Attractors of viscous balance laws: Uniform estimates for the dimension

Jörg Härterich
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin
Germany

July 3, 1997

1 Introduction

This paper is devoted to the study of global attractors of a class of singularly perturbed scalar parabolic equations depending on a small parameter ε. These equations possess a special structure allowing for a detailed description of the global attractor. Many properties of the attractor can be deduced using information on equilibria and their variational equations only. This leads to the study of certain singularly perturbed boundary value problems which in general have many solutions. As proposed by Allen and O'Malley [1] for problems where qualitative information on solutions is sought rather than high order approximations we use phase plane methods to describe the solutions of the boundary value problem.
As ε tends to zero one typically expects that the global attractor has either a very simple structure (e.g. consists of one stable equilibrium only) or that its dimension tends to infinity. The rather surprising result of this paper consists of the proof that for a large class of nonlinearities the dimension of the global attractors stays bounded as ε tends to zero.

1.1 Global attractors of scalar parabolic equations

Semiflows generated by scalar semilinear parabolic equations are a class of infinite-dimensional dynamical systems whose qualitative behavior has been an object of intensive research during the last fifteen years. It has been shown that the equation

$$u_t = u_{xx} + h(x, u, u_x)\ ,\ \ h \in C^2 \tag{1}$$

with Neumann boundary conditions

$$u_x(0) = u_x(1) = 0$$

and initial condition

$$u(0, x) = u_0(x) \tag{2}$$

gives rise to a (local) C^1-semiflow on the Sobolev space $X \subseteq W^{2,2}([0, 1], \mathbb{R})$ of functions satisfying the boundary conditions at $x = 0$ and $x = 1$. The associated semigroup T assigns to each pair (t, u_0) the solution profile $u(t, \cdot)$ of (1) at time $t > 0$ that satisfies the initial condition (2) at time $t = 0$. If the nonlinearity h satisfies some growth and sign conditions, the semiflow is global and dissipative, i.e. solutions exist for all (positive) times and there exists a large ball $B \subseteq X$ such that every solution $u(t)$ will eventually stay in this ball. Due to the smoothing properties of the Laplacian, $T(t,\cdot)$ is compact for all $t > 0$. Under these conditions a global attractor $\mathcal{A}$ exists, defined as a maximal compact invariant set in B that attracts all bounded subset of X, see e.g. the monograph of Hale [16] for theorems in this rather general setting of compact and dissipative semigroups. This global attractor consists of all orbits that are defined for all (positive and negative) times t and that are uniformly bounded.

There are two special features of scalar parabolic equations that allow for a more precise description of the global attractor:

(a) a gradient structure and

(b) nodal properties.

Concerning (a), Zelenyak [28] showed already in the sixties that equation (1) possesses a Lyapunov functional. Except at equilibria, this Lyapunov functional decreases along trajectories. Since this Lyapunov functional can be shown to be bounded on bounded sets, any orbit that stays uniformly bounded for all $t \geq 0$ will tend to the set of equilibria of (1), i.e. the set of time independent solutions. In other words, the ω-limit set of a single point $u_0 \in X$ is contained in the set E of all equilibria. Matano [25] showed that it even consists of exactly one equilibrium. The same arguments hold for negative t if a trajectory is defined there. So, for every trajectory that is defined and bounded for all $t < 0$, the α-limit set is also an equilibrium. Using the above characterization of the global attractor $\mathcal{A}$ as the union of all uniformly bounded trajectories that are defined also for all negative t, we obtain the following description of the global attractor:

Proposition 1.1 $\mathcal{A} = \bigcup_{e \in E} W^u(e)$, *where E is the set of all equilibria and $W^u(e)$ is the unstable set of e. It consists of*

- *the set E of equilibria and*
- *heteroclinic orbits connecting different equilibria.*

To refine this description, consider now the eigenvalue problem associated with the linearization of (1) at an equilibrium v:

$$\left.\begin{aligned} w_{xx} + \partial_u h(x, v(x), v_x(x))w + \partial_p h(x, v(x), v_x(x))w_x &= \lambda w \\ w_x(0) = w_x(1) &= 0 \end{aligned}\right\} \tag{3}$$

Definition 1.2 *An equilibrium v is called* **hyperbolic** *if 0 is not an eigenvalue of the linearization at v, i.e. if (3) has no nontrivial solution for $\lambda = 0$.*

Definition 1.3 *The* **Morse index** *$i(v)$ of a hyperbolic equilibrium v is the number of positive eigenvalues of the linearization at v.*

In other words, $i(v)$ is the dimension of the unstable manifold $W^u(v)$. If all equilibria are hyperbolic then the global attractor is the finite union of equilibria and their unstable manifolds.

It is a classical observation by Sturm that the eigenvalues are connected to the oscillation properties of the eigenfunctions. There is a sequence of simple eigenvalues

$$\lambda_0 > \lambda_1 > \ldots, \ \lambda_n \to -\infty \ \text{ as } n \to \infty$$

and the eigenfunction w_k associated with λ_k has exactly k zeroes in the open interval $(0, 1)$.

If $h = h(u, u_x)$ does not depend explicitly on x, there is an important relation between the Morse index of an equilibrium v and the number $z(v_x)$ of strict sign changes of v_x defined as

$$\begin{aligned} z(v) &:= \sup\{n \in \mathbb{N}; \exists 0 < x_1 < \ldots < x_n < 1 \text{ with } v(x_i) \cdot v(x_{i+1}) < 0 \ , \ 1 \le i < n\} \\ z(0) &:= 0. \end{aligned}$$

Proposition 1.4 *If $h = h(u, u_x)$ does not depend on x, then:*

(i) the Morse index of any nonconstant hyperbolic equilibrium v is either $z(v_x)$ or $z(v_x) + 1$.

(ii) the linearization at a nonconstant non-hyperbolic equilibrium v possesses one zero eigenvalue and $z(v_x) - 1$ or $z(v_x)$ positive eigenvalues.

This can be proved by a simple application of the Sturm comparison theorem to v_x and the eigenfunctions w_{n-1} and w_n where $n = z(v_x) + 1$.
In case (ii), Henry [22] has shown that the center-unstable manifold $W^{cu}(v)$ is a manifold with boundary which has dimension $z(v_x)$ or $z(v_x) + 1$.

The second peculiarity of scalar parabolic equations is the existence of a discrete Lyapunov functional connected to the zero number $z(u(\cdot))$ of the solution u in $(0, 1)$.

Already Sturm recognized that the zero number $z(u^1(t,\cdot)-u^2(t,\cdot))$ of the difference of two solutions u^1 and u^2 is non-increasing in time. In a recent version, Angenent [3] shows that $z(u^1(t,\cdot)-u^2(t,\cdot))$ drops strictly at time t if $u^1(t,\cdot)-u^2(t,\cdot)$ possesses a multiple zero. One consequence of those nodal properties is the Morse-Smale property of the attractor: There can only exist heteroclinic connections from equilibria with higher Morse index to such with lower Morse index.

1.2 Equilibria and connections

Since the global attractor is the union of equilibria and connecting orbits, given a specific equation, one may want to find the equilibrium solutions first and think about connecting orbits afterwards. The equilibria are simply solutions of the boundary value problem

$$\left.\begin{array}{rcl} u_{xx}-h(x,u,u_x) & = & 0 \\ u_x(0)=u_x(1) & = & 0. \end{array}\right\} \tag{4}$$

The next step in the description of the attractor consists of finding criteria whether two given equilibria are connected by a heteroclinic orbit or not. This question was first adressed by Brunovský and Fiedler [7, 8] in the case of a nonlinearity depending on u only. Later, Fiedler and Rocha [13], could show that also in case $h=h(x,u,u_x)$ all information on the connections can already be derived from the equilibrium solutions. Their work uses the observation of Fusco and Rocha [15] that all information on the Morse indices of the equilibria as well as on zero numbers of differences of equilibria is contained in the ordering of the equilibrium solutions at $x=0$ and $x=1$. Their statement is a constructive one: for a given nonlinearity h, from the knowledge of all equilibrium solutions, one can determine explicitly the Morse indices and the zero numbers of the difference of two equilibria.
Based on this, Fiedler and Rocha [13] gave explicit criteria to decide whether two equilibria are connected.

1.3 Singular perturbations

We introduce now a small positive parameter ε in front of the diffusivity term of the parabolic equation (1), which thereby becomes

$$\left.\begin{array}{l} u_t=\varepsilon u_{xx}+h(x,u,u_x) \\ u_x(0,t)=u_x(1,t)=0. \end{array}\right\} \tag{5}$$

It is easy to check that most of the statements above do not depend on ε. It neither influences the global existence nor the dissipativeness and compactness of the (ε-dependent) semigroup. Thus, for each fixed $\varepsilon > 0$ equation (5) possesses a global attractor $\mathcal{A}_\varepsilon$. In general, this attractor $\mathcal{A}_\varepsilon$ will vary with ε. In particular, hyperbolicity of all equilibria will not hold for all ε as the following 'classical' example shows.

Consider a nonlinearity $h = h(u)$ not depending on x and u_x, e.g. the cubic $h(u) = u(1 - u^2)$. Chafee and Infante [9] showed that the equilibrium $u \equiv 0$ undergoes a sequence of pitchfork bifurcations at values $\varepsilon = (l\pi)^{-2}$, $l = 1, 2, \ldots$. At each of these pitchfork bifurcations two new equilibria appear and the Morse index of $u \equiv 0$ is increased by one. Hence there are two problems in getting a description of the attractor as ε tends to 0: the number of equilibria and the dimension of the attractor both tend to infinity. It is not difficult to see that this behaviour is typical for nonlinearities depending on u only. In this case the equilibria satisfy the equation

$$\varepsilon u_{xx} + h(u) = 0,$$

which has a Hamiltonian structure. By rescaling $x = \sqrt{\varepsilon}\xi$ the equilibrium equation becomes

$$u_{\xi\xi} + h(u) = 0, \tag{6}$$

and does not depend on ε any more, only the boundary condition at $x = 1$ is transformed into $u_\xi(\varepsilon^{-1/2}) = 0$ and contains the parameter ε. If h has at least two zeroes then (6) admits for families of periodic orbits that accumulate onto a homoclinic orbit or a pair of heteroclinic orbits. A solution of (6) following one of these periodic orbits with period p, say, $\frac{k}{2}$ turns is a solution of the boundary value problem if $\varepsilon^{-1/2} = \frac{k}{2} \cdot p$. It is now easy to see that the following alternative holds, depending on whether h has exactly one or more than one zero: Either there is only one (spatially homogenous) equilibrium that does not depend on ε, or the attractor $\mathcal{A}_\varepsilon$ blows up in the way described above.

A natural question to ask is, whether the same is true for more general nonlinearities h. There are some results indicating that for $h = h(x, u)$ the behaviour is rather more complicated. While for h not depending on x and the gradient u_x all nonconstant equilibria are unstable, Angenent, Mallet-Paret and Peletier [5] found stable solutions which develop a transition layer. Later, Hale and Sakamoto [17] described also unstable equilibrium solutions with transition layers.

Theorem 1.1 below shows that for a class of nonlinearities h of the special form $h(x, u, u_x) = (f(u))_x + g(u)$ a different behaviour of $\mathcal{A}_\varepsilon$ occurs: The dimension of $\mathcal{A}_\varepsilon$

remains bounded for all small ε. This shows that convection can prevent, at least in some cases, the attractor from blowing up.

Theorem 1.1 *Consider the singularly perturbed parabolic equation of the special form*

$$u_t + (f(u))_x = \varepsilon u_{xx} + g(u), \quad f, g \in C^3. \tag{7}$$

with Neumann boundary conditions.
Assume that

(H1) *g is a dissipative function, i.e.*

$$u \cdot g(u) < 0 \quad \forall \, |u| > R \tag{8}$$

with some (large) constant R.

(H2) *the critical points of f are quadratic folds, i.e. $f'(u) = 0 \Rightarrow f''(0) \neq 0$*

(H3) *the derivative of f does not vanish at zeroes of g.*

Then the dimension of $\mathcal{A}_\varepsilon$ remains bounded as $\varepsilon \to 0$.

Note that condition **(H1)** is open with respect to the strong Whitney topology while the conditions **(H2)** and **(H3)** persist under C^2-small perturbations. Hence theorem 1.1 is a rather general statement.
The rest of the paper is organized in the following way: Chapter 2 deals with viscous balance laws. Chapter 3 contains an investigation of equilibrium solutions to the viscous balance law and the proof of theorem 1.1. The paper concludes with a short discussion.

2 Balance Laws

Viscous balance law is a term used for a scalar parabolic equation of the form (7). We will study this equation on the unit interval with Neumann boundary conditions

$$u_x(0) = u_x(1) = 0$$

and initial data

$$u(0, \cdot) = u_0(\cdot) \in W^{1,2}.$$

Here f and g are of class C^2 and g satisfies the dissipativeness assumption **(H1)**. The parameter ε is very small and adds some viscosity to the usual "balance law"

$$u_t + (f(u))_x = g(u). \tag{9}$$

Balance laws are a generalization of conservation laws

$$u_t + (f(u))_x = 0$$

where a source term $g(u)$ is present. As with conservation laws, for balance laws there are in general no global smooth solutions even for arbitrarily smooth initial data. After a finite time, shocks are formed. For x on the whole real line, Kruzhkov [23] showed that under some admissibility condition for any bounded measurable u_0 there is a unique solution of the hyperbolic equation (9). On finite time intervals the solution of the viscous balance law (7) converges to this unique solution as ε tends to 0.

Only recently conservation as well as balance laws on an interval have attracted more attention. This is mostly due to the occurence of steep transition layers that move very slowly [26]. With these property, these second order scalar equations serve as crude models for phase transitions or semiconductors [6]. Especially, they are used as test problems for numerical analysists who are interested in the numerical treatment of more complicated (and hopefully more realistic) problems, e.g. higher dimensional equations describing phase transitions.

One difference between scalar conservation laws and balance laws is the fact that solutions of balance laws need not decay to a spatially homogenous state as time tends to infinity. In the case of $x \in \mathbb{R}$ and periodic initial data it was shown 1970 by Glimm and Lax that solutions of scalar conservation laws decay like $\mathcal{O}(t^{-1})$ to their spatial average when the time t tends to infinity. This is not true for balance laws: There may exists periodic solutions and at least in the case of f being convex, a Poincaré-Bendixson type result holds: Every solution either tends to an equilibrium or to a periodic orbit, which is a traveling wave then. This result was shown by Lyberopoulos [24], Fan & Hale [10] and also by Sinestrari [27] and ressembles very much the results in the parabolic case treated in Angenent & Fiedler [4] and Fiedler & Mallet-Paret [12] although the methods are quite different.

Local existence of weak solutions of solutions for the parabolic equation

$$u_t + (f(u))_x = g(u) + \varepsilon u_{xx} \tag{10}$$

with Neumann boundary conditions can be shown by semigroup methods as in the book of Henry [21]. The (unbounded) linear operator εu_{xx} together with the Neumann boundary conditions generates an analytical semigroup on the space $W^{1,2}(0,1)$ and the smoothness assumptions on f and g are sufficient to guarantee a local solution of (10). This solution lies in the domain of the infinitesimal generator, i.e. in the space $X \subseteq W^{2,2}$ of functions which satisfy the Neumann boundary conditions. Furthermore, the time derivative $u_t(t,\cdot)$ is Hölder continuous so by elliptic Schauder regularity theory the solution is a classical solution. This allows to talk about derivatives of the solution, to use maximum principles involving u_{xx}, etc..
Although this could also be derived from general theorems by Amann [2], it will be shown here in an elementary way that the local solutions of the viscous balance law (10) exist globally in time and that dissipativeness is guaranted by the sign condition (8) on g.
Global existence of solutions will be shown via some a-priori estimates on u and the derivative u_x:

Lemma 2.1 *If u satisfies equation (7) for all $t \in [0,T]$, then:*
(i) $\|u(T,\cdot)\|_{L^\infty} \leq C(u_0)$ independent of ε
(ii) $\|u_x(T,\cdot)\|_{L^2} \leq C(u_0,\varepsilon)$

Proof: (i) follows from a maximum principle. Due to the dissipativeness condition (8), in any positive maximum $u(x_0,t_0)$ with $0 < t_0 \leq T$ and $|u(x_0,t_0)| > R$

$$u_t = \varepsilon \underbrace{u_{xx}}_{\leq 0} - \underbrace{f'(u)u_x}_{=0} + \underbrace{g(u)}_{<0} < 0.$$

In the same way we can conclude that in any negative minimum $u(x_0,t_0)$ with $0 < t_0 \leq T$ and $u(x_0,t_0) < -R$ we have $u_t > 0$. Therefore, the L^∞-norm decreases as long as $u(t,\cdot)$ takes values outside $[-R,R]$ and hence

$$\|u\|_{L^2} \leq \|u\|_{L^\infty} \leq \max\{R, \|u(0,\cdot)\|_{L^\infty}\} =: C.$$

To proof claim (ii), first note that for $u \in W^{2,2}$

$$\|u_x\|_{L^2}^2 = \int_0^1 u_x^2 dx \leq \int_0^1 |u u_{xx}| dx \leq \|u\|_{L^2} \|u_{xx}\|_{L^2}$$

hence for any solution u

$$-\|u_{xx}\|_{L^2}^2 \leq -C_0 \cdot \|u_x\|_{L^2}^4$$

where the constant

$$C_0 := \frac{1}{C^2}$$

depends on u_0. Furthermore,

$$\begin{aligned}\int_0^1 f'(u)u_x u_{xx}\,dx &\leq \max_{|u|\leq C}|f'(u)|\cdot\|u_x\|_{L^2}\|u_{xx}\|_{L^2}\\ &\leq \frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2+\frac{C}{2\varepsilon}\|u_x\|_{L^2}^2.\end{aligned}$$

Multiplying equation (7) by u_{xx} one obtains after integrating

$$\begin{aligned}\frac{d}{dt}\,\frac{1}{2}\|u_x\|_{L^2}^2 &= -\varepsilon\|u_{xx}\|_{L^2}^2-\int_0^1 f'(u)u_x u_{xx}\,dx+\int_0^1 g'(u)u_x^2\,dx\\ &\leq -\frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2-\frac{\varepsilon C_0}{2}\|u_x\|_{L^2}^4+\frac{\varepsilon}{2}\|u_{xx}\|_{L^2}^2+\frac{C}{2\epsilon}\|u_x\|_{L^2}^2+C(u_0)\|u_x\|_{L^2}^2\\ &\leq 0 \qquad\qquad \text{for } \ \|u_x\|_{L^2}\geq C(u_0,\varepsilon)\end{aligned}$$

$$\Rightarrow\quad \|u_x(t,\cdot)\|_{L^2}\leq\max\{\|u_x(0,\cdot)\|_{L^2},C(u_0,\varepsilon)\}$$

□

This lemma implies immediately that all solutions exist globally in time and that (forward) orbits are bounded in $W^{1,2}$. Due to the variational structure of equation (7), for any $u_0\in W^{1,2}$ the ω-limit set of u_0 is contained in the union of the equilibrium solutions of (7). To prove dissipativeness, it remains only to show that for any fixed ε the equilibrium solutions form a bounded set. Below we will study equilibrium solutions in detail and for this reason we postpone the proof of boundedness to lemma 3.4. Here we only note that the boundedness of the set of equilibrium solutions implies the dissipativeness of the semiflow.

3 Equilibrium solutions

3.1 A singularly perturbed boundary value problem

Since, by definition, equilibrium solutions do not depend on time t, we will write for these solutions simply $u(x)$ instead of $u(x,t)$. Equilibrium solutions of (7) are solutions of the boundary value problem

$$\left.\begin{aligned}\varepsilon u_{xx}-(f(u))_x+g(u) &= 0\\ u_x(0)=u_x(1) &= 0.\end{aligned}\right\}\tag{11}$$

This singularly perturbed boundary value problem can be written as a first order system

$$\left.\begin{array}{rcl} \varepsilon u_x &=& v + f(u) \\ v_x &=& -g(u) \\ u_x(0) &=& u_x(1) = 0, \end{array}\right\} \tag{12}$$

a choice of coordinates which is sometimes called the "Liénard plane" in contrast to the more common "phase plane" where $v = u_x$.

Note that the boundary condition in (12) could also be written in the form

$$v(x) + f(u(x)) = 0 \quad \text{at } x = 0 \text{ and } x = 1 \quad ,$$

so, geometrically speaking, we are looking for trajectories of (12) which take exactly "time" $\Delta x = 1$ to join two points on the curve $v + f(u) = 0$. To avoid too much confusion between time and space variables, we recast (12) in the form

$$\left.\begin{array}{rcl} \varepsilon u' &=& v + f(u) \\ v' &=& -g(u) \\ x' &=& 1 \end{array}\right\} \tag{13}$$

$$\begin{gathered} 0 < s < 1 \\ u'(0) = u'(1) = x(0) = 0 \end{gathered}$$

where $'$ denotes differentiation with respect to a new variable s that looks more like a time variable than x does, although obviously $x \equiv s$. Below, methods from singular perturbation theory are used that compare system (13) for small ε with some limiting systems for $\varepsilon = 0$. There are different possibilities to perform this limit, leading to the so called "slow" and "fast" systems which both describe a part of the limiting behaviour of system (12). The difference originates in a different scaling of the variable s.
Setting $\varepsilon = 0$ in equation (12), we arrive at the "slow system"

$$\begin{array}{rcl} 0 &=& v + f(u) \\ v' &=& -g(u). \end{array}$$

Here the motion is confined to a curve given by the first of the two equations, while the second one describes the flow along this curve. Since this curve will play a special rôle later on it deserves a name on its own:

Definition 3.1 *The curve $\mathcal{C}$ given by the equation $v + f(u) = 0$ in the (u, v)-plane is called the* **singular curve**.

Note that $u_x = 0$ exactly where the trajectory hits or crosses the curve $\mathcal{C}$. Later, when we have to determine $z(u_x)$ to use lemma 1.4 we will use this property: Instead of counting the extrema of u, we can simply count the number of intersections between the trajectory and the curve $\mathcal{C}$.

System (12) can also be scaled in another way. If the second equation is multiplied by ε and the variable s is rescaled according to $s = \varepsilon\sigma$, we arrive at

$$\begin{aligned} \dot{u} &= v + f(u) \\ \dot{v} &= -\varepsilon g(u). \end{aligned}$$

with $\dot{}$ denoting the derivative with respect to the fast variable σ. Putting $\varepsilon = 0$, the "fast system"

$$\begin{aligned} \dot{u} &= v + f(u) \\ \dot{v} &= 0 \end{aligned}$$

is obtained. Here, the singular curve consists of equilibrium points only. According to the stability of these equilibria, parts of $\mathcal{C}$ where $f' > 0$ are called unstable arcs of $\mathcal{C}$, while the parts with $f' < 0$ are called stable arcs.

3.2 Proof of theorem 1.1

From the three assumptions of theorem 1.1, we have already used one: The dissipativeness **(H1)** of g was necessary for the existence of a global attractor. The condition **(H2)** that all zeroes of f' be simple is not the best possible. It can probably be weakened on the price of longer and more complicated calculations. Let us just mention in short that the assumption **(H3)** which states that no zeroes of g lie on the fold points of the singular curve $\mathcal{C}$ is necessary to prevent a blow-up of $\mathcal{A}_\varepsilon$ similar to the Chafee-Infante example presented in the introduction.

We begin with a short outline of the proof. Recall the characterization of $\mathcal{A}_\varepsilon$ as the unstable manifold of the set E of equilibria. If all equilibria are hyperbolic then they are isolated and the attractor is contained in the union of the unstable manifolds of all single equilibria. The hyperbolicity of all equilibria is not guaranteed under the weak assumptions of theorem 1.1, such that we have to consider the possibility of a non-hyperbolic attractor, too. In this case a bound on the dimension of the center-unstable manifolds of the equilibria will be derived. We will distinguish between spatially homogenous and non-homogenous equilibrium solutions. It will turn out, that for the homogenous solutions the linearization has at most one non-negative eigenvalue. For the spatially non-homogenous equilibria, in view of lemma 1.4 it is necessary to prove that for some integer N and all small ε any equilibrium has at most N extrema. This will be an immediate consequence of lemma 3.6 where we will prove that three extrema of an equilibrium solution u cannot be arbitrarily close to each other. On the way to this lemma, we collect some properties of the two-dimensional system

$$\left.\begin{array}{rcl} \varepsilon u' & = & v + f(u) \\ v' & = & -g(u) \end{array}\right\} \tag{14}$$

for small ε.

Observe first that the equilibrium points of system (14) lie on the curve $\mathcal{C}$ and have as u-coordinates exactly the zeroes of g. Due to assumption **(H3)**, the eigenvalues

$$\mu_{1,2} = \frac{f'(u_0) \pm \sqrt{f'(u_0)^2 - 4\varepsilon g'(u_0)}}{2\varepsilon}$$

of the linearization at such an equilibrium (u_0, v_0) turn out to be real for small ε and behave asymptotically like

$$\mu_1 \sim \frac{f'(u_0)}{\varepsilon} \quad \text{and} \quad \mu_2 \sim \frac{g'(u_0)}{f'(u_0)}.$$

If $g'(u_0) \neq 0$ the corresponding equilibrium of (14) is hyperbolic and a saddle exactly if $g'(u_0) < 0$. If all zeroes of g are simple, the equilibria are alternately saddles and sources or sinks.

Each of these zeroes of g corresponds to one homogenous equilibrium solution. The next step consists of showing that (e.g. in contrast to the Chafee-Infante case) these homogenous equilibrium solutions cannot become very unstable as ε decreases.

Lemma 3.2 *If $g(\tilde{u}) = 0$ then for ε sufficiently small the first eigenvalue λ_0 of the equilibrium solution $u \equiv \tilde{u}$ is*

$$\lambda_0 = g'(\tilde{u})$$

and all other eigenvalues are strictly negative. In particular, $u \equiv \tilde{u}$ is hyperbolic iff $g'(\tilde{u}) \neq 0$ and in this case the Morse index is 0 for $g'(\tilde{u}) < 0$ resp. 1 for $g'(\tilde{u}) > 0$.

Proof: For a homogenous equilibrium solution the eigenvalue equation is just a linear second-order boundary-value problem with constant coefficients:

$$\begin{array}{rcl} \varepsilon w_{xx} - f'(\tilde{u}) w_x + g'(\tilde{u}) w & = & \lambda w \\ w_x(0) = w_x(1) & = & 0. \end{array}$$

The solutions of this second-order equation can be found easily to be of the form

$$w(x) = c_1 e^{\mu_-(\varepsilon)x} + c_2 e^{\mu_+(\varepsilon)x}$$

where

$$\mu_\pm = \frac{f'(\tilde{u}) \pm \sqrt{f'(\tilde{u})^2 - 4\varepsilon(g'(\tilde{u}) - \lambda)}}{2\varepsilon}.$$

Thus, $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ are real if

$$f'(\tilde{u})^2 - 4\varepsilon(g'(\tilde{u}) - \lambda) > 0.$$

It is easy to check that for real $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ the boundary value problem can only have a solution if either $\mu_+(\varepsilon) = 0$ or $\mu_-(\varepsilon) = 0$. This in turn holds exactly for $\lambda = g'(\tilde{u})$. Other eigenvalues require $\mu_-(\varepsilon)$ and $\mu_+(\varepsilon)$ to be complex which is only possible for

$$\lambda < -\frac{f'(\tilde{u})^2}{4\varepsilon} + g'(\tilde{u}).$$

Thus, for all sufficiently small ε, $\lambda = g'(\tilde{u})$ has to be the first eigenvalue.

□

The next definition keeps track of the fact that our interest is not in any trajectory of (14), but mainly in those that correspond to solutions of the Neumann boundary value problem.

Definition 3.3 *An* **admissible trajectory** *of system (14) is a trajectory that corresponds to a solution of the boundary value problem, i.e. it is a finite piece of a trajectory $u(s)$ that satisfies $u'(0) = u'(1) = 0$.*

Let us now state a simple lemma that allows us to restrict our attention to a finite range of u.

Lemma 3.4 *Let f and g be as in theorem 1.1 and denote with u_{min} and u_{max} the minimal, resp. maximal zero of g. Then:*

(i) For any admissible trajectory $(u(s), v(s))$ of (14)

$$u_{min} \le u(s) \le u_{max} \quad \forall s \in [0,1].$$

(ii) There are constants C, ε_0 such that for $0 < \varepsilon \le \varepsilon_0$ along any admissible trajectory of (14)

$$u'(s) \le \frac{C}{\varepsilon} \qquad \forall s \in [0,1].$$

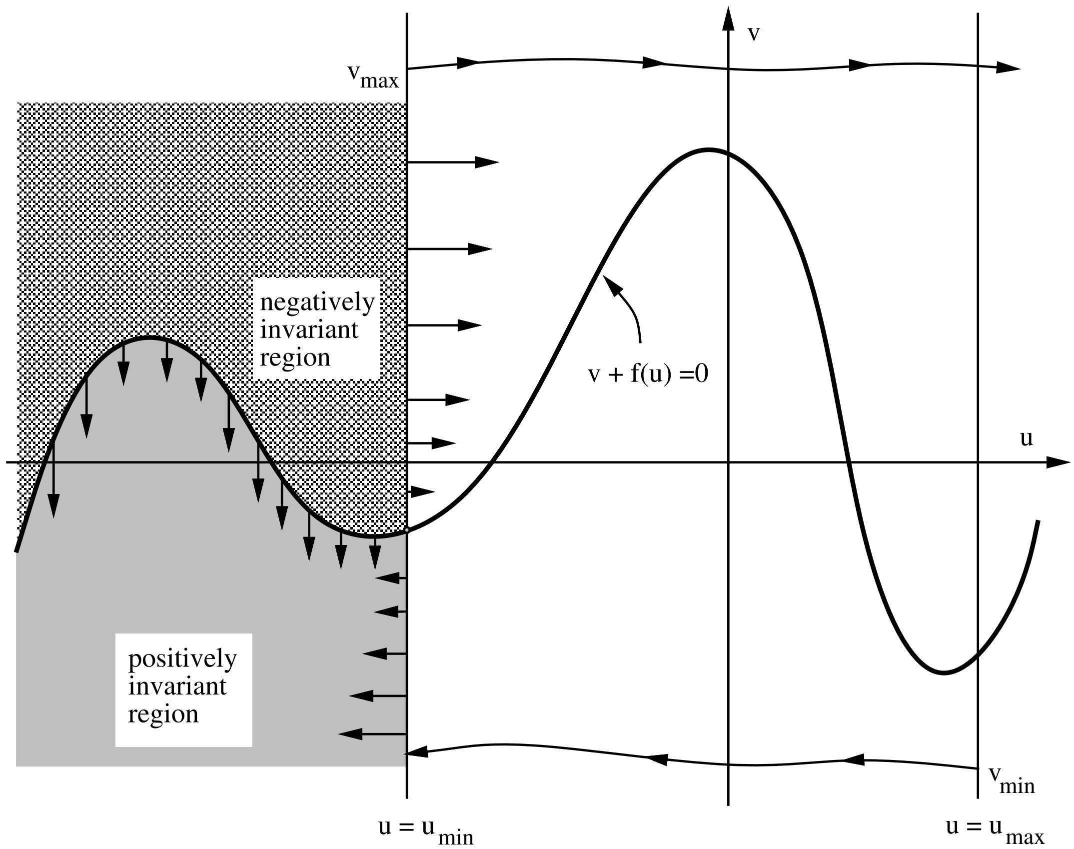

Figure 1: Admissible solutions are confined to a bounded region in the Liénard plane

Proof : To prove part (i) indirectly, suppose that for some $s \in [0,1]$ we have $u(s) < u_{min}$. The curve $v + f(u) = 0$ divides the region $\{(u,v); u < u_{min}\}$ into two parts. Since the vector field is vertical on the curve $\{v + f(u) = 0\}$ and horizontal on the line $u = u_{min}$ the region $\{(u,v);\ u < u_{min}, v < f(u)\}$ is positively invariant while $\{(u,v);\ u < u_{min}, v > f(u)\}$ is negatively invariant, see figure 1.

If $(u(s), v(s))$ lies in the positively invariant region for some s then the right boundary condition can obviously not be satisfied. If $(u(s), v(s))$ lies in the upper, negatively invariant region there is no chance of satisfying the left boundary condition. So a solution of the boundary value problem may not enter one of the two regions. It cannot stay on the curve $\{v + f(u) = 0\}$ either since by assumption u_{min} is the leftmost zero of g. Thus, there can be no solution of the boundary value problem that takes a value less than u_{min}. The argument excluding $u(s) > u_{max}$ is similar.

Claim (ii) obviously follows by showing that

$$\varepsilon u'(s) = v(s) + f(u(s)) \leq C.$$

To see this take

$$v_{max} > \sup\{-f(u)\ ;\ u_{min} \leq u \leq u_{max}\} + 1$$

and

$$v_{min} < \inf\{-f(u)\ ;\ u_{min} \leq u \leq u_{max}\} - 1.$$

For ε small enough the trajectory starting in (u_{min}, v_{max}) will be almost horizontal and intersects the line $u = u_{max}$ without having hit the singular curve before. Similarly, the trajectory starting in (u_{max}, v_{min}) stays below the singular curve until it hits the line $u = u_{min}$. Therefore, all admissible trajectories are for small ε confined to a bounded region of the (u, v)-plane.

□

The restriction of admissible trajectories to a bounded region also yields an upper bound on v':
Along any admissible trajectory we have

$$|v'(s)| \leq \max_{u_{min} \leq u \leq u_{max}} |g(u)| =: C_g.$$

There are also some restrictions on trajectories that are close to the curve $\mathcal{C}$:

Lemma 3.5 *Consider again the system (14) and assume that $f'(u) < 0$ ($f'(u) > 0$) for all $u \in [u_-, u_+]$.*
Then:

(i) For all sufficiently large k and all small ε trajectories can leave (enter) a region

$$\{(u, v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

only at $u = u_-$ or $u = u_+$.

(ii) If furthermore u_- and u_+ are two adjacent zeroes of g, then there is a positively invariant region between the curves $\mathcal{C}$ and $v + f(u) - k\varepsilon g(u) = 0$ for k large and all ε small (a negatively invariant region between $\mathcal{C}$ and a curve $v + f(u) + k\varepsilon g(u) = 0$).

(iii) In this case the two equilibria $(u_-, -f(u_-))$ and $(u_+, -f(u_+))$ on the singular curve are connected by a heteroclinic orbit.

Remark: The heteroclinic orbits of (iii) are part of the **slow manifold**, an invariant manifold that exists for $\varepsilon > 0$ near the singular curve $\mathcal{C}$ except in a neighborhood of the fold points, cf. [11].

Proof:

(i) For definiteness, we suppose that $f'(u) < 0$ for $u \in [u_-, u_+]$. The case $f'(u) > 0$ can be treated in the same way.

Along a curve $v + f(u) + k\varepsilon = 0$ the trajectories written as $v = v(u)$ satisfy

$$\left|\frac{dv(u)}{du}\right| = \left|\frac{g(u)}{k}\right|.$$

Since g is bounded on $[u_-, u_+]$, by choosing

$$k > \max_{u_- \leq u \leq u_+} \left|\frac{g(u)}{f'(u)}\right|$$

one can achieve that the vector field is directed into the region

$$\{(u,v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

along the whole curve so that trajectories can leave the region only via the left or the right boundary. Especially, trajectories that enter a neighborhood of a stable arc of $\mathcal{C}$ can leave such a neighborhood only near a fold.

(ii): To establish the invariant regions, one has to check that along curves $v + f(u) - k\varepsilon g(u) = 0$ trajectories satisfy

$$\left|\frac{dv}{du}\right| = \frac{1}{k}.$$

which is for k large enough and all small ε certainly smaller than the infimum

$$\inf_{u_- \leq u \leq u_+} |f'(u) - k\varepsilon g'(u)|$$

of the slope of the curve.

Then the existence of invariant regions is easily established by distinguishing the two cases $g > 0$ and $g < 0$ on (u_-, u_+). In both cases one finds a positively invariant region either above or below $\mathcal{C}$. It is easy to check that for $f' > 0$ there are negatively invariant regions.

(iii): A simple argument proves the heteroclinic connection between the two equilibria: Since the two equilibria are adjacent, one of them is a saddle and the other a sink. Consider the eigenvector to the unstable eigenvalue λ_2 of the saddle. A short computation shows that the (one-dimensional) invariant manifold corresponding to

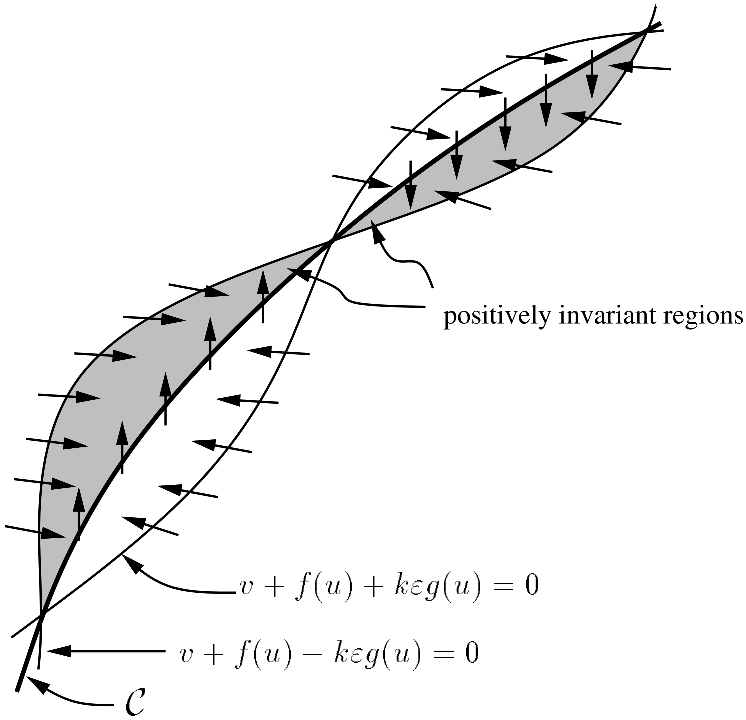

Figure 2: Invariant regions near the singular curve $\mathcal{C}$

λ_2 is directed into the invariant region if only k is chosen large enough. Since this region contains in its interior neither equilibria nor periodic orbits (u' has a definite sign there) the invariant manifold must connect to the sink lying on the boundary of the invariant region.

□

Figure 2 shows a situation with $f' < 0$ and two positively invariant regions. There are heteroclinic orbits from the upper and lower equilibrium to the one in between which are not shown in the figure.
Note also that admissible trajectories may not enter one of the regions enclosed by $\mathcal{C}$ and such a heteroclinic orbit because once inside such a region they cannot reach the curve $\mathcal{C}$ again in finite time to satisfy the boundary condition.
Now we state the lemma which is crucial for proving theorem 1.1 since it shows that for all small ε and any solution u of the boundary value problem (13) the number $z(u_x)$ will not exceed a certain bound.

Lemma 3.6 *Let $(u(s), v(s))$, $0 \leq s \leq 1$ be a nonconstant admissible trajectory. Then there exists a $\varepsilon_0 > 0$ and some $\delta > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the following holds: If $s_1 < s_2 < s_3$ are three different zeroes of u', then $s_3 - s_1 > \delta$.*

Proof: The proof concentrates on s_2 and shows that $s_3 - s_2 > \delta$ or $s_2 - s_1 > \delta$. We may without restriction assume, that there are no other zeroes of u' in the intervals (s_1, s_2) and (s_2, s_3). The arguments differ depending on whether $(u(s_2), v(s_2))$ lies near a fold or on a stable or unstable arc of $\mathcal{C}$. To this end the interval $[u_{min}, u_{max}]$ is divided into several parts. It was assumed that all zeroes of f' are simple, so they cannot accumulate and there is a finite number of points $\overline{u}_1 < \overline{u}_2 < \ldots < \overline{u}_F$ in $[u_{min}, u_{max}]$ with

$$f'(\overline{u}_i) = 0.$$

The case of $g(\overline{u}_i) < 0$ and $f''(\overline{u}_i) > 0$ is treated here in detail, but all other combinations of signs for $g(\overline{u}_i)$ and $f''(\overline{u}_i)$ lead to similar results.
It is possible to find a neighborhood $[\alpha_i, \beta_i]$ of $\overline{u}_i$ such that

$$f(\alpha_i) = f(\beta_i)$$

and both

$$g(u) > c \;\text{ and }\; f''(u) > c$$

hold for all u in the whole interval $[\alpha_i, \beta_i]$ and some $c > 0$.
The condition $g(u) > c$ implies that all trajectories that cross $\mathcal{C}$ between α_i and β_i will do this from above. It is easy to see that these trajectories can leave the region enclosed by $\mathcal{C}$ and the horizontal line $v = -f(\alpha_i)$ only through just this line.
Between α_i and β_i, one can find $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ with

$$\alpha_i < \tilde{\alpha}_i < \overline{u}_i < \tilde{\beta}_i < \beta_i,$$

$$f(\tilde{\alpha}_i) = f(\tilde{\beta}_i)$$

and

$$f(\tilde{\alpha}_i) - f(\alpha_i) = \frac{1}{2}(f(\overline{u}_i) - f(\alpha_i)),$$

compare figure 3.
Consider the case that an admissible trajectory hits the curve $\mathcal{C}$ between $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ at the time $s = s_2$. Since $|v'|$ was bounded by C_g independent of ε, it will take the trajectory a time of at least

$$\Delta s_i := \frac{f(\tilde{\alpha}_i) - f(\alpha_i)}{C_g}$$

to leave the region enclosed by $\mathcal{C}$ and the line $v = -f(\alpha_i)$. During this time it cannot hit the singular curve again. Thus, if s_2 lies in the interval $[\tilde{\alpha}_i, \tilde{\beta}_i]$, we have

$$s_3 - s_2 \geq \Delta s_i.$$

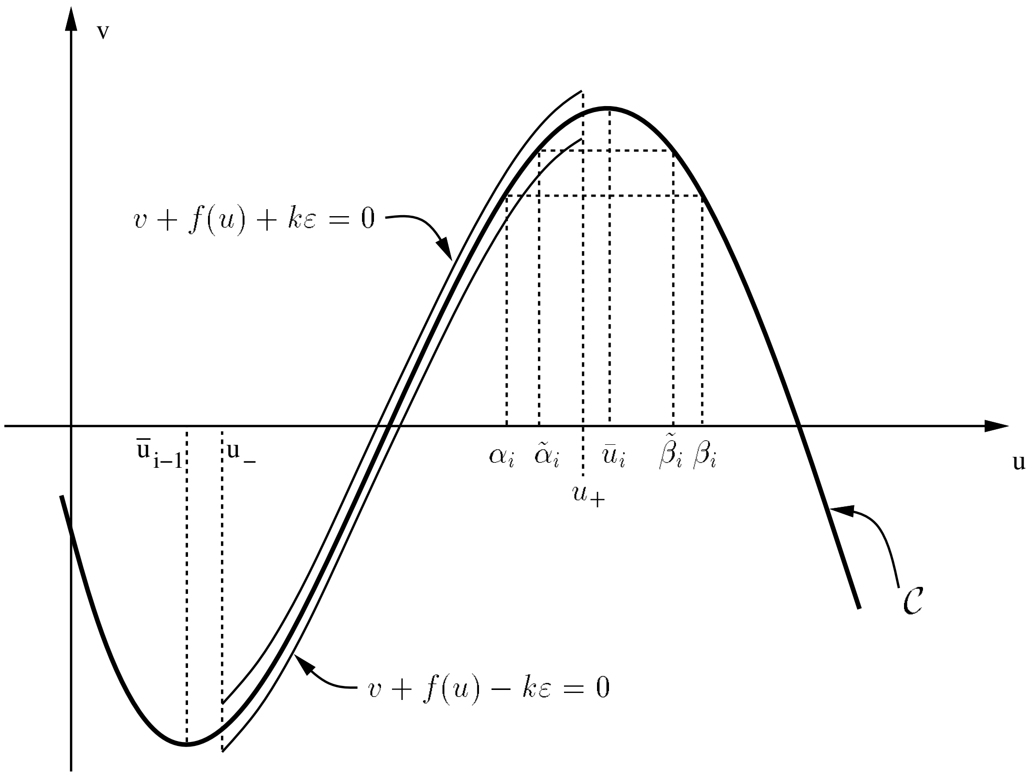

Figure 3

The same situation appears if $g(\overline{u}_i) > 0$ and $f''(\overline{u}_i) < 0$. For the other two cases where $g(\overline{u}_i) > 0$ and $f''(\overline{u}_i) > 0$ have the same sign a similar reasoning shows that a trajectory that hits $\mathcal{C}$ near the fold cannot have hit it for a certain time before and hence

$$s_2 - s_1 \geq \Delta s_i.$$

Thus, if s_2 lies in some interval $[\tilde{\alpha}_i, \tilde{\beta}_i]$ the lemma is proved by chosing δ smaller than the infimum of the Δs_i.
If the trajectory intersects the singular curve on a stable arc in some interval $[\tilde{\beta}_{i-1}, \tilde{\alpha}_i]$ and ε is sufficiently small then the trajectory is in one of the strips

$$\{(u,v); u_- \leq u \leq u_+, -k\varepsilon \leq v + f(u) \leq k\varepsilon\}$$

of lemma 3.5 that can only be left at their left or right boundary.
Two things can be shown:
1) A trajectory cannot intersect the singular curve again before leaving the strip.
2) The trajectory stays inside the strip for at least the time $\Delta s_i/2$.
To prove the first claim, remember that an admissible trajectory may not enter one of the regions enclosed by $\mathcal{C}$ and a heteroclinic orbit connecting two adjacent equilibria on the same (stable or unstable) arc of $\mathcal{C}$. Therefore, an admissible trajectory can

cross the curve $\mathcal{C}$ only between a fold and the nearest equilibria on either side. By crossing $\mathcal{C}$ there, the trajectory enters a region enclosed by $\mathcal{C}$ and a curve $v + f(u) + k\varepsilon g(u) = 0$ with large $|k|$. As was shown in lemma 3.5(ii), this region can be left only at $u = u_-$ or $u = u_+$.

The second claim, concerning the time it takes a trajectory to leave the invariant strip, is proved here for the case $f'(u_+) < 0$ and $g(u_+) < 0$, i.e. of a trajectory that follows a stable arc of $\mathcal{C}$ to its right end. However, all other cases can be treated in the same way changing signs appropriately and reversing time if necessary. The idea is simple again: By choosing ε small enough, the trajectory has to cover a certain v-distance near $\mathcal{C}$ and since the velocity in v-direction is bounded by C_g this will take a certain amount of time. More precisely, if u_- is chosen close to $\overline{u}_{i-1}$ and u_+ close to $\overline{u}_i$, and furthermore ε is sufficiently small, then

$$-f(u_+) - k\varepsilon + f(\alpha_i) \geq \frac{1}{4}\left(-f(\overline{u}_i) + f(\alpha_i)\right).$$

If a trajectory intersects the singular curve on the stable arc below $v = -f(\alpha_i)$ and leaves the invariant strip at $u = u_+$ the v-coordinate has to increase by at least

$$\frac{1}{4}\left(-f(\overline{u}_i) + f(\alpha_i)\right).$$

This implies that a trajectory needs at least the time $\Delta s_i/2$ from a horizontal section $v = -f(\alpha_i)$ to the point where it leaves the invariant strip.

So, in the case that $(u(s_2), v(s_2))$ lies on a stable arc of $\mathcal{C}$ we have shown that

$$s_3 - s_2 \geq \inf_i \Delta s_i/2$$

independent of ε. The case that $(u(s_2), v(s_2))$ lies on an unstable arc is similar and leads to the result that

$$s_2 - s_1 \geq \inf_i \Delta s_i/2.$$

Choosing δ smaller than the infimum of the $\Delta s_i/2$ completes the proof of lemma 3.6.

□

We are now able to finish the proof of theorem 1.1. It was shown that for any non-homogenous equilibrium solution u of the viscous balance law the zero number of u_x can be estimated by

$$z(u_x) \leq \frac{2}{\delta} + 1,$$

where δ is the constant from the previous lemma 3.6.
This ensures that the dimension of the center-unstable manifold of any non-homogenous equilibrium solution of the viscous balance law does not exceed a certain ε-independent bound $\frac{2}{\delta}+2$. The spatially homogenous equilibria have been shown in lemma 3.2 to have an at most one-dimensional unstable resp. center manifold.
Assume first that for a given ε there is finite number of equilibria which are all hyperbolic. Then the global attractor is the union of the unstable manifolds of these equilibria. Each of these manifolds has a dimension not exceeding $\frac{2}{\delta}+2$ so this gives an upper bound on the dimension of $\mathcal{A}_\varepsilon$ as well and the theorem is proved for this case.
In the other cases with non-hyperbolic and possibly infinitely many equilibria the arguments have to be refined.
The set of equilibria is a closed subset of $\mathcal{A}_\varepsilon$ in the space X where the semiflow is defined and hence a compact set. Consider a local center-unstable manifold $W^{cu}_{loc}(u)$ of an equilibrium u. Despite of non-uniqueness, it contains all the solutions from some neighborhood $\mathcal{N}(u)$ that are bounded backward in time, especially all the equilibria contained in $\mathcal{N}(u)$. Due to compactness of the set of equilibria a finite union of local center-unstable manifolds

$$\bigcup_{i=1}^{k} W^{cu}_{loc}(u_i)$$

suffices to cover all equilibria. We claim that the global attractor is contained in the set

$$\mathcal{W} := \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{k} T(n, W^{cu}_{loc}(u_i))$$

where $T(n,\cdot)$ is the time-n-map of the semiflow. The dimension of $\mathcal{W}$ is not greater than the maximal dimension of the center-unstable manifolds since applying the semiflow to a set over a finite time does not increase the dimension. Recall that the global attractor consists of equilibria and heteroclinic orbits only, so it remains only to show that every heteroclinic orbit $h(t)$ is contained in $\mathcal{W}$. The $\alpha-$limit set of h is an equilibrium $h_{-\infty}$ and there is a i_0 such that $h_{-\infty}$ lies in $W^{cu}_{loc}(u_{i_0})$. Moreover, there exists a time $t_- < 0$ such that $h(t)$ lies in $\mathcal{N}(u_{i_0})$ for all times $t < t_-$. Therefore, $h(t) \in W^{cu}_{loc}(u_{i_0})$ for $t < t_-$. So it is clear that the whole heteroclinic orbit h is contained in

$$\bigcup_{n=1}^{\infty} T(n, W^{cu}_{loc}(u_{i_0}))$$

finishing thereby the proof of theorem 1.1.

□

4 Discussion

We have demonstrated that the dimension of the global attractor remains bounded as ε tends to 0. By a more delicate analysis of the equilibrium solutions, it can be shown that the dimension stabilizes in the following sense: There is an integer d such that the dimension of $\mathcal{A}_\varepsilon$ is either d or $d+1$ if ε is sufficiently small. This will be shown in a subsequent article [20]. It seems in fact that the dimension tends to a limit in most cases, although we are not able to prove this by now. A question we have not adressed in the present article is the number of equilibrium solutions. It is not yet clear under which assumptions not only the dimension of $\mathcal{A}_\varepsilon$ but also the number of equilibria tends to a limit. It has been shown that in the rather simple case of convex f the number of equilibria tends to a limit and moreover the attractors $\mathcal{A}_\varepsilon$ are C^0-equivalent for all small ε, see [18, 19]. For a definition of C^0-equivalence we refer to [14].

Another issue is the limiting hyperbolic case $\varepsilon = 0$. Unfortunately, our results do not have straight-forward implications to the hyperbolic case, since it is not clear how to perform the singular limit $\varepsilon \searrow 0$. The hyperbolic equation might not be well defined on the unit interval (characteristics may enter from the boundary) and there is no direct equivalent of Neumann boundary conditions in the hyperbolic case. However, it can easily be seen that our results do carry over to other boundary conditions. So, there is some hope at least that finite-dimensionality of the attractor holds for some hyperbolic initial-boundary value problems if they are well-defined on the unit interval.

References

[1] J. D. Allen and R. E. O'Malley. Singularly perturbed boundary value problems viewed in the Liénard plane. In *Asymptotic and computational analysis, Winnipeg 1989*, volume **124** of *Lect. Notes Pure Appl. Math.*, pages 357–378. Dekker, New York, 1990.

[2] H. Amann. Global existence for semilinear parabolic systems. *J. Reine Angew. Math.*, **360**:47–83, 1985.

[3] S. Angenent. The zero set of a solution of a parabolic equation. *J. Reine Angew.Math.*, **390**:79–96, 1988.

[4] S. Angenent and B. Fiedler. The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. AMS*, **307**:545–568, 1988.

[5] S. Angenent, J. Mallet-Paret, and L. A. Peletier. Stable transition layers in a semilinear boundary value problem. *J. Diff. Eq.*, **67**:212–242, 1987.

[6] L. L. Bonilla and F. J. Higuera. The onset and end of the Gunn effect in extrinsic semiconductors. *SIAM J. Appl. Math.*, **55**:1625–1649, 1995.

[7] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported*, **1**:57–89, 1988.

[8] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Diff. Eq.*, **81**:106–135, 1989.

[9] N. Chafee and E. Infante. A bifurcation problem for a nonlinear parabolic equation. *J. Applicable Analysis*, **4**:17–37, 1974.

[10] H. Fan and J. K. Hale. Attractors in inhomogeneous conservation laws and parabolic regularizations. *Trans. Amer. Math. Soc.*, 1995.

[11] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J.Diff.Eq.*, **31**:53–98, 1979.

[12] B. Fiedler and J. Mallet-Paret. A Poincaré-Bendixson Theorem for Scalar Reaction Diffusion Equations. *Arch. Rational Mech. Anal.*, **107**:325–345, 1989.

[13] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. *J. Diff. Eq.*, **125**:239–281, 1996.

[14] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. Preprint, 1996.

[15] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar pde. *J.Diff.Eq.*, **91**:111–137, 1991.

[16] J. K. Hale. *Asymptotic behavior of dissipative systems*, volume **25** of *Mathematical surveys and monographs.* AMS, 1988.

[17] J. K. Hale and K. Sakamoto. Existence and stability of transition layers. *Japan J. Appl. Math.*, **5**:367–405, 1988.

[18] J. Härterich. Attractors of viscous balance laws. Ph. D. thesis, Freie Universität Berlin, 1996.

[19] J. Härterich. Equilibrium solutions of viscous balance laws with a convex flux. Preprint, Freie Universität Berlin, in preparation, 1997.

[20] J. Härterich. Singularly perturbed boundary value problems and the dimension of global attractors. Preprint, Freie Universität Berlin, in preparation, 1997.

[21] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume **840** of *Lect. notes Math.* Springer-Verlag, New York, 1981.

[22] D. Henry. Some infinite dimensional Morse-Smale systems defined by parabolic differential equations. *J.Diff.Eq.*, **59**:165–205, 1985.

[23] S. N. Kruzhkov. First order quasilinear equations in several independent variables. *Math. USSR-Sbornik*, **10**:217–243, 1970.

[24] A. N. Lyberopoulos. A Poincaré-Bendixson theorem for scalar balance laws. *Proc. Roy. Soc. Edinburgh*, **124A**:589–607, 1994.

[25] H. Matano. Convergence of solutions of one-dimensional semilinear parabolic equations. *J. Math. Kyoto Univ.*, **18**:221–227, 1978.

[26] L. B. Reyna and M. J. Ward. On the exponentially slow motion of a viscous shock. *Comm. Pure Appl. Math*, **48**:79–120, 1995.

[27] C. Sinestrari. Large time behaviour of solutions of balance laws with periodic initial data. *Nonlin. Diff. Equ. Appl.*, **2**:111–131, 1995.

[28] T. J. Zelenyak. Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. *Differential equations*, **4**:17–22, 1968.