DYNAMICS ON DIFFERENT TIME SCALES IN THE YAMADA MODEL *

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The basic dynamics of the Yamada model consists of a slow drift along a slow manifold which changes its stability along a not normally hyperbolic line and a fast jump from the stable part of the slow manifold to the unstable part. Depending on parameters, a homoclinic orbit and a transcritical bifurcation occur. We present the main ideas of the proof that in the parameter space a smooth curve of homoclinic orbits exists exponentially close to a curve of transcritical bifurcations. In the singular limit, where the two curves meet, a singular homoclinic orbit exists. The proof is based on the blow-up method in combination with the geometric singular perturbation theory.

The Yamada model for lasers with a saturable absorber shows interesting dynamics on widely different time scales. The system

$$\begin{aligned} J' &= (G - Q - 1)J \\ G' &= -GJ + \varepsilon(A - G) \\ Q' &= -aQJ + \varepsilon(B - Q), \end{aligned} \tag{1}$$

(see [6] with $J = \varepsilon I$) is given in scaled dimensionless variables, where J denotes the laser intensity, G and Q the population densities of the amplifying and absorbing media and A, a, B and ε are parameters. The parameter ε is typically of the order of $10^{-3} - 10^{-4}$, due to the different time scales of dynamics of the laser intensity and the media in the cavity, which makes system (1) a singular perturbed differential equation.

In the following we keep the parameters a and B constant. We are interested in the case $a > 1$, $B > \frac{1}{a-1}$ and (A, ε) close to $(B + 1, 0)$. Numerical

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calculations[6] for $\varepsilon > 0$ indicate the existence of a curve of homoclinic orbits $A = A_{Hom}(\varepsilon)$ which lies very close to a curve of transcritical bifurcations $A = A_T(\varepsilon) = B + 1$ in the parameter space (A, ε). In the singular limit the curves seem to meet at $(A, \varepsilon) = (B + 1, 0)$. We proof analytically that the numeric results are indeed correct and can be extended to $\varepsilon = 0$:

Theorem 1. *For $a > 1$, $B > \frac{1}{a-1}$ there exists a $\varepsilon_0 > 0$ such that*

(1) there exists a C^1-function $A_{Hom}(\varepsilon)$, $\varepsilon \in [0, \varepsilon_0]$, such that for $A = A_{Hom}(\varepsilon)$ an orbit homoclinic to N^u exists,
(2) the functions $A_T(\varepsilon)$ and $A_{Hom}(\varepsilon)$ are exponentially close, that means, there exists $c > 0$ such that $|A_{Hom}(\varepsilon) - A_T(\varepsilon)| < O(e^{-c/\varepsilon})$ for $\varepsilon \in [0, \varepsilon_0]$.

The basic dynamics of system (1) consists of a slow drift close to the invariant slow manifold $S_\varepsilon := \{J = 0\}$, which changes its stability along the not normally hyperbolic line $L := \{J = 0,\ G - Q - 1 = 0\}$, and a fast jump from the unstable part of the slow manifold $S_\varepsilon^u := \{J = 0,\ G - Q - 1 > 0\}$ to the stable part of the slow manifold $S_\varepsilon^s := \{J = 0,\ G - Q - 1 < 0\}$. Fenichel theory[3] and regular perturbation theory tell us how to combine the solutions of the layer problem and the reduced problem, the two limiting problems of the fast and the slow dynamics, to describe the dynamics of the full problem, if we stay away from a neighbourhood of the line L, where the crucial hyberbolicity condition fails.

System (1) has a transcritical bifurcation at the parameter value $A = A_T = B+1$ and $\varepsilon > 0$, where the equilibria $N^s(A, \varepsilon) \in S_\varepsilon$ and the equilibria $N^u(A, \varepsilon)$ with $J = O(\varepsilon) \geq 0$ intersect. This is exactly the parameter value of A, $A = A_T$, where $N^s(A_T, \varepsilon) \in L$. For $\varepsilon = 0$ the situation degenerates an the equilibria $N^u(A, 0)$ and $N^s(A, 0)$ are part of S_0.

We have to show that for certain parameter values $A = A_{Hom}$ an orbit homoclinic to N^u exists. For $(A, \varepsilon) = (A_T, 0)$ a singular homoclinic orbit γ, consisting of an orbit of the layer problem and a solution of the reduced problem exists. This orbit γ, which is homoclinic to $N^s(A_T, 0) = N^u(A_T, 0) \in L$ is the organizing center of the proof. We will consider the unstable manifold $M^u(A, \varepsilon)$ of $N^u(A, \varepsilon)$ and calculate its intersection with the stable manifold $M^s(A, \varepsilon)$ of $N^u(A, \varepsilon)$, which are up to now only well defined for $\varepsilon > 0$.

To describe the dynamics close to the line L, where an important part of the dynamics takes place, we use a blow-up transformation[4] of the extended phase space $(J, G, Q, \varepsilon, A)$, which maps the 'line' $(L, \varepsilon, A) \subset R^3$ to the 'cylinder' $\bar{L} \subset S_2 \times R^2$.

In the new coordinates it is possible to desingularize the vector field and we obtain a nontrivial flow on $\bar{L}$. The great benefit of this transformations is that now we have obtained more regularity close to $\bar{L}$ and can describe the flow close to it by using standard theory again. One problem is that by this transformation the parameters inherit dynamics and we have to study a five dimensional system of differential equations instead of a three dimensional system with two parameters.

For the blown-up system we obtain a regular transcritical bifurcation of the smooth manifolds of equilibria N^s and N^u, which are defined for $\varepsilon \geq 0$ and we can now define regular smooth stable and unstable manifolds M^s and M^u of N^u for $\varepsilon \geq 0$, where N^s etc. satisfy $N^s|_{(A,\varepsilon)} = N^s(A,\varepsilon)$ for $\varepsilon > 0$. The singular homoclinic orbit satisfies $\gamma = M^u|_{(A,\varepsilon)=(A_T,0)} \subset M^s|_{(A,\varepsilon)=(A_T,0)}$.

Next we study the return map close to γ and describe how the smoothness of M^u changes during the transition using centermanifold theory, regular perturbation theory, Fenichel coordinates, etc. When M^u returns to a neighborhood of N^u again, it is a C^1 manifold which is exponentially close to S_ε. The manifold M^s lies transversally to S_ε. So the two manifolds intersect transversally and we can use the Implicit Function Theorem to see that the homoclinic orbit exists for a C^1 curve $A = A_{Hom}(\varepsilon)$ which satisfies $|A_{Hom}(\varepsilon) - A_T| < O(e^{-C/\varepsilon})$.

In the last years the blow-up transformation was applied succesfully to interesting planar singular perturbed differential equations involving not normally hyperbolic equilibria[1,2,7,8]. This work shows that the blow-up method is also an effective tool to analyze higher dimensional problems. For the complete proof see[4,5]. This article is based on my PhD thesis[4] under the supervision of P. Szmolyan at the Technical University of Vienna.

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