 6.2,

$$\|b(\cdot,t)\|_{L_2} \to 0, \quad t \to \infty. \tag{6.7}$$

Since the semigroup generated by the Laplacian with the Neumann boundary conditions is not exponentially decreasing, we cannot directly use (6.7). We shall use the Fourier representation of the solution $u$.

Let

$$\lambda_k = \left(\frac{\pi k}{\overline{x} - \underline{x}}\right)^2, \quad k = 0, 1, 2, \ldots,$$

$$e_0(x) = \frac{1}{\sqrt{\overline{x} - \underline{x}}}, \quad e_k(x) = \sqrt{\frac{2}{\overline{x} - \underline{x}}} \cos \frac{\pi k(x - \underline{x})}{\overline{x} - \underline{x}}, \quad k = 1, 2, \dots,$$

denote the sequence of eigenvalues and the corresponding system of eigenfunctions (orthonormal in $L_2$) of the spectral problem

$$-e_k''(x) = \lambda_k e_k(x) \quad (x \in (\underline{x}, \overline{x})), \qquad e_k'(\underline{x}) = e_k'(\overline{x}) = 0.$$

In particular, we will use that any function $\psi \in L_2$ can be expanded into the Fourier series with respect to $e_k$, which converges in $L_2$:

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k e_k(x), \qquad \|\psi\|_{L_2}^2 = \sum_{k=0}^{\infty} |\psi_k|^2, \tag{6.8}$$

where $\psi_k = \int\limits_{\underline{x}}^{\overline{x}} \psi(x) e_k(x)\, dx$.

**Remark 6.1.** The semigroup $\mathbf{S}_t$, $t \geq 0$ (see Sec. 5.4), can be represented as follows:

$$\mathbf{S}_t \psi = \sum_{j=0}^{\infty} e^{-\lambda_k t} \psi_k e_k(x) \quad (t \geq 0),$$

where the series converges in $L_2$ ($W_2^1$) if $\psi \in L_2$ ($\psi \in W_2^1$).

Now we prove (6.5) and thus complete the proof of Theorem 6.1.

*Proof of Theorem* 6.1. We represent the solution $u$ of equation (6.6) with the Neumann boundary conditions and the initial condition $u(x, 0) = u_0(x)$ as the series

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) e_k(x), \tag{6.9}$$

which converges in $W_2^{2,1}(Q_T)$ for any $T > 0$, provided $u_0 \in W_2^1$ (see, e.g., [21]). Here $u_k(t)$ are the Fourier coefficients of $u(x, t)$.

It follows from Lemma 6.2 that

$$u_0(t) e_0(x) = \int\limits_{\underline{x}}^{\overline{x}} u(y, t) e_0(y)\, dy \cdot e_0(x) = \frac{1}{\overline{x} - \underline{x}} U(t) \to \frac{U_0 + v_0}{\overline{x} - \underline{x}}. \tag{6.10}$$

Denote

$$m(x, t) = \sum_{k=1}^{\infty} u_k(t) e_k(x). \tag{6.11}$$

It remains to show that

$$\|m(\cdot,t)\|_{C[\underline{x},\overline{x}]} \to 0, \quad t \to \infty. \tag{6.12}$$

Fix an arbitrary $\varepsilon > 0$. It follows from (6.7) that there is $t_0 > 0$ such that

$$\|b(\cdot,t)\|_{L_2} \le \varepsilon, \quad t \ge t_0. \tag{6.13}$$

Due to the Fourier method, the coefficients $u_k(t)$, $k = 1, 2, \ldots$, satisfy the Cauchy problems for the ordinary differential equations

$$u_k' = -\lambda_k u_k + b_k(t), \quad u_k(t_0) = u_{k0}, \quad k = 1, 2, \ldots,$$

with

$$b_k(t) = \int\limits_{\underline{x}}^{\overline{x}} b(x,t) e_k(x)\, dx, \quad u_{k0} = \int\limits_{\underline{x}}^{\overline{x}} u(x,t_0) e_k(x)\, dx, \quad k = 1, 2, \ldots.$$

Note that (6.13) implies

$$|b_k(t)| \le \varepsilon, \quad t \ge t_0. \tag{6.14}$$

By explicitly solving the Cauchy problems, we have

$$u_k(t) = u_{k0} e^{-\lambda_k (t - t_0)} + \int\limits_{t_0}^{t} e^{-\lambda_k (t-s)} b_k(s)\, ds. \tag{6.15}$$

First, we estimate $u_k(t)$, using (6.14):

$$\begin{aligned} |u_k(t)| &\le |u_{k0}| e^{-\lambda_k (t-t_0)} + \frac{\varepsilon}{\lambda_k} \left(1 - e^{-\lambda_k (t-t_0)}\right) \\ &\le |u_{k0}| e^{-\lambda_k (t-t_0)} + \frac{\varepsilon}{\lambda_k}. \end{aligned} \tag{6.16}$$

Now we can estimate $m(x,t)$ given by (6.11), using (6.16) and the fact that $|e_k(x)| \le \sqrt{2/(\overline{x} - \underline{x})}$:

$$\begin{aligned} \|m(\cdot,t)\|_{C[\underline{x},\overline{x}]} &\le \sqrt{\frac{2}{\overline{x} - \underline{x}}} \left( \sum_{k=1}^{\infty} |u_{k0}|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} e^{-2\lambda_k (t-t_0)} \right)^{1/2} \\ &\quad + \varepsilon \sqrt{\frac{2}{\overline{x} - \underline{x}}} \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \end{aligned} \tag{6.17}$$

for $t \geq 2t_0$.

Taking into account Lemma 6.2, we see that there exists $t_1 \geq 2t_0$ such that

$$\left(\sum_{k=1}^{\infty} |u_{k0}|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} e^{-2\lambda_k(t-t_0)}\right)^{1/2} \leq \|u(\cdot, t_0)\|_{L_2} \left(\sum_{k=1}^{\infty} e^{-2\lambda_k(t-t_0)}\right)^{1/2} \leq \varepsilon \tag{6.18}$$

for all $t \geq t_1$. Then (6.17) and (6.18) yield

$$\|m(\cdot, t)\|_{C[\underline{x}, \overline{x}]} \leq \varepsilon \sqrt{\frac{2}{\overline{x} - \underline{x}}} \left(1 + \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\right)$$

for all $t \geq t_1$, which proves (6.12). ☐

# 7 Discussion

## 7.1 Large time behavior

We have shown that the variable $v$ measuring the total amount of nutrients in the system uniformly converges to zero and the population density $u$ converges to a uniform distribution over the interval $[\underline{x}, \overline{x}]$ as $t \to \infty$. This is to be expected as there is no supply of nutrients in the system. When the density of nutrients vanishes as a result of consumption by bacteria, the equation for the density $u$ "approaches" the homogeneous heat equation with zero flux boundary conditions and $u$ converges to the uniform profile.

Another important characterization of the large time behavior is the distribution of phenotypes over the range of available threshold values $[\underline{x}, \overline{x}]$, which results from the evolution of system (3.4)–(3.6) as $t \to \infty$. Numerical calculations presented in [12] suggest that the binary function $r(x, t) = \mathcal{R}^x(w)(t)$ describing the distribution of two phenotypes converges to a stationary binary pattern $r_*(x) = \lim_{t \to \infty} \mathcal{R}^x(w)(t)$ and each of the sets

$$A_1 = \{x \in [\underline{x}, \overline{x}] : r_*(x) = 1\}, \qquad A_{-1} = \{x \in [\underline{x}, \overline{x}] : r_*(x) = -1\} = [\underline{x}, \overline{x}] \setminus A_1$$

is a union of finitely many disjoint intervals. However, our simulations indicate that the sign changing pattern of $r^*$ is different for different initial data. For example, the number of disjoint intervals in each of the limit sets $A_{\pm 1}$

increases with the increase of the initial value $v_0$ (initial food supply) and with the decrease of the diffusion rate. That is, there is no single winner in the competition of the two phenotypes, or a single limit distribution. The attractor seems to be a connected continual set of stationary distributions. Rigorous analysis of the attractor will be the subject of future work.

## 7.2 Relation to systems with spatially distributed hysteresis

In [15], a reaction-diffusion system with discontinuous hysteresis depending on a diffusing component of the unknown vector-valued function was introduced and numerical analysis was performed. The thresholds of hysteresis were fixed, but the hysteresis itself was defined at every spatial point, i.e., the input was a function of $x$ and $t$, where $x$ refers to a spatial position of a diffusive substance. Existence of solutions for such systems was proved in [1, 17, 30] for a modified version of hysteresis (multi-valued hysteresis) as well as in [16] for some special case. Existence, uniqueness, and continuous dependence of solutions on initial data for the original system was treated in [14], where an important notion of spatial transversality was introduced.

It turns out that the model of the present paper is related to that with spatially distributed hysteresis. For example, by introducing the new unknown function $\tilde{w}(x,t) = w(t)/x$, we see that $\mathcal{R}^x(w)(t) \equiv \mathcal{R}^1(\tilde{w}(x,\cdot))(t)$, where $\mathcal{R}^1$ is the non-ideal relay with the fixed thresholds ±1. The operator $\mathcal{R}^1$ can now be treated as spatially distributed, since its input $\tilde{w}(x,t)$ depends on the "spatial" point $x$. It would be interesting to further study the connection between spatially distributed hysteresis and hysteresis with diffusing thresholds as well as consider a combination of both.

## 7.3 Variations of model

It would be interesting to consider variations of model (3.4)–(3.6) and their effect on dynamics, the attractor and the pattern formation. Possible modifications might account for the death process in bacteria; permanent or variable supply of nutrients; switching off the diffusion process; inclusion of non-ideal relays $\mathcal{R}^{y,x}$ with asymmetric switching thresholds $x, y$, $y \neq -x$; variations of the boundary conditions. Well-posedness of these models can be established by a slight modification of the proof presented in this paper (the case $\underline{x} = 0$ might require additional effort). Preliminary simulation results indicate that different dynamical scenarios, such as Hopf bifurcation, are possible in a model with permanent supply of nutrients.

An important assumption we made in (3.4)–(3.6) was that bacteria, when sporadically changing their threshold $x'$ to a new value $x$, simultaneously copy the state from their peers who have the same threshold $x$. It would be natural to explore a model where the state remains unchanged when the threshold changes. Such a model should have simultaneous nonzero populations of bacteria with the same threshold in two phenotypes. This is also a subject of future work.

# Acknowledgments

Dmitrii Rachinskii acknowledges the support of the Alexander von Humboldt Foundation (Germany) and the Russian Foundation for Basic Research through grant 10-01-93112. Pavel Gurevich acknowledges the support of Collaborative Research Center 910 (Germany) and the Russian Foundation for Basic Research through grant 10-01-00395. The authors are grateful to Sergey Tikhomirov who created a software for a number of numerical experiments.

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