DISSERTATION

Singularly Perturbed Laser Equations - Slow-Fast Dynamics in the Yamada Model

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Chapter 1

Introduction

Lasers (Light Amplification by Stimulated Emission of Radiation), since their first experimental verification 1960 by Maiman, have undergone an astonishing development. Nowadays they play an essential role in many fields of technology, ranging from communication and information technology to measurement engineering and medicine. One reason for their great usefulness is the rich dynamical behaviour of the various types of lasers.

The most basic lasers consist of a cavity filled with an amplifying medium, two mirrors at the ends, one of which is semipermeable, and energy is applied by external pumping. A laser is designed such that only light of certain wavelengths and energies is amplified.

Adding other features such as different materials, a saturable absorber, feedback, optical injection or coupling of two lasers leads to lasers which show rich dynamical properties such as various types of oscillatory and periodic behaviour, pulse-trains of extremely short pulses of high intensity, and chaotic behaviour. For a good introduction to the physics of lasers and the derivation of the different laser equations see [21] whereas the books [24],[37] give an overview of the dynamics of laser.

In the past decade dynamical system approaches to the analysis of lasers have been quite successful [37], [24]. One characteristic is that by changing the material or - even more simple - by solely changing the pump power, the qualitative behaviour of the laser beam can change dramatically [15],[25],[31],[33]. The occurrence of vastly different time scales makes numerical simulations of lasers challenging and time consuming. On the other hand it allows to use perturbation methods to get simple approximate equations which are easier to analyze [11],[12],[13],[14].

Many systems which show dynamics on different time scales can be transformed to a singularly perturbed system in standard form

$$\begin{aligned} x' &= \varepsilon f(x,y,\varepsilon) \\ y' &= g(x,y,\varepsilon). \end{aligned} \tag{1.1}$$

The variable $x \in \mathbb{R}^n$ is the slow variable, the variable $y \in \mathbb{R}^m$ is the fast variable, f, g are of class C^k and $0 < \varepsilon \ll 1$. By transforming time to the slow time scale

$\tau = \varepsilon t$ we get the equivalent system

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y, \varepsilon) \\ \varepsilon \dot{y} &= g(x, y, \varepsilon). \end{aligned} \tag{1.2}$$

Setting $\varepsilon = 0$ in both systems we get the two limiting systems, the layer problem

$$\begin{aligned} x' &= 0 \\ y' &= g(x, y, 0) \end{aligned} \tag{1.3}$$

and the reduced problem

$$\begin{aligned} \dot{x} &= f(x, y, 0) \\ 0 &= g(x, y, 0). \end{aligned} \tag{1.4}$$

The manifold of equilibria of the layer problem is the critical manifold S_0. A normally hyperbolic part S_H of S_0 consists of a connected compact subset of S_0 where $\frac{\partial g}{\partial y}|_{S_0}$ has no eigenvalues on the imaginary axes. There we can solve for $y = \varphi_0(x)$ to obtain the dynamics on the critical manifold

$$\dot{x} = f(x, \varphi_0(x), 0). \tag{1.5}$$

For $\varepsilon > 0$ the manifold S_H perturbs to a locally invariant manifold of the full problem, the slow manifold S_ε. This geometric theory goes back to the pioneering work of N.Fenichel [17].

The restriction of the flow of (1.2) to S_ε is as small smooth perturbation of the flow of the reduced problem (1.4). In many applications the dynamics of the limiting problems (1.3) and (1.4) can be analyzed in sufficient detail to understand the existence and bifurcations of global singular structures, such as singular periodic and heteroclinic orbits. Under certain hyperbolicity and transversality conditions, this allows one to describe the persistence of these global structures for the full system. For a good introduction to geometric singular perturbation theory, see [22].

For normally hyperbolic critical manifolds this theory is fairly complete, but it breaks down near points where normal hyperbolicity is lost. This corresponds to equilibria where $\frac{\partial g}{\partial y}|_{S_0}$ has eigenvalues on the imaginary axes [27],[30], to points of self-intersection [28] of the slow manifold, or S_0 becoming unbounded. In the first two cases it is often possible to use the blow-up transformation, a clever coordinate transformation, to desingularize the vectorfield [9],[27]. With this transformation it is possible obtain some hyperbolicity to use methods from geometric theory, such as center manifold and normal form theory to describe the dynamics.

In this work we deal with laser equations which show a combination of the dynamical properties described above. In Chapter 2 which constitutes the main part of this thesis we analyze the Yamada equations [15],[11], a particularly interesting model for lasers with saturable absorber. The governing equations are

$$\begin{aligned} \dot{I} &= (G - Q - 1)I \\ \dot{G} &= \varepsilon(A - G - GI) \\ \dot{Q} &= \varepsilon(B - Q - aQI). \end{aligned} \tag{1.6}$$

They correspond to lasers of type B with an additional equation for the saturable absorber. Here I describes the intensity of the electromagnetic field, G and Q denote the amplifying and the absorbing medium, and we have the parameters A, B, a and ε, where A corresponds to the pump power and $0 < \varepsilon \ll 1$ describe the dynamics on the different time scales. In [15] an interesting bifurcation analysis with respect to the parameters a, B, A, ε has been carried out. However, the most relevant case of very small values of ε has been investigated only numerically. The singular limit $\varepsilon \to 0$ was not considered in [15]. It is the goal of this work to fill in some analytical details on the dynamics of system (1.6) for small values of ε.

Lasers with saturable absorber show pulse-trains of very short pulses with high intensity and high frequency as characteristic output. This behaviour corresponds to a family of periodic orbits of the Yamada model which start in a homoclinic bifurcation.

It is known that a curve of transcritical bifurcation points exists in $A = A_T(\varepsilon) = B$ for constant values $a > 1, B > \frac{1}{a-1}$. From numerical calculations it is known that a curve of homoclinic bifurcations in $A = A_L(\varepsilon)$ exists nearby. These results show that the two curves are extremely close for ε small and the local bifurcations seem to meet at the point $(A, \varepsilon) = (B, 0)$ in the singular limit $\varepsilon \to 0$.

In this work we give a detailed geometric analysis of the complicated global dynamics and local bifurcations near the point $(A, \varepsilon) = (B, 0)$. We show analytically that the curve of homoclinic orbits exists and that the two curves $A_L(\varepsilon)$ and $A_T(\varepsilon)$ indeed intersect for $\varepsilon = 0$. Further we show that the curves are exponentially close for $\varepsilon \to 0$, that is $|A_L(\varepsilon) - A_T(\varepsilon)| = O(e^{-c/\varepsilon})$.

For the proof we transform to $J = \varepsilon I$ because this is the natural scale for the intensity I. We get a singularly perturbed differential equation (not in standard form) where the critical manifold is given by $S_0 = \{J = 0\}$. Away from the non-hyperbolic line $G - Q - 1 = 0$ we can use regular and geometric singular perturbation theory. On this line the transcritical bifurcation takes place. The equilibrium of the homoclinic orbit lies nearby and for $\varepsilon = 0$ exactly on the non-hyperbolic line. We use a blow-up transformation of the non-hyperbolic line to describe the dynamics there. In the blown-up problem we find a regular transcritical bifurcation at $\varepsilon = 0$ and a singular homoclinic orbit at $\varepsilon = 0$. To describe the homoclinic orbit near $G - Q - 1 = 0$ for $\varepsilon > 0$ we have to use methods from center manifold theory, invariant foliations, normal form theory, Fenichel coordinates, and Gronwall type of estimates.

In Appendix A we collect some results of center manifold theory and geometric singular perturbation theory we need throughout this thesis.

Appendix B is about a method, called the adiabatic elimination, which is frequently used in the physics literature. This method is used to reduce the number of equations by eliminating equations for the (fast) stable dynamics [19],[20],[37]. We briefly recall the classification of lasers into lasers of type A, B, and C according to their slow-fast structure. We show that under certain conditions the adiabatic elimination corresponds to a reduction of the dynamics to a center manifold or to a slow manifold. For the semiclassical laser equations of type A and B - which show dynamics on different timescales - this method leads to the simpler rate equations [29]. In this case the adiabatic elimination corresponds to a reduction to a slow manifold, and we give a geometric interpretation of the dynamics on this slow manifold.

Chapter 2

Lasers with saturable absorber

A particulary interesting class of lasers which show self-pulsations are lasers with a saturable absorber. The characteristic behaviour of the laser beam are pulse trains of short pulses of high intensity with a frequency of several gigahertz. Whereas the laser is driven by a constant pumping power, self-pulsations is a result of the nonlinear interaction of the slowly responding amplifying and absorbing medium and the fast response of the electric field.

When the laser is turned on, the amplifying medium, the gain, is excited through some pumping process as usual. But now the absorber absorbs the free photons in the laser and so the electric field intensity stays low and the saturation of the gain continues. When the absorbing medium saturates, the ordinary laser process starts with a strongly excited gain which leads to a high intensity of the electric field and therefore a greatly enhanced output power. During this process the gain and the absorber turn to ground state again and the process starts anew.

Self-pulsation was observed in all types of lasers. We concentrate on semiconductor lasers, which show self-pulsation in a reproducible way. Applications are possible in telecommunication systems, the reduction of optical feedback noise and for optical timing extraction. (For references and a longer explanation, see [15]).

An interesting model for lasers with saturable absorber is the Yamada model ([15]):

$$
\begin{aligned}
\dot{I} &= (G - Q - 1)I \\
\dot{G} &= \varepsilon(A - G - GI) \\
\dot{Q} &= \varepsilon(B - Q - aQI).
\end{aligned}
\tag{2.1}
$$

This system corresponds to a model of lasers of type B with an additional equation for the saturable absorber (see Appendix B.4) and consits of normalized equations for the laser intensity I, the gain G and the absorber Q. A, B, a, ε are parameters. A describes the pump current, B and a are fixed in the experiment and specify the relative absorption. For physical reasons we restrict the parameter space to $A > 0$, $B > 0$ and $a \geq 1$. ε is positive and describes the different time scales of the dynamics of the field intensity and the media in the cavity and is very small, typically in the order of $10^{-3} - 10^{-4}$.

Solutions of these equations show a slow drift along $I \approx 0$ and large jumps from higher values of G and Q to lower values. The large pulses have maximum amplitudes of order $O(\frac{1}{\varepsilon})$. Therefore we make a rescaling $I = \frac{J}{\varepsilon}$ to capture these pulses. We get the equations

$$\begin{aligned} J' &= (G - Q - 1)J \\ G' &= -GJ + \varepsilon(A - G) \\ Q' &= -aQJ + \varepsilon(B - Q), \end{aligned} \tag{2.2}$$

which will be the starting point of our analysis.

2.1 Geometric singular perturbation analysis

Setting $\varepsilon = 0$ in (2.2) we get the layer problem

$$\begin{aligned} J' &= (G - Q - 1)J \\ G' &= -GJ \\ Q' &= -aQJ. \end{aligned} \tag{2.3}$$

The equilibria of these equations are given by the 2−dimensional critical manifold $S_0 := \{J = 0\}$. The set of stable equilibria S_0^s of S_0, with $G - Q - 1 < 0$, and the set of unstable equilibria S_0^u, with $G - Q - 1 > 0$, are separated by the line $G - Q - 1 = 0$ of non hyperbolic equilibria. For $J \neq 0$ and $G \neq 0$ we can change to G as independent variable

$$\begin{aligned} \frac{d}{dG}J &= \frac{\frac{d}{dt}J}{\frac{dG}{dt}} = (\frac{1}{G} + \frac{Q}{G} - 1) \\ \frac{d}{dG}Q &= \frac{\frac{d}{dt}Q}{\frac{dG}{dt}} = a\frac{Q}{G} \end{aligned} \tag{2.4}$$

where increasing t corresponds to decreasing G because of $\frac{dG}{dt} < 0$. Integrating these equations with initial conditions $G = G_{out}, Q = Q_{out}, J = J_{out}$ leads to the orbits of the layer problem

$$\begin{aligned} Q(G) \quad &= \frac{Q_{out}}{G_{out}^a}G^a \\ J(G, Q) \quad &= ln(G) + \frac{1}{a}Q - G + J_{out} - ln(G_{out}) - \frac{1}{a}Q_{out} + G_{out}. \end{aligned} \tag{2.5}$$

We see from (2.4) that the orbits $Q(G)$ are independent from J. Figure 2.1 shows different solutions for G_{out}, Q_{out} fixed, $a > 1$ and different values of J_{out}. For $a > 1$ $Q(G)$ describes parabolas through $(G, Q) = (0, 0)$ and $(G, Q) = (G_{out}, Q_{out})$ (see Fig. 2.1(a)). After plugging $Q = Q(G)$ into the second equation of (2.5) we see that $J \to -\infty$ for $G \to 0$ (see Fig. 2.1(b)). For $a > 1$ $J \to \infty$ for $G \to \infty$ and J has a maximum and a minimum in between where the solutions satisfy $G - Q - 1 = 0$. (For $a = 1$ $J \to \infty$ if $Q_{out} \geq G_{out}$ and $J \to -\infty$ with one maximum if $Q_{out} < G_{out}$.) We are interested in solutions with $G_{out} - Q_{out} - 1 > 0$ for $J_{out} = 0$, which jump from S_0^u to S_0^s (see Fig. 2.2, green curves).

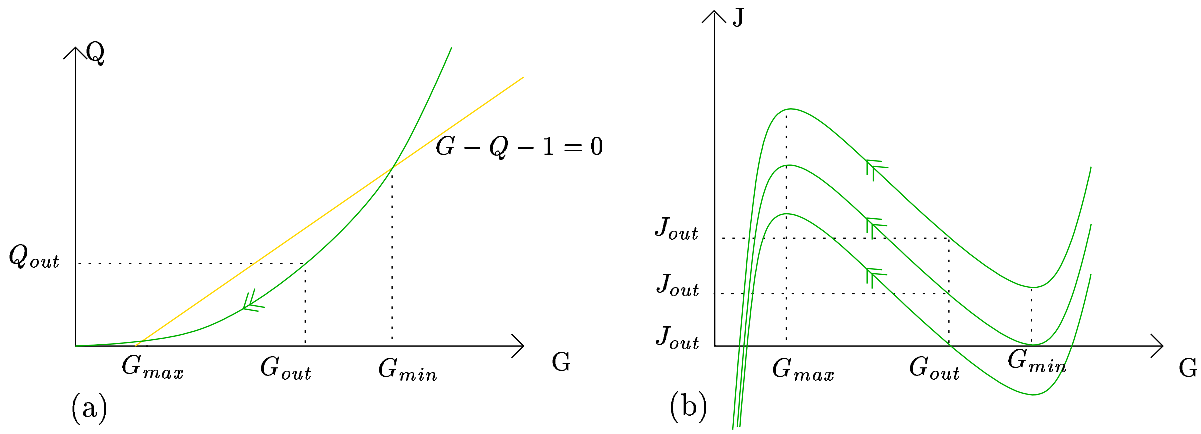

Figure 2.1: Dynamics of the layer problem, (a) Q versus G and (b) J versus G

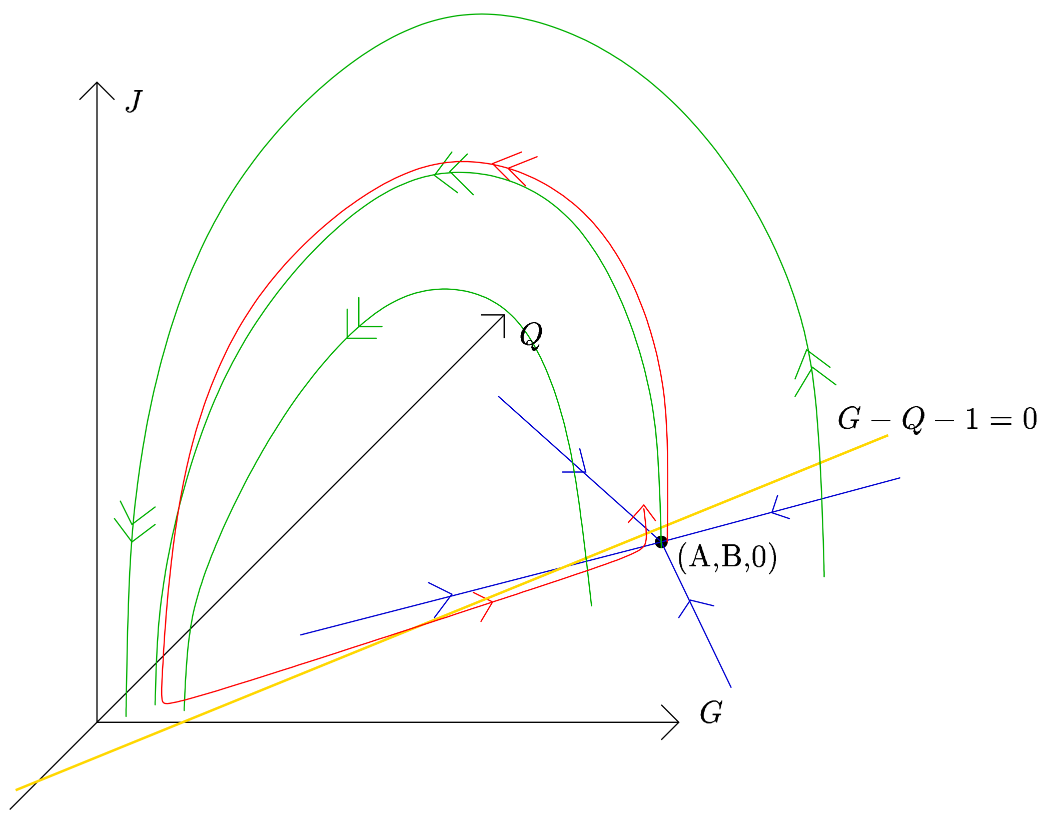

Figure 2.2: The basic dynamics of the Yamada model

To get the reduced system for the laser equations (2.2), which describes the slow flow on S_0, we change to the slow time scale and set $J = 0$ and $\varepsilon = 0$ (see [17]):

$$\begin{aligned} \dot{J} &= 0 \\ \dot{G} &= A - G \\ \dot{Q} &= B - Q. \end{aligned} \tag{2.6}$$

For this system of differential equations, $(G, Q) = (A, B)$ is the globally stable equilibrium, which attracts all solutions, and the solutions follow straight lines (see Fig. 2.2, blue lines). We are interested in solutions where this equilibrium is near the non-hyperbolic line, i.e. $A - B - 1 \approx 0$.

We define the stable hyperbolic parts of S_0, S_H^s, by $G - Q - 1 \leq \delta_2$ and the unstable hyperbolic parts of S_0, S_H^u, by $\delta_1 \leq G - Q - 1$ for some fixed small $\delta_1, \delta_2 > 0$.

Now we can use the Theory of Fenichel (see Appendix A.2) to describe the solutions of the full system near compact subsets S_H^s and S_H^u by combining solutions of the layer problem with solutions of the reduced equations. Since $J = 0$ is invariant, the slow manifold is equal to the critical manifold $S_\varepsilon = S_0$ and we can describe the flow on the whole slow manifold, even for the non-hyperbolic points.

Solutions which start near S_H^s will be attracted to the slow manifold S_ε with an exponential rate. Near the non-hyperbolic line, $\delta_1 \leq G - Q - 1 \leq \delta_2$, we have to use a blow-up transformation to describe the flow. The solutions are attracted to the stable slow manifold, follow the slow manifold till it becomes unstable and are repelled from the unstable part of the slow manifold after a distance of order $O(\varepsilon^0)$ (see [32]).

For $J \geq \delta$, $\delta > 0$ small, the solutions of the original equations are regular perturbations of the layer equation. That means that for $t \in [0, T], T < \infty$ the solutions of the original equation $x(t, \varepsilon)$ and the solution of the layer equation $y(t)$ differ by a term of order $O(\varepsilon)$, $\|x(t, \varepsilon) - y(t)\| = O(\varepsilon)$ if the initial conditions satisfy $\|x(0, \varepsilon) - y(0)\| = O(\varepsilon)$.

Combining these descriptions we get the behaviour of the solutions, a fast jump from the unstable part of the slow manifold to the stable part of the slow manifold, a slow drift along the slow manifold and (exponential) attraction till it becomes unstable and then after some time of following the unstable part it jumps up again (see fig 2.2, red orbit). This basic reinjection mechanism allows for oscillatory, periodic and homoclinic behaviour.

2.2 Bifurcation diagram

We now briefly summarize the bifurcation analysis of [15]. Bifurcations occur by variation of the parameters A, B and a. We are interested in $B > \frac{1}{a-1}$ and $0 \leq \varepsilon \ll 1$. The equilibrium $(G, Q, J) = (A, B, 0)$ exists for all values of A and is attracting for $A - B - 1 < 0$. For $A = A_{SN}(\varepsilon)$ two equilibria (with $J > 0$) of saddle type appear in a saddle node bifurcation. The equilibria for $J > 0$ will remain in the manifold with $G - Q - 1 = 0$. For $A = A_{Hom}(\varepsilon)$ a homoclic orbit exists. For $A > A_{Hom}(\varepsilon)$ there is a family of periodic orbits which started at the

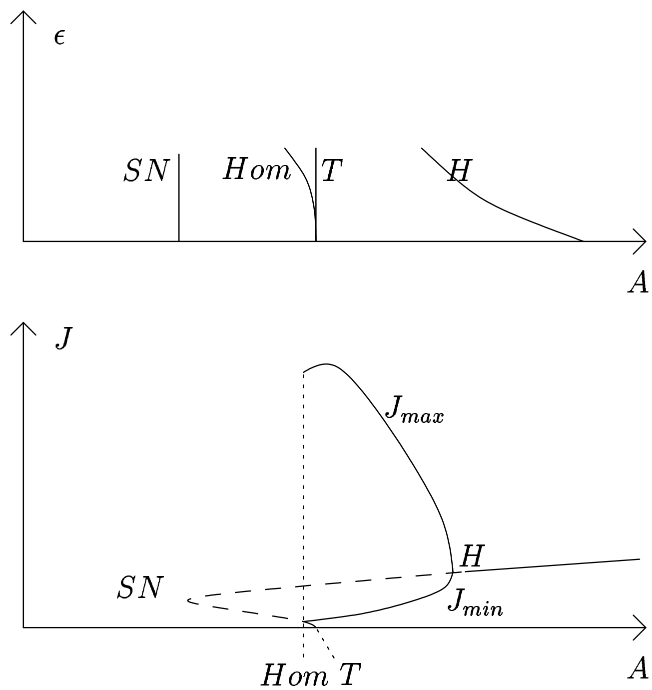

Figure 2.3: The bifurcations of the Yamada model

homoclic orbit at $A = A_{Hom}(\varepsilon)$. At $A = A_T(\varepsilon) = B + 1$ one of the equilibria with $J > 0$ coalesces with the equilibrium in $J = 0$ in a transcritical bifurcation and the equilibrium $(A, B, 0)$ loses stability in the J direction. The periodic orbit becomes smaller and then disappears in a Hopf bifurcation at $A = A_H(\varepsilon)$ and the equilibrium with $J > 0$ becomes attracting. The bifurcation curves are ordered by $A_{SN} < A_{Hom} \leq A_T < A_H$. This behaviour is illustrated in Figure 2.3. In the lower picture, the equilibria and the envelope of the periodic orbits (J_{min}, J_{max}) is shown.

We are interested in understanding the bifurcations for small ε based on the behaviour in the singular limit $\varepsilon = 0$. We focus on understanding the behaviour near $A_{Hom}(0) = A_T(0)$, where the transcritical and the homoclinic bifurcation curves intersect. We give a detailed analysis of local and global bifurcations near this point. In this analysis the blow-up method for singularly perturbed differential equations ([9], [27]) plays an essential role. Our main results are that the homoclinic orbit indeed exists for $\varepsilon \ll 1$ and that the bifurcation curves of the homoclinic bifurcation and the transcritical bifurcation are exponentially close, i.e. $|A_T(\varepsilon) - A_{Hom}(\varepsilon)| < O(e^{-c/\varepsilon})$ with $c > 0$.

2.3 Exponential closeness of the homoclinic and the transcritical bifurcation curve

We show the following Theorem for the Yamada Equations:

Theorem 1. *For $a > 1$, $B > \frac{1}{a-1}$ the transcritical bifurcation curve $A_T(\varepsilon)$ and the homoclinic bifurcation curve $A_{Hom}(\varepsilon)$ are exponentially close for system (2.2).*

That means, there exists $c > 0$ *and* $\varepsilon_0 > 0$ *such that* $|A_T(\varepsilon) - A_{Hom}(\varepsilon)| < O(e^{-c/\varepsilon})$ *for* $\varepsilon \in [0, \varepsilon_0]$.

The transcritical bifurcation occurs for $(A_T, B, 0)$ on the non-hyperbolic line $G - Q - 1 = 0$ and parts of the homoclinic orbit are near also. Therefore we need a detailed analysis of the dynamics near the non-hyperbolic line. The parameters a, B are kept constant in the following. Since the transcritical bifurcation occurs at $A = B + 1$ and the homoclinic bifurcation occurs nearby we replace the parameter A by the new parameter $\Delta = A - B - 1$. The homoclinic bifurcation curve is described now by $\Delta_{Hom}(\varepsilon) = A_{Hom}(\varepsilon) - B - 1$ and we show that $|\Delta_{Hom}(\varepsilon)| < O(e^{-c/\varepsilon})$.

We make the coordinate transformation

$$\begin{aligned} D &= G - Q - 1 \\ H &= G - 1 \\ \Delta &= A - B - 1, \end{aligned} \tag{2.7}$$

and obtain the equations

$$\begin{aligned} J' &= DJ \\ D' &= ((a-1)H - 1 - aD)J + \varepsilon(\Delta - D) \\ H' &= -(H+1)J + \varepsilon(B + \Delta - H). \end{aligned} \tag{2.8}$$

In these variables $D = 0$ describes the non-hyperbolic line. $\{J = 0, D < 0\}$ is now the stable part of the slow manifold and $\{J = 0, D > 0\}$ describes the unstable part. We will use a blow-up transformation of the line $\{D = 0, J = 0, \Delta = 0, \varepsilon = 0\}$ in the extended phase space $(J, D, H, \varepsilon, \Delta)$ to understand the dynamics in a neighborhood of the non-hyperbolic line.

2.3.1 The blow-up transformation

We use the blow-up transformation

$$\begin{aligned} \psi: \quad & \mathbb{R}^3 \times S_2 && \to \mathbb{R}^5 \\ & (\bar{r}, \bar{H}, \bar{\Delta}, \bar{J}, \bar{D}, \bar{\varepsilon}) && \to (J, D, H, \varepsilon, \Delta) \end{aligned}$$

defined by

$$J = \bar{r}^2 \bar{J}, \ D = \bar{r}\bar{D}, \ H = \bar{H}, \ \varepsilon = \bar{r}\bar{\varepsilon}, \ \Delta = \bar{r}\bar{\Delta} \tag{2.9}$$

with $(\bar{J}, \bar{D}, \bar{\varepsilon}) \in S_2$. Since ψ is a C^∞ diffeomorphism for $\bar{r} > 0$ we get an equivalent system of differential equations for $\bar{r} > 0$. The main reason to use this transformation is that in the new system it is possible to desingularize the vector field on $\{\bar{r} = 0\} = \psi^{-1}(0, 0, \mathbb{R}, 0, 0)$, and to use the flow on $\{\bar{r} = 0\}$ in the analysis of the dynamics for $\bar{r}$ small.

To make calculations as easy as possible we introduce directional charts as follows:
Chart K_1, which is defined by

$$J = r_1^2 J_1, \ D = -r_1, \ H = H_1, \ \varepsilon = r_1 \varepsilon_1, \ \Delta = r_1 \Delta_1$$

on $\mathbb{R}^5$ with $r_1 \geq 0$, $\varepsilon_1 \geq 0$, describing the flow for $\bar{D} < 0$.
Chart K_2, which is defined by

$$J = r_2^2 J_2,\ D = r_2 D_2,\ H = H_2,\ \varepsilon = r_2,\ \Delta = r_2 \Delta_2$$

on $\mathbb{R}^5$ with $r_2 \geq 0$, describing the flow for $\bar{\varepsilon} > 0$.
Chart K_4, which is defined by

$$J = r_4^2,\ D = r_4 D_4,\ H = H_4,\ \varepsilon = r_4 \varepsilon_4,\ \Delta = r_4 \Delta_4$$

on $\mathbb{R}^5$ with $r_4 \geq 0$, $\varepsilon_4 \geq 0$, describing the flow for $\bar{J} > 0$.

The laser equations have five time-dependent variables after the blow-up transformation, hence it is not easy to visualize the blown-up system. The non-hyperbolic line (for $\varepsilon = 0$) is blown up to the manifold $S_2 \times \mathbb{R}^3$. This is illustrated in Fig. 2.4(b),(d). In Fig. 2.4(b) the ε and the Δ direction are neglected. For the flow near $J = 0$, $H = B$ is attracting, further $H = B$ is invariant on $\bar{r} = 0$. Therefore in Fig. 2.4(a),(c) the H direction is neglected. The dynamics takes place near $H = B$ except the dashed part of $\gamma(\varepsilon, \Delta)$. In these two figures we see, which chart is used to describe the dynamics in each region. Figure 2.4(a) shows the relevant behaviour for $\Delta < 0$ and Figure 2.4(c) for $\Delta = 0$.

2.3.2 Idea of the proof

We will show that system (2.8) has a homoclinic orbit for $\Delta = \Delta_{Hom}$. Each of these homoclinic orbits $\gamma(\varepsilon, \Delta_{Hom})$ starts (for $t \to -\infty$) at an equilibrium in the manifold of equilibria N^u in $H = B$ (see Fig.2.4 (a)). Then $\gamma(\varepsilon, \Delta_{Hom})$ follows the flow near the cylinder $\bar{r} = 0$ to the exit line (near $H = B$) of equilibria l^o where it jumps away from the cylinder to the stable part of the slow manifold S_ε^s with a positive distance to the singular line. During this jump it follows the solutions of the layer problem (dashed line). Then $\gamma(\varepsilon, \Delta_{Hom})$ is attracted to S_ε^s exponentially close and near to $H = B$. After that the orbit follows the flow near the cylinder again until it ends (for $t \to \infty$) in the equilibrium in N^u again.

To show this we will follow all orbits $\gamma(\varepsilon, \Delta)$ of the unstable manifold of N^u with ε, Δ small. For $\varepsilon = 0$ there exists a singular homoclinic orbit for $\Delta_{Hom} = 0$, which starts in $o \in N^s|_{\Delta=0} = N^u|_{\Delta=0}$ (see Fig. 2.4(c)). The singular orbit $\gamma(0,0)$ consists of 4 different parts:

- the orbit on the cylinder connecting o to the exit point p^o,
- the orbit of the layer problem which connects p^o to an equilibrium in the critical manifold S_0,
- the solution in the critical manifold to the equilibrium p on the cylinder and
- the orbit on the cylinder from p to o.

For $r \geq 0, \Delta \geq 0$ the 2-parameter family of orbits $\gamma(r, \Delta)$ is a 3-dimensional manifold close to the singular orbit $\gamma(0,0)$. The orbits of this manifold follow $\gamma(0,0)$ to the stable part of the slow manifold S_0^s and are attracted exponentially close to $J = 0$. Afterwards they follow $\gamma(0,0)$ close to the cylinder.

We will proof that for each ε there exists a unique $\Delta = \Delta_{Hom} = \Delta_{Hom}(\varepsilon)$ such that $\gamma(\varepsilon, \Delta_{Hom})$ lies in the stable manifold of N^u and therefore converges to the equilibrium in N^u. For $\Delta < \Delta_{Hom}$ the orbits $\gamma(\varepsilon, \Delta)$ converge to the stable equilibria in N^s and for $\Delta > \Delta_{Hom}$ the orbits pass near N^u and jump away near l^o.

At the end of this chapter the proof of (1) is finished by collecting the results we show in the following in the different charts.

2.3.3 Dynamics in chart K_1, the incoming flow

We begin the proof in chart K_1 where the orbit $\gamma(\varepsilon, \Delta)$, in this chart described by $\gamma_1(\varepsilon_1, \Delta_1)$, passes near the point p (now p_1) (see last section and Fig. 2.4). We describe the transition of $\gamma_1(\varepsilon_1, \Delta_1)$ from the section

$$\Sigma_1^{in} := \{r_1 = \rho, \varepsilon_1 \in [0, e]\}$$

to the section

$$\Sigma_1^{out} := \{r_1 \in [0, \rho], \varepsilon_1 = e\},$$

with small constants $\rho, e > 0$. We denote the coordinates in Σ_1^{in} by subscript $_{1i}$ and all coordinates in Σ_1^{out} by subscript $_{1o}$.

We make the following assumtion for the parametrization $J_{1i} = J_{1i}(\varepsilon_{1i}, \Delta_{1i})$ and $H_{1i} = H_{1i}(\varepsilon_{1i}, \Delta_{1i})$ and the parameter Δ_{1i} for $\gamma_1(\varepsilon_1, \Delta_1) \cap \Sigma_1^{in}$:

Assumptions for γ_1:

- $J_{1i}(\varepsilon_{1i}, \Delta_{1i})$ is a C^1 function satisfying $J_{1i}(\varepsilon_{1i}, \Delta_{1i}) = O(e^{-c/\varepsilon_{1i}})$ with a constant $c > 0$.
- $H_{1i}(\varepsilon_{1i}, \Delta_{1i})$ is continous for $\varepsilon_{1i} \geq 0$ and C^1 for $\varepsilon_{1i} > 0$. The derivatives satisfy the bounds:

$$\frac{\partial}{\partial \varepsilon_{1i}} H_{1i} = O(ln(\varepsilon_{1i})), \ \frac{\partial}{\partial \Delta_{1i}} H_{1i} = O(ln(\varepsilon_{1i})).$$

- The parameter Δ_{1i} satisfies $-\frac{\varepsilon_{1i}}{2e} \leq \Delta_{1i} < 0$ and Δ_{1i} sufficientally small.

At the end of this chapter in Section 2.3.8 we will proof that $\gamma_1(\varepsilon_1, \Delta_1) \cap \Sigma_1^{in}$ indeed satisfies these assumptions.

In the coordinates of chart K_1 equation (2.8) has the form

$$\begin{aligned} J_1' &= r_1(-J_1 + 2J_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1)) \\ r_1' &= -r_1^2 h(J_1, r_1, H, \varepsilon_1, \Delta_1) \\ H' &= r_1(-(H+1)r_1 J_1 + \varepsilon_1(B + \Delta_1 r_1 - H)) \\ \varepsilon_1' &= r_1(\varepsilon_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1)) \\ \Delta_1' &= r_1 \Delta_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1), \end{aligned} \tag{2.10}$$

with $h(J_1, r_1, H, \varepsilon_1, \Delta_1)) := ((a-1)H - 1)J_1 + ar_1 J_1 + \varepsilon_1(\Delta_1 + 1)$ and time $\tilde{t}$. The equations for ε_1 and Δ_1 are obtained by differentiation of the defining equations:

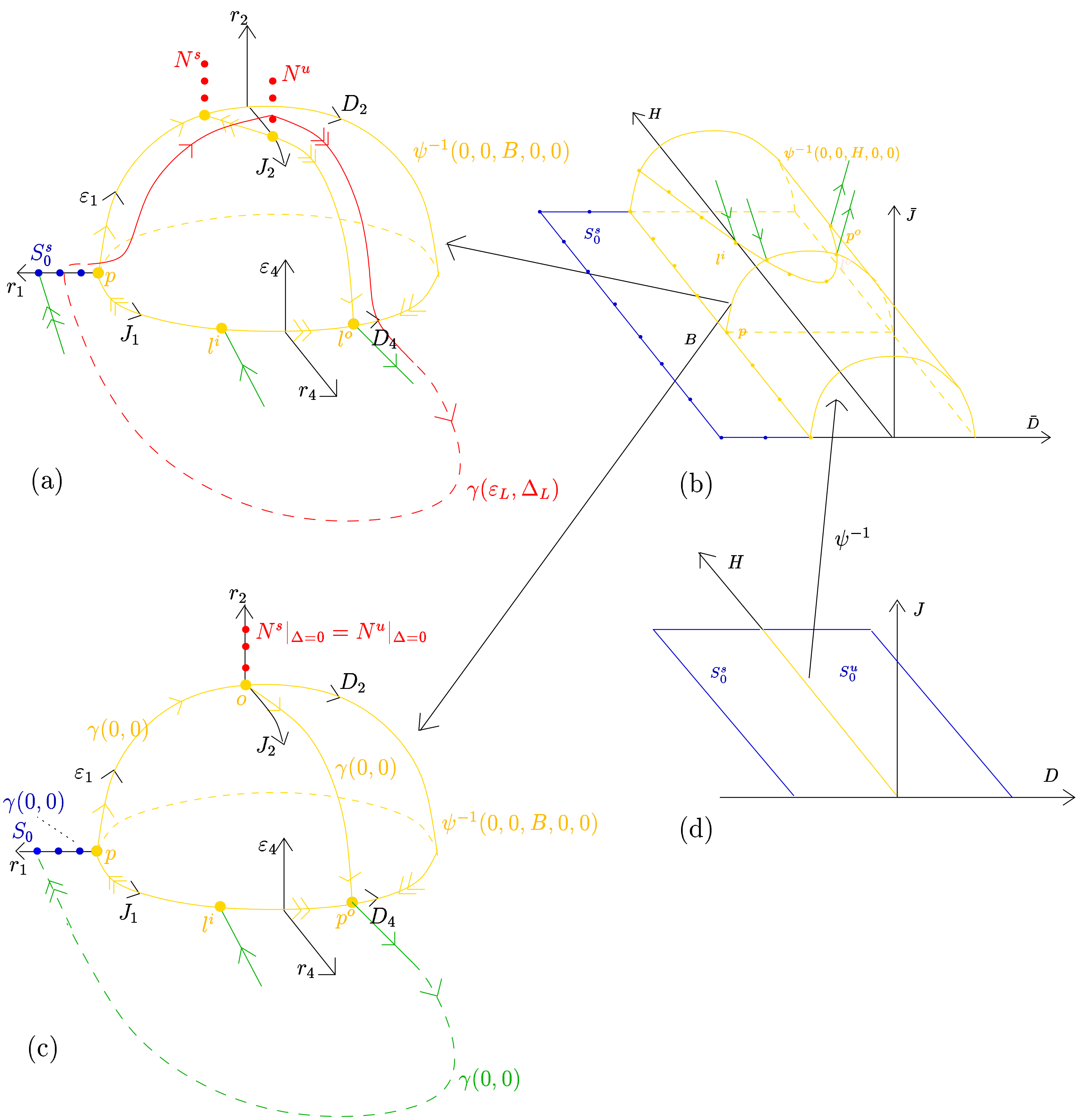

Figure 2.4: Blow-up of the laser equations

$\varepsilon' = (r_1\varepsilon_1)'$ and $\Delta' = (r_1\Delta_1)'$. System (2.10) is well defined for $r_1 = 0$ and has a trivial flow there because of the common factor r_1. We now desingularize the flow on $\{r_1 = 0\}$ by dividing the right side of (2.10) by r_1 which corresponds to a r_1-dependent time transformation $dt = r_1 d\tilde{t}$ for $r_1 > 0$:

$$\begin{aligned} J_1' &= -J_1 + 2J_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1) \\ r_1' &= -r_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1) \\ H' &= -(H+1)r_1 J_1 + \varepsilon_1(B - H) \\ \varepsilon_1' &= \varepsilon_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1) \\ \Delta_1' &= \Delta_1 h(J_1, r_1, H, \varepsilon_1, \Delta_1), \end{aligned} \tag{2.11}$$

where $'$ denotes $\frac{d}{dt}$. This time transformation only changes the time parametrisation and takes orbits to orbits for $r_1 > 0$.These equations show now a non trivial dynamics for $r_1 = 0$ and carries information which can be used in the analysis for $r_1 > 0$.

This system has the invariant manifold $\{J_1 = 0\}$, which contains the critical manifold S_0^s in chart K_1:

$$S_{01}^s = \{J_1 = 0, r_1, H, \varepsilon_1 = 0, \Delta_1\}.$$

The other manifold of equilibria

$$N_1^s := \{J_1 = 0, r_1, H = B - r_1, \varepsilon_1, \Delta_1 = -1\}$$

corresponds to the attracting equilibria of the slow manifold, $\{(G, Q, J) = (A, B, 0)\}$. There exists a third manifold of equilibria,

$$l_1^i := \{J_1 = \frac{1}{2((a-1)H) - 1}, r_1 = 0, H, \varepsilon_1 = 0, \Delta_1 = 0\},$$

which lies on the invariant 'sphere' $\{r_1 = 0\}$. These equilibria correspond to 'entrance points' to the sphere (and will not be important for us).

It is sufficient to describe the behaviour in a small neighborhood of $\{J_1 = 0\}$, so we calculate the invariant manifolds of the equilibrium $(J_1, r_1, H, \varepsilon_1, \Delta_1) = (0, 0, B, 0, 0)$:

Theorem 2. *At $p_1 := (0, 0, B, 0, 0)$ system (2.11) has a one-dimensional stable manifold M^s corresponding to the eigenvalue -1 tangent to $(1, 0, 0, 0, 0)^T$ and a 4-dimensional center manifold M^c corresponding to the eigenvalue 0 defined by $\{J_1 = 0\}$ in a small neighborhood of p_1 (see Fig.2.5(a)). There is a $c_1 > 0$, with c_1 close to 1 such that orbits near $\{J_1 = 0\}$ are attracted to $\{J_1 = 0\}$ by an exponential rate of order $O(e^{-c_1 t})$.*

Proof: Because $\{J_1 = 0\}$ is invariant and tangent to the center directions, it is a center manifold. The rest follows from standard center manifold theory [8]. □

For sufficiently small constants ρ, e the two sections Σ_1^{in} and Σ_1^{out} intersect the center manifold M^c of Theorem 2 transversally. Since $r_1\varepsilon_1 = \varepsilon$ and $r_1\Delta_1 = \Delta$ are constant we see that all orbits which start in Σ_1^{in} with $\varepsilon_{1i} > 0$ will exit in Σ_1^{out} with $r_{1o} = \varepsilon_{1i}\frac{\rho}{e}$ and $\Delta_{1o} = \Delta_{1i}\frac{\rho}{r_{1o}}$.

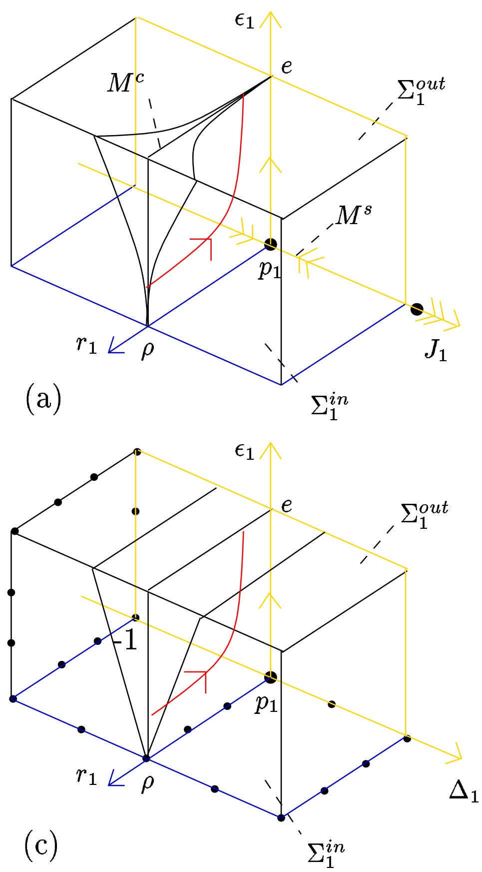

Figure 2.5: Dynamics in chart K_1

Lemma 1. *The leading term of the transition time T from Σ_1^{in} to Σ_1^{out} for orbits which start with $\varepsilon_{1i} \neq 0$ is of order $T \sim \varepsilon_{1i}^{-1}$.*

Proof: Since $\Delta_1 = \frac{\Delta_{1i}}{\varepsilon_{1i}}\varepsilon_1$ and $J_1 = 0$ the equation for ε_1 becomes

$$\varepsilon_1' = \varepsilon_1^2(\frac{\Delta_{1i}}{\varepsilon_{1i}}\varepsilon_1 + 1).$$

Integrating this equation for $-1 < \frac{\Delta_{1i}}{\varepsilon_{1i}}\varepsilon_1 < 0$ from $t = 0$ to $t = T$ gives

$$T = \{-\varepsilon_1^{-1} - \frac{\Delta_{1i}}{\varepsilon_{1i}}ln(\varepsilon_1) + ln(\frac{\Delta_{1i}}{\varepsilon_{1i}}\varepsilon + 1)\frac{\Delta_{1i}}{\varepsilon_{1i}}\} |_{\varepsilon_{1i}}^{e}$$

Because $\Delta_{1i} = O(\varepsilon_{1i})$ as we will see later, the dominating term in T is ε_{1i}^{-1}. □

From this we get estimates for the contraction of orbits which start in Σ_1^{in} near $J_1 = 0$:

Lemma 2. *Orbits which start with $J_{1i} = O(\varepsilon_{1i}^3)$ in section Σ_1^{in} will be attracted to $J_1 = 0$ exponentially: $J_{1o} = J_{1i}O(e^{-c_1/\varepsilon_{1i}})$.*

Proof: If $J_{1i} = O(\varepsilon_{1i}^3)$ and H and r_1 smaller than a constant, the additional terms for $J_1 \neq 0$ are only perturbations of higher order when integrating the ε_1' equation. The rest follows from Lemma 1 and Theorem 2. □

On the center manifold M^c the dynamics happens on the slower time scale $\tilde{t}_1$ and we get the following differential equation after dividing the right side by a common factor ε_1 for $\varepsilon_1 > 0$ (time transformation $\varepsilon_1 dt = dt_1$) in time t_1:

$$\begin{aligned} \dot{r}_1 &= -r_1(\Delta_1 + 1) \\ \dot{H} &= B + r_1\Delta_1 - H \\ \dot{\varepsilon}_1 &= \varepsilon_1(\Delta_1 + 1) \\ \dot{\Delta}_1 &= \Delta_1(\Delta_1 + 1). \end{aligned} \tag{2.12}$$

Because $\Delta = r_1\Delta_1 = r_{1i}\Delta_{1i}$ is constant we see that the dynamics in H is independent and can be integrated easily.

Lemma 3. *The transition time T_1 for equation (2.12) from Σ_1^{in} to Σ_1^{out} is of order $T_1 \sim -ln(\varepsilon_{1i})$.*

Proof: analogous to Lemma 1 for equation $\dot{\varepsilon}_1 = \varepsilon_1(\Delta_1 + 1)$ □

Since $H(t_1) = B + r_1\Delta_1 + (H_{1i} - (B + r_1\Delta_1))e^{-t_1}$ is the solution for H in the center manifold we get with Lemma 3:

Lemma 4. *In Σ_1^{out} H satisfies the relation $|H_{1o} - B| \sim |r_{1o}\Delta_{1o} + (H_{1i} - (B + r_{1o}\Delta_{1o}))\varepsilon_{1i}|$.*

The dynamics of the variables $r_1, \varepsilon_1, \Delta_1$ is independent of H. We see that the equations have an equilibrium in $(r_1, \varepsilon_1, \Delta_1) = (0, 0, 0)$ and the manifold of equilibria N_1^s for $\Delta_1 = -1$. For $\Delta_1 \neq -1$ we can again divide the right-hand side

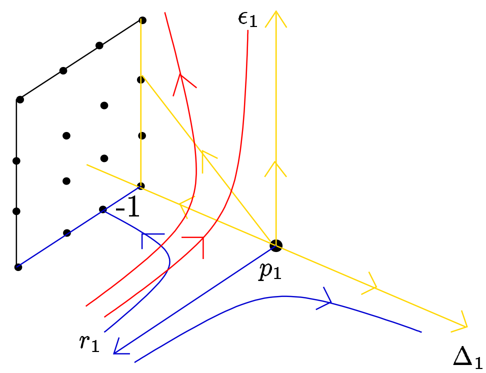

Figure 2.6: Dynamics in chart K_1 for r_1, ε_1 and Δ_1

of equation (2.12) by $\Delta_1 + 1$ and get a linear system and therefore can draw the phase portrait. Here we see geometrically what happens to orbits which do not satisfy $-1 < \frac{\Delta_{1i}}{\varepsilon_{1i}}\varepsilon_1 < 0$. They will converge to an equilibrium in N_1^s for $\varepsilon_1 < e$. We are interested in orbits which exit at Σ_1^{out} with small $-1 < \Delta_{1o} < 0$ and therefore have to restrict our calculations to sufficiently small $\Delta_{1i} = O(\varepsilon_{1i})$.

Proposition 1. *If $J_{1i} = O(\varepsilon_{1i}^3)$, $-\frac{\varepsilon_{1i}}{2e} < \Delta_{1i} < 0$ there exist constants $c, \tilde{c}$ such that the map from Σ_1^{in} to Σ_1^{out} satisfies the following relations:*

1. *A wedge of size $|J_{1i}| \le O(\varepsilon_{1i}^3)$ is mapped to a wedge of size $|J_{1o}| \le O(e^{-c/r_{1o}})$.*
2. *The wedge $-\frac{\varepsilon_{1i}}{2e} < \Delta_{1i} \le 0$ is mapped to the domain $-\frac{1}{2} < \Delta_{1o} \le 0$.*
3. *A domain with $|H_{1i} - B| < \tilde{c}_1$ is mapped to a wedge of size $|H_{1o} - B| \le \tilde{c}_2 r_{1o}$.*

Proof: Follows directly from the previous Lemmas with $r_{1o} = \varepsilon_{1i}\frac{\delta}{e}$, $\Delta = r_{1o}\Delta_{1o}$. □

Since $\gamma_1(\varepsilon_1, \Delta_1)$ satisfies the assumptions of Proposition 1 we get the following Theorem for γ_1 in Σ_1^{out}:

Theorem 3. *The manifold $\gamma_1(\varepsilon_1, \Delta_1)$ satisfies the following estimates in Σ_1^{out}:*

1. $|J_{1o}| \le O(e^{-c/r_{1o}})$,
2. $-\frac{1}{2} < \Delta_{1o} \le 0$, *and*
3. $|H_{1o} - B| \le \tilde{c}_2 r_{1o}$.

2.3.4 Dynamics in chart K_2, intersection of the manifolds

The change of coordinates from chart K_1 to chart K_2 in the overlap domain of K_1 and K_2, which is a diffeomorphism from $\{\varepsilon_1 > 0, r_1 > 0\}$ to $\{D_2 < 0, r_2 > 0\}$, is given by

$$J_2 = \varepsilon_1^{-2} J_1,\ D_2 = -\varepsilon_1^{-1},\ r_2 = r_1\varepsilon_1,\ \Delta_2 = \varepsilon_1^{-1}\Delta_1.$$

The out-section in chart K_1, Σ_1^{out}, is mapped to the in-section of chart K_2, Σ_2^{in}, defined by

$$\Sigma_2^{in} := \{D_{2i} = -e^{-1}, r_2 \in [0, \rho e]\}.$$

In chart K_2 the manifold $\gamma_1(\varepsilon_1, \Delta_1)$ is described by $\gamma_2(r_2, \Delta_2)$:

Lemma 5. *For sufficiently small r_{2i}, Δ_{2i} the manifold $\gamma_2(r_{2i}, \Delta_{2i})$ satisfies the following bounds in Σ_2^{in}:*

1. $|J_{2i}| \leq O(e^{-c/r_{2i}})$,
2. $|H_{2i} - B| \leq \frac{\tilde{c}_2}{e} r_{2i}$.

Proof: By combining the change of coordinates from K_1 to K_2 with Theorem 3. □

The dynamics in chart K_2 is given by the system

$$\begin{aligned} J_2' &= r_2 D_2 J_2 \\ D_2' &= r_2(((a-1)H - 1 - ar_2 D_2)J_2 + \Delta_2 - D_2) \\ H' &= r_2(B + r_2\Delta_2 - H - (H+1)r_2 J_2) \\ r_2' &= 0 \\ \Delta_2' &= 0. \end{aligned} \tag{2.13}$$

Dividing out the common factor r_2 yields the desingularized system

$$\begin{aligned} J_2' &= D_2 J_2 \\ D_2' &= ((a-1)H - 1 - ar_2 D_2)J_2 + \Delta_2 - D_2 \\ H' &= B + r_2\Delta_2 - H - (H+1)r_2 J_2 \\ r_2' &= 0 \\ \Delta_2' &= 0. \end{aligned} \tag{2.14}$$

Lemma 6. *System (2.14) has the 2-dimensional manifolds of equilibria*

1. $N_2^s = \{J_2 = 0, D_2 = \Delta_2, H = B + r_2\Delta_2, r_2, \Delta_2\}$ *and*
2. $N_2^u = \{J_2 = \frac{-\Delta_2}{aB-B-1} + \Delta_2 r_2 O(1), D_2 = 0, H = B + r_2\Delta_2 O(1), r_2, \Delta_2\}$.

Proof: For $J_2 = 0$ we get the manifold of equilibria N_2^s. $J_2 > 0$ implies $D_2 = 0$ and we get the system of equations

$$\begin{aligned} f_1 &= ((a-1)H - 1)J_2 + \Delta_2 = 0 \\ f_2 &= B + r_2\Delta_2 - H - (H+1)r_2 J_2 = 0. \end{aligned}$$

If $(r_2, \Delta_2) = (0, 0)$ we get $J_2 = 0$ and $H = B$. Since

$$Det(\frac{\partial(f_1, f_2)}{\partial(J_2, H)})|_{(r_2,\Delta_2)=(0,0)} = -aB + B + 1 \neq 0$$

the Implicit Function Theorem implies that we can solve for J_2 and H with C^k functions $J_2 = j(r_2, \Delta_2, B, a)$ and $H = h(r_2, \Delta_2, B, a)$. Plugging these solutions into the equations gives $j(0, \Delta_2, B, a) = \frac{-\Delta_2}{aB-B-1}$, $h(0, \Delta_2, B, a) = 0$ and $j(r_2, 0, B, a) = 0$, $h(r_2, 0, B, a) = 0$. Therefore, we get

$$\begin{aligned} J_2 = j(r_2, \Delta_2, B, a) &= \frac{-\Delta_2}{aB - B - 1} + \Delta_2 r_2 O(1), \\ H = h(r_2, \Delta_2, B, a) &= B + r_2\Delta_2 O(1). \end{aligned}$$

□

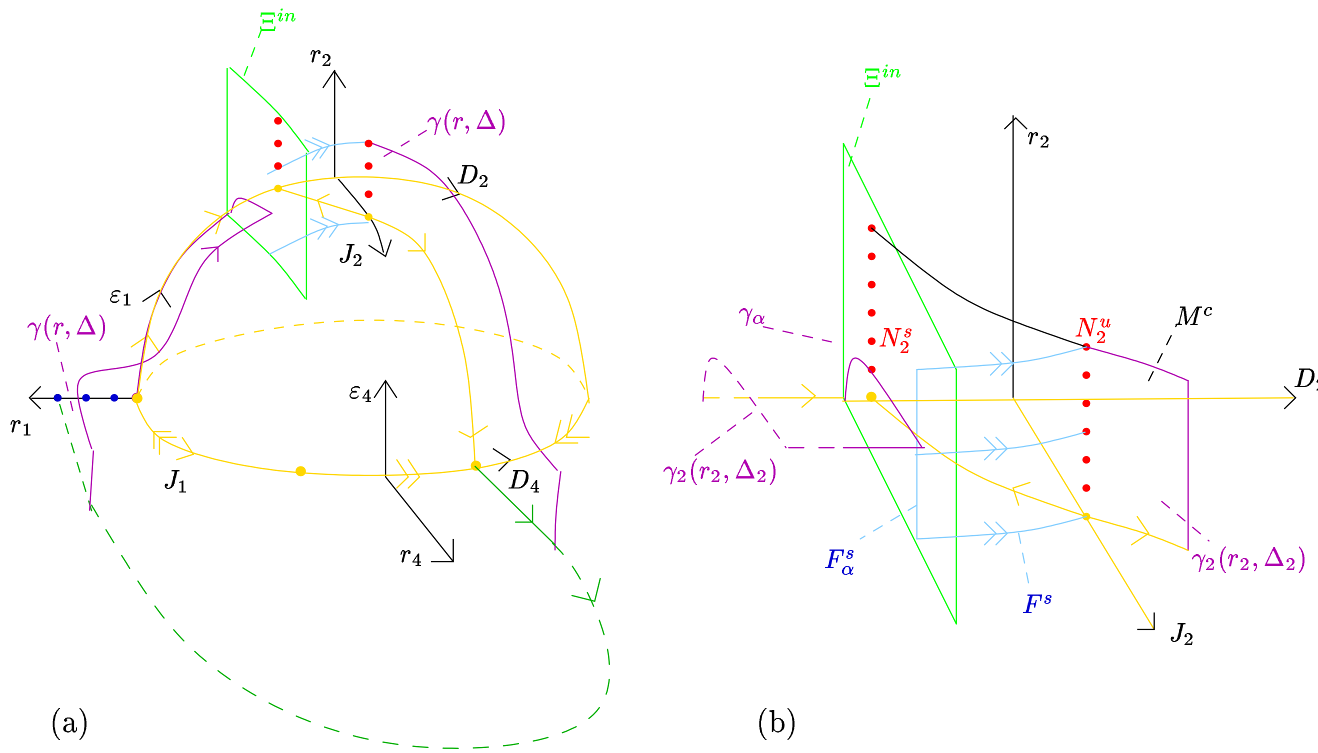

Figure 2.7: The manifolds F^s and $\gamma_2(r_2, \Delta_2)$ for $\Delta_2 = \tilde{\Delta}_2 < 0$ (a) in the blow-up picture, (b) the intersection with Ξ^{in}.

One important feature of these equilibria is the common factor Δ_2 for J_2. For $\Delta_2 < 0$ N_2^s corresponds to the stable equilibria N_1^s in K_1. For $\Delta_2 = 0$ the two manifolds of equilibria coincide.

The equilibria of the homoclinic orbits lie in N_2^u. The orbit $\gamma_2(r_2, \Delta_2)$ starts at the equilibrium in N_2^u with parameter values r_2, Δ_2. We will show that after a large excursion γ returns to Σ_2^{in} (see Fig. 2.7). To show the existence of an homoclinic orbit we have to show that for certain values of r_2 and Δ_2 the orbits $\gamma_2(r_2, \Delta_2)$ lie in the stable manifold of N_2^u.

We will first describe the dynamics near the intersection of N_2^u and N_2^s:

Proposition 2. *The equilibrium of system (2.14)*

$$\{J_2 = 0, D_2 = 0, H = B, r_2 = 0, \Delta_2 = 0\}$$

possesses

1. *a 3-dimensional center manifold M^c corresponding to the eigenvalue 0 with eigenvectors $(\frac{1}{aB-B-1}, 1, 0, 0, 0)^T, (0, 0, 0, 1, 0)^T, (0, 1, 0, 0, 1)^T$,*

2. *a 2-dimensional stable manifold M^s corresponding to the eigenvalue -1 with eigenvectors $(0, 1, 0, 0, 0)^T, (0, 0, 1, 0, 0)^T$*

3. *and a 2-dimensional stable foliation F^s over M^c with a contraction rate of $O(e^{-ct})$ with a $c \approx 1$.*

Proof: Follows from standard center manifold theory [8]. □

With the transformation

$$\begin{aligned} J_2 &= j_n \\ D_2 &= (aB - B - 1)j_n + d_n + \delta_n \\ H &= B + h_n \\ r_2 &= r_n \\ \Delta_2 &= \delta_n \end{aligned} \tag{2.15}$$

we transform system (2.14) to a system with diagonal linear part:

$$\begin{aligned} j_n' &= j_n(j_n(aB - B - 1) + d_n + \delta_n) \\ d_n' &= -d_n + j_n O(\|(j_n, d_n, h_n, r_n, \delta_n)\|) \\ h_n' &= -h_n - r_n j_n(B + 1 + h_n) + r_n \delta_n \\ r_n' &= 0 \\ \delta_n' &= 0. \end{aligned} \tag{2.16}$$

Lemma 7. *The 3-dimensional center manifold of system (2.16) has the following structure*

$$\begin{aligned} d_n &= d(j_n, r_n, \delta_n) = j_n O(\|(j_n, r_n, \delta_n)\|) \\ h_n &= h(j_n, r_n, \delta_n) = j_n r_n O(\|(j_n, r_n, \delta_n)\|^0) + r_n \delta_n. \end{aligned}$$

Proof: According to standard center manifold theory ([8], [38]) the 3-dimensional center manifold can be described as a C^k graph over (j_n, r_n, δ_n):

$$\begin{aligned} d_n &= d(j_n, r_n, \delta_n) = O(\|(j_n, r_n, \delta_n)\|^2) \\ h_n &= h(j_n, r_n, \delta_n) = O(\|(j_n, r_n, \delta_n)\|^2). \end{aligned}$$

Since for $j_n = 0$ the center manifold becomes trivial: $d(0, r_n, \delta_n) = 0, h(0, r_n, \delta_n) = r_n \delta_n$ and $h(j_n, 0, \delta_n) = 0$, the assertions follow. □

The dynamics restricted to the center manifold is described by

$$\begin{aligned} j_n' &= j_n(j_n(aB - B - 1) + j_n O(\|(j_n, r_n, \delta_n)\|) + \delta_n) \\ r_n' &= 0 \\ \delta_n' &= 0. \end{aligned} \tag{2.17}$$

which has the manifolds of equilibria N_2^s and N_2^u which are given in these coordinates by

$$N_2^s = \{j_n = 0, r_n, \delta_n\}, \tag{2.18}$$

$$N_2^u = \{j_n = \frac{-\delta_n}{aB - B - 1} + \delta_n r_n O(1), r_n, \delta_n\}. \tag{2.19}$$

Lemma 8. *For each small $r_n > 0$ system (2.17) has a transcritical bifurcation in $\delta_n = 0$. For $\delta_n < 0$ N_2^s is stable and N_2^s is unstable. They exchange stability in $\delta_n = 0$ (see Fig. 2.8).*

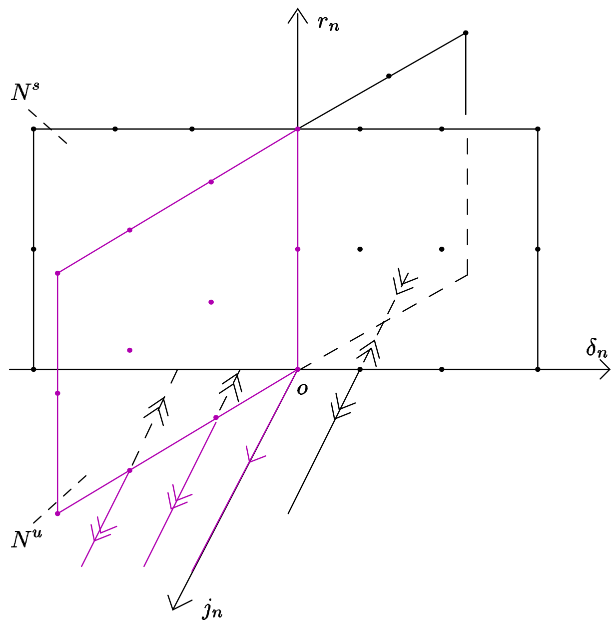

Figure 2.8: Dynamics in the center manifold in chart K_2

Proof: The manifolds intersect transversally in $\delta_n = 0$ and if $f := j_n(j_n(aB - B - 1) + j_n O(\|(j_n, r_n, \delta_n)\|) + \delta_n)$ we see that

$$\frac{\partial f}{\partial j_n}|_{N_2^s} = \delta_n$$
$$\frac{\partial f}{\partial j_n}|_{N_2^u} = -\delta_n(1 + O(\|(r_n, \delta_n)\|)).$$

Therefore wee see that we have a transcritical bifurcation ([18], [38]) in $\delta_n = 0$. □

To describe the behaviour of the full system, we use the dynamics on the center manifold together with the stable foliation. Because r_n and δ_n are constant, we draw the behaviour for certain values in Fig.2.9 in the original coordinates of chart K_2. Since N_2^s is attracting in the center manifold, orbits which start near $j_n = 0$ will be attracted to these equilibria. The orbits lie in the stable manifold of N_2^u for certain values of j_n and grow for larger j_n.

Next we describe the stable manifold M^s over N_2^u which consists of stable fibers of F^s with basepoint in N_2^u and describe the intersection with the manifold

$$\Xi^{in} := \{D_2 = -\alpha\}$$

with α small.

Proposition 3. *The stable fibers over the equilibria N_2^u can be parameterized by*

$$\begin{pmatrix} j_n \\ d_n \\ h_n \\ r_n \\ \delta_n \end{pmatrix} = \begin{pmatrix} \delta_n(\frac{-1}{aB-B-1} + \overline{f}_1(d_s, h_s, r_c, \delta_c)) \\ d_s \\ h_s \\ r_c \\ \delta_c \end{pmatrix}$$

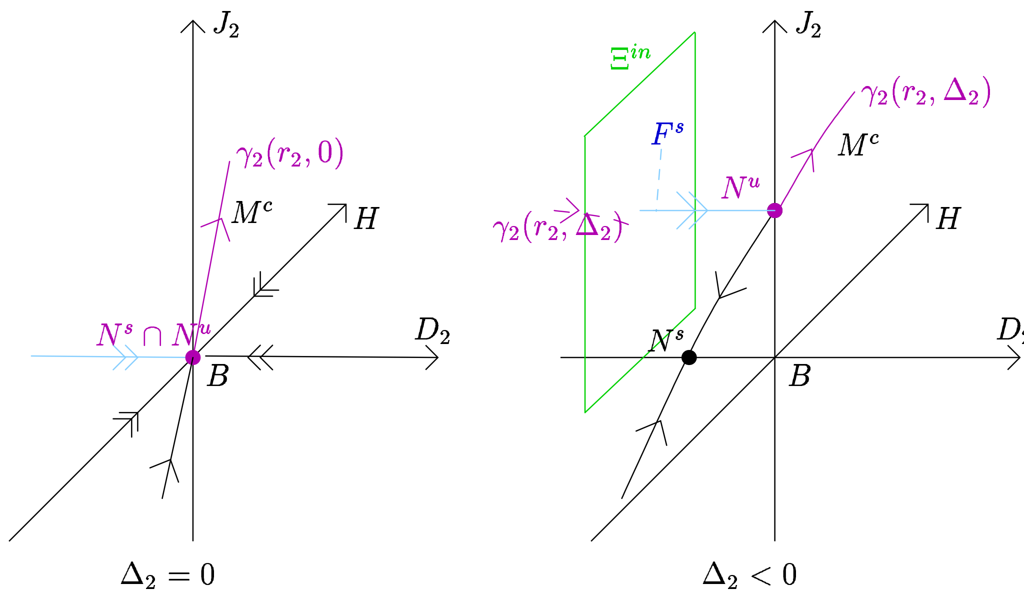

Figure 2.9: Center Manifold and stable foliation in chart K_2 with a constant $r_2 \geq 0$.

with $\overline{f}_1 = O(\|(d_s, h_s, r_c, \delta_c)\|)$ *a* C^{k-1} *function.*

Proof: According to (Appendix A.1) it is possible to write the foliation as

$$\begin{pmatrix} j_n \\ d_n \\ h_n \\ r_n \\ \delta_n \end{pmatrix} = \begin{pmatrix} j_c \\ 0 \\ 0 \\ r_c \\ \delta_c \end{pmatrix} + \begin{pmatrix} f_1(j_c, d_s, h_s, r_c, \delta_c) \\ d_s \\ h_s \\ f_4(j_c, d_s, h_s, r_c, \delta_c) \\ f_5(j_c, d_s, h_s, r_c, \delta_c) \end{pmatrix}$$

with f_i a C^k function of order $O(\|(j_c, d_s, h_s, r_c, \delta_c)\|^2)$. Since r_n and δ_n are constant $f_4 = 0$ and $f_5 = 0$. For $j_c = 0$ the flow of (2.16) and the fibers are trivial,

$$f_1(0, d_s, h_s, r_c, \delta_c) = 0, \ \forall d_s, h_s, r_c, \delta_c.$$

Therefore

$$f_1(j_c, d_s, h_s, r_c, \delta_c) = j_c \tilde{f}_1(j_c, d_s, h_s, r_c, \delta_c)$$

with a C^{k-1} function $\tilde{f}_1 = O(\|(j_c, d_s, h_s, r_c, \delta_c)\|^0)$ and $j_n = j_c(1 + \tilde{f}_1)$. On the center manifold there is $d_s = d(j_c, r_c, \delta_c), h_s = h(j_c, r_c, \delta_c)$ (see Lemma 7) and $\tilde{f}_1 = 0$. So we get $\tilde{f}_1 = O(\|(j_c, d_s, h_s, r_c, \delta_c)\|^1)$. The coordinates j_c, r_c, δ_c, of the center manifold, follow the dynamics of (2.17). Because we are interested in fibers over N_2^u (2.19) we can substitute $j_c = \frac{-\delta_c}{aB - B - 1} + \delta_c r_c O(1)$ and get

$$j_n = \delta_c\left(\frac{-1}{aB - B - 1} + \overline{f}_1(d_s, h_s, r_c, \delta_c)\right)$$

with a C^{k-1} function $\overline{f}$ of order $O(\|(d_s, h_s, r_c, \delta_c)\|)$. □

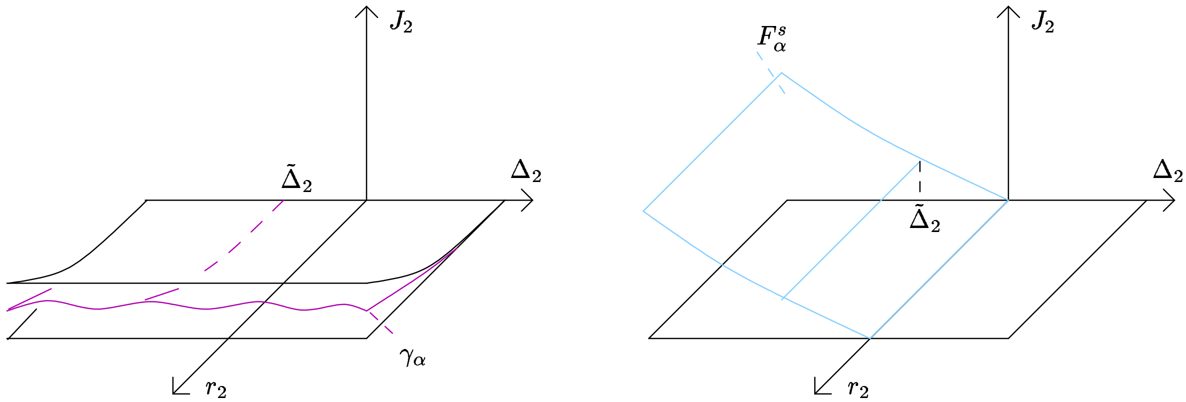

Figure 2.10: The manifolds F^s_α and $\gamma_\alpha(r_2, \Delta_2)$ in Ξ^{in}

Proposition 4. *The intersection of the fibers with Ξ^{in} in original coordinates is (see Fig. 2.10):*

$$\begin{pmatrix} J_2 \\ D_2 \\ H \\ r_2 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \delta_n(\frac{-1}{aB-B-1} + \overline{f}_1(-\alpha + O(\delta_n), h_n, r_n, \delta_n)) \\ -\alpha \\ B + h_n \\ r_n \\ \delta_n \end{pmatrix}.$$

Proof: Transforming to the original coordinates of K_2 gives

$$\begin{pmatrix} J_2 \\ D_2 \\ H \\ r_2 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \delta_n(\frac{-1}{aB-B-1} + \overline{f}_1(d_n, h_n, r_n, \delta_n)) \\ \delta_n(\overline{f}_1(d_n, h_n, r_n, \delta_n)(aB - B - 1) + d_n) \\ B + h_n \\ r_n \\ \delta_n \end{pmatrix}.$$

Because $D_2 = -\alpha$ in Ξ^{in} we can solve this equation according to the Implicit Function Theorem in $\delta_n = 0$ for d_n and get $d_n = -\alpha + \delta_n O(\|(\alpha, h_n, r_n, \delta_n)\|)$. Substitution of d_n into the first equation gives the statement. $\square$

Next we will look how the manifold γ_2 evolves under the flow and describe its intersection with Ξ^{in}.

Proposition 5. *In Ξ^{in} the following bounds for $\gamma_2(r_2, \Delta_2)$ are satisfied(see Fig. 2.10 (a)):*

$$0 \le J_2(r_{2i}, \Delta_{2i}) \le O(e^{-c/r_{2i}})$$

$$|H(r_{2i}, \Delta_{2i}) - B| \le K_2 r_{2i} \alpha e + r_{2i}\Delta_{2i} + O(e^{-c/r_{2i}}).$$

Proof: Because $D_2 \in [-\frac{1}{e}, -\alpha]$ and $J_2 > 0$, $J_2' = D_2 J_2 < 0$ and therefore $J_2 < J_{2i} = O(e^{-c/r_{2i}})$. Let $H \in [B-\beta, B+\beta]$ be an interval around B with $\beta > 0$ sufficiently small and $H_{2i} \in [B - \frac{\beta}{2}, B + \frac{\beta}{2}]$. Then H satisfies

$$H' = B + r_{2i}\Delta_{2i} - H + O(e^{-c/r_{2i}})$$

as long as it stays in the small interval.

$$H(t) = B + r_{2i}\Delta_{2i} + (H_{2i} - B - r_{2i}\Delta_{2i})e^{-t} + tO(e^{-c/r_{2i}})$$

is the solution for $t \in [0, T]$ with $T < \infty$. We see that for

$$|H(t, r_{2i}, \Delta_{2i}) - B| \leq (H_{2i} - B - r_{2i}\Delta_{2i})e^{-t} + r_{2i}\Delta_{2i} + tO(e^{-c/r_{2i}})$$

the first term is smaller than $\frac{\beta}{2}$ and decreases for increasing t, $tO(e^{-c/r_{2i}}) = O(e^{-c/r_{2i}}) < \frac{\beta}{4}$ and $|r_{2i}\Delta_{2i}| < \frac{\beta}{4}$ for small r_{2i}. Therefore we can find for each T an upper bound for r_{2i} such that H remains in the interval. For $D_2 \in [-\frac{1}{e}, -\alpha]$ and $-\alpha < \Delta_2$, D_2 satisfies

$$D_2' = \Delta_2 - D_2 + O(e^{-c/r_{2i}}).$$

The solution is

$$D_2(t) = \Delta_2 + (-e^{-1} - \Delta_2)e^{-t} + O(e^{-c/r_{2i}})$$

for $t \in [0, T]$, $D_2 \leq -\alpha$ and $D_{2i} = -e^{-1}$. The equation

$$D_2(T_\alpha) = -\alpha$$

has the solution

$$e^{-T_\alpha} = \frac{\alpha + \Delta_2}{e^{-1} + \Delta_2} + O(e^{-c/r_{2i}}).$$

With $|H_i - B - r_{2i}\Delta_{2i}| < K_2 r_{2i}$ and $|\frac{\alpha+\Delta_2}{e^{-1}+\Delta_2}| < \alpha e$ we get the bounds for $|H(r_{2i}, \Delta_{2i}) - B|$. □

Now that we know the intersection M^s_α of the stable manifold M^s, which is equal to the stable foliation F^s over N^u_2, with Ξ^{in} and the intersection γ_α of the returning flow $\gamma(r_2, \Delta_2)$ with Ξ^{in} we can describe the intersection of M^s and $\gamma(r_2, \Delta_2)$ in Ξ^{in}. Because Ξ^{in} is transversal to it is sufficient to look for which parameter values this intersection occurs in Ξ^{in}.

Theorem 4. *The intersection of $\gamma(r_2, \Delta_2)$ and the stable foliation F^s over N^u_2 (see Fig. 2.11) exists for*

$$\Delta_2 = \Delta_2(r_2) = O(e^{-c/r_2}).$$

Proof: From Propositions 4 and 5 we obtain equations for the intersection of the manifolds:

$$\begin{pmatrix} \delta_n(\frac{-1}{aB-B-1} + \overline{f}_1(-\alpha + O(\delta_n), h_n, r_n, \delta_n)) \\ -\alpha \\ B + h_n \\ r_n \\ \delta_n \end{pmatrix} = \begin{pmatrix} J_2(r_2, \Delta_2) \\ -\alpha \\ H(r_2, \Delta_2) \\ r_2 \\ \Delta_2 \end{pmatrix}$$

with $J_2(r_2, \Delta_2) = O(e^{-c/r_2})$ and $H(r_2, \Delta_2) = B + O(r_2)$. This leads to the equations

$$0 = g_1 = \Delta_2(\frac{-1}{aB - B - 1} + \overline{f}_1(-\alpha + O(\Delta_2), h_n, r_2, \Delta_2)) + O(e^{-c/r_2})$$
$$0 = g_2 = h_n + O(r_2).$$

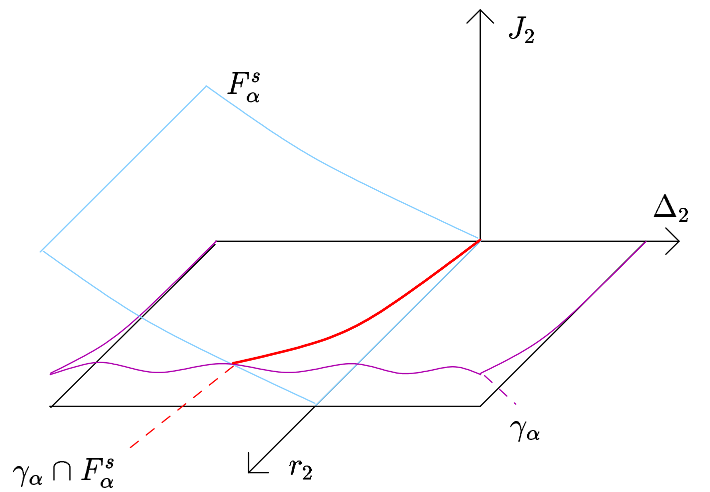

Figure 2.11: Intersection of F^s and $\gamma_2(r_2, \Delta_2)$ in Ξ^{in}

In $\Delta_2 = r_2 = h_n = 0$ the Jacobian

$$\frac{\partial(g_1, g_2)}{\partial(\Delta_2, h_n)} = \begin{pmatrix} (\frac{-1}{aB-B-1} + \overline{f}_1(-\alpha, 0, 0, 0))) & 0 \\ 0 & 1 \end{pmatrix}$$

is regular if α is sufficiently small. By the Implicit Function Theorem the equations can be solved for Δ_2 and h_n which gives:

$$\Delta_2 = \Delta_2(r_2, B, a, \alpha) = O(e^{-c/r_2})$$
$$h_n = h_n((r_2, B, a, \alpha)) = O(r_2).$$

Because Δ_2 is independent of α we get $\Delta_2 = \Delta_2(r_2, B, a) = O(e^{-c/r_2})$. $\square$

This can be easily understood geometrically. For the stable foliation M^s_α the manifold

$$J_2(r_2, \Delta_2) = \Delta_2(\frac{-1}{aB - B - 1} + \overline{f}_1(-\alpha + O(\Delta_2), h_n, r_2, \Delta_2))$$

is transversal to $\{J_2 = 0\}$ because $|\overline{f}_1| < \frac{1}{aB-B-1}$ for $\alpha, h_n, r_2, \Delta_2$ sufficiently small. On the other side γ_α satisfies

$$J_2(r_2, \Delta_2) = O(e^{-c/r_2})$$

for the returning manifold and is tangent to $\{J_2 = 0\}$ in $\{r_2 = 0\}$ (see Fig.2.10). Therefore the two manifolds intersect transversally (see Fig. 2.11) along $\Delta_2 = O(e^{-c/r_2})$ and the orbits of the manifold $\gamma_2(r_2, \Delta_2)$ tend to N^u_s for these parameter values.

This implies Theorem 1 provided that $\gamma(\varepsilon, \Delta)$ satisfies the assumptions for γ_1 (see Chapter 2.3.3) needed in chart K_1. This will be done in the following:
We start by describing the part of the manifold $\gamma_2(r_2, \Delta_2)$ corresponding to the local unstable manifolds M^u of the equilibria N^u_2. The outgoing orbits lie in the

center manifold M^c. We define the outgoing section of chart K_2, which intersects the unstable manifold M^u transversally, by

$$\Xi^{out} = \{J_2 = \alpha\}$$

with $\alpha > 0$ as above. We get:

Theorem 5. *The intersection of Ξ^{out} and $\gamma_2(r_2, \Delta_2)$ is given by*

$$\begin{pmatrix} J_2 \\ D_2 \\ H \\ r_2 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ ((aB - B - 1) + O(\|(\alpha, r_n, \delta_n)\|))\alpha + \Delta_2 \\ B + r_2 O(\|(\alpha, r_n, \delta_n)\|) \\ r_n \\ \Delta_n \end{pmatrix}$$

with $D_2 = D_2(r_2, \Delta_2) > 0$ and $H = H(r_2, \Delta_2) = B + r_2 O(\|(\alpha, r_2, \Delta_2)\|)$.

Proof: Because r_n and δ_n are constant, the unstable manifold for $\delta_n \leq 0$ of the equilibria N_2^u of equation (2.17) can be described by

$$j_n > \delta_n(\frac{-1}{aB - B - 1} + O(\|(r_n, \delta_n)\|)).$$

The other variables are (Lemma 7)

$$\begin{aligned} d_n &= j_n O(\|(j_n, r_n, \delta_n)\|) \\ h_n &= j_n r_n O(1). \end{aligned}$$

Therefore in original coordinates this manifold is given by

$$\begin{pmatrix} J_2 \\ D_2 \\ H \\ r_2 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} j_n \\ ((aB - B - 1)j_n + j_n O(\|(j_n, r_n, \delta_n)\|) + \delta_n \\ B + j_n r_n O(1) \\ r_n \\ \delta_n \end{pmatrix}.$$

Setting $j_n = \alpha$ proves the Theorem. □

Next we follow $\gamma_2(r_2, \Delta_2)$ through the region $\{J_2 > 0\}$. We choose to describe the dynamics in chart K_4, which has certain computational advantages.

2.3.5 Dynamics in chart K_4 near the exit point

The change of coordinates from chart K_2 to chart K_4, which is a diffeomorphism from $\{J_2 > 0, r_2 > 0\}$ to $\{\varepsilon_4 > 0, r_4 > 0\}$, is given by

$$r_4 = r_2 J_2^{1/2},\ D_4 = J_2^{-1/2} D_2,\ \varepsilon_4 = J_2^{-1/2},\ \Delta_4 = J_2^{-1/2} \Delta_2.$$

The outgoing section in chart K_2, Ξ^{out}, is mapped to the incoming section of chart K_4, Σ_4^{in}, defined by

$$\Sigma_4^{in} := \{\varepsilon_4 = \alpha^{-1/2}\}.$$

In chart K_4 the manifold $\gamma_2(r_2, \Delta_2)$ is described by $\gamma_4(r_4, \Delta_4)$:

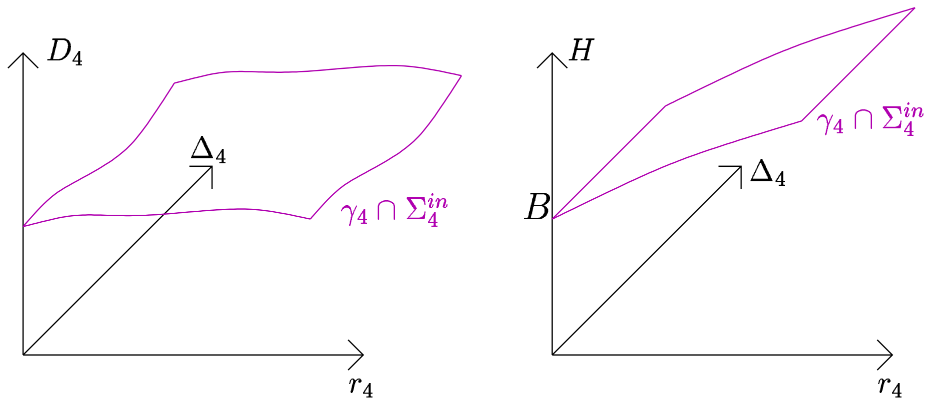

Figure 2.12: The manifold $\gamma_4(r_4, \Delta_4)$ in Σ_{in}^4

Lemma 9. *For sufficiently small Δ_{4i}, r_{4i} the manifold γ_4 has the following parametrisation in Σ_4^{in}:*

$$D_4 = \tilde{D}(r_4, \Delta_4), \; H = B + r_4 \tilde{H}(r_4, \Delta_4)$$

with C^k-functions $\tilde{D}$ and $\tilde{H}$ (see Fig. 2.12).

The dynamics in chart K_4 is given by

$$\begin{aligned}
r_4' &= \frac{1}{2} r_4 D_4 \\
D_4' &= (a-1)H - 1 - a r_4 D_4 + \varepsilon_4(\Delta_4 - D_4) - \frac{1}{2} D_4^2 \\
H' &= -(H+1) r_4 + \varepsilon_4(B + r_4 \Delta_4 - H) \\
\varepsilon_4' &= -\frac{1}{2} \varepsilon_4 D_4 \\
\Delta_4' &= -\frac{1}{2} \Delta_4 D_4,
\end{aligned} \tag{2.20}$$

after dividing out the common factor r_4. This system has the invariant manifolds $\{r_4 = 0\}$, $\{\varepsilon_4 = 0\}$ and $\{\Delta_4 = 0\}$. Further it has the curve of equilibria

$$l_4^o := \{r_4 = 0, \; D_4 = \sqrt{2((a-1)H - 1)}, \; H, \; \varepsilon_4 = 0, \; \Delta_4 = 0\}$$

which are the 'exit' points of the cylinder $\{r_4 = 0\}$. Along this line orbits, which pass nearby, jump away from $\{r_4 = 0\}$. The other line of equilibria,

$$l_4^i := \{r_4 = 0, \; D_4 = -\sqrt{2((a-1)H - 1)}, \; H, \; \varepsilon_4 = 0, \; \Delta_4 = 0\},$$

corresponds to the entrance points. The manifold of unstable equilibria from chart K_2, N_2^u, has the parametrisation

$$N_4^u := \{r_4 = \varepsilon_4 \frac{B - H}{H + 1 - \Delta_4 \varepsilon_4}, \; D_4 = 0, \; H, \; \varepsilon_4, \; \Delta_4 = \frac{H + 1 - aH}{\varepsilon_4}\}.$$

Near the equilibrium p^o,

$$p^o := l_4^o \cap \{H = B\},$$

which satisfies $\{D_o = \sqrt{aB - B - 1}\}$ we have the following invariant manifolds:

Proposition 6. *In p^o system (2.20) has*

1. *a 4-dimensional center-stable manifold $M^{cs} \subseteq \{r_4 = 0\}$, which consists of a 3-dimensional stable manifold $M^s \subseteq \{r_4 = 0,\ H = B\}$ corresponding to the eigenvalues $-D_0$, $-\frac{1}{2}D_0$ and $-\frac{1}{2}D_0$ and a 1-dimensional center manifold M^c corresponding to the eigenvalue 0, which is the line of equilibria l_4^o,*

2. *a 1-dimensional unstable manifold M^u corresponding to the eigenvalue $\frac{1}{2}D_0$ transversal to $\{r_4 = 0\}$.*

Proof: Standard center manifold theory □

Let

$$\Sigma_4^e := \{\varepsilon_4 = e\},$$
$$\Sigma_4^{out} := \{r_4 = \rho\}$$

be sections near p^o with sufficiently small $e, \rho > 0$. Further the intersection of the unstable manifold M^u with Σ_4^{out} defines

$$q^o := M^u \cap \Sigma_4^{out}.$$

Transition from Σ_4^{in} to Σ_4^e

In Σ_4^{in} the orbit $\gamma_4(0, \Delta_4)$ lies in the invariant manifold $\{r_4 = 0, H = B\}$ of system (2.20). Therefore we look at the subsystem:

$$\begin{aligned} D_4' &= aB - B - 1 + \varepsilon_4(\Delta_4 - D_4) - \frac{1}{2}D_4^2 \\ \varepsilon_4' &= -\frac{1}{2}\varepsilon_4 D_4 \\ \Delta_4' &= -\frac{1}{2}\Delta_4 D_4. \end{aligned} \tag{2.21}$$

Lemma 10. *For sufficiently small $|\Delta_4|$ with $\Delta_4 \leq 0$ the orbit $\gamma_4(0, \Delta_4)$ converges to p_4^o as $t \to \infty$ (see Fig. 2.13).*

Proof: We prove this Theorem by constructing invariant regions for system (2.21). In the octant

$$C := \{D_4 \geq 0, \varepsilon \geq 0, \Delta_4 \leq 0\}$$

the vector field has the following properties:

- $\varepsilon_4' = 0$ for $\varepsilon_4 = 0 \vee D_4 = 0$ and $\varepsilon_4' < 0$ in the interior of C, C^o,
- $\Delta_4' = 0$ for $\Delta_4 = 0 \vee D_4 = 0$ and $\Delta_4' > 0$ in C^o,
- $D_4' = 0$ for $\varepsilon_4 = F(D_4, \Delta_4) := \frac{D_4/2 - aB + B + 1}{\Delta_4 - D_4}$, $D_4' < 0$ for $\varepsilon_4 < F(D_4, \Delta_4)$ and $D_4' > 0$ for $\varepsilon_4 > F(D_4, \Delta_4)$.

(For D_4' we get the relations because $aB - B - 1 + \varepsilon_4(\Delta_4 - D_4) - \frac{1}{2}D_4^2) = 0$ defines a surface $\varepsilon_4 = F(D_4, \Delta_4)$ which satisfies $\varepsilon_4 = \frac{D_4/2 - aB + B + 1}{\Delta_4 - D_4}$. On $C \setminus C^o$ we have that $F(0, \Delta_4) = \frac{-aB + B + 1}{\Delta_4}$. We get $D_4' > 0$ for $\varepsilon_4 < F(D_4, \Delta_4)$ and $D_4' < 0$ for

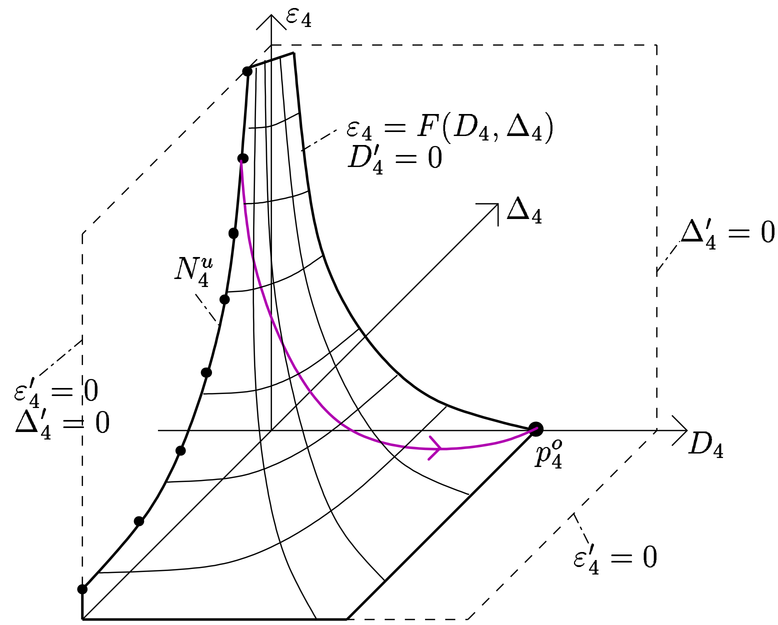

Figure 2.13: Dynamics of system (2.21)

$\varepsilon_4 > F(D_4, \Delta_4)$.
Because the vector field on $\varepsilon = F(D_4, \Delta_4)$ is $(0, -\frac{1}{2}\varepsilon_4 D_4 - \frac{1}{2}\Delta_4 D_4)^T$, it points in the direction $\varepsilon_4 < F(D_4, \Delta_4)$. Therefore the region

$$C^* := C \cap \{\varepsilon_4 \leq F(D_4, \Delta_4)\}$$

is positive invariant under the flow. Since $\gamma(0, \varepsilon_4)$ is part of the unstable manifold of N_4^u (which lies in $\varepsilon_4 = F(0, \Delta_4)$) with $D_4 > 0$, it has the following characteristics:

- $\Delta_4' > 0$ for $\Delta_4 < 0$ and therefore $\varepsilon_4 < F(D_4, \Delta_4)$, so we get $\gamma(0, \varepsilon_4) \subset C^*$,
- $\Delta_4' \geq 0$ for $\Delta_4 = 0$ and therefore $\varepsilon_4 \leq F(D_4, \Delta_4)$, so we get $\gamma(0, \varepsilon_4) \subset C^*$ (further p^o is attracting for all orbits in $C^* \cap \{\Delta_4 = 0\}$).

$\Delta_4' > 0$, $\varepsilon_4 < 0$ and $\Delta_4 < 0$ in the interior of C^* and therefore all orbits apart from N_4^u converge to p^o for $t \to \infty$. (Similar arguments imply that all orbits of C with $\varepsilon_4 \leq F(0, \Delta_4)$ converge to p^o.) □

Further the proof shows that $\gamma_4 \cap \Sigma_4^{in}$ is in C^* for $r_4 = 0$. We denote by subscript $_i$ the initial values in Σ_4^e and by subscript $_o$ the variables in Σ_4^{out}. If $r_4 > 0$ in Σ_4^{in} is small enough $\gamma_4(r_4, \Delta_4)$ is a small perturbation and if Δ_4 is small enough the map from Σ_4^{in} to Σ_4^e is a diffeomorphism. We get:

Lemma 11. *The manifold γ_4 in Σ_4^e is described by*

$$D_4 = D_i(r_i, \Delta_i) = d^*(r_i, \Delta_i)$$
$$H = H_i(r_i, \Delta_i) = B + r_i h^*(r_i, \Delta_i)$$

with C^k functions d^*, h^*.

Proof: All orbits of $\gamma_4(0, \Delta_4)$ need a finite time for the transition from Σ_4^{in} to Σ_4^e, if Δ_4 is sufficiently small. For $r_4 > 0$ $\gamma_4(r_4, \Delta_4)$ is a small perturbation. Therefore the transition of $\gamma_4(r_4, \Delta_4)$ needs only finite time and therefore is a C^k diffeomorphism. With $H \equiv B$ for $r_4 = 0$ we get the relations above. □

Dynamics near a non-hyperbolic equilibrium

We need to describe the transition $\Pi : \Sigma_4^e \to \Sigma_4^{out}$,

$$(r_o, D_o, H_o, \varepsilon_o, \Delta_o) = \Pi(r_i, D_i, H_i, \varepsilon_i, \Delta_i),$$

of γ_4 near the equilibrium p^o. We need that $\gamma_4 \cap \Sigma_4^{out}$ is a 2-dimensional smooth manifold with some bounds on the derivatives. For hyperbolic equilibria it is possible to use normal form transformations in combination with cut off techniques for the higher order terms [34], [5]. These cut-off techniques in general work for hyperbolic equilibria only.

For partially hyperbolic equilibria satisfying certain nonresonance conditions it is possible to find a C^k change of coordinates to a standard form [35]:

$$\begin{aligned} x_c' &= \varphi(x_c) \\ x_s' &= a(x_c)x_s \\ x_u' &= b(x_c)x_u. \end{aligned} \tag{2.22}$$

(Here $x_c \in \mathbb{R}^{n_c}$, $x_s \in \mathbb{R}^{n_s}$, $x_u \in \mathbb{R}^{n_u}$ describes the center, stable or unstable directions and $\varphi(x_c) \in \mathbb{R}^{n_c}$, $a(x_c) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$ and $b(x_c) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$ are C^k.) Due to the resonances of the eigenvalues of our system it is not possible to apply this method here.

Without needing any resonance conditions we can use Fenichel coordinates [23] to transform our system to the form

$$\begin{aligned} x_s' &= \Lambda(x_s, x_u, x_c)x_s \\ x_u' &= \Gamma(x_s, x_u, x_c)x_u \\ x_c' &= h(x_c) + H(x_s, x_u, x_c) \otimes x_s \otimes x_u, \end{aligned} \tag{2.23}$$

with $x_s, x_u, x_c, \Lambda, \Gamma, h$ as before, and H a tensor in $\mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$ where $\otimes$ is the tensor product.

Transformation to Fenichel coordinates

It will be important in later calculations that H of $\gamma_4 \cap \Sigma_4^e$ has the same form after the transformation to Fenichel coordinates and normal form transformations. Therefore we have to look at the transformations step by step.

The first step (for $B > \frac{a}{a-1}$) is

$$D_4 = \sqrt{2((a-1)H - 1)} + \tilde{d}, H = B + \tilde{h}$$

which shifts the line of equilibria to $r_4 = \tilde{d} = \varepsilon_4 = \Delta_4 = 0$ and only changes the equation for $\tilde{d}$. Because $D_4 = \sqrt{2(aB - B - 1 + (a-1)\tilde{h})} + \tilde{d} \neq 0$ near $H = B$

we can divide the right-hand side of (2.20) by D_4 which again corresponds to a space dependent time transformation. This simplifies the first, fourth and fifth equation. For the other two equations we develop D_4^{-1} into a series

$$D_4^{-1} = \frac{1}{D_0}\frac{1}{1+O(\|(\tilde{d},\tilde{h})\|)} = \frac{1}{D_0}(1+O(\|(\tilde{d},\tilde{h})\|)$$

which simplifies the linear terms and adds some higher order terms. If $f(x)$ is the right side of a system of differential equations with Jacobian $J(x)$, $f(x_0)=0$ and $B(x)$ a common factor, $\tilde{f} := Bf$ at x_0 has the Jacobian $\tilde{J}(x_0) = B(x_0)J(x_0)$. Therefore multiplying our system with D_4^{-1} means changing the Jacobian at the origin by a common factor $\frac{1}{D_0}$. To transform our system to diagonal form we use the transformation

$$\begin{pmatrix} r_4 \\ \tilde{d} \\ \tilde{h} \\ \varepsilon_4 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-2(aD_0^2+2a(B+1)-2(B+1))+6(B+1)(a-1)D_0}{3D_0^3} & 1 & \frac{a-1-(a-1)D_0}{D_0^2} & 1 & 0 \\ \frac{-2(B+1)}{D_0} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_d \\ d_d \\ h_d \\ \varepsilon_d \\ \delta_d \end{pmatrix}$$

and we finally get a system of the form

$$\begin{aligned} r_d' &= \frac{1}{2}r_d \\ d_d' &= -d_d + O(2) \\ h_d' &= \varepsilon_d O(1) + r_d O(1) \\ \varepsilon_d' &= -\frac{1}{2}\varepsilon_d \\ \Delta_d' &= -\frac{1}{2}\Delta_d, \end{aligned} \tag{2.24}$$

where $O(n)$ denotes C^k functions of order $O(\|(r_d, d_d, h_d, \varepsilon_d, \Delta_d)\|^n)$. So far the general form has not changed for γ_4 in Σ_4^e, there still exist some C^k functions d^*, h^* such that $d_d = d^*(r_d, \Delta_d)$, $h_d = r_d$, $h^*(r_d, \Delta_d)$.

As a next step we will transform (2.24) to Fenichel coordinates (see Fig. 2.14). For this we first straighten out the center stable and center-unstable manifolds and afterwards straighten out the fibers in these two manifolds, as in [23]. For $r_d = 0$ we are in the center-stable manifold M^{cs} and for $r_d = d_d = \varepsilon_d = \Delta_d = 0$ we are in the center manifold M^c, so they are flat already. The center-unstable manifold M^{cu} is tangent to $E^{cu} = \{d_d = \varepsilon_d = \Delta_d = 0\}$ and according to center manifold theory it can be described by C^k functions ψ_i of order $O(\|(r_d, h_d)\|^2)$:

$$d_d = \psi_1(r_d, h_d),\ \varepsilon_d = \psi_2(r_d, h_d),\ \Delta_d = \psi_3(r_d, h_d).$$

Because of the simple form of our equations, we have $\psi_2 \equiv 0$ and $\psi_3 \equiv 0$. For $r_d = 0$ we are in the center manifold and therefore $d_d = \psi_1(0, h_d) = 0$ and there exists some $\tilde{\psi_1}$ such that

$$d_d = r_d\tilde{\psi_1}(r_d, h_d),\ \varepsilon_d = 0\ \Delta_d = 0.$$

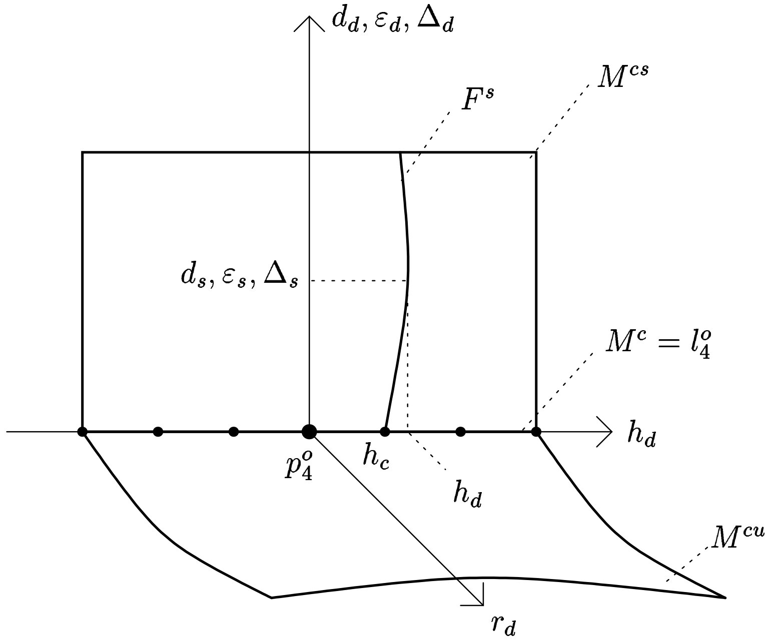

Figure 2.14: Invariant manifolds and foliation of system (2.24)

We change to the new coordinates $\tilde{d}$ defined by $d_d = r_d\tilde{\psi}_1(r_d, h_d) + \tilde{d}$ where $\tilde{d} = r_d = \varepsilon_d = 0$ corresponds to M^{cu}. In this system we have

$$\tilde{d}' = d' - r'\tilde{\psi}_1 - r\tilde{\psi}_{1r}r' - r\tilde{\psi}_{1h} = -\tilde{d} + O(2)$$

again. Next we will straighten the fibers in the center-stable manifold $M^{cs} \subseteq \{r_d = 0\}$. The fibers can be described by

$$\begin{pmatrix} \tilde{d} \\ h_d \\ \varepsilon_d \\ \Delta_d \end{pmatrix} = \begin{pmatrix} 0 \\ h_c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} d_s \\ f(h_c, d_s, \varepsilon_s, \Delta_s) \\ \varepsilon_s \\ \Delta_s \end{pmatrix}$$

with a C^{k-1} function f. For $d_s = \varepsilon_s = \Delta_s = 0$ we are in the center manifold and $f(h_c, 0, 0, 0) = 0$. p_4^o is in the invariant submanifold $h_d = 0$. The fiber of $h_c = 0$ and $d_s = \varepsilon_s = \Delta_s = 0$, which is described by $f(0, d_s, \varepsilon_s, \Delta_s)$ lies in $h_d = 0$. Therefore $f(0, d_s, \varepsilon_s, \Delta_s) = 0$ and there exists a function F such that

$$\begin{pmatrix} \tilde{d} \\ h_d \\ \varepsilon_d \\ \Delta_d \end{pmatrix} = \begin{pmatrix} 0 \\ h_c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} d_s \\ h_c(d_s, \varepsilon_s, \Delta_s)^T F(h_c, d_s, \varepsilon_s, \Delta_s) \\ \varepsilon_s \\ \Delta_s \end{pmatrix}.$$

Next we define a map P by

$$P(\tilde{d}, h_d, \varepsilon_d, \Delta_d) := h_c,$$

which maps each point of a fiber to its intersection with M^c. This map we get by solving

$$h_d = h_c + h_c(d_s, \varepsilon_s, \Delta_s)^T F(h_c, d_s, \varepsilon_s, \Delta_s)$$

for h_c and we get

$$h_c = P(d_d, h_d, \varepsilon_d, \Delta_d) = h_d + h_d(d_s, \varepsilon_s, \Delta_s)^T \tilde{P}(h_d, d_s, \varepsilon_s, \Delta_s)$$

with the Implicit Function Theorem. We change to the new variable

$$\tilde{h} := h_d + h_d(\tilde{d}, \varepsilon_d, \Delta_d)^T \tilde{P}(h_d, \tilde{d}, \varepsilon_d, \Delta_d),$$

which leaves M^{cu} invariant. In the new variables $(r_d, \tilde{d}, \tilde{h}, \varepsilon_d, \Delta_d)$ the fibers are flat in M^{cs}. As a next step we will do the same procedure in $M^{cu} \subseteq \{d_d = \varepsilon_d = \Delta_d = 0\}$ and we get a transformation to a new variable $\tilde{\tilde{h}}$:

$$\tilde{\tilde{h}} := \tilde{h} + \tilde{h} r_d \tilde{\tilde{P}}(r_d).$$

This transformation leaves M^{cs} invariant. In the coordinates $r = r_d, d = \tilde{d}, h = \tilde{\tilde{h}}, \varepsilon = \varepsilon_d, \Delta = \Delta_d$ we get a system of the form (2.23):

$$\begin{aligned} r' &= \frac{1}{2} r \\ d' &= -d + f_2(r, d, h, \varepsilon, \Delta) \\ h' &= f_3(r, d, h, \varepsilon, \Delta) \\ \varepsilon' &= -\frac{1}{2}\varepsilon \\ \Delta' &= -\frac{1}{2}\Delta, \end{aligned} \tag{2.25}$$

with f_2, f_3 C^{k-1} functions of order $O(2)$. Due to the above transformations we know: $f_2 = 0$ for $d = \varepsilon = \Delta = 0$, which means that $f_2 = \Lambda(r, d, h, \varepsilon, \Delta)(d, \varepsilon, \Delta)^T$, f_3 is independent of $r, d, \varepsilon, \Delta$ for $d = \varepsilon = \Delta = 0$ or $r = 0$, which means $f_3 = f_3(0, 0, h, 0, 0) + H(r, d, h, \varepsilon, \Delta)(d, \varepsilon, \Delta)^T r$. Because M^c is a line of equilibria we have $f_3(0, 0, h, 0, 0) \equiv 0$. Applying the change of coordinates to our initial conditions, we see that the form of Lemma 11 has not changed. So we get:

Theorem 6. *In Fenichel coordinates, system (2.20) has the form*

$$\begin{aligned} r' &= \frac{1}{2} r \\ d' &= -d + \Lambda(r, d, h, \varepsilon, \Delta)(d, \varepsilon, \Delta)^T \\ h' &= H(r, d, h, \varepsilon, \Delta)(d, \varepsilon, \Delta)^T r \\ \varepsilon' &= -\frac{1}{2}\varepsilon \\ \Delta' &= -\frac{1}{2}\Delta, \end{aligned} \tag{2.26}$$

with the initial conditions in Σ_4^e*:*

$$r = r_i, d = d_i = d^*(r_i, \Delta_i), h = h_i = r_i h^*(r_i, \Delta_i),$$

and $\Lambda \in C^{k-2}$ *of order* $O(1)$, $H \in C^{k-3}$ *of order* $O(0)$, $d^* \in C^{k-1}$ *and* $h^* \in C^{k-2}$, *both of order* $O(0)$.

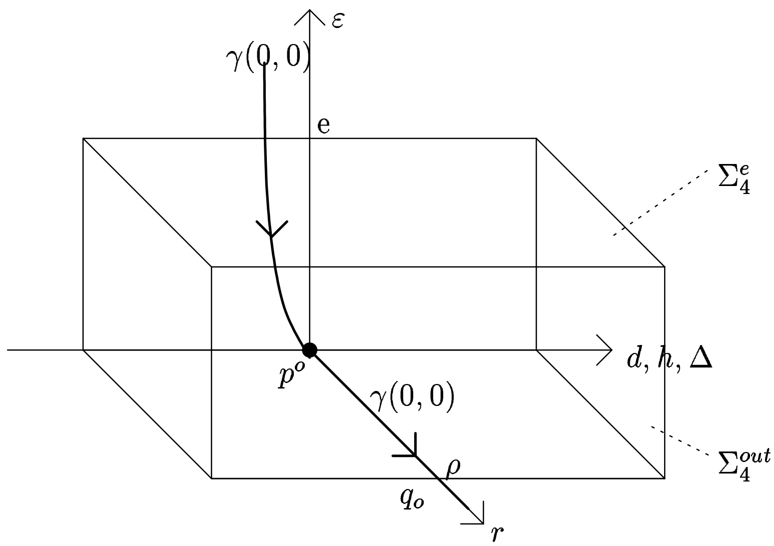

Figure 2.15: The singular orbit $\gamma(0,0)$ in Fenichel coordinates near p^o.

Transition Π from Σ_4^e to Σ_4^{out}

This section is very technical, the main Theorem is at the end of the section. We will describe the transition by first integrating the equations for $r = 0$ from 0 to the transition time T and calculating some bounds for the solution $d(t)$ with $r = 0$. Afterwards we use these results to calculate bounds for $\Pi(\gamma_4)$ for $r > 0$. As a last step we derive bounds for the derivatives of $\Pi(\gamma_4)$ to show that this manifold is smooth enough. For the transition time T we get:

Lemma 12. *The transition time T is*

$$T = 2ln(\frac{\rho}{r_i}).$$

Proof: By integration of r' as in K_1. □
First we will look at system (2.26) for $r = 0$ and $d = d_i, h = h_i, \varepsilon = \varepsilon_i, \Delta = \Delta_i$:

$$\begin{aligned} d' &= -d + \Lambda_0(d,h,\varepsilon,\Delta)(d,\varepsilon,\Delta)^T \\ h' &= 0 \\ \varepsilon' &= -\frac{1}{2}\varepsilon \\ \Delta' &= -\frac{1}{2}\Delta, \end{aligned} \tag{2.27}$$

where h is a constant. When we apply the transformation to normal form we get:

Lemma 13. *In normal form coordinates of order $O(p)$ the equation for d' is:*

$$d' = -d + dg_1(h) + \varepsilon^2 g_2(h) + \varepsilon\Delta g_3(h) + \Delta^2 g_4(h) + O(p),$$

with g_1 a polynomial of order 1 in h and the other g_i of order 0 in h. The term $O(p)$ consists of functions of order $O(\|(d,h,\varepsilon,\Delta)\|^p)$ (which have Fenichel form still).

Proof: It is easy to see that the only resonant terms are dh^{k+1}, $\varepsilon^2 h^k$, $\varepsilon\Delta h^k$, $\Delta^2 h^k$ with $k \geq 0$. Further the normal form transformation is of the form $d = \bar{d} + \psi(\bar{d}, h, \varepsilon, \Delta)$ with $\psi = O(\|(\bar{d}, \varepsilon, \Delta)\|)O(\|(\bar{d}, h, \varepsilon, \Delta)\|)$. This and the special form of the equation for the other variables leads to a new equation for $\bar{d}$ which again has Fenichel form. (We omit the $^-$ in the following.) □

We can solve the equation for

$$\varepsilon = \varepsilon_i e^{-t/2},\ \Delta = \Delta_i e^{-t/2}$$

explicitly. Next we make the assumption that all solutions for $t \in [0, T]$ are in a small area

$$U := \{r, d, h, \Delta \in [-\rho, \rho],\ \varepsilon \in [0, e]\}.$$

We will show later that this is satisfied if the initial conditions are appropriate and ρ, e are sufficiently small. This we use to find bounds for higher order terms. For the notation, we use C_i for bounds which are simple constants and K_i for bounds for terms of order $O(\|(\bar{d}, d, h, \varepsilon, \Delta, \varepsilon_i, \Delta_i)\|)$, because they get smaller when ρ, e get smaller. We use the Fenichel form of the higher order terms $O(p)$ to get:

Lemma 14. *The solution $d(t)$ of system (2.27) is a solution of an equation of the form*

$$d' = -d + df_1(d, h, \varepsilon, \Delta) + e^{-t} f_2(d, h, \varepsilon, \Delta, \varepsilon_i, \Delta_i) + e^{-1/2t} h^{p-1} f_3(d, h, \varepsilon, \Delta, \varepsilon_i, \Delta_i)$$

with $\|f_1\| \leq K_1,\ \|f_2\| \leq K_2,\ \|f_3\| \leq K_3$ on U with the initial condition $d(0) = d_i$.

Proof: For the equation in Lemma 13 we can separate the resonant and nonresonant terms in two different groups:

- terms of order $O(\|d\|)$, which are

$$dg_1(h),\ dO(\|(d, h, \varepsilon, \Delta)\|^{p-1})$$

- terms of order $O(\|d\|^0)$, which are

$$\varepsilon^2 g_2(h),\ \varepsilon\Delta g_3(h),\ \Delta^2 g_4(h),\ O(\|(\varepsilon, \Delta)\|^2)O(\|(h, \varepsilon, \Delta)\|^{p-2}),\ O(\|(\varepsilon, \Delta)\|)|h|^{p-1}.$$

The first kind can be described by $df_1(d, h, \varepsilon, \Delta)$ with $f_1 = O(\|(d, h, \varepsilon, \Delta)\|)$. If we plug the solutions $\varepsilon(t)$ and $\Delta(t)$ in the second kind of terms we can collect the first four terms to $e^{-t} f_2(d, h, \varepsilon, \Delta, \varepsilon_i, \Delta_i)$ with $f_2 = O(\|\varepsilon_i, \Delta_i)\|^2)$ and the last term has the form $e^{-1/2t} h^{p-1} f_3(d, h, \varepsilon, \Delta, \varepsilon_i, \Delta_i)$ with $f_3 = O(\|\varepsilon_i, \Delta_i)\|)$. Because the f_i are higher order terms there exists small constants K_i such that $f_i \leq K_i$ (which can be chosen smaller when ρ, e get smaller). □

We will use the following two relations:

Lemma 15. *The differential equation*

$$d' = -d + f(t, d),\ d(0) = d_i$$

has the equivalent integral formulation

$$d(t) = e^{-t} d_i + \int_0^t e^{-t+\tau} f(\tau, d(\tau)) d\tau.$$

Lemma 16. *(Lemma of Gronwall) The assumptions* $J := \{t \in [0,T]\}$, $g(t), v(t) \in C(J)$, $0 \le h(t)$ *Lebesgue integrable on* J *together with the inequality*

$$v(t) \le g(t) + \int_0^t h(\tau)v(\tau)d\tau$$

imply

$$v(t) \le g(t) + \int_0^t g(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau$$

with $H(t) = \int_0^t h(\tau)d\tau$ *(see [36] p.15).*

With these Lemmas we get:

Lemma 17. *The solution to the equation of Lemma 14 has the form*

$$d(t) = (d_0(t) + d_1(t)t + d_2(t)t^2)e^{-t} + h^{p-1}d_3(t)e^{-t/2}$$

where $|d_0(t)| \le |d_i|$, $|d_1(t)| \le |d_i|K_1e^{K_1} + K_2$, $|d_2(t)| \le K_2K_1e^{K_1}$ *and* $|d_3(t)| \le 2K_3(1 + K_1e^{K_1})$ *for* $t \in [0,T]$.

Proof: We first transform the equation of Lemma 14 to a integral equation according to Lemma 15 and then take the absolute value. So we get

$$|d| \le e^{-t}|d_i| + \int_0^t e^{-t+\tau}|df_1 + e^{-\tau}f_2 + e^{-\tau/2}h^{p-1}f_3|d\tau.$$

Taking into account the bounds on the f_i we get the inequality

$$|d| \le e^{-t}|d_i| + \int_0^t e^{-t}(K_2 + e^{\tau/2}|h|^{p-1}K_3)d\tau + \int_0^t e^{-t}K_1|d|d\tau.$$

The terms in the Gronwall inequality are
$g(t) = (|d_i| + K_2t)e^{-t} + K_3|h|^{p-1}2(e^{-t/2} - e^{-t})$ and
$H(t) - H(\tau) = K_1(e^{-\tau} - e^{-t}) \le K_1$. Plugging these terms into the Gronwall inequality and integration yields

$$|d| \le (|d_i|+K_2t+(|d_i|t+K_2t^2/2)K_1e^{K_1})e^{-t}+K_3|h|^{p-1}2(e^{-t/2}-e^{-t}+2(e^{-t/2}-e^{-t}-te^{-t})K_1e^{K_1}),$$

which leads to the form of our Theorem. Next we have to show that $d(t)$ indeed satisfies our assumption $d(t) \in [-\rho, \rho]$. Taking into account our form of solution and that $e^{-t} \le 1, te^{-t} \le 1, t^2e^{-t} \le 1$ we obtain

$$|d(t)| \le |d_i| + |d_i|K_1e^{K_1} + K_2 + K_2K_1e^{K_1} + 2K_3(1 + K_1e^{K_1}) < \rho.$$

If we take $K_2 := sup_{U,|\bar{\varepsilon}_i| \le |\varepsilon_i|, |\bar{\Delta}_i| \le |\Delta_i|} f_2(d, h, \varepsilon, \Delta, \bar{\varepsilon}_i, \bar{\Delta}_i) = O(\|(\varepsilon_i, \Delta_i)\|^2)$ and analog $K_3 = O(\|(\varepsilon_i, \Delta_i)\|)$ we see that we can choose $d_i, \varepsilon_i, \Delta_i$ small enough, such that the above inequality is satisfied for $t \in [0,T]$. □

As a next step we will calculate the solution for the equation for $r > 0$.

Lemma 18. *For $r > 0$ system (2.26) is equivalent to*

$$\begin{aligned} r' &= \frac{1}{2}r \\ d' &= -d + \Lambda_N(d,h,\varepsilon,\Delta)(d,\varepsilon,\Delta)^T + r\Lambda_1(r,d,h,\varepsilon,\Delta)(d,\varepsilon,\Delta)^T \\ h' &= H(r,d,h,\varepsilon,\Delta)(d,\varepsilon,\Delta)^T r \\ \varepsilon' &= -\frac{1}{2}\varepsilon \\ \Delta' &= -\frac{1}{2}\Delta, \end{aligned} \tag{2.28}$$

where Λ_N is the nonlinear part of Lemma (13) and Λ_1 of order $O(1)$. The initial conditions have the form $h_i = r_i h^*(r_i, \Delta_i)$, $d = d_i^*(r_i, \Delta_i)$.

Proof: In equation (2.26) Λ_1 would be of order $O(0)$. Because all terms $rd, r\varepsilon, r\Delta$ are not resonant we can apply one step of normal form transformation and get a new system where Λ_1 is of order $O(1)$. Further the Fenichel form and the special form of the initial conditions will be preserved. □

If we plug the solutions for $r(t), d(t), \Delta(t)$ into equation (2.28) we get a 2-dimensional system for d' and h'. Because it is easier to apply the Lemma of Gronwall to 1-dimensional systems, we make the assumption that

$$h(t) = h_i + r_i t h_1(t) \tag{2.29}$$

with $|h_1| \le C$, which we have to prove later.

For $d(t)$ we make the Ansatz

$$d(t) = \bar{d}(t) + r_i d_4(t), \tag{2.30}$$

where $\bar{d}(t)$ is the solution for $r = 0$. Further we make the assumptions $|d_4| \le C$, $|h| \le \rho$ on $[0, T]$.

Lemma 19. *$d_4(t)$ is a solution to the equation*

$$d_4' = -d_4 + d_4 f_1 + g_f + r_i d_4 g_1$$

with $f_1 = f_1(\bar{d}, h, \varepsilon, \Delta, r_i d_4) \le L_0$, $g_f = g_f(\bar{d}, h, \varepsilon, \Delta, r) \le K_L$ and $g_1 = g_1(\bar{d}, h, \varepsilon, \Delta, r, r_i d_4) \le L_1 e^{t/2}$ with the three small constants L_0, L_1, K_L.

Proof: We define $f := \Lambda_N(d,\varepsilon,\Delta)^T$ and $g := e^{t/2}\Lambda_1(d,\varepsilon,\Delta)^T$. Because $|f_d| = |\Lambda_{Nd}(d,\varepsilon,\Delta)^T + \Lambda_N(1,0,0)^T| = O(1)$ and $|g_d| = |e^{t/2}\Lambda_{1d}(d,\varepsilon,\Delta)^T + e^{t/2}\Lambda_1(1,0,0)^T| = e^{t/2}O(1)$ we get the two small Lipschitz constants $|f_d| \le L_0$, $|g_d| \le L_1 e^{t/2}$ on U small enough. From the Mean Value Theorem we have that there exists some $\vartheta \in (0, s)$ such that

$$\frac{f(d+s,h,\varepsilon,\Delta) - f(d,h,\varepsilon,\Delta)}{s} = f_d(d+\vartheta,h,\varepsilon,\Delta) =: f_1(d,h,\varepsilon,\Delta,s)$$

with $|f_1| \le L_0$ and analogously

$$\frac{g(d+s,h,\varepsilon,\Delta,r) - g(d,h,\varepsilon,\Delta,r)}{s} = g_d(d+\vartheta,h,\varepsilon,\Delta,r) =: g_1(d,h,\varepsilon,\Delta,r,s)$$

with $|g_1| \leq L_1 e^{t/2}$. Further we have that because

$$(\bar{d}, \varepsilon, \Delta)^T = e^{-t/2}((d_0 + td_1 + t^2 d_2)e^{-t/2} + h_i^{p-1} d_3, \varepsilon_i, \Delta_i)^T$$

and $\Lambda_1 = O(1)$ there exists some small constant K such that

$$|g(\bar{d}, h, \varepsilon, \Delta, r)| = |e^{t/2}\Lambda_1(\bar{d}, h, \varepsilon, \Delta, r)(\bar{d}, \varepsilon, \Delta)^T| \leq K.$$

Analogously we get for $h = h_i + r_i t h_1$ that

$$\frac{f(d, h, ...) - f(d, h + s, ...)}{s} =: f_2(d, h_i, ..., s)$$

with $|f_2| \leq L_2$. We use this on the equation

$$d' = -d + f(d, h, \varepsilon, \Delta) + r_i g(d, h, \varepsilon, \Delta, r)$$

and get

$$\begin{aligned} \bar{d}' + r_i d_4' &= -(\bar{d} + r_i d_4) + f(\bar{d} + r_i d_4, h_i + r_i t h_1, ...) + r_i g(\bar{d} + r_i d_4, ...) \\ &= -(\bar{d} + r_i d_4) + f(\bar{d}, h_i ...) + f_2(\bar{d}, h_i, ..., r_i t h_1) r_i t h_1 + r_i d_4 f_1(\bar{d}, h, ..., r_i d_4) + \\ &\quad + r_i g(\bar{d}, ...) + r_i^2 d_4 g_1(\bar{d}, ..., r_i d_4). \end{aligned}$$

Because $\bar{d}$ solves the equation $\bar{d}' = -\bar{d} + f(\bar{d}, h_i, ...)$ and we substract these terms and divide by r_i and so finally get the equation

$$d_4' = -d_4 + f_2 r_i t h_1 + d_4 f_1 + g + r_i d_4 g_1$$

Because $f_2 r_i t h_1 \leq L_2 e^{-t/2} t C \leq L_2 C$ and $|g| \leq K$ we can define a new function

$$g_f := f_2 r_i t h_1 + g, |g_f| \leq K_L.$$

□

Proposition 7. *The solution of system (2.28) has the form*

$$\begin{aligned} d(t) &= (d_0(t) + d_1(t)t + d_2(t)t^2)e^{-t} + h_i^p d_3 e^{-t/2} + r_i d_4(t) \qquad (2.31) \\ h(t) &= h_i + r_i t h_1(t) \end{aligned}$$

with $t \in [0, T]$, $|d_4(t)| \leq K_L + 2K_L(L_0 + \rho L_1)e^{L_0 + \rho L_1}$ *and* $|h_1(t)| \leq K_4$.

Proof: With Lemma 19 we get the solution $d(t)$ by transforming to a integral equation and applying the Lemma of Gronwall analogously to Lemma 17. For h we plug the solution for d into equation (2.28). For $h' = H(d, \varepsilon, \Delta)r$ we get that $|H| \leq C$ and as above

$$\|(d, \varepsilon, \Delta)^T r\| = \|((d_0 + td_1 + t^2 d_2)e^{-t/2} r_i + h_i^{p_1} d_3 r_i + r_i d_4 r_i e^{-t/2})\| \leq r_i K_4.$$

The integral equation for h leads to

$$|h| \leq |h_i + \int_0^t H(d, \varepsilon, \Delta)r| \leq h_i + \int_0^t C r_i K_4 = h_i + r_i t K_4$$

and we get $h(t) = h_i + r_i t h_1(t)$ with $|h_1| \leq K_4$. Because $|d_4| \leq K_L + 2K_L(L_0 + \rho L_1)e^{L_0+\rho L_1}$ and $|h_1| \leq K_4$ the small constants K_4, K_L get small enough such that the assumptions $|d_4| \leq C$, $|h_1| \leq C$ are satisfied for $t \in [0,T]$ for h_i, r_i, ε, Δ_i, d_i small enough. Analogously $|\bar{d}| \in \rho/2$ and $|d| \leq \rho$, $h \leq \rho$ is satisfied too. □

For the transition map Π we need to show that $d_{\Delta_i}(T), h_{\Delta_i}(T), d_{r_i}(T), h_{r_i}(T)$ is sufficiently smooth. Therefore we look at the variational equations. If we write the equations as

$$\begin{aligned} d' &= -d + \tilde{D} = -d + (\Lambda_0 + r\Lambda_1)(d, \varepsilon, \Delta)^T \\ h' &= \tilde{H} = H(d, \varepsilon, \Delta)^T, \end{aligned}$$

we get the variational equations

$$\begin{aligned} d'_{\Delta_i} &= -d_{\Delta_i} + \tilde{D}_{\Delta_i} = -d_{\Delta_i} + \tilde{D}_d d_{\Delta_i} + \tilde{D}_h h_{\Delta_i} + \tilde{D}_\Delta e^{-t/2} \\ h'_{\Delta_i} &= \tilde{H}_{\Delta_i} = \tilde{H}_d d_{\Delta_i} + \tilde{H}_h h_{\Delta_i} + \tilde{H}_\Delta e^{-t/2} \end{aligned}$$

and

$$\begin{aligned} d'_{r_i} &= -d_{r_i} + \tilde{D}_{r_i} = -d_{r_i} + \tilde{D}_d d_{r_i} + \tilde{D}_h h_{r_i} + \tilde{D}_r e^{t/2} \\ h'_{r_i} &= \tilde{H}_{r_i} = \tilde{H}_d d_{r_i} + \tilde{H}_h h_{r_i} + \tilde{H}_r e^{t/2}. \end{aligned}$$

Because $\tilde{D}_d = (\Lambda_{0d} + r\Lambda_{1d})(d, \varepsilon, \Delta)^T + (\Lambda_0 + r\Lambda_1)(1,0,0)^T = O(1)$, similarly $\tilde{D}_\Delta = O(1)$ and with $\|(d, \varepsilon, \Delta)\| = \|((d_0 + td_1 + t^2 d_2)e^{-t/2} + h_i^{p-1} d_3 + r_i e^{t/2} d_4, \varepsilon_i, \Delta_i)\| e^{-t/2} \leq K_4 e^{-t/2}$, $\tilde{D}_h = O(1)e^{-t/2}$ and $\tilde{D}_r = O(1)e^{-t/2}$ we get the bounds:

$$|\tilde{D}_d| \leq K_5,\ |\tilde{D}_h| \leq K_6 e^{-t/2},\ |\tilde{D}_\Delta| \leq K_7,\ |\tilde{D}_r| \leq K_8 e^{-t/2}.$$

In a similar way, because $r = r_i e^{t/2}$, we get

$$|\tilde{H}_d| \leq C_1 r,\ |\tilde{H}_h| \leq K_9 r_i,\ |\tilde{H}_\Delta| \leq C_2 r,\ |\tilde{H}_r| \leq K_{10} r_i + K_{11} e^{-t/2}.$$

We again need some assumptions for h on $[0,T]$: there exists some C, $\alpha \in (0,1)$ such that

$$|h_{\Delta_i}| \leq C,\ |r_i^\alpha h_{r_i}| \leq C.$$

Proposition 8. *With $|d_i(r_i, \Delta_i)_{\Delta_i}| \leq C_3$ and $|h_i(r_i, \Delta_i)_{\Delta_i}| \leq r_i C_4$ the derivatives satisfy*

$$\begin{aligned} |d_{\Delta_i}(t)| &\leq e^{-t} C_3 + K_{13} t e^{-t} + K_{14}(e^{-t/2} - e^{-t}) \\ |h_{\Delta_i}(t)| &\leq r_i C_5 + C_6 r_i t \end{aligned}$$

for some small constants K_{13}, K_{14} and constants C_5, C_6.

Proof: An equivalent integral equation for $d_{\Delta_i}(t)$ is

$$\begin{aligned} d_{\Delta_i} &= e^{-t} d_{i\Delta_i} + \int_0^t e^{-t+\tau} \tilde{D}_{\Delta_i} d\tau \\ &= e^{-t} d_{i\Delta_i} + \int_0^t e^{-t+\tau} (\tilde{D}_d d_{\Delta_i} + \tilde{D}_h h_{\Delta_i} + \tilde{D}_\Delta e^{-\tau/2}) d\tau. \end{aligned}$$

With the assumption $|h_{\Delta_i}| \leq C$ and the bounds on the derivatives we get the inequality

$$|d_{\Delta_i}| \leq e^{-t}C_3 + \int_0^t e^{-t}(K_6 e^{\tau/2}C + K_7 e^{\tau/2} + e^{\tau}K_5|d_{\Delta_i}|)d\tau$$

where we can apply the Lemma of Gronwall. With $K_{12} := 2(K_6C + K_7)$, $K_{13} := K_5e^{K_5}C_3$ and $K_{14} := K_5e^{K_5}K_{12}2$ we get the bounds on $|d_{\Delta_i}(t)|$. From this we know that d_{Δ_i} can be written as

$$d_{\Delta_i}(t) = e^{-t}(\bar{d}_1(t) + t\bar{d}_2(t)) + \bar{d}_3(t)(e^{-t/2} - e^{-t})$$

with $|\bar{d}_1(t)| \leq C_3,\ |\bar{d}_2(t)| \leq K_{13},\ |\bar{d}_3(t)| \leq K_{14}$. We plug this into the equations for h_{Δ_i} and get

$$\begin{aligned} h_{\Delta_i} &= h_{i\Delta_i} + \int_0^t \tilde{H}_{\Delta_i} d\tau \\ &= h_{i\Delta_i} + \int_0^t (\tilde{H}_d d_{\Delta_i} + \tilde{H}_h h_{\Delta_i} + \tilde{H}_{\Delta} e^{-\tau/2})d\tau. \end{aligned}$$

Taking the absolute value and plugging the bounds into the equation leads to

$$|h_{\Delta_i}| \leq r_iC_4 + \int_0^t (C_1r_i(e^{-\tau/2}C_3 + K_{13}\tau e^{-\tau/2} + K_{14}(1-e^{-\tau/2})) + C_2r_i + K_9r_i|h|_{\Delta_i})d\tau,$$

where we can apply the Lemma of Gronwall again. If $C_5 := C_1(C_32 + K_{13}4)$ and because there is a $C_6 \geq C_2 + r_itC_5 + r_it^2/2C_2$ for $t \in [0,T]$ we get the bounds on $|h_{\Delta_i}|$. We see that for r_i sufficiently small, $T = 2ln(\rho/r_i)$ the assumption $|h_{\Delta_i}(t)| \leq r_1C_5 + C_6r_iT \leq C$ is satisfied. □

For the derivatives with respect to r_i we get a similar result:

Proposition 9. *With $|d_i(r_i,\Delta_i)_{r_i}| \leq C_7$ and $|h_i(r_i,\Delta_i)_{r_i}| \leq C_8$ the derivatives satisfy*

$$\begin{aligned} &|d_{r_i}(t)| \leq e^{-t}C_7 + K_{16}r_i^{-\alpha}e^{-t/2} + K_{17} \\ &|h_{r_i}(t)| \leq C_{10} + K_{18}t \end{aligned}$$

for some small constants K_{16}, K_{17}, K_{20} and constants C_7, C_{10}.

Proof: The proof is analogous to the previous Theorem. Especially we get the equations

$$|d_{r_i}| \leq e^{-t}C_7 + \int_0^t e^{-t+\tau}(K_6e^{\tau/2}Cr_i^{\alpha} + K_8 + K_5|d_{r_i}|)d\tau$$

and

$$|h_{r_i}| \leq C_8 + \int_0^t (K_{18} + K_{19}r_ie^{-\tau/2} + K_9r_i|h|_{r_i})d\tau.$$

With new constants $K_{15} := K_6C2$, $K_{16} := K_{15} + K_5e^{K_5}K_{15}2$, $K_{17} := K_8 + K_5e^{K_5}(C_7 + K_8)$, $C_9 := C_8 + C_1\rho 2C_7$, $K_{19} := C_1K_{16}$ and because there exists a constant $C_{10} \geq C_9 + r_1^{1-\alpha} K_{19} + K_9 e^{K_9 \rho} r_i (C_9 t + K_{18} t^2/2 + r_i K_{19} t^2/2)$ for $t \in [0, T]$, we get the statement of the Theorem. Again we see that for r_i sufficient small and $T = ln(\rho/r_i)$, the assumption $|r_i^\alpha h_{r_i}(t)| \leq r_i^\alpha (C_{10} + K_{18} T) \leq C$ is satisfied. □

Because T only depends on r_i we get with

$$\frac{\partial}{\partial \Delta_i}(d(T)) = d_{\Delta_i}(T),\ \frac{\partial}{\partial \Delta_i}(h(T)) = h_{\Delta_i}(T)$$

and

$$\frac{\partial}{\partial r_i}(d(T)) = d_{r_i}(T) + d'(T) T_{r_i},\ \frac{\partial}{\partial r_i}(h(T)) = h_{r_i}(T) + h'(T) T_{r_i}$$

the following result:

Proposition 10. *For r_i, Δ_i small enough $\gamma_4(r_i, \Delta_i) \cap \Sigma_4^{out}$ satisfies*

$$\begin{array}{lll} d(T) = O(r_i), & \frac{\partial}{\partial \Delta_i}(d(T)) = O(r_i), & \frac{\partial}{\partial r_i}(d(T)) = O(r_i^0) \\ h(T) = O(r_i ln(r_i)), & \frac{\partial}{\partial \Delta_i}(h(T)) = O(r_i ln(r_i)), & \frac{\partial}{\partial r_i}(h(T)) = O(ln(r_i)). \end{array}$$

The combination of coordinate transformation from the coordinates of chart K_4, $(r_4, D_4, H, \varepsilon_4, \Delta_4)$, to the final coordinates in Fenichel normal form with some steps of normal form transformation, $(r, d, h, \varepsilon, \Delta)$, is a C^k diffeomorphism:

$$r_4 = r, D_4 = \Phi_1(r, d, h, \varepsilon, \Delta), H = \Phi_2(r, d, h, \varepsilon, \Delta), \varepsilon_4 = \varepsilon, \Delta_4 = \Delta.$$

For $r = 0$ the intersection of $\gamma(0,0)$ with Σ_4^{out}, which is equal to $q^o = M^u \cap \Sigma_4^{out}$ is

$$(r, d, h, \varepsilon, \Delta) = (\rho, 0, 0, 0, 0),$$

which is in original coordinates of K_4

$$q^o = (\rho, \Phi_1(\rho, 0, 0, 0, 0), \Phi_2(\rho, 0, 0, 0, 0), 0, 0)^T.$$

Theorem 7. *For the intersection of γ_4 with Σ_4^{out} we get*

$$\begin{pmatrix} r_o \\ D_o \\ H_o \\ \varepsilon_o \\ \Delta_o \end{pmatrix} = q^o + \begin{pmatrix} 0 \\ D_*(\varepsilon_o, \Delta_o) \\ H_*(\varepsilon_o, \Delta_o) \\ \varepsilon_o \\ \Delta_o \end{pmatrix},$$

where D_*, H_* are C^k for $\varepsilon > 0$, which satisfy

$$D_*(\varepsilon_o, \Delta_o) = O(\varepsilon_o ln(\varepsilon_o)),\ H_*(\varepsilon_o, \Delta_o) = O(\varepsilon_o ln(\varepsilon_o))$$

and

$$\frac{\partial}{\partial \varepsilon_o} D_* = O(ln(\varepsilon_o)),\ \frac{\partial}{\partial \varepsilon_o} H_* = O(ln(\varepsilon_o)),\ \frac{\partial}{\partial \Delta_o} D_* = O(ln(\varepsilon_o)),\ \frac{\partial}{\partial \Delta_o} H_* = O(ln(\varepsilon_o)).$$

for $\varepsilon_o \to 0$.

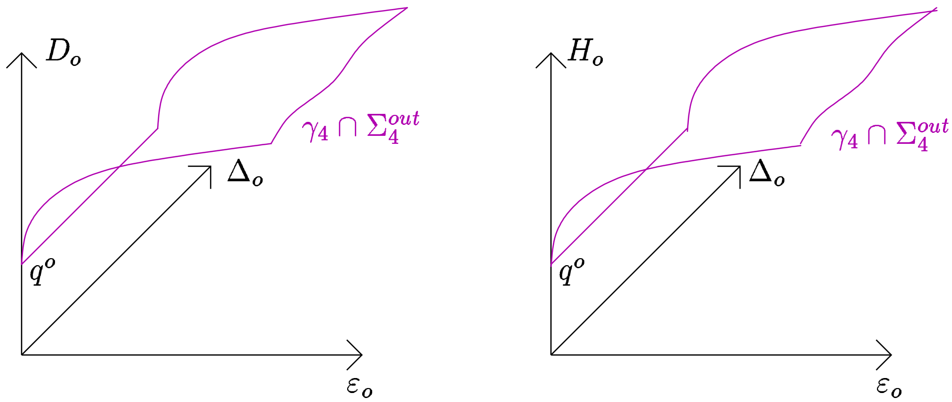

Figure 2.16: Intersection of γ_4 with Σ_4^{out}

Proof: We can write the transformations Φ_i as

$$D_4 = \Phi_1(r,0,0,0,0) + A_1(r,d,h,\varepsilon,\Delta)(d,h,\varepsilon,\Delta)^T$$
$$H = \Phi_2(r,0,0,0,0) + A_2(r,d,h,\varepsilon,\Delta)(d,h,\varepsilon,\Delta)^T$$

with two functions $A_1, A_2 \in C^k(\mathbb{R}^5, \mathbb{R}^4)$. So we get in Σ_4^{out} that

$$\begin{aligned}\Phi_i(\rho,d,h,\varepsilon,\Delta) - \Phi_i(\rho,0,0,0,0) &= A_1(\rho,d,h,\varepsilon,\Delta)(O(r_i), O(r_i ln(r_i)), O(r_i), O(r_i)) \\ &= O(r_i ln(r_i)).\end{aligned}$$

For the derivatives we get

$$\frac{\partial}{\partial \Delta_i}\Phi_i(\rho,d,h,\varepsilon,\Delta) = O(r_i ln(r_i)),\ \frac{\partial}{\partial r_i}\Phi_i(\rho,d,h,\varepsilon,\Delta) = O(ln(r_i)).$$

So we have in Σ_4^{out}:

$$\begin{pmatrix} r_o \\ D_o \\ H_o \\ \varepsilon_o \\ \Delta_o \end{pmatrix} = q^o + \begin{pmatrix} 0 \\ O(r_i ln(r_i)) \\ O(r_i ln(r_i)) \\ er_i/\rho \\ \Delta_i r_i/\rho \end{pmatrix}$$

Changing the parametrisation to $\varepsilon_o = \frac{er_i}{\rho}, \Delta_o = \frac{\Delta_i r_i}{\rho}$ we get with

$$\frac{\partial}{\partial \varepsilon_o} D_4 = \frac{\partial D_4}{\partial r_i}\frac{\rho}{e} + \frac{\partial D_4}{\partial \Delta_i}\frac{-\Delta_i}{\varepsilon_o},\ \frac{\partial}{\partial \Delta_o} D_4 = \frac{\partial D_4}{\partial \Delta_i}\frac{e}{\varepsilon_o}.$$

our Theorem. □

2.3.6 Jump back to $D < 0$ of γ

By now we have obtained a full description of the orbit $\gamma(\varepsilon, \Delta)$ in a neighborhood of the cylinder $\{\bar{r} = 0\}$. We now describe how the orbit returns from K_4 (with $D > 0$) to K_1. In original coordinates we have

$$J_{out} = \rho^2,\ D_{out} = \rho D_o,\ H_{out} = H_o,\ \varepsilon = \rho\varepsilon_o,\ \Delta = \rho\Delta_o.$$

Therefore we get for $\gamma(\varepsilon,\Delta)$ that $D_{out} = D_{out}(\varepsilon,\Delta), H_{out} = H_{out}(\varepsilon,\Delta)$ with the bounds of Theorem 7. The jump back (see chapter 2.1) to

$$\Sigma^{in} := \{J = \rho^2\}$$

is a C^k diffeomorphism:

$$D = \Psi_1(D_{out}, H_{out}, \varepsilon, \Delta),\ H = \Psi_2(D_{out}, H_{out}, \varepsilon, \Delta).$$

For $\varepsilon = 0$ the jump back and D_{out}, H_{out} are independent of Δ and

$$\Psi_i(D_{out}(0,\Delta), H_{out}(0,\Delta), 0, \Delta) = \Psi_i(D_{out}(0,0), H_{out}(0,0), 0, 0),$$

and $q_i := \Psi(q^o)$.

Theorem 8. *In Σ^{in} γ satisfies*

$$J_i = \rho^2,\ D_i = D(\varepsilon,\Delta),\ H_i = H(\varepsilon,\Delta)$$

with $D(\varepsilon,\Delta)$ and $H(\varepsilon,\Delta)$ are C^k for $\varepsilon > 0$ and continuous on $\varepsilon \geq 0$ with

$$\frac{\partial}{\partial\varepsilon}D = O(ln(\varepsilon)),\ \frac{\partial}{\partial\varepsilon}H = O(ln(\varepsilon)),\ \frac{\partial}{\partial\Delta}D = O(ln(\varepsilon)),\ \frac{\partial}{\partial\Delta}D = O(ln(\varepsilon)).$$

2.3.7 Exponential attraction to the slow manifold $J = 0$

As a last step we describe the dynamics near the slow manifold S_ε. Because for $\gamma_4 \cap \Sigma_4^{out}$ we have $D_4 \geq c > 0$ with a constant c, we have that $D \geq \rho c > 0$ and therefore $\Sigma^{in} \cap \gamma$ satisfies $supD \leq -\tilde{c} < 0$ for some constant $\tilde{c} > 0$. For $\rho < \tilde{c}$ we need the transition to

$$\Sigma^\rho := \{D = -\rho\}$$

which maps to Σ_1^{in} in chart K_1.

Theorem 9. *$\gamma \cap \Sigma^{out}$ satisfies the relations*

$$J_o = J_o(\varepsilon,\Delta),\ D_o = -\rho,\ H_o = H_o(\varepsilon,\Delta)$$

with $J_o, H_o \in C^{k-1}$ for $r > 0$ and $J_o, H_o \in C$ for $r \geq 0$, satisfying the bounds

$$J_o = O(e^{-c/\varepsilon}),\ \frac{\partial}{\partial\varepsilon}J_o = O(e^{-c/\varepsilon}),\ \frac{\partial}{\partial\Delta}J_o = O(e^{-c/\varepsilon}),$$

$$H_o = O(1),\ \frac{\partial}{\partial\varepsilon}H_o = O(ln(\varepsilon)),\ \frac{\partial}{\partial\Delta}H_o = O(ln(\varepsilon)).$$

Proof: According to Fenichel's theory we have a stable foliation F^s over the slow manifold $S_\varepsilon = \{J = 0\}$. The dynamics separates in a slow drift in direction of the slow manifold and an exponential attraction of order $O(e^{-c/\varepsilon})$ to S_ε along the stable fibers. Because the dynamics in the slow manifold is C^k and the foliation is C^k the map Π is C^k. For

$$(J_o, D_o, H_o, \varepsilon, \Delta) = \Pi(J_i, D_i, H_i, \varepsilon, \Delta)$$

we have:

$$J_o = e^{-c/\varepsilon}j(J_i, D_i, H_i, \varepsilon, \Delta),\ H_o = h(J_i, D_i, H_i, \varepsilon, \Delta),\ D_o = -\rho,$$

with j, h C^k. Because $(J_o)_\varepsilon = e^{-c/\varepsilon}O(ln(\varepsilon)),\ (H_o)_\varepsilon = O(1)O(ln(\varepsilon))$ and analogously for the derivatives with respect to Δ we get this Theorem. □

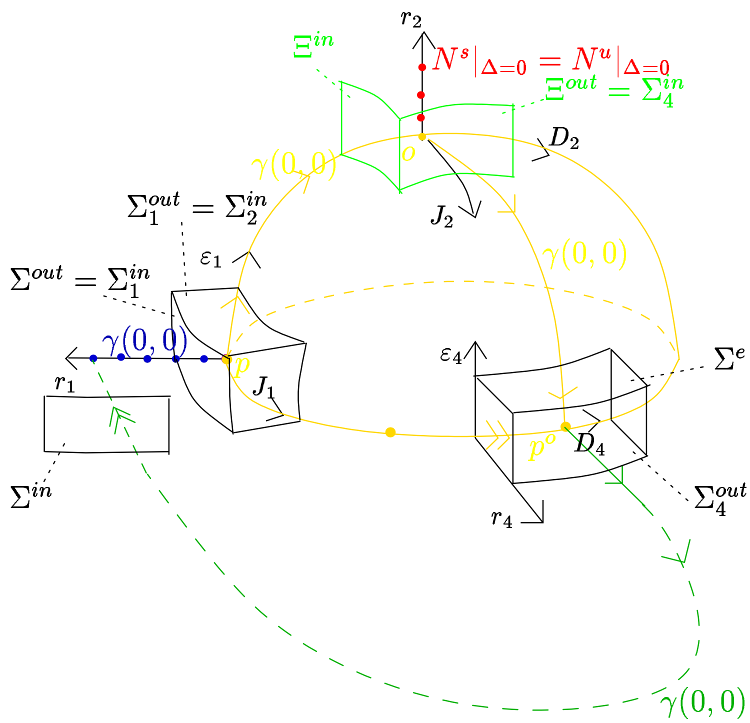

Figure 2.17: The different sections Σ transversal to the homoclinic orbit γ of the proof

2.3.8 Proof of Theorem 1

Now we are able to finish the proof of Theorem 1 by collecting the results we have established in the various charts. An overview over the different sections Ξ and Σ we have used is given in Fig. 2.17. We describe the evolution of the 2-dimensional manifold $\gamma(\varepsilon, \Delta)$ attached to the singular orbit $\gamma(0,0)$ which contains the homolinic orbits $\gamma(\varepsilon, \Delta_{Hom})$ as a submanifold.

In chart K_2 the manifold $\gamma_2(\varepsilon_2, \Delta_2)$ starts as the invariant unstable C^k manifold M^u of N_2^u. Its intersection with Ξ^{out} is given in Theorem 5.

In chart K_4 the transition from Σ_4^{in} to Σ_4^{out} is described in two steps. The transition from Σ_4^{in} to Σ_4^e is a diffeomorphism. The intersection $\gamma_4 \cap \Sigma_4^e$ is still C^k (Lemma 9). During the transition from Σ_4^e to Σ_4^{out} it looses some of its smoothness. This is due to resonances of the nonhyperbolic equilibrium p_4^o. This transition is described in Theorem 7 by transformation to Fenichel coordinates and Gronwall-like estimates to get bounds on the variables and derivatives of $\gamma_4 \cap \Sigma_4^{out}$. The bounds on the derivatives are of order $O(ln(\varepsilon_4))$.

Then the transition from Σ_4^{out} to Σ^{in} is described in original coordinates as a regular perturbation of the layer problem (Theorem 8).

The transition from Σ^{in} to Σ^{out} is described by the Fenichel theory in Theorem 9. Due to the attraction of $\gamma(\varepsilon, \Delta)$ to $J = 0$ by an exponential rate of order $O(e^{-c/\varepsilon})$ the J component of $\gamma(\varepsilon, \Delta)$ regains differentiability: the derivatives of $J = O(e^{-c/\varepsilon})$ can be continuously extended to $\varepsilon = 0$.

In chart K_1 the manifold $\gamma_1 \cap \Sigma_1^{in}$ satisfies the assumptions we have made for

γ_1 in $\Sigma_1^{in} = \Sigma^{out}$. The transition from Σ_1^{in} to Σ_1^{out} is described with the center manifold in p and its foliation (Theorem 2). The intersection $\gamma_1 \cap \Sigma_1^{out}$ is described in Theorem 3. Due to the relation $|H_{1o} - B| \sim |r_{1o}\Delta_{1o} + (H_{1i} - B)\varepsilon_{1i}|$ (Lemma 3) the derivatives of the variable H_{1o} of $\gamma_1 \cap \Sigma_1^{out}$ are continously extendable, too. So manifold $\gamma_1 \cap \Sigma_1^{out}$ is a C^1 manifold.

Next we describe the trasition from $\Sigma_2^{in} = \Sigma_1^{out}$ to Ξ^{in} which is a diffeomorphism (Proposition 5). A transcritical bifurcation of the curves N_2^s and N_2^u which takes place in the 3-dimensional center manifold in q_2 (Lemma 7, Lemma 8). A homoclinic orbits exists for the parameter values of ε_2, Δ_2 where the orbit $\gamma_2(\varepsilon_2, \Delta_2)$ converges to an equilibrium in N_2^u. This means that $\gamma_2(\varepsilon_2, \Delta_2)$ is part of a stable fiber F^s of N_2^u. To show this we describe the intersection of the stable fibers F^s of N_2^u and the manifold $\gamma_2(\varepsilon_2, \Delta_2)$ in Ξ^{in}. In Theorem 4 we have shown that this intersection is transversal and takes place for a unique curve of parameters $\Delta_2 = \Delta_2(r_2) = O(e^(- c/r_2))$. Because $\varepsilon = r_2$ and $\Delta = r_2\Delta_2$ this completes the proof of Theorem 1. □

Appendix A

Invariant manifold theory

In this appendix we present some results from the geometric theory of dynamical systems we need in this thesis. For a general introduction we refer to [2],[3],[18],[38].

A.1 Center manifold theory and invariant foliations

Center manifold theory describes the local dynamics near nonhyperbolic equilibria of dynamical systems. The standard form of dynamical systems to apply this theory is

$$\begin{aligned} x_c' &= A_c x_c + f_c(x_s, x_u, x_c) \\ x_s' &= A_s x_s + f_s(x_s, x_u, x_c) \\ x_u' &= A_u x_u + f_u(x_s, x_u, x_c), \end{aligned} \tag{A.1}$$

where $x_s \in \mathbb{R}^{n_s}$, $x_u \in \mathbb{R}^{n_u}$, $x_c \in \mathbb{R}^{n_c}$, and $f_s, f_u, f_c \in C^k$ are of order $O(\|(x_s, x_u, x_c)\|^2)$. The eigenvalues λ_s of the matrix A_s have a negative real part, the eigenvalues of A_c are on the imaginary axes and the eigenvalues λ_u of A_u have a positive real part.

The linear part of system (A.1) has the following invariant subspaces: the stable manifold $E^s = \{x_u = x_c = 0\}$ (which is the eigenspace corresponding to the eigenvalues λ_s), a n_u-dimensional unstable manifold $E^u = \{x_s = x_c = 0\}$ (which is the eigenspace corresponding to the eigenvalues λ_u) and a n_c-dimensional manifold $E^c = \{x_s = x_u = 0\}$ (which is the eigenspace to the eigenvalues λ_c). Analogously we get the invariant subspaces $E^{cs} = \{x_u = 0\}$ and $E^{cu} = \{x_s = 0\}$. To get the dynamics of the full linear system we only have to combine the dynamics of the invariant sub-systems.

For the nonlinear system exists invariant sub-manifolds tangent to the invariant manifolds of the linear system ([8, 18]):

Theorem 10. *System (A.1) has an n_c-dimensional invariant manifold M^c tangential to E^c, an n_s-dimensional invariant manifold M^s tangential to E^s and an n_u-dimensional invariant manifold M^u tangential to E^u. Furthermore there exists an $(n_c + n_s)$-dimensional invariant manifold M^{cs} tangential to $E^c + E^s$ and an $(n_c + n_u)$-dimensional invariant manifold M^{cu} tangential to $E^c + E^u$. These invariant manifolds are C^k.*

In the neighborhood of the center-stable manifold M^{cs} or of the center-unstable manifold M^{cu} we can describe the dynamics by the dynamics on invariant foliations.

A foliation F of an n-dimensional manifold M consists of a family of m-dimensional connected submanifolds ('leaves' or 'fibers') $F(x)$ where $m < n$ and $x \in M$ which are identical or have an empty intersection ($F(x_1) = F(x_2)$ or $F(x_1) \cap F(x_2) = \emptyset$) and the foliation covers M, $M = \bigcup_x F(x)$.

An invariant foliation is a foliation that is invariant under the flow, which means that each point $y_0 \in F(x_0)$ is mapped to the same fiber, $y(t) \in F(x(t))$ with $x(0) = x_0, y(0) = y_0$, by the flow.

Let the real parts of the eigenvalues satisfy $Re(\lambda_s) < -\beta$, $Re(\lambda_u) > \beta$, further there exist constants $\alpha, \gamma > 0$ such that $\alpha < \beta$ and $\alpha < \gamma < k\gamma < \beta$ with the k from C^k.

Theorem 11. *There exists a stable (unstable) foliation F^s (F^u) of $\mathbb{R}^n$ near the center-unstable (center-stable) manifold M^{cu} (M^{cs}) which is invariant under the flow. The foliation has the following properties [8]:*

1. *Each fiber has an unique transversal intersection with M^{cu} (M^{cs}).*
2. *Each fiber $F^s(x)$ ($F^u(x)$) is a C^k manifold but only Hölder continuous in the base point x.*
3. *The distance of two points - which start on the same fiber of F^s (F^u) - is of order $O(e^{-\gamma t})$ ($O(e^{\gamma t})$).*

For $n_u = 0$ ($n_s = 0$) the invariant foliation F^s (F^u) to the center-stable manifold M^{cs} (center-unstable manifold M^{cu}) is C^k in the basepoint. In this case the distance between two point - which start in the same fiber of F^s (F^u) - is of order $O(e^{-\beta t})$ ($O(e^{\beta t})$).

A parameterization of the foliation F^s if $n_u = 0$

We use these results to obtain a better description of the stable foliation F^s which we will need in the proof of Proposition 3. In the case of $n_u = 0$ system (A.1) has the form:

$$\begin{aligned} x_c' &= A_c x_c + f_c(x_s, x_c) \\ x_s' &= A_s x_s + f_s(x_s, x_c). \end{aligned} \tag{A.2}$$

Because the center-manifold M^c can be written as a graph

$$x_s = \varphi(x_c)$$

with $\varphi(x_c) = (|x_c|^2)$ [38], the dynamics restricted to the center manifold is described by

$$x_c' = A_c x_c + f_c(\varphi(x_c), x_c). \tag{A.3}$$

The C^k fibers of the stable foliation F^s over M^c have a transversal intersection with M^c. Each fiber $F^s(x)$ can be written as a graph over $(0, x_s)^T$ for $x_s \in U \subset$

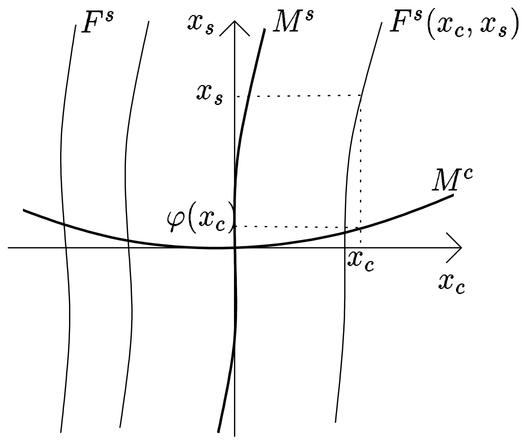

Figure A.1: The invariant manifolds and the invariant stable foliation F^s of system (A.2)

$\mathbb{R}^{n_s}$ a small neighborhood of the origin (see Fig. A.1). We can describe the fiber in the point (x_c, x_s) by

$$F^s(x_c, x_s) = \left\{ \begin{pmatrix} \tilde{f}(s, x_c) \\ s \end{pmatrix}, s \in U \right\}, \tag{A.4}$$

with a function $\tilde{f}(s, x_c) \in \mathbb{R}^{n_c}$ of class C^k.

We write the foliation in a another parameterization to make the difference between the coordinates (x_c, x_s) and the parameterization described by parameters $y_c \in \mathbb{R}^{n_c}, y_s \in \mathbb{R}^{n_s}$ clear:

$$\begin{pmatrix} x_c \\ x_s \end{pmatrix} = \begin{pmatrix} \tilde{f}(y_s, y_c) \\ y_s \end{pmatrix}. \tag{A.5}$$

Taking y_c constant describes one fiber. The parameterization is chosen such that the relation $y_c = 0$ corresponds to $x_c = 0$ and and

$$\tilde{f}(\varphi(y_c), y_c) = y_c.$$

Therefore a function $f \in C^r$ exists such that

$$\begin{pmatrix} x_c \\ x_s \end{pmatrix} = \begin{pmatrix} y_c + f(y_s, y_c) \\ y_s \end{pmatrix} \tag{A.6}$$

with $f(\varphi(y_c), y_c) = 0$. The stable manifold M^s, which can also be written as a graph [38] by $x_c = \varphi_s(x_s) = O(\|x_s\|^2)$ is the stable fiber of the origin with $(y_c, y_s) = (0, 0)$. So we get $f(y_s, 0) = O(\|y_s\|^2)$.

A.2 Fenichel theory

In the Introduction we gave a short description of singularly perturbed dynamical systems which can be written as

$$\begin{aligned} x' &= \varepsilon f(x, y, \varepsilon) \\ y' &= g(x, y, \varepsilon), \end{aligned} \tag{A.7}$$

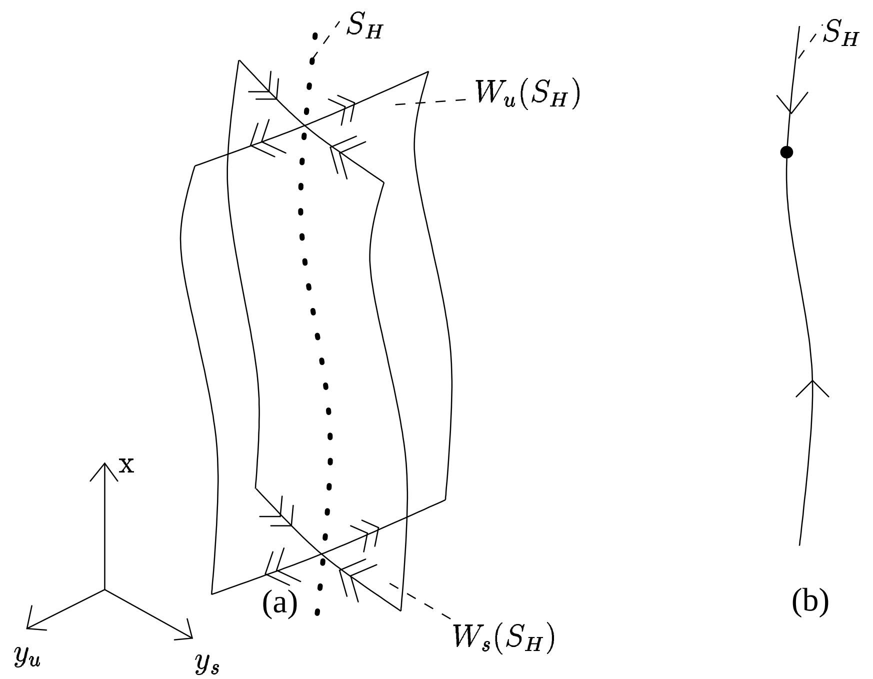

Figure A.2: The invariant manifolds of S_H (a) and the slow dynamics on S_H (b)

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, f, g of class C^k, time t and $0 < \varepsilon \ll 1$. Here we want to state the theorems founded on the pioneering work of N.Fenichel [17] which are essential for describing the invariant manifolds and the dynamics of the full system.

In the Inroduction we have described the two limiting systems for the fast and the slow dynamics, the layer problem (1.3) and the reduced problem (1.4). Further we have introduced the critical manifold S_0, its normally hyperbolic compact submanifold $S_H \subset S_0$ and the slow manifold S_ε.

On each manifold S_H the number of positive eigenvalues n_u and the number of negative eigenvalues n_s is constant. Near the manifold S_H the layer problem (1.3) has two manifolds, a $(m + n_s)$-dimensional stable manifold $W^s(S_H)$ and a $(m + n_u)$-dimensional unstable manifold $W^u(S_H)$ (Fig.A.2(a)), which intersect in S_H.

Usually it is very difficult to analyze nonlinear differential equations in more than two dimensions. In the case of slow-fast dynamics it is sometimes possible to reduce the analysis of the the original system of differential equations to the analysis of the the two lower-dimensional limiting problems, the layer and the reduced problem.

The following three theorems give a precise description of the relation between the dynamics of the full problem (A.7) and the combined dymamics of the reduced problem (1.3) and the layer problem (1.4).

Theorem 12. *[17, 22] If f, g are C^k in (x, y, ε) and S_H is a compact hyperbolic subset of S_0 which can be described by $S_H = \{(x, \varphi_0(x)) | x \in U, U compact\}$ there is an ε_0 such that for $\varepsilon \in (0, \varepsilon_0]$ there is a locally invariant m-dimensional C^k manifold S_ε given as a graph $S_\varepsilon = \{(x, \varphi(x, \varepsilon)), x \in U\}$ with $\varphi(x, \varepsilon)$ of class C^k*

in x and ε and $\varphi(x,0)=\varphi_0(x)$.

Due to this Theorem we can describe the dynamics on the slow manifold by the equation

$$\dot{x}=f(x,\varphi(x,\varepsilon),\varepsilon) \tag{A.8}$$

which is a smooth perturbation of the reduced problem (1.4). Structurally stable properties of the reduced problem persist for sufficiently small values of ε for the restriction of the full problem (A.7) to the slow manifold S_ε.

With the the assumptions of Theorem 12 and if the real parts of the eigenvalues λ_s and λ_u satisfy $Re(\lambda_s)<-a<0$ and $Re(\lambda_u)>b>0$ on S_H we can describe the the dynamics in a neighborhood of S_H in the invariant manifolds $W^s(S_\varepsilon)$ and $W^u(S_\varepsilon)$:

Theorem 13. *[17, 22] There exists a locally invariant stable $(m+n_s)$-dimensional C^k manifold $W^s(S_\varepsilon)$ and a locally invariant unstable $(m+n_u)$-dimensional C^k manifold $W^u(S_\varepsilon)$ which are C^k close to $W^s(S_H)$ and $W^u(S_H)$.*
The dynamics in $W^s(S_\varepsilon)$ is described by a C^k invariant stable foliation F^s of $W^s(S_\varepsilon)$ such that the distance between orbits which start in the same leaf of F^s is decaying exponentially fast with rate e^{-at} $(a>0)$. Analogously the dynamics in $W^u(S_\varepsilon)$ is described by a C^k invariant unstable foliation F^u of $W^u(S_\varepsilon)$ such that the distance between orbits which start in the same leaf of F^u is growing exponentially fast with rates e^{bt} $(b>0)$.
The leaves of the foliation F^s (F^u) are invariant under the flow, that means, each leaf $F^s(x,y)$ $(F^u(x,y))$ is mapped to another leaf $F^s(x(t),y(t))$ $(F^u(x(t),y(t))))$ by the flow in forward (backward) time t (see Fig. A.3).

For a good introduction and proofs see [22],[17].

Theorem 12 and 13 can be described in a more general setting [17]:

$$x'=f(x,\varepsilon), \tag{A.9}$$

where $x\in\mathbb{R}^N$, f is a C^k function of x and ε, $k\geq 2$ and $\varepsilon\in[0,\varepsilon_0]$. Assume that system (A.9) has a compact n_0-dimensional C^k manifold S_0 consisting entirely of equilibria of

$$x'=f(x,0). \tag{A.10}$$

Let $\frac{\partial f(x,0)}{\partial x}|_{S_0}$ have n_s eigenvalues in the left half plane, n_u eigenvalues in the right half plane and n_c eigenvalues on the imaginary axis in addition to the n_0 eigenvalues zero corresponding to the manifold of equilibria S_0 and $n_0+n_s+n_c+n_u=N$. Then the extended system

$$\begin{aligned} x'&=f(x,\varepsilon)\\ \varepsilon'&=0 \end{aligned}$$

has an (n_c+n_0+1)-dimensional locally invariant C^k manifold W^c with $S_0\subset W^c$, an $(n_c+n_0+1+n_s)$-dimensional C^k center-stable manifold W^{cs} and an $(n_c+n_0+1+n_u)$-dimensional C^k center-unstable manifold W^{cu} near S_0. These manifolds are invariant under the flow. As in Theorem 13 there exists an invariant

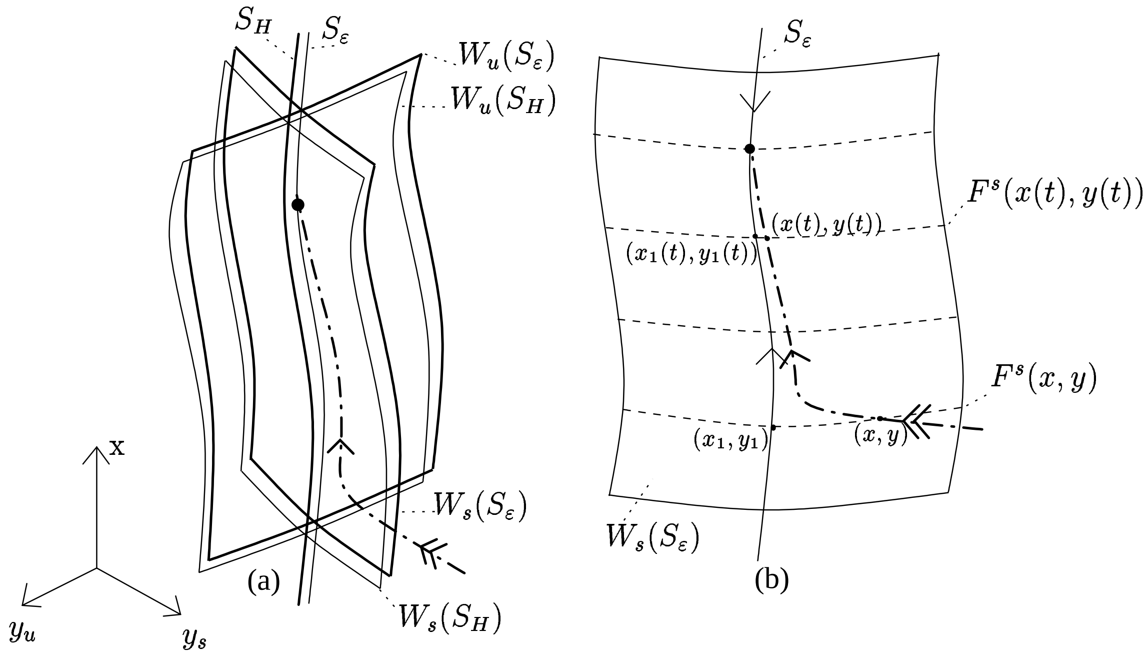

Figure A.3: Invariant manifolds of S_ϵ and the foliation of $W^s(S_\epsilon)$

C^k foliation F^s of W^{cs} and an invariant C^k foliation F^u of W^{cu}, the fibers of F^s, F^u intersect W^c transversally [17].

There exists a certain standard form for system (A.9) which is often useful in explicit calculations:

Theorem 14. *For system (A.9) there exists a C^k transformation to Fenichel coordinates:*

$$\begin{aligned} x_s' &= \Lambda(x_s, x_u, x_c, \varepsilon) x_s \\ x_u' &= \Gamma(x_s, x_u, x_c, \varepsilon) x_u \\ x_c' &= h(x_c, \varepsilon) + H(x_s, x_u, x_c, \varepsilon) \otimes x_s \otimes x_u, \end{aligned} \tag{A.11}$$

where $x_s \in \mathbb{R}^{n_s}$ are the coordinates of the stable, $x_u \in \mathbb{R}^{n_u}$ of the unstable and $x_c \in \mathbb{R}^{n_c+n_0}$ the coordinates of the slow manifold, Λ, Γ, h, H are C^k functions with $\Lambda(x_s, x_u, x_c, \varepsilon) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}$, $\Gamma(x_s, x_u, x_c, \varepsilon) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_u}$, $h(x_c, \varepsilon) \in \mathbb{R}^{n_c}$ and $H(x_s, x_u, x_c, \varepsilon) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$.

The Fenichel coordinates are characterized by the requirement that the invariant manifolds $W^c = \{x_s = x_u = 0\}$, $W^s(S_\varepsilon) = \{x_u = 0\}$, $W^u(S_\varepsilon) = \{x_s = 0\}$ and the foliations F^s and F^u of the manifolds W^{cs}, W^{cu} are flat. The condition of flat fibers in W^{cs} and W^{cu} is satisfied because the last equations becomes

$$x_c' = h(x_c, \varepsilon)$$

for $x_u \vee x_s = 0$.

If $n_c = 0$ we the assumptions of Theorem 12 and 13 are satisfied. Then the sections $\varepsilon = const.$ of W^c, W^{cs}, W^{cu} are the manifolds S_ε, $W^s(S_\varepsilon)$, $W^u(S_\varepsilon)$. Further the dynamics on W^c (S_ε) is slow and the last equation in Fenichel coordinates becomes

$$h(x_c, \varepsilon) = \varepsilon h_1(x_c, \varepsilon).$$

Appendix B

Adiabatic elimination for the semiclassical laser equations

We start with the semiclassical equations

$$\begin{aligned} E' &= -(i\omega_E + \gamma_E)E + ngP \\ P' &= -(i\omega_P + \gamma_P)P + gED \\ D' &= -\gamma_D(D - D_0) - 2g(E\bar{P} + \bar{E}P). \end{aligned} \tag{B.1}$$

for a single-mode, homogeneously broadened, unidirectional ring laser (see [37] pp.81). The complex variables E and P describe the electromagnetic field and the atomic dipole moment. D is the population inversion of the atomic energy levels. The first terms in the first two equations describe the fast oscillation of E and P. The terms containing the factors γ describe the damping of the corresponding variables and the last terms describe the coupling between the variables by the matter-radiation interaction. The variable n denotes the total number of atoms in the laser medium and g is a constant.

To reduce the number of parameters we perform the scaling

$$E \to \frac{E}{\sqrt{2}g}, P \to \frac{P}{\sqrt{2}ng^2}, D \to \frac{D}{g^2 n}, D_0 \to \frac{D_0}{g^2 n}$$

to obtain

$$\begin{aligned} E' &= -(i\omega_E + \gamma_E)E + P \\ P' &= -(i\omega_P + \gamma_P)P + ED \\ D' &= -\gamma_D(D - D_0) - (E\bar{P} + \bar{E}P). \end{aligned} \tag{B.2}$$

This system has two complex and one real independent variables and, therefore is a five-dimensional system of real nonlinear differential equations. It is known that these equations for certain values of parameters contain the Lorenz equations and therefore show chaotic behaviour ([37] pp.87). Therefore it is impossible to describe the behaviour of the general equations. Fortunately many types of lasers show dynamics on different time scales and it is possible to describe the essential dynamics by lower dimensional approximating systems ([37] p.14).

Following the usual classification (see [37] pp.13) we distinguish between three types of lasers. In type A lasers $\gamma_P, \gamma_D \gg \gamma_E$, and the dynamics can be described by a one-dimensional equation for the amplitude of the electrical field. In type B lasers $\gamma_P \gg \gamma_D, \gamma_E$ and the equations can be reduced to a two-dimensional system for the amplitude of the electrical field and the population inversion. Type C lasers are lasers where none of these relations are satisfied.

In the physics literature the heuristic method of adiabatic elimination is commonly used to derive simplified systems from system (B.2) in cases of lasers of type A and B. We will give a geometric interpretation of this method and state assumptions under which adiabatic elimination is mathematically justified.

B.1 Adiabatic elimination

Adiabatic elimination is applied in systems with a strong damping for one or more directions and a slow evolution in the other directions. The system of equations

$$\begin{aligned} x' &= \lambda_u x + f(x,y) \\ y' &= \lambda_s y + g(x,y) \end{aligned} \tag{B.3}$$

is often used as a model problem with the assumtions $\lambda_s < 0$, $\lambda_u \approx 0$ or $\lambda_s \ll -|\lambda_u|$ and different functions f, g, which have in common that they are fuctions of second order in x and y (see [20] pp.187, pp.195, [19] pp.194). The idea is that y relaxes fast, i.e. $y' \approx 0$ after a short time and y follows x adiabatically. This approximation is obtained by setting $y' = 0$ and solving the equation

$$\lambda_s y + g(x,y) = 0 \tag{B.4}$$

for $y = \psi(x)$. Then plugging $y = \psi(x)$ into the remaining equation

$$x' = \lambda_u x + f(x, \psi(x)) \tag{B.5}$$

describes the slow motion. This equation should describe the essential dynamics at least after a short time. (For x and y in higher dimensions the procedure is analogous. In this more general situation λ_s and λ_u are matrices and their eigenvalues should satisfy the relations above.) Most authors refer to the books of Haken ([20] p.187, [19] p.194.), where he explained the adiabatic elimination procedure for different cases and gave some justification. A short description is found in ([37] p.38).

To describe the fact that the system is governed by different time scales we write the system in a different form

$$\begin{aligned} x' &= \varepsilon\lambda_u x + f(x,y,\varepsilon) \\ y' &= \lambda_s y + g(x,y,\varepsilon). \end{aligned} \tag{B.6}$$

f and g are C^k with $k \geq 1$ and of second order in x and y, ε is a small positive parameter and describes the different scales, $\lambda_s < 0, \lambda_u$ are $O(1)$.

Obviously a system of the form (B.3) can be transformed to (B.6) by a time transformation if the different scales for the eigenvalues are described by the small parameter ε: $|\lambda_u/\lambda_s| = O(\varepsilon)$.

In this notation, the adiabatic solution is described by $y = \psi(x,\varepsilon)$ which solves the equation

$$\lambda_s \psi(x,\varepsilon) + g(x, \psi(x,\varepsilon), \varepsilon) = 0$$

and the differential equation

$$x' = \varepsilon\lambda_u x + f(x, \psi(x,\varepsilon), \varepsilon).$$

B.1.1 Adiabatic elimination via center manifold theory

Under the assumption $f, g = O(1)$ and f, g is of second order in x, y we can use center manifold theory to approximate the system

$$\begin{aligned} x' &= \varepsilon\lambda_u x + f(x, y, \varepsilon) \\ y' &= \lambda_s y + g(x, y, \varepsilon). \\ \varepsilon' &= 0. \end{aligned} \tag{B.7}$$

As in Appendix A.1 we get:

Theorem 15. *If f, g are C^k there exists a C^k center manifold M^c described by $y = \varphi(x,\varepsilon) = O(\|(x,\varepsilon)\|^2)$. There exists a C^k foliation F^s transversal to the center manifold. There is contraction of order $O(e^{-ct})$ with $-c \sim \lambda_s$ along the fibers.*

The function $\varphi(x,\varepsilon)$ is determined by the equation (see [38])

$$\lambda_s\varphi + g(x, \varphi, \varepsilon) = \frac{\partial\varphi}{\partial x}(\varepsilon\lambda_u x + f(x, \varphi, \varepsilon). \tag{B.8}$$

After a short time orbits near M^c are attracted to the center manifold and follow the dynamics on M^c. Developing the equations for the center manifold and the adiabatic solution in a series, we obtain

$$\varphi(x,\varepsilon) = \sum_{i+j=h} \varphi_{ij} x^i \varepsilon^j + O(\|(x,\varepsilon)\|^{h+1})$$

and

$$\psi(x,\varepsilon) = \sum_{l+m=n} \psi_{lm} x^l \varepsilon^m + O(\|(x,\varepsilon)\|^{n+1})$$

where h and n are the order of the lowest order terms not equal 0 in the sum. Then we get

Theorem 16. *Let f, g be C^k. If there exists a term of lowest order $h < k$ not equal 0 in the series φ, then $h = n$ and*

$$\varphi_{ij} = \psi_{ij}$$

for $i + j = h$. Otherwise both series are 0 up to order $O(\|(x,\varepsilon)\|^k)$:

$$\varphi(x,\varepsilon) = O(\|(x,\varepsilon)\|^k), \ \psi(x,\varepsilon) = O(\|(x,\varepsilon)\|^k).$$

Proof: For the slow solution we have to look at equation (B.8). Developing g into a series we get

$$g(x, y, \varepsilon) = \sum_{a+b+c=d} g_{abc} x^a y^b \varepsilon^c + O(\|(x, y, \varepsilon)\|^{d+1}),$$

with lowest order terms $d < k$ not equal 0. Plugging $y = x^i\varepsilon^j$ with $i + j \geq 2$ into the equation we get

$$x^a y^b \varepsilon^c = x^{a+ib} y^{c+jb}$$

is of order d only if $b = 0$. On the other side we have

$$\frac{\partial\varphi(x,\varepsilon)}{\partial x} f(x, \varphi, \varepsilon) = O(\|(x,\varepsilon)\|^{h+1})$$

because $f = O(\|(x,y)\|^2)$. So we get the equation

$$\lambda_s \sum_{i+j=h} \varphi_{ij} x^i \varepsilon^j + O(\|(x,\varepsilon)\|^{h+1}) + \sum_{a+c=d} g_{a0c} x^a \varepsilon^c + O(\|(x,\varepsilon)\|^{d+1}) = O(\|(x,\varepsilon)\|^{h+1})$$

Comparing the coefficients of $x^a\varepsilon^c$ we get $h = d$ and

$$\varphi_{ac} = -\frac{g_{a0c}}{\lambda_s}.$$

For $g_{a0c} = 0$ for $a + c < k$ we get $\varphi_{ac} = 0$ for $a + c < k$. Analogously we get for the adiabatic solution

$$\psi_{ac} = -\frac{g_{a0c}}{\lambda_s},$$

and $\psi_{ac} = 0$ for $a + c < k$ if $g_{a0c} = 0$. □

The leading order terms of the adiabatic solution and the center manifold are equal. If the leading order terms determine the dynamics of equation

$$x' = \varepsilon\lambda_u x + f(x, \sum_{i+j=h} \varphi_{ij} x^i \varepsilon^j + O(\|(x,\varepsilon)\|^{h+1}), \varepsilon),$$

then the adiabatic solution shows the same dynamics as the dynamics on the center manifold.

B.1.2 Adiabatic elimination via Fenichel theory

In the case that f is of order ε, i.e. $f(x,y,\varepsilon) = \varepsilon\tilde{f}(x,y,\varepsilon)$, and g of order $O(1)$, we rewrite sytem (B.6) as a singularly perturbed sytem in standard form:

$$\begin{aligned} x' &= \varepsilon\lambda_u x + \varepsilon\tilde{f}(x,y,\varepsilon) \\ y' &= \lambda_s y + g(x,y,\varepsilon). \end{aligned} \tag{B.9}$$

The corresponding layer problem is

$$\begin{aligned} x' &= 0 \\ y' &= \lambda_s y + g(x,y,0). \end{aligned} \tag{B.10}$$

and the reduced system is

$$\begin{aligned} \dot{x} &= \lambda_u x + \tilde{f}(x,y,0) \\ 0 &= \lambda_s y + g(x,y,0). \end{aligned} \tag{B.11}$$

Because $\lambda_s < 0$ the layer problem has only stable equilibria for (x, y) in a neighborhood of the origin. Therefore the critical manifold S_0, defined by the equation

$$0 = \lambda_s y + g(x, y, 0),$$

is normally hyperbolic and can be described as a graph $y = \varphi_0(x)$. Fenichel Theory implies (see Appendix A.2):

Theorem 17. *If $g, \tilde{f} \in C^k$ there exists a slow manifold S_ε, C^k-close to S_0, which can be described by $y = \varphi(x, \varepsilon)$, and a stable C^k foliation F^s over the slow manifold S_ε. Along the stable fibers there is a contraction of order e^{-ct} with $-c \sim \lambda_s$.*

The dynamics on S_ε on the slow time scale is described by

$$\dot{x} = \lambda_u x + \tilde{f}(x, \varphi(x, \epsilon), \varepsilon).$$

Near the slow manifold all solutions converge to the slow manifold exponentially fast. Because of this fact the dynamics on the slow manifold is a good approximation for the dynamics of the full system (B.9) after a short period of time. Because the critical manifold satisfies the equation of the adiabatic solution for $\varepsilon = 0$, we have

$$\psi(x, \varepsilon) = \varphi_0(x) + O(\varepsilon) = \varphi(x, \varepsilon) + O(\varepsilon).$$

Thus the adiabatic solution approximates the slow manifold up to errors of order $O(\varepsilon)$.

More generally it is possible that the adiabatic approximation and/or the expansion of the slow manifold vanish to a certain order in ε:

$$\varphi(x, \varepsilon) = \varphi_i(x)\varepsilon^i + O(\varepsilon^{i+1}) \tag{B.12}$$
$$\psi(x, \varepsilon) = \psi_j(x)\varepsilon^j + O(\varepsilon^{j+1}). \tag{B.13}$$

The following theorem shows that the two approximations always agree to leading order in ε, in particular either $i = j$ or all terms are zero.

Theorem 18. *If system (B.9) is of class C^k with $k \geq 1$ then the slow solution $\varphi(x, \varepsilon)$ and the adiabatic approximation $\psi(x, \varepsilon)$ do exist near $(x, \varepsilon) = (0, 0)$.*
If there is some l with $0 \leq l < k$ such that $\frac{\partial^l g(x,0,\varepsilon)}{\partial \varepsilon^l} \neq 0$ the lowest terms not equal to zero of the expansions (B.12) and (B.13) are of order l and they are equal, $\varphi_l(x) = \psi_l(x)$. If there is no such $l < k$ then both expansions are equal to zero with an error of order ε^k: $\varphi(x, \varepsilon) = O(\varepsilon^k), \psi(x, \varepsilon) = O(\varepsilon^k)$.

Proof: Because of Fenichel theory $\varphi(x, \varepsilon)$ does exist and is C^k. Since $g(x, y) = O(\|(x, y)\|^2)$, the equation

$$\lambda_s y + g(x, y, \varepsilon) = 0 \tag{B.14}$$

satisfies $\frac{\partial \lambda_s y + g(x,y,\varepsilon)}{\partial y} < 0$ near $(x, y, \varepsilon) = (0, 0, 0)$ and by the Implicit Function Theorem there exists $y = \psi(x, \varepsilon)$ of class C^k which solves the equation. Because both $\varphi(x, \varepsilon)$ and $\psi(x, \varepsilon)$ are C^k, it is possible to calculate the expansions up to order $k - 1$ with an error of order $O(\varepsilon^k)$.

For the adiabatic case we expand $g(x,y,\varepsilon) = g(x,0,\varepsilon) + g_y(x,0,\varepsilon)y + O(y^2)$ in equation (B.14) and plug (B.13) into the resulting equation, which leads to

$$\lambda_s \psi_j(x)\varepsilon^j + g(x,0,\varepsilon) + g_y(x,0,\varepsilon)\psi_j(x)\varepsilon^j + O(\varepsilon^{j+1}) = 0. \tag{B.15}$$

If there is some l such that $\frac{\partial^l g(x,0,\varepsilon)}{\partial \varepsilon^l} \neq 0$ then $g(x,0,\varepsilon) = \varepsilon^l \tilde{g}(x,0,\varepsilon) = \varepsilon^l \tilde{g}(x,0,0) + O(\varepsilon^{l+1})$ with $\tilde{g}(x,0,0) \neq 0$. Further we expand $g_y(x,0,\varepsilon) = g_y(x,0,0) + O(\varepsilon)$. Plugging these terms into the equation we get

$$(\lambda_s + g_y(x,0,0))\psi_j(x)\varepsilon^j + O(\varepsilon^{j+1}) + \varepsilon^l \tilde{g}(x,0,0) + O(\varepsilon^{l+1}) = 0. \tag{B.16}$$

Comparing the coefficients before the ε terms we see that because $(\lambda_s + g_y(x,0,0)) \neq 0$ near $x = 0$ the lowest j for which $\psi_j(x) \neq 0$ is $j = l$ and

$$\psi_l(x) = \frac{-\tilde{g}(x,0,0)}{\lambda_s + g_y(x,0,0)}. \tag{B.17}$$

Further we see that if there is no such $\tilde{g}(x,0,0) \neq 0$ for $l < k$ than all $\psi_l(x) = 0$ for $l < k$.

In the singularly perturbed case we know $y = \varphi(x,\varepsilon)$ is an invariant solution of equation (B.9), further it satisfies $y' = \frac{\partial \varphi}{\partial x} x'$ and therefore is a solution of

$$\lambda_s \varphi(x,\varepsilon) + g(x,\varphi(x,\varepsilon),\varepsilon) = \frac{\partial \varphi(x,\varepsilon)}{\partial x}(\varepsilon \lambda_u x + \varepsilon \tilde{f}(x,\varphi(x,\varepsilon),\varepsilon)). \tag{B.18}$$

Following the same line of argumens as in the adiabatic case we get for the right side of (B.18) the same expansion as the right side of (B.16) with i, φ instead of j, ψ. Since $\varphi_x(x,\varepsilon) = (\varphi_i)_x(x)\varepsilon^i + O(\varepsilon^{i+1})$ is $O(\varepsilon^i)$ and $x, \tilde{f}(x,\varphi(x,\varepsilon),\varepsilon)$ are of order $O(1)$ equation (B.18) transforms to

$$(\lambda_s + g_y(x,0,0))\varphi_i(x)\varepsilon^i + O(\varepsilon^{i+1}) + \varepsilon^l \tilde{g}(x,0,0) + O(\varepsilon^{l+1}) = O(\varepsilon^{i+1}). \tag{B.19}$$

Again we see that in the expansion of $\varphi(x,\varepsilon)$ the first non-zero term is $\varphi_l(x)$ and again

$$\varphi_l(x) = \frac{-\tilde{g}(x,0,0)}{\lambda_s + g_y(x,0,0)}. \tag{B.20}$$

If there is no such $\tilde{g}(x,0,0) \neq 0$ for $l < k$ then all $\varphi_l(x) = 0$ for $l < k$. □

We get

$$\dot{x} = \lambda_u x + \tilde{f}(x, \varphi_i(x)\varepsilon^i + O(\varepsilon^{i+1}), \varepsilon). \tag{B.21}$$

for the dynamics of the slow and of the adiabatic solution. If the $O(\varepsilon^i)$ terms in (B.21) determine the dynamics then the dynamics on the slow manifold and of the adiabatic solution is the same.

B.2 Dynamics of lasers with $\gamma_P, \gamma_D \gg \gamma_E$

This section is a short description of lasers of type A. We will describe the adiabatic solution and the slow solution and we will see that the lowest order terms are equal. Furthermore we will describe the dynamics on the slow manifold. Fenichel theory describes the relation between the semiclassical laser equations and the rate equations of lasers of type A.

To describe the fact $\gamma_P, \gamma_D \gg \gamma_E$ we substitute

$$\gamma_P \to \frac{\gamma_P}{\varepsilon},\ \gamma_D \to \frac{\gamma_D}{\varepsilon},$$

further we assume that all other parameters are $O(1)$. This leads to the new system of equations

$$\begin{aligned} E' &= -(i\omega_E + \gamma_E)E + P \\ P' &= -(i\omega_P + \frac{\gamma_P}{\varepsilon})P + ED \\ D' &= -\frac{\gamma_D}{\varepsilon}(D - D_0) - (E\bar{P} + \bar{E}P). \end{aligned} \tag{B.22}$$

The adiabatic solution of this system is

$$\begin{aligned} P &= \varepsilon\frac{(1 - i\varepsilon\alpha)}{\gamma_P(1 + \varepsilon^2\alpha^2)}\frac{D_0 E}{1 + \frac{2\varepsilon^2|E|^2}{\gamma_P\gamma_D(1+\varepsilon^2\alpha^2)}} \\ D &= \frac{D_0}{1 + \frac{2\varepsilon^2|E|^2}{\gamma_P\gamma_D(1+\varepsilon^2\alpha^2)}}, \end{aligned} \tag{B.23}$$

where $\alpha = \frac{\omega_P}{\gamma_P}$ and the dynamics of the adiabatic solution is

$$E' = E(-\gamma_E - i\omega_E + \varepsilon\frac{(1 - i\varepsilon\alpha)}{\gamma_P(1 + \varepsilon^2\alpha^2)}\frac{D_0}{1 + \frac{2\varepsilon^2|E|^2}{\gamma_P\gamma_D(1+\varepsilon^2\alpha^2)}}). \tag{B.24}$$

This equation is an equation for the electric field only and corresponds to the result in [29]. To describe the dynamics of E we substitute $E = Re^{i\varphi}$ and get for $R = |E|$:

$$R' = R(-\gamma_E + \varepsilon\frac{1}{\gamma_P(1 + \varepsilon^2\alpha^2)}\frac{D_0}{1 + \frac{2\varepsilon^2 R^2}{\gamma_P\gamma_D(1+\varepsilon^2\alpha^2)}}). \tag{B.25}$$

This equation has two equilibria,

$$\begin{aligned} R &= 0 \\ R &= \frac{1}{\varepsilon}\sqrt{\frac{\gamma_D}{2\gamma_E}}\sqrt{\varepsilon D_0 - \gamma_E\gamma_P(1 + \varepsilon^2\alpha^2)}. \end{aligned} \tag{B.26}$$

The first corresponds to the 'off-state' of the laser. The second one corresponds to the 'on-state'. It exists only for $D_0 > \frac{1}{\varepsilon}\gamma_E\gamma_P(1 + \varepsilon^2\alpha^2)$, which is equivalent to the 'laser condition'. Here we see for $\gamma_E\gamma_P = O(1)$ that $D_0 \sim O(\frac{1}{\varepsilon})$ for the equilibrium $R > 0$, which shows $R, E, D, P \sim O(\frac{1}{\varepsilon})$. The assumptions for the adiabatic solution ($P' \sim 0, D' \sim 0$) is satisfied because $P', D' = O(1)$ compared to $R, E, D, P \sim O(\frac{1}{\varepsilon})$.

We can justify this reasoning by Fenichel theory. By scaling

$$D = \frac{D_1}{\varepsilon},\ D_0 = \frac{D_{01}}{\varepsilon},\ E = \frac{E_1}{\varepsilon},\ P = \frac{P_1}{\varepsilon},\ \tau = \varepsilon t$$

with $D_{01}, E_1, D_1, P_1 \sim O(1)$ we get the system

$$\begin{aligned} E_1' &= \varepsilon(-(i\omega_E + \gamma_E)E_1 + P_1 \\ P_1' &= -(i\omega_P + \gamma_P)P_1 + E_1 D_1 \\ D_1' &= -\gamma_D(D_1 - D_{01}) - (E_1\bar{P}_1 + \bar{E}_1 P_1), \end{aligned} \tag{B.27}$$

which has the form (B.9). For the proof it is necessary to transform our system to real coordinates $E = a + ib, P = p + iq$. The results are the same as when applying Fenichel theory formally and we write our results in the complex variables E_1, P_1.

The critical manifold is

$$\begin{aligned} P_1 &= \frac{1}{\gamma_P}\frac{D_{01}E_1}{(1+\frac{2|E_1|^2}{\gamma_P\gamma_D})} \\ D_1 &= \frac{D_{01}}{1+\frac{2|E_1|^2}{\gamma_P\gamma_D}}, \end{aligned} \tag{B.28}$$

with the reduced dynamics

$$\dot{E}_1 = E_1(-i\omega_E - \gamma_E + \frac{1}{\gamma_P}\frac{D_{01}}{(1+\frac{2|E_1|^2}{\gamma_P\gamma_D})}). \tag{B.29}$$

The equations for the adiabatic solution and the reduced problem have the same terms of lowest order as shown in the previous section. For the dynamics on S_0 we change to coordinates $E_1 = R_1 e^{i\varphi_1}$ and obtain

$$\dot{R}_1 = (-\Gamma_E + \frac{1}{\gamma_P}\frac{D_{01}}{(1+\frac{2R_1^2}{\gamma_P\gamma_D})})R_1. \tag{B.30}$$

This equation has again the equilibria

$$\begin{aligned} R_1 &= 0, \\ R_1 &= \sqrt{\frac{\gamma_D}{2\gamma_E}}\sqrt{D_{01} - \gamma_E\gamma_P}. \end{aligned} \tag{B.31}$$

A pitchfork bifurcation occurs in $D_{01} = \gamma_E\gamma_P$. For $D_{01} < \gamma_E\gamma_P$ only the attracting equilibrium $R_1 = 0$ exists. For $D_{01} > \gamma_E\gamma_P$, the 'laser condition' is satisfied, the attracting equilibrium with $R_1 > 0$ exists and the equilibrium $R_1 = 0$ becomes unstable (see Fig. B.1(a)).

For the layer problem we get

$$\begin{aligned} E_1' &= 0 \\ P_1' &= \gamma_P P_1 + E_1 D_1 \\ D_1' &= -\gamma_D(D_1 - D_{01}) - (E_1\bar{P}_1 + \bar{E}_1 P_1). \end{aligned} \tag{B.32}$$

The eigenvalues of the linearizations along the critical manifold are (calculated for the corresponding real system)

$$\begin{aligned} \lambda_1 &= -\gamma_P \\ \lambda_{2,3} &= -\frac{1}{2}(\gamma_P + \gamma_D) \pm \sqrt{(\gamma_D - \gamma_P)^2 - 8R^2}. \end{aligned} \tag{B.33}$$

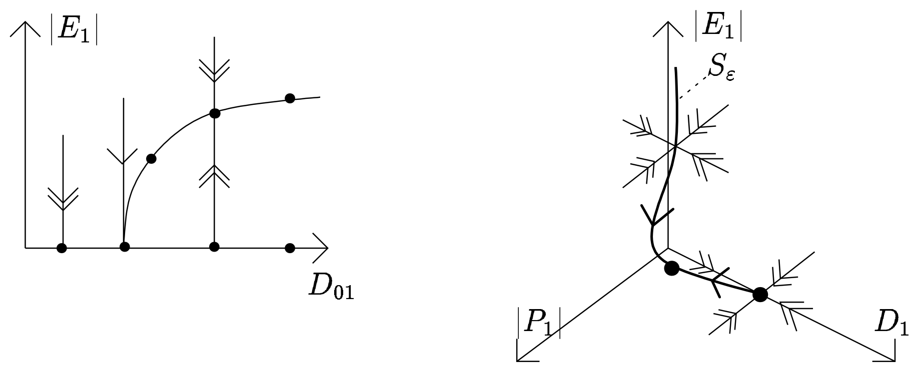

Figure B.1: Dynamics of lasers of type A: (a) on the critical manifold; (b) for system (B.27) for $D_{01} > \gamma_E\gamma_P$

Obviously the real parts of all eigenvalues λ_i satisfy $Re(\lambda_i) < 0$. Therefore S_0 is normally hyperbolic and attracting (Fig.B.1(b)), and we can apply Fenichel's theory. The slow manifold S_ε is a small perturbation of order $O(\varepsilon)$ of the critical manifold. The pitchfork bifurcation with parameter D_0 is structurally stable ([18]) to small perturbations. There exists a stable foliation transversal to S_ε and the solutions will follow the dynamics on the slow manifold after a short time. Again the adiabatic solution shows the characteristic behaviour of the solutions on the slow manifold. Thus the adiabatic solution is a good approximation of the full equations.

B.3 Dynamics of lasers with $\gamma_P \gg \gamma_E, \gamma_D$

As in the previous section we describe the adiabatic solution. We will see that the adiabatic solution and a slow manifold have the same lowest order terms and again Fenichel theory is the link between the semiclassical laser equations and the rate equations for lasers of type B, which can be obtained by adiabatic elimination. Further we analyse the dynamics on the slow manifold.

Because $\gamma_P \gg \gamma_E, \gamma_D$, we substitute $\gamma_P \to \frac{\gamma_P}{\varepsilon}$ and assume that the other parameters are $O(1)$. We get the system of equations

$$\begin{aligned} E' &= -(i\omega_E + \gamma_E)E + P \\ P' &= -(i\omega_P + \frac{\gamma_P}{\varepsilon})P + ED \\ D' &= -\gamma_D(D - D_0) - (E\bar{P} + \bar{E}P). \end{aligned} \tag{B.34}$$

The adiabatic elimination leads to

$$P = \frac{\varepsilon ED}{\gamma_P + i\omega_P\varepsilon} \tag{B.35}$$

and the dynamics

$$\begin{aligned} E' &= E(-(i\omega_E + \gamma_E) + \frac{\varepsilon D}{\gamma_P + i\omega_P \varepsilon}) \\ D' &= -\gamma_D(D - D_0) - \frac{2\varepsilon\gamma E\bar{E}D}{\gamma_P^2 + \omega_P^2\varepsilon^2}, \end{aligned} \tag{B.36}$$

which corresponds to the solution in [29]. Changing coordinates again we obtain with $E = Re^{i\varphi}$ and $\beta = \frac{\gamma_P}{\gamma_P^2+\omega_P^2\varepsilon^2}$ the system

$$\begin{aligned} R' &= R(-\gamma_E + \varepsilon\beta D) \\ D' &= -\gamma_D(D - D_0) - \varepsilon 2\beta R^2 D, \end{aligned} \tag{B.37}$$

which has the equilibria

$$\begin{aligned} (R, D) &= (0, D_0) \\ (R, D) &= (\sqrt{\frac{\gamma_D}{2\gamma_E}(D_0 - \frac{\gamma_E}{\varepsilon\beta})}, \frac{\gamma_E}{\varepsilon\beta}). \end{aligned} \tag{B.38}$$

The equilibrium $(R, D) = (0, D_0)$ is again the 'off-state' of the laser. The equilibria with $R > 0$ exists only if $D_0 > \frac{\gamma_E}{\varepsilon\beta}$ which again corresponds to the 'laser condition'. Further $D_0 \sim \frac{1}{\varepsilon}$ suggests the scaling

$$D = \frac{D_1}{\varepsilon},\ D_0 = \frac{D_{01}}{\varepsilon},\ E = \frac{E_1}{\sqrt{\varepsilon}},\ P = \frac{P_1}{\sqrt{\varepsilon}},\ \tau = \varepsilon t$$

with $D_{01}, E_1, D_1, P_1 \sim O(1)$. We get the system

$$\begin{aligned} E_1' &= \varepsilon(-(i\omega_E + \gamma_E)E_1 + P_1 \\ P_1' &= -(i\varepsilon\omega_P + \gamma_P)P_1 + E_1 D_1 \\ D_1' &= \varepsilon(-\gamma_D(D_1 - D_{01}) - (E_1\bar{P}_1 + \bar{E}_1 P_1)), \end{aligned} \tag{B.39}$$

which has the form (B.9). In this case the layer system is

$$\begin{aligned} E_1' &= 0 \\ P_1' &= -\gamma_P P_1 + E_1 D_1 \\ D_1' &= 0, \end{aligned} \tag{B.40}$$

with the critical manifold S_0

$$P_1 = \frac{E_1 D_1}{\gamma_P}.$$

For the critical manifold the layer problem has the negative eigenvalues

$$\lambda_{1,2} = -\gamma_P,$$

which shows that the critical manifold is hyperbolic and we can apply Fenichel theory. The dynamics on the critical manifold is with $E_1 = R_1 e^{i\varphi}$, $\beta_1 = \frac{1}{\gamma_P}$

$$\begin{aligned} R_1' &= R_1(-\gamma_E + D_1\beta_1) \\ D_1' &= -\gamma_D(D - D_0) - 2\beta_1 R_1^2 D_1, \end{aligned} \tag{B.41}$$

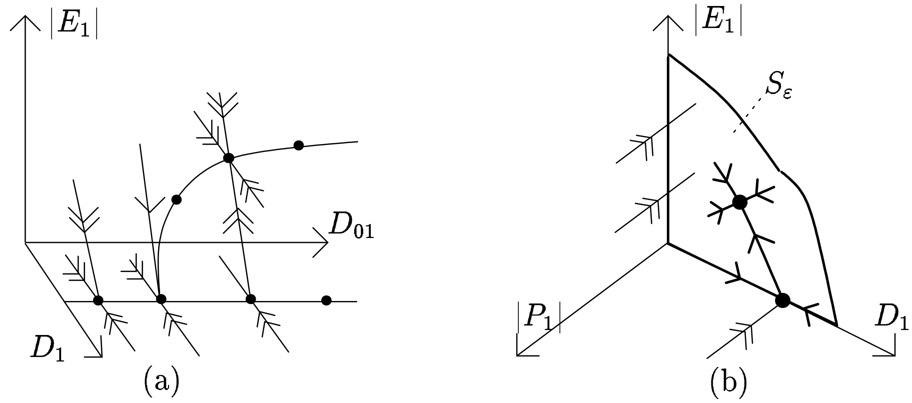

Figure B.2: Dynamics of lasers of type B: (a) on the critical manifold; (b) for system (B.39) for $D_{01} > \gamma_E \gamma_P$

and we see that the lowest order terms of the adiabatic solution and of the slow manifold are the same again.

On the critical manifold we have again the equilibria

$$(R_1, D_1) = (0, D_{01}) \tag{B.42}$$
$$(R_1, D_1) = (\sqrt{\frac{\gamma_D}{2\gamma_E}(D_{01} - \frac{\gamma_E}{\beta_1})}, \frac{\gamma_E}{\beta_1}).$$

A pitchfork bifurcation occurs at $D_0 = \frac{\gamma_E}{\beta_1} = \gamma_E \gamma_P$. For $D_0 < \frac{\gamma_E}{\beta_1}$ there only exists the equilibrium $(R_1, D_1) = (0, D_{01})$ with the eigenvalues

$$\lambda_1 = -\gamma_D, \ \lambda_2 = -\gamma_E + \beta_1 D_{01}. \tag{B.43}$$

In $D_0 = \frac{\gamma_E}{\beta_1}$ the eigenvalue λ_2 changes from negative to positive. Further the branch of equilibria with $R_1 > 0$ starts to exist, which have the eigenvalues

$$\lambda_{3,4} = -\frac{D_{01}\gamma_D\beta_1}{2\gamma_D} \pm \sqrt{(\frac{D_{01}\gamma_D\beta_1}{2\gamma_D})^2 - 2\gamma_D\beta_1(D_{01} - \frac{\gamma_E}{\beta_1})}. \tag{B.44}$$

For $D_0 > \frac{\gamma_E}{\beta_1}$ the real part of satisfies $Re(\lambda_{3,4}) < 0$ and the equilibria are stable. Near the bifurcation point $D_{01} - \frac{\gamma_D}{\beta_1} \sim 0$ and the eigenvalues $\lambda_{3,4} \in \mathbb{R}$. So we get again that the dynamics in S_0 is structurally stable and describes the dynamics of the slow manifold and of the adiabatic manifold (see Fig. B.2).

B.4 The connection of lasers with saturable absorbers to lasers of class B

In the following we give a brief overview about laser models which are related to the Yamada model which is examined in Capter 2. A special class of lasers of type B are lasers where $\gamma_E \gg \gamma_D$. By the scaling $\gamma_D \to \varepsilon\gamma_D$ in system (B.41) we get

$$R_1' = R_1(-\gamma_E + D_1\beta_1) \tag{B.45}$$
$$D_1' = -\varepsilon\gamma_D(D - D_0) - 2\beta_1 R_1^2 D_1.$$

By applying the scaling

$$I = \frac{2\beta R^2}{\varepsilon\gamma_D},\ D = \frac{D_1}{D_0},\ \tilde{\varepsilon} = \frac{\varepsilon\gamma_D}{2\beta_1},\ \tilde{t} = 1\beta_1 t,\ A = \frac{D_0\beta}{\gamma_E}$$

and assuming $2\beta_1 = \gamma_D$ with a new small parameter $\tilde{\varepsilon}$ and a new time $\tilde{t}$ we get the system

$$\begin{aligned} I' &= 2I(-1 + AD) \\ D' &= \tilde{\varepsilon}(1 - D - DI), \end{aligned} \tag{B.46}$$

which has been studied in [11]. The solution of this equation shows relaxation oscillations to the stable equilibrium, if the 'laser condition' is satisfied. Adding an equation for an absorbing medium leads to

$$\begin{aligned} I' &= 2I(-1 + AD + \bar{A}\bar{D}) \\ D' &= \tilde{\varepsilon}(1 - D - DI) \\ \bar{D}' &= \bar{\tilde{\varepsilon}}(-1 - \bar{D} - \alpha\bar{D}I) \end{aligned} \tag{B.47}$$

which describes lasers with saturable absorber [11].

Because of

$$I \sim \frac{R_1^2}{\varepsilon} \sim \frac{|E_1|^2}{\varepsilon} \sim |E|^2$$

with E from system (B.34) and E_1 from system (B.39) I is proportional to the intensity $|E|^2$ of the original system. But $J = I\varepsilon = O(1)$ is the natural scale to describe the dynamics, because the intensity of the solutions is of order $I \sim O(\frac{1}{\varepsilon})$ (see [11] for system (B.47)), which is the scale we needed to apply Fenichel theory. System (B.47) is equivalent to the Yamada model (2.1) for $\tilde{\varepsilon} = \bar{\tilde{\varepsilon}}$ with

$$D = \frac{G}{A},\ \bar{D} = -\frac{Q}{B},\ \bar{A} = B,\ \alpha = a,\ \gamma = \varepsilon.$$

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