Admissibility of traveling waves for scalar balance laws

Jörg Härterich, Freie Universität Berlin
Arnimallee 2-6, D-14195 Berlin, Germany

We study traveling wave solutions of scalar hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R} \tag{1}$$

and their viscous counterpart

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R} \tag{2}$$

where the viscosity ε is assumed to be small. We assume the following about f and g:

(F) f is convex: $f \in C^3$, $f''(u) > 0$

(G) $g \in C^2$ with simple zeroes

and look for *entropy traveling waves* of (1).

Def. 1 *An* **entropy traveling wave** *is a solution of (1) of the form $u(x,t) = u(\xi)$ where $\xi := x - st$ for some wave speed $s \in \mathbb{R}$ with the following properties:*

(i) u is piecewise C^1, i.e. $u \in C^1(\mathbb{R} \setminus J)$ with a set J that has only isolated accumulation points. At points where u is continuously differentiable it satisfies the ordinary differential equation

$$(f'(u(\xi)) - s)\ u'(\xi) = g(u(\xi)). \tag{3}$$

(ii) At points of discontinuity the one-sided limits $u(\xi+)$ and $u(\xi-)$ of u satisfy both the Rankine-Hugoniot condition and the entropy condition $u(\xi+) \leq u(\xi-)$.

A classification of all traveling waves has been given by Mascia [**?**] and can also be found in [**?**]. An interesting question is whether all traveling waves of the hyperbolic balance law can be obtained as the limit of traveling waves of the viscous balance law in the following sense.

Def. 2 *An entropy traveling wave u_0 of (1) with wave speed s_0 is called* **admissible**, *if there is a sequence (u^{ε_n}) of traveling wave solutions of (2) s. t. $\varepsilon_n \searrow 0$, $s_n \to s_0$ and $\|u^{\varepsilon_n} - u_0\|_{L^1(\mathbb{R})} \to 0$.*

The results on admissibility can be summarized as:

Thm. 3 *Most types of entropy traveling waves are admissible. However, there exist classes of entropy traveling waves which are not admissible.*

Proofs can befound in [**?**] for the cases where classical singular perturbation theory, e.g. [**?**] applies, and in [**?**] where blow-up techniques as in [**?**, **?**] are used. [**?**] contains also two cases of entropy traveling waves for which no viscous profile exists.
Here we prove existence of a viscous profile for one particular class of waves: Let u_1, u_2, u_3 be three consecutive zeroes of g with $g'(u_1) < 0$, $g'(u_2) > 0$ and $g'(u_3) < 0$. Then there is a unique differentiable entropy traveling wave $u_0(\xi)$ with monotone increasing profile and wave speed $s = f'(u_2)$. Note that both sides of (3) vanish simultanously when $u(\xi) = u_2$, but $\lim_{u \to u_2} \frac{g(u)}{f'(u) - s} = \frac{g'(u_2)}{f''(u_2)}$ exists.

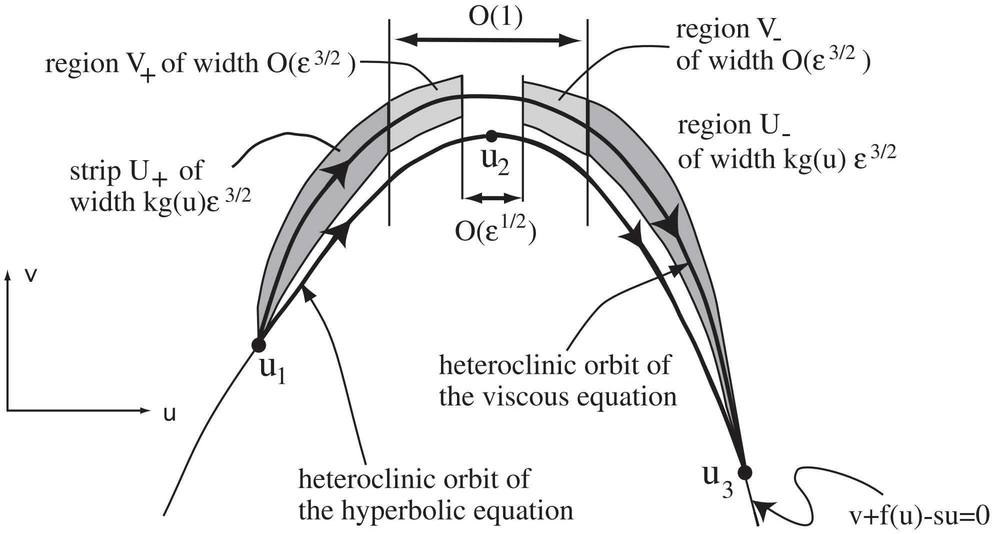

Figure 1: The phase portrait for small ε and wave speed $s = s(\varepsilon)$

Thm. 4 *This type of entropy traveling wave admits a viscous profile.*

Writing the second order equation one obtains from (2) with the traveling wave ansatz $x - st =: \xi$ as a first order system gives

$$\left.\begin{array}{rcl} \varepsilon u' &=& v + f(u) - su \\ v' &=& -g(u). \end{array}\right\} \tag{4}$$

We collect some properties of this system:

Lemma 5 *Given d, $\delta > 0$ small, there exists some k, $\varepsilon_1 > 0$, such that for all $0 < \varepsilon \leq \varepsilon_1$ and all $|s - f'(u_2)|$ sufficiently small all trajectories of (4) leave the regions*

$$\begin{array}{rcl} U_+ &:=& \left\{(u,v)\ ;\ u_1 \leq u \leq u_2 - d,\ \left|v + f(u) - su + \varepsilon\dfrac{g(u)}{f'(u) - s}\right| \leq k\varepsilon^{3/2}|g(u)|\right\} \text{ and} \\ V_+ &:=& \left\{(u,v)\ ;\ u_2 - d \leq u \leq u_2 - \delta\sqrt{\varepsilon},\ \left|v + f(u) - su + \varepsilon\dfrac{g(u)}{f'(u) - s}\right| \leq k|g(u_2 - d)|\varepsilon^{3/2}\right\} \end{array}$$

on the right side. Moreover, a branch of the unstable manifold of u_1 passes through U_+ and V_+. Similar regions U_- and V_- exist to the right of u_2, see figure 1.

Proof: One needs to compare the slope of the upper and lower boundary of U_+, V_+ with the vector field on these boundaries and with the slope of the unstable manifold at u_1. See [**?**] for details of the calculation. □

Lemma 6 *There exists a unique wave speed $s = s(\varepsilon)$ with*

$$s(\varepsilon) = f'(u_2) + \frac{f'''(u_2)g'(u_2) - g''(u_2)f''(u_2)}{2f''(u_2)^2}\varepsilon + h.o.t.$$

such that a heteroclinic connection $u_\varepsilon(\xi)$ of (4) from u_1 to u_3 exists.

Proof: Existence and uniqueness can be shown by the method of rotated vector fields [**?**]. For the asymptotics the blow-up method as in [**?**, **?**] is used. Details of the calculation can be found in [**?**]. □

We parametrize the heteroclinic orbits such that $u_0(0) = u_\varepsilon(0) = u_2$. Let $\eta_\pm = \eta_\pm(\varepsilon)$ be such that $u_\varepsilon(\eta_\pm) = u_2 \pm \delta\sqrt{\varepsilon}$.

Lemma 7 *$u_\varepsilon'(\xi) \geq c$ for $\xi \in [\eta_-, \eta_+]$ with a constant $c > 0$ independent of ε.*

Proof: This follows from the fact that the heteroclinic orbit leaves the strip V_+ at a height $v(\eta_-) \geq -f(u) + s(\varepsilon)u + \frac{\varepsilon g(u)}{(f'(u)-s)} - k\varepsilon^{3/2} = -\frac{\delta}{2}f''(u_2)\varepsilon + \frac{\varepsilon g'(u_2)}{f''(u_2)} + \mathcal{O}(\varepsilon^{3/2})$ which is bigger than $c\varepsilon$ if δ is chosen sufficiently small. Similarly, $v(\eta_+) \geq c\varepsilon$. Moreover, $v(\xi) + f(u(\xi)) - su(\xi) \geq v(\xi) \geq min\{v(\eta_-), v(\eta_+)\}$ for $\xi \in [\eta_-, \eta_+]$. So, on this small part of the heteroclinic orbit we have $u' = \frac{1}{\varepsilon}(v + f(u) - s(\varepsilon)u) \geq c$. □

Proof of theorem 4: Let $n \in \mathbb{N}$ be given. We want to show that $\|u_0 - u_\varepsilon\|_{L^1(\mathbb{R})} \leq \frac{1}{n}$ for ε small. Since the heteroclinic orbits of both the hyperbolic and the viscous traveling wave converge to u_1 resp. u_3 exponentially fast as $\xi \to \pm\infty$, we can find $\xi_\pm$ such that

$$\int_{-\infty}^{\xi_-} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n} \quad \text{and} \quad \int_{\xi_+}^{+\infty} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n}.$$

By choosing ε small, we have due to lemma 7

$$\int_{\eta_-}^{\eta_+} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \int_{\eta_-}^{\eta_+} |u_\varepsilon(\xi)| + u_0(\xi)|\, d\xi \leq \frac{1}{5n}.$$

Also, $u_0(\eta_\pm)$ and $u_\varepsilon(\eta_\pm)$ are $\mathcal{O}(\sqrt{\varepsilon})$-close. From the fact that the heteroclinic solution passes through the invariant region U_+ and V_+ we know that u_0' and u_ε' are also $\mathcal{O}(\sqrt{\varepsilon})$-close on $[\xi_-, \eta_-]$ and on $[\eta_+, \xi_+]$. The remaining estimates

$$\int_{\xi_-}^{\eta_-} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n} \text{ and } \int_{\eta_+}^{\xi_+} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n}$$

follow then by a use of the Gronwall inequality. □

References

[Duf53] G. F. D. Duff. Limit cycles and rotated vector fields. *Ann. Math.*, **57**:15–31, 1953.

[DR96] F. Dumortier and R. Roussarie. *Canard cycles and center manifolds*, volume **121** of *AMS Memoirs.* 1996.

[Fen79] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J.Diff.Eq.*, **31**:53–98, 1979.

[Här99a] J. Härterich. Viscous profiles for traveling waves of scalar balance laws: The canard case. Preprint, 1999.

[Här99b] J. Härterich. Viscous profiles for traveling waves of scalar balance laws: The uniformly hyperbolic case. Preprint, 1999.

[Mas97] C. Mascia. Travelling wave solutions for a balance law. *Proc. Roy. Soc. Edinburgh*, **127 A**:567–593, 1997.

[MS98] A. Milik and P. Szmolyan. Multiple time scales and canards in a chemical oscillator. IMA Preprint 1558, 1998.