Hopf-Bifurcation in Systems with spherical Symmetry: Invariant Tori and connecting Orbits

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Contents

1 Introduction **3**

2 Analytical methods **5**
2.1 Representation of the group $O(3) \times S^1$ on $V_2 \oplus iV_2$ 5
2.2 Restriction to $\mathrm{Fix}(\mathbb{Z}_2, 1)$ 9
2.2.1 Poincaré-series, invariants and equivariants 9
2.2.2 Orbit space reduction 12
2.2.3 Lattice of isotropy subgroups 14
2.2.4 Critical points of the reduced vector field 19
2.2.5 Stability of the critical points of the reduced vector field 27
2.2.6 Fifth order terms 30
2.2.7 Singular perturbation theory 33
2.2.8 Invariant tori 37
2.2.9 Stability of the invariant tori 41

3 Topological methods **43**
3.1 Homotopy and homology index 43
3.2 Calculation of the homology groups 43
3.3 Morse-decompositions and connection graphs 50
3.4 Topological results 51

1 Introduction

An interesting problem in the theory of ordinary differential equations is the generalisation of the Hopf-bifurcation from the two dimensional case to higher dimensional systems having a symmetry. In this connection [GoSt] and [GoStSch] investigated problems on a vector space X that can be decomposed into a direct sum of absolutely irreducible representations of the group O(3) in the form $X = V_l \oplus \mathrm{i}V_l$. Here V_l denotes the space of homogeneous harmonic polynomials $P : \mathbb{R}^3 \to \mathbb{R}$ of degree l. This is the simplest case where purely imaginary eigenvalues (of high multiplicity) in the bifurcation point are possible. Using Lie-group theory the authors showed the existence of branches of periodic solutions with certain symmetries. Here in addition to the spatial O(3)-symmetry a temporal S^1-symmetry occurs. This symmetry corresponds to a time shift along the periodic solutions. In order to get their results the authors made a Lyapunov-Schmidt-reduction on the space of periodic functions. The reduced system then has $\mathrm{O}(3) \times \mathrm{S}^1$-symmetry and solutions correspond to periodic solutions of the original system with spatial-temporal symmetry. Under certain transversality assumptions periodic solutions with symmetry $\tilde{H} \subset \mathrm{O}(3) \times \mathrm{S}^1$ bifurcate if $\mathrm{Dim}\,\mathrm{Fix}(\tilde{H}) = 2$ for the induced representation of the group $\mathrm{O}(3) \times \mathrm{S}^1$ on the space X (cf. [GoSt] resp. [GoStSch]). [Fi] has shown that it is sufficient that $\tilde{H}$ is a maximal subgroup for periodic solutions with symmetry $\tilde{H}$ to bifurcate. Using this methods only the existence of periodic solutions can be investigated. Via normal form theory (cf. [EletAl]) one gets $\mathrm{O}(3) \times \mathrm{S}^1$-equivariant polynomial vector fields up to every finite order for our systems. This additional S^1-symmetry comes from the fact that the normal form commutes with the one parameter group $e^{L^{\mathrm{T}} t}$ which is generated by the linearisation L in the bifurcation point. For a Hopf-bifurcation L has purely imaginary eigenvalues (of high multiplicity) and the group generated is a rotation. [IoRo], [HaRoSt] and [MoRoSt] did analytic calculations for the normal form up to fifth order for $l = 2$. They gave conditions for the stability of the five branches of periodic solutions predicted by [GoSt] resp. [GoStSch] in terms of coefficients of the normal form. Quasiperiodic solutions found by [IoRo] in the normal form up to third order could not be confirmed in this paper. We will show a mechanism for quasiperiodic solutions to bifurcate in the fifth order.

[FiMi] introduced the Conley-index, a generalisation of the Morse-index due to [Co], in connection with the notion of a connection graph in order to prove the existence of heteroclinic orbits between different fixed points of a dynamical system. The authors investigated the fixed point bifurcation in O(3)-equivariant systems that have an additional gradient structure. A gradient structure in the neighbourhood of the bifurcation point means a flow defined order on the isolated invariant sets (Morse-sets) in this neighbourhood. Motion between this Morse-sets is only possible in downward direction. Especially recurrent structure is impossible. Now one assumes to know all bifurcating objects which should be isolated invariant sets. Then it is possible to draw conclusion from the connection graph defined orders between the Morse-sets to the unknown flow defined order. Statements on the existence of connecting orbits between different fixed points become possible. For the representations V_l, $l = 2, 3$, [FiMi] gave the corresponding bifurcation diagrams with connecting orbits between fixed points with different symmetry. Algebraic results in this connection have been achieved by [La1] and [ChLaMe].

In the third chapter we will apply the methods introduced by [FiMi] to the case of Hopf-bifurcation. This systems do not possess a natural gradient structure. This has to be assumed. Investigating the normal form due to [IoRo] one finds a region in parameter space where two of the branches of periodic solutions bifurcating supercritically are stable simultaneously. We will show that here the topological methods lead to a contradiction, it is not possible to fill in correct connection graphs. This means either recurrent structure has to occur (the assumption has been wrong) or the set of Morse-sets underlying the investigations was not complete. In this case other invariant objects have to bifurcate. This contradiction actually was the starting point of the analytical investigations that are described in the second chapter.

In the second chapter we want to show that in the region in parameter space in question other solutions bifurcate. This solutions (quasiperiodic solutions) make it possible, again assuming we have gradient like structure, to fill in correct connection graphs. So by inclusion of the quasiperiodic solutions into the topological argument the contradiction disappears. A possible bifurcation scenario is found and statements on the existence of connecting orbits between different invariant sets become possible. In order to get our results we will proceed the following way.

First the representation of the group $\Gamma = \mathrm{O}(3) \times \mathrm{S}^1$ on the ten dimensional space $X = V_2 \oplus \mathrm{i}V_2$ is introduced. The lattice of isotropy subgroups of this representation is given according to [MoRoSt] and the results of [IoRo] are quoted. The smallest invariant subspace containing both solutions that are stable simultaneously has isotropy $\Sigma = (\mathbb{Z}_2, 1)$.

Then our considerations are restricted to this six dimensional subspace. The normaliser of Σ is $\mathrm{N}(\Sigma) = \mathrm{O}(2) \times \mathrm{S}^1 \subset \Gamma$. This is the biggest subgroup of Γ leaving $\mathrm{Fix}(\Sigma)$ invariant as a subspace. Now we will

look at the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$.
Dealing with differential equations with symmetries one has to do with group orbits of solutions because a solution $x(t)$ gives solutions $\gamma\, x(t)$ with $\gamma \in \Gamma$. This redundancy induced by the action of the group will be removed by identifying points that lie on a group orbit. I.e. one studies the orbit space that is homeomorphic to the image of the Hilbert-map $\Pi : \mathrm{Fix}(\Sigma) \to \mathbb{R}^k : z \to \pi_i(z)$ (cf. [La2] and [Bi]). Here k denotes the minimal number of generators of the ring of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ invariant polynomials $P : \mathrm{Fix}(\Sigma) \to \mathbb{R}$ and let π_i, $i = 1, \ldots, k$, be such a system of generators. Thus the original differential equation is reduced to a differential equation on $\Pi\big(\mathrm{Fix}(\Sigma)\big)$ of the form $\dot{\pi} = g(\pi)$, $\pi = (\pi_1, \ldots, \pi_k)$. In order to perform this reduction for a given equation one first of all has to know the number of independent invariants and equivariants for a given representation. Then one has to calculate them explicitly. Statements on the number of independent invariants and equivariants and possible relations between them are given by the Poincaré-series. This are formal power series $\sum_{i=0}^{\infty} a_i\, t^i$ in t. Here a_i denotes the dimension of the vector space of homogeneous invariant polynomials of degree i resp. the dimension of the vector space of homogeneous equivariant mappings of degree i. This series can be determined just by knowledge of the representation of the group on the space.
The lattice of isotropy subgroups of the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$ and the image of the Hilbert-map are determined. This is a stratified space which consists of manifolds (strata) that are images of points of some isotropy type of the representation of $\frac{\mathrm{N}(\Sigma)}{\Sigma}$ on $\mathrm{Fix}(\Sigma)$ each. Thus each stratum is flow invariant with respect to the reduced vector field on $\Pi\big(\mathrm{Fix}(\Sigma)\big)$.
Then we will carry out the orbit space reduction for the normal form up to third order. The critical points of the reduced vector field in $\Pi\big(\mathrm{Fix}(\Sigma)\big)$ are determined. As expected by inspection of the lattice of isotropy subgroups of Γ on $V_2 \oplus \mathrm{i}V_2$ we will find images of periodic solutions of isotropy $(\mathrm{O}(2), 1)$, $(\mathrm{D}_4, \mathbb{Z}_2)$, $\widetilde{\mathrm{SO}(2)}^2$ and $(\mathbb{T}, \mathbb{Z}_3)$. Moreover there exists a stratum F in $\Pi\big(\mathrm{Fix}(\Sigma)\big)$. On this stratum lie connected via a curve g of fixed points the fixed points having isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$ in the original system. There are no further fixed points. The preimage of the stratum F consists of points having isotropy $(\mathbb{Z}_2, 1)$ in the restricted system. Perturbations that respect the symmetry will therefore respect the stratum. The curve g is stable for the reduced vector field restricted to the stratum F. Small perturbations of the original vector field in fifth order of magnitude ε will therefore preserve a curve. By use of singular perturbation theory (cf. [Fe]) one gets a resulting drift on the curve. This explains the observation made by [IoRo] that the stability of the fixed points of isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_2, \mathbb{Z}_2)$ is determined in fifth order.
According to the relative choice of the coefficients of the third order normal form in the region in parameter space in question there is a point on the curve g where the linear stability of the curve in the direction of the principle stratum changes. Linearisation of the reduced vector field in this point yields a nontrivial two dimensional Jordan-block to the eigenvalue zero. The second dimension comes from the linearisation along the curve. Now the flow on the two dimensional center manifold in this point is determined for small ε. The persistence of the curve g for small ε, knowledge of the direction of the drift, the change of stability in the direction of the principle stratum and the existence of a nontrivial two dimensional Jordan-block to the eigenvalue zero are sufficient to prove for small ε the bifurcation of a fixed point of the reduced equation in the direction of the principle stratum using the implicit function theorem. Fixed points of the reduced system on the stratum F correspond to periodic solutions, fixed points in the principle stratum correspond to quasiperiodic solutions in the original system.
In the third chapter this quasiperiodic solutions will be included into the topological argument. In order to do so the stability and the homology groups of the group orbit of quasiperiodic orbits have to be determined. Again we will assume that in a neighbourhood of zero besides the trivial solution, the group orbits of periodic resp. quasiperiodic solutions and possible connecting orbits between this solutions no further bounded solutions exist for small values of the bifurcation parameter. Moreover the flow should have gradient like structure with respect to the trivial solution and the group orbits of periodic resp. quasiperiodic solutions. Then it is possible, including the quasiperiodic solutions, to fill in correct connection graphs. Thus a possible bifurcation scenario is given, an explanation of the contradiction of the topological argument for the periodic solutions is found.
There a several possibilities to fill in connection graphs in our case. We will list the corresponding graph defined orders for the Morse-sets. Comparing them we will show that under our assumptions a connecting orbit from the group orbit of quasiperiodic solutions to the group orbit of solutions with isotropy $(\mathrm{O}(2), 1)$ has to exist.

2 Analytical methods

2.1 Representation of the group $\mathrm{O}(3) \times \mathrm{S}^1$ on $V_2 \oplus \mathrm{i}V_2$

In this paper we investigate systems of ODE's of the form

$$\dot{x} \quad = \quad f(\lambda, x)$$

in the ten dimensional space

$$X \quad = \quad V_2 \oplus \mathrm{i}V_2.$$

Let V_2 be the five dimensional space of homogeneous harmonic polynomials

$$p \quad : \quad \mathbb{R}^3 \quad \to \quad \mathbb{R}$$

of degree two. We have

$$V_2 \quad = \quad \left\langle 2x_3^2 - (x_1^2 + x_2^2), x_1x_3, x_2x_3, x_1^2 - x_2^2, x_1x_2 \right\rangle .$$

Let us introduce the following coordinates $(z, \overline{z})$,

$$z \quad = \quad (z_{-2}, z_{-1}, z_0, z_1, z_2), \ z_m \in \mathbb{C}, m = -2, \ldots, 2,$$

in the space X:

$$x \in X \quad \Leftrightarrow \quad x = \sum_{m=-2}^{2} z_m \, Y_m.$$

Here

$$\begin{aligned} Y_0 \quad &= \quad \sqrt{\tfrac{5}{16\pi}} \left(2x_3^2 - (x_1^2 + x_2^2)\right), \\ Y_{\pm 1} \quad &= \quad \sqrt{\tfrac{15}{8\pi}} \left(x_1x_3 \pm \mathrm{i}x_2x_3\right), \\ Y_{\pm 2} \quad &= \quad \sqrt{\tfrac{15}{32\pi}} \left((x_1^2 - x_2^2) \pm \mathrm{i}2x_1x_2\right) \end{aligned}$$

denote spherical harmonics. Moreover let

$$f \quad : \quad \mathbb{R} \times X \quad \to \quad X$$

be a smooth map that commutes with the following representaion of the compact Lie-group

$$\Gamma \quad = \quad \mathrm{O}(3) \times \mathrm{S}^1$$

on the space X.
The group

$$\mathrm{O}(3) \quad = \quad \mathrm{SO}(3) \oplus \mathbb{Z}_2^c$$

with

$$\mathbb{Z}_2^c \quad = \quad \{\pm \mathrm{Id}\}$$

acts via the natural representation absolutely irreducible on V_2. For $p \in V_2$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} \gamma \, p(\cdot) \quad &= \quad p(\gamma^{-1} \cdot) \text{ for } \gamma \in \mathrm{SO}(3), \\ -\mathrm{Id} \, p(\cdot) \quad &= \quad p(\cdot). \end{aligned}$$

This representation is a special case of the representation of the group $\mathrm{O}(3)$ on the space V_l, $l \geq 1$. For l even the subgroup $\mathbb{Z}_2^c$ acts trivially in the natural representation. On the space X the group $\mathrm{O}(3)$ acts diagonal. For the general representation theory of $\mathrm{O}(3)$ we refer to [StiFä] and [GoStSch].
The group S^1 acts as a rotation in the coordinates

$$\begin{aligned} \phi \, z \quad &= \quad e^{i\phi} z, \\ \phi \, \overline{z} \quad &= \quad e^{-i\phi} \overline{z} \end{aligned}$$

with $\phi \in \mathrm{S}^1$.
So we have

$$f(\lambda, \gamma\, x) \quad = \quad \gamma\, f(\lambda, x),\ \forall \gamma \in \Gamma.$$

In their paper concerning Hopf-bifurcation with O(3)-Symmetry [GoSt] and [GoStSch] look at systems of the form

$$\dot{x} \quad = \quad f(\lambda, x).$$

Here

$$x \in X \quad = \quad V_l \oplus \mathrm{i}V_l$$

and let

$$f \quad : \quad \mathbb{R} \times X \quad \to \quad X$$

be a smooth mapping. This direct sum of two absolutely irreducible representations of the group O(3) is the simplest case allowing imaginary eigenvalues, however of high multiplicity, in the bifurcation point. Let us assume:

- f is equivariant with respect to the diagonal representation of O(3) on X.
- $f(\lambda, 0) \equiv 0$.
- $(\mathrm{D}f)_{\lambda,0}$ has a pair of complex conjugate eigenvalues $\sigma(\lambda) \pm \mathrm{i}\rho(\lambda)$ with $\sigma(0) = 0$, $\dot{\sigma}(0) \neq 0$ and $\rho(0) = \omega$ of multiplicity $(2l+1) = \mathrm{Dim}(V_l)$ with smooth functions σ and ρ.

The authors now look at subgroups

$$\bar{H} \quad \subset \quad \Gamma.$$

Here the group $\mathrm{S}^1 \subset \Gamma$ acts as a time shift on the periodic solutions. Therefore subgroups $\bar{H}$ consist of spatial and temporal symmetries. For subgroups $\bar{H}$ with

$$\mathrm{Dim}\,\mathrm{Fix}(\bar{H}) \quad = \quad 2$$

with respect to our representation of the group Γ on $V_l \oplus \mathrm{i}V_l$ the authors prove the existence of exactly one branch of periodic solutions with small amplitude of period near $\frac{2\pi}{\omega}$ and the group of symmetries $\bar{H}$. In order to do this the authors make a Lyapunov-Schmidt-reduction on the space of periodic functions. The reduced system has the full $\mathrm{O}(3) \times \mathrm{S}^1$-symmetry and solutions correspond to periodic solutions with spatial-temporal symmetries in the original system.
For $l = 2$ [IoRo] applied normal form theory (cf. [EletAl]) to this systems. Up to every finite order they got $\mathrm{O}(3) \times \mathrm{S}^1$-equivariant systems of the form described above. This additional S^1-symmetry up to every finite order comes from the fact that the normal form of f commutes with the one parameter group $e^{(\mathrm{D}f)^{\mathrm{T}}_{0,0}\, t}$. Due to our conditions on the eigenvalues this is just a complex rotation.
The following calculations are done using the normal form up to fifth order due to [IoRo]. The explicit normal form up to fifth order is very lengthy and will not be given here. The parts important for our calculations will be cited when necessary.
Let G be a compact Lie-group acting on a space X. The most general form of a G-equivariant polynomial mapping $g : X \to X$ is

$$g(x) \quad = \quad \sum_{i=1}^{n} p_i(x)\, e_i(x).$$

Here

$$p_i \quad : \quad X \quad \to \quad \mathbb{R}$$

denote G-invariant polynomials and

$$e_i \quad : \quad X \quad \to \quad X$$

G-equivariant, polynomial mappings.
In order to determine the most general G-equivariant, polynomial mapping up to a fixed order one first of all has to know the number of independend invariants and equivariants and possible relations between them. On this occasion the Poincaré-series described in the next chapter are useful. The next problem is to find the polynomials. In the case of the group $\mathrm{O}(3)$ using raising and lowering operators (cf. [Sa],[Mi]) one can check whether a specific polynomial is invariant or not. The raising and lowering operators are in close relationship to the infinitesimal generators of the Lie-algebra of the group. So the problem is to construct and check all possible polynomials resp. polynomial mappings. Dealing with high order polynomials and large dimensions of the problem this is a very difficult task that is only accessible via symbolic numerics. At least using the Poincaré-series one knows when everything is found.

The lattice of isotropy subgroups of the representation of the group Γ on $V_2 \oplus \mathrm{i}V_2$ has been determined by [MoRoSt].

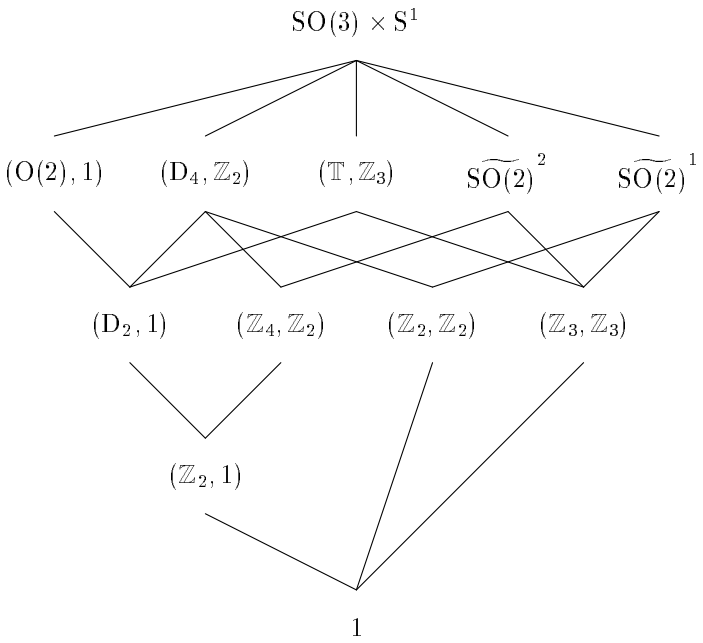


Figure 1: Lattice of isotropy subgroups of Γ on $V_2 \oplus \mathrm{i}V_2$.

The subgroups $\bar{H} \subset \Gamma$ are given as twisted subgroups

$$\bar{H} \quad = \quad \big(H, \Theta(H)\big)$$

with $H \subset \mathrm{SO}(3)$ and $\Theta(H) \subset \mathrm{S}^1$. In this connection

$$\Theta \quad : \quad H \quad \to \quad \mathrm{S}^1$$

is a group homomorphism. Every isotropy subgroup $\bar{H} \subsetneq \Gamma$ can be written in this form (cf. [GoStSch]). In the case of the isotropy subgroups $\widetilde{\mathrm{SO}(2)}^1$ resp. $\widetilde{\mathrm{SO}(2)}^2$ we have $H = \mathrm{SO}(2) \subset \mathrm{SO}(3)$ and $\Theta(H) = \mathrm{S}^1$ with $\Theta(\phi) = \phi$ resp. $\Theta(\phi) = \phi^2$.
In [MoRoSt] the authors investigate Hamiltonian systems of the form

$$\dot{v} \quad = \quad J\,\mathrm{D}H(v)$$

with $v \in \mathbb{R}^{10} = V_2 \oplus \mathrm{i}V_2$,

$$J \quad = \quad \begin{pmatrix} 0 & -I_5 \\ I_5 & 0 \end{pmatrix}$$

and $\mathrm{O}(3) \times \mathrm{S}^1$ invariant Hamiltonian $H : \mathbb{R}^{10} \to \mathbb{R}$. This leads to restrictions on the coefficients of the normal form of the vector field. Like [IoRo] for the general vector field [MoRoSt] analytically prove

the existence of periodic solutions of isotropy $(\mathrm{O}(2),1)$, $(\mathrm{D}_4,\mathbb{Z}_2)$, $(\mathbb{T},\mathbb{Z}_3)$, $\widetilde{\mathrm{SO}(2)}^1$ and $\widetilde{\mathrm{SO}(2)}^2$. This are exactly the subgroups of Γ having a two dimensional fixed point space for our representation, i.e. the subgroups for which [GoSt] and [GoStSch] predicted the bifurcation of periodic solutions using group theoretical methods. Moreover the authors give conditions for the stability of the different branches of periodic solutions by means of regions in the parameter space of the normal form.
In the following we will look only at the situation where all solutions bifurcate supercritically. In this case there is a region in parameter space where the periodic solutions of isotropy $(\mathrm{O}(2),1)$ resp. $\widetilde{\mathrm{SO}(2)}^2$ are stable simultaneously, see [IoRo]. Using topological methods we will show that in this region in parameter space besides the trivial solution and the different group orbits of periodic solutions other isolated invariant objects have to exist, or that there is recurrent structure between the trivial solution and the different group orbits of periodic solutions. Recurrent structure means that it is possible to go from a group orbit via connecting orbits that connect different group orbits in the direction of the flow back to the group orbit one started at.
In this paper we will prove the existence of quasiperiodic solutions in the region in parameter space in question. The quasiperiodic solutions found by [IoRo] using the third order normal form can't be confirmed. We will prove that the quasiperiodic solutions bifurcate in fifth order from a curve of periodic solutions that is degenerate up to third order.
In order to reduce the dimension of the problem we will restrict our calculations in the following to the smallest invariant subspace containing the two stable solutions. This is a subspace of isotropy $(\mathbb{Z}_2,1)$ due to the lattice of isotropy subgroups. Now we want to fix a specific subgroup

$$\mathrm{O}(2) \quad \subset \quad \mathrm{SO}(3)$$

because it is well suited for our coordinates:

$$\mathrm{O}(2) \quad = \quad \left\{ r_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};\ \phi \in [0,2\pi) \right\}.$$

It acts (cf. [GoStSch]) in the following form on our coordinates z:

$$\begin{aligned} r_\phi\,(z_{-2}\,,z_{-1},z_0\,,z_1,z_2) &= (e^{-2i\phi}z_{-2},e^{-i\phi}z_{-1},z_0,e^{i\phi}z_1,e^{2i\phi}z_2), \\ \kappa\,(z_{-2}\,,z_{-1},z_0\,,z_1,z_2) &= (z_2\,,-z_1,z_0\,,-z_{-1},z_{-2}). \end{aligned}$$

Finally let

$$\Sigma \quad = \quad (\mathbb{Z}_2,1)$$

with

$$\mathbb{Z}_2 \quad = \quad \{1,r_\pi\}.$$

2.2 Restriction to $\mathrm{Fix}(\mathbb{Z}_2, 1)$

Lemma 2.2.1:

$$\mathrm{Fix}(\Sigma) = \mathrm{Span}\{(z_{-2}, 0, z_0, 0, z_2)\} \cong \mathbb{C}^3$$

Lemma 2.2.2:

$$\Xi = \frac{\mathrm{N}(\Sigma)}{\Sigma} = \mathrm{O}(2) \times \mathrm{S}^1.$$

The group $\mathrm{O}(2) \times \mathrm{S}^1$ *acts on* $\mathbb{C}^3$*:*

$$\begin{aligned} r_\theta(z_{-2}, z_0, z_2) &= (e^{-i\theta} z_{-2}, z_0, e^{i\theta} z_2), \\ \kappa(z_{-2}, z_0, z_2) &= (z_2, z_0, z_{-2}), \\ \phi(z_{-2}, z_0, z_2) &= (e^{i\phi} z_{-2}, e^{i\phi} z_0, e^{i\phi} z_2). \end{aligned}$$

The group $\mathrm{O}(2)$ *is generated by the rotations* r_θ *and the reflection* κ *and the group* S^1 *by the rotations* ϕ*.*

Proof: We have $\mathrm{N}_{\mathrm{SO}(3)}(\mathbb{Z}_2) = \mathrm{O}(2)$. The representation of $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{C}^3$ is given by restriction of the representation of $\mathrm{SO}(3) \times \mathrm{S}^1$ on $\mathrm{Fix}(\Sigma)$. ⋈

Let $z = (z_{-2}, z_0, z_2) \in \mathbb{C}^3$. The definition

$$\sigma \overline{z} = \overline{\sigma z}, \text{ with } \sigma \in \Xi,$$

gives rise to an unitary representation of Ξ on the space

$$\mathbb{C}^3 \oplus \mathbb{C}^3 \supset \{(z, \overline{z}),\ z \in \mathbb{C}^3\} = \mathbb{R}^6.$$

2.2.1 Poincaré-series, invariants and equivariants

The number of generators of the ring of Ξ-invariant polynomials $P : \mathbb{R}^6 \to \mathbb{R}$ and of the module of Ξ-equivariant polynomial mappings $Q : \mathbb{R}^6 \to \mathbb{R}^6$ over the ring of invariant polynomials can be determined using Poincaré-series.
For a unitary representation T of a compact Lie-group G on a vector space V we have

$$\begin{aligned} P_I(t) &= \int_G \frac{1}{\det(I - tT(g))} \mathrm{d}g = \sum_{i=0}^{\infty} c_i t^i, \\ P_{Eq}(t) &= \int_G \frac{\overline{\chi(g)}}{\det(I - tT(g))} \mathrm{d}g = \sum_{i=0}^{\infty} d_i t^i. \end{aligned}$$

Here $c_i,\ i > 0$, denotes the dimension of the vector space of homogeneous invariant polynomials of degree i and $d_i,\ i > 0$, the dimension of the vector space of homogeneous equivariant mappings of degree i. Let $c_0 = d_0 = 1$. The integral appearing in the formulas is the Haar-integral associated to the compact Lie-group G (cf. [BrtD]), $\chi(g),\ g \in G$, denotes the character of g relative to the representation T. The theory of Poincaré-series is presented in [La2].

Lemma 2.2.3:

$$\begin{aligned} P_I(t) &= \frac{1 + t^4}{(1 - t^2)^2 (1 - t^4)^2}, \\ P_{Eq}(t) &= \frac{2t + 3t^3 + t^5}{(1 - t^2)^2 (1 - t^4)^2}. \end{aligned}$$

Proof: The group $\Xi = \mathrm{O}(2) \times \mathrm{S}^1$ can be written as the disjoint union of two sets in the following form

$$\mathrm{O}(2) \times \mathrm{S}^1 = \mathrm{SO}(2) \times \mathrm{S}^1 \dot{\cup}\ \kappa \mathrm{SO}(2) \times \mathrm{S}^1.$$

Therefore the integrals appearing in the formulas split in two parts.
a. $\Xi_1 = \mathrm{SO}(2) \times \mathrm{S}^1$ acts on the space $\mathbb{C}^3 \oplus \mathbb{C}^3$. So we get

$$\begin{aligned} P_I^1(t) &= \int_{\Xi_1} \frac{1}{\det(I - tT(g))} \mathrm{d}g \\ &= \frac{1}{(2\pi)^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{2\pi} \frac{1}{\det(I - tT(\theta,\phi))} \mathrm{d}\theta \, \mathrm{d}\phi. \end{aligned}$$

For our representation we have

$$\begin{aligned} &\det(I - tT(\theta,\phi)) = \\ &\quad (1 - te^{i(\theta-\phi)})(1 - te^{-i\phi})(1 - te^{-i(\theta+\phi)})(1 - te^{i(-\theta+\phi)})(1 - te^{i\phi})(1 - te^{i(\theta+\phi)}). \end{aligned}$$

A transformation of variables

$$e^{i\theta} \to y_1, \; e^{i\phi} \to y_2$$

leads to

$$\begin{aligned} P_I^1(t) &= \frac{1}{(2\pi i)^2} \oint_{y_1} \oint_{y_2} \frac{1}{y_1 y_2 \det(I - tT(y_1, y_2))} \mathrm{d}y_1 \, \mathrm{d}y_2 \\ &= \frac{1}{(2\pi i)^2} \oint_{y_1} \oint_{y_2} \frac{y_1 y_2^2}{(y_2 - ty_1)(y_2 - t)(y_1 y_2 - t)(y_1 - ty_2)(1 - ty_2)(1 - ty_1 y_2)} \mathrm{d}y_1 \, \mathrm{d}y_2. \end{aligned}$$

Using the residue theorem two times one gets

$$P_I^1(t) = \frac{1 + t^4}{(1 - t^2)^3 (1 - t^4)}.$$

b. For the set $\kappa\mathrm{SO}(2) \times \mathrm{S}^1$ we have

$$\det(I - tT(\kappa,\theta,\phi)) = (1 - te^{-i\phi})^2 (1 + te^{-i\phi})(1 - te^{i\phi})^2 (1 + te^{i\phi}).$$

A transformation of variables gives

$$P_I^2(t) = \frac{1}{2\pi i} \oint_{y_2} \frac{y_2^2}{(y_2 - t)^2 (y_2 + t)(1 - ty_2)^2 (1 + ty_2)} \mathrm{d}y_2.$$

Using the residue theorem one gets

$$P_I^2(t) = \frac{1 + t^4}{(1 - t^4)^2 (1 - t^2)}.$$

c. Because of the normalization of the Haar-integral we have

$$\begin{aligned} P_I(t) &= \tfrac{1}{2} \left(P_I^1(t) + P_I^2(t) \right) \\ &= \frac{1 + t^4}{(1 - t^2)^2 (1 - t^4)^2}, \end{aligned}$$

proving the first formula.
d. We want to calculate

$$P_{Eq}^1(t) = \int_{\Xi_1} \frac{\overline{\chi(g)}}{\det(I - tT(g))} \mathrm{d}g.$$

Here we get

$$\begin{aligned} \chi(\theta,\phi) &= \mathrm{Tr}(T(\theta,\phi)) \\ &= e^{i(-\theta+\phi)} + e^{i\phi} + e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{-i\phi} + e^{-i(\theta+\phi)} \\ &= \left(e^{i\phi} + e^{-i\phi} \right) \left(e^{i\theta} + 1 + e^{-i\theta} \right). \end{aligned}$$

This leads to

$$\begin{aligned} P^1_{Eq}(t) &= \\ &\frac{1}{(2\pi i)^2}\oint_{y_1}\oint_{y_2}\frac{y_2(1+y_1+y_1^2)(1+y_2^2)}{(y_2-ty_1)(y_2-t)(y_1y_2-t)(y_1-ty_2)(1-ty_2)(1-ty_1y_2)}\mathrm{d}y_1\,\mathrm{d}y_2 \\ &= 2\frac{3t+3t^3}{(1-t^2)^3(1-t^4)}. \end{aligned}$$

e. Correspondingly for the set $\kappa\mathrm{SO}(2)\times\mathrm{S}^1$ one gets

$$\chi(\kappa,\theta,\phi) = e^{i\phi}+e^{-i\phi}.$$

This leads to

$$\begin{aligned} P^2_{Eq}(t) &= \frac{1}{2\pi i}\oint_{y_2}\frac{y_2(1+y_2^2)}{(y_2-t)^2(y_2+t)(1-ty_2)^2(1+ty_2)}\mathrm{d}y_2 \\ &= 2\frac{t}{(1-t^2)^2(1-t^4)}. \end{aligned}$$

f. Therefore we have

$$\begin{aligned} \tilde{P}_{Eq}(t) &= \tfrac{1}{2}\left(P^1_{Eq}(t)+P^2_{Eq}(t)\right) \\ &= 2\frac{2t+3t^3+t^5}{(1-t^2)^2(1-t^4)^2}. \end{aligned}$$

Doing this we used the diagonal representation of Ξ on $\mathbb{C}^3\oplus\mathbb{C}^3$. But we are only interested in the subspace $\{(z,\overline{z}),\ z\in\mathbb{C}^3\}\subset\mathbb{C}^3\oplus\mathbb{C}^3$. Therefore the number of equivariants given by the formula is twice as big as it should be, counting also equivariants with one component being zero. ⋈

The Poincaré-series can be interpreted in the following way.

Lemma 2.2.4: *The polynomials*

$$\begin{aligned} \pi_1 &= |z_0|^2 \\ \pi_2 &= |z_{-2}|^2+|z_2|^2 \\ \pi_3 &= |z_{-2}|^2|z_2|^2 \\ \pi_4 &= \frac{1}{2}\left(\overline{z_0}^2z_{-2}z_2+z_0^2\overline{z_{-2}}\,\overline{z_2}\right) \\ \pi_5 &= \frac{i}{2}\left(\overline{z_0}^2z_{-2}z_2-z_0^2\overline{z_{-2}}\,\overline{z_2}\right) \end{aligned}$$

are a minimal set of generators of the ring of invariant polynomials.

$$P : \mathbb{R}^6 \to \mathbb{R}.$$

The only relation between them is

$$\pi_4^2+\pi_5^2 = \pi_1^2\pi_3.$$

Proof: One easily sees that the given polynomials $\pi_1,\ldots,\pi_5$ are invariant and meet just the given relation. Therefore the Poincaré-series of this polynomials is identical to the calculated one. Because of this there are no additional generators and relations. ⋈

Introducing polar coordinates in the following form

$$z_j = r_je^{i\phi_j},\ j\in\{-2,0,2\},$$

and defining

$$\theta = 2\phi_0-\phi_{-2}-\phi_2$$

one gets

$$\pi_4 \quad = \quad r_0^2 r_{-2} r_2 \cos\theta$$

and

$$\pi_5 \quad = \quad r_0^2 r_{-2} r_2 \sin\theta.$$

Consequently the invariants π_4 and π_5 mean phase relations between the different coordinates.

Lemma 2.2.5: *Let $\pi : \mathbb{R}^6 \to \mathbb{R}$ be an invariant polynomial for the representation of Ξ on $\mathbb{R}^6$. Then*

$$p(z,\overline{z}) = \overline{\nabla_{z,\overline{z}}\pi(z,\overline{z})}$$

is a Ξ-equivariant polynomial mapping for this representation.

Proof: We have

$$p(\sigma(z,\overline{z})) = \overline{\nabla_{\sigma(z,\overline{z})}\pi(z,\overline{z})} = \overline{\nabla_{z,\overline{z}}\pi(z,\overline{z})}\,\overline{\sigma^{-1}} = \sigma p(z,\overline{z}).$$

The last equality is correct because the representation is unitary. ⋈

Lemma 2.2.6: *The independent, Ξ-equivariant, polynomial mappings*

$$Q \quad : \quad \mathbb{R}^6 \quad \to \quad \mathbb{R}^6$$

up to fifth order are

$$e_1 = \begin{pmatrix} 0 \\ z_0 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} z_{-2} \\ 0 \\ z_2 \end{pmatrix},\ e_3 = \begin{pmatrix} z_{-2}|z_2|^2 \\ 0 \\ z_2|z_{-2}|^2 \end{pmatrix},$$

$$e_4 = \frac{1}{2}\begin{pmatrix} z_0^2\overline{z_2} \\ 2z_{-2}z_2\overline{z_0} \\ z_0^2\overline{z_{-2}} \end{pmatrix},\ e_5 = -\frac{i}{2}\begin{pmatrix} z_0^2\overline{z_2} \\ -2z_{-2}z_2\overline{z_0} \\ z_0^2\overline{z_{-2}} \end{pmatrix}.$$

Here e_i, $i = 1,\ldots,5$, always denote the first component of the equivariant. The second is given by complex conjugation of the first one.

Proof: Using the previous lemma one knows that the mappings $e_j = \overline{\nabla_{z,\overline{z}}\pi_j}$, $j = 1,\ldots,5$, are equivariant. Power series expansion of $P_{Eq}(t)$ leads to

$$P_{Eq}(t) = 2t + 7t^3 + 17t^5 + \mathrm{O}(t^7).$$

There are $2, 7$ resp. 18 different possibilities to construct equivariant mappings of degree $1, 3$ resp. 5 from invariant polynomials $\pi_1,\ldots,\pi_5$ and equivariant mappings $e_1,\ldots,e_5$ by multiplication of invariants with an equivariant. In the fifth order one gets the relation

$$e_1(\pi_4 - \mathrm{i}\pi_5) \quad = \quad \frac{1}{2}\pi_1(e_4 - \mathrm{i}e_5).$$

All other combinations can't be generated this way. Therefore the Poincaré-series belonging to $\pi_1,\ldots,\pi_5$ and $e_1,\ldots,e_5$ is identical to the calculated one up to fifth order. Because of this there are no more generators or relations up to fifth order. ⋈

2.2.2 Orbit space reduction

The most general $\mathrm{O}(2)\times\mathrm{S}^1$-equivariant Hopf-bifurcation problem on $\mathbb{R}^6$ up to third order has the form

$$\dot{z} \quad = \quad (\lambda + \mathrm{i}\omega)(e_1 + e_2) + a_1\pi_1 e_1 + a_2\pi_1 e_2 + a_3\pi_2 e_1 + a_4\pi_2 e_2 + a_5 e_3 + a_6 e_4 + a_7 e_5\,,$$

with $a_j \in \mathbb{C}$, $j = 1,\ldots,7$, $\lambda,\omega \in \mathbb{R}$ and $z = (z_{-2}, z_0, z_2)$.
We want to study bifurcation problems on $\mathbb{R}^6$ resulting from a $\mathrm{SO}(3)\times\mathrm{S}^1$-equivariant problem on $\mathrm{V}_2 \oplus \mathrm{i}\mathrm{V}_2$. This gives the following restrictions for the coefficients $a_1,\ldots,a_7$:

$$\begin{aligned}\dot{z} \quad = \quad & (\lambda + \mathrm{i}\omega)\,(e_1 + e_2) + \left(a - \frac{1}{2}b - \sqrt{\frac{3}{2}}c\right)\pi_1 e_1 + \left(a - \sqrt{\frac{8}{3}}c\right)\pi_1 e_2 + \left(a - \sqrt{\frac{8}{3}}c\right)\pi_2 e_1 \\ & + a\pi_2 e_2 - \left(b + \sqrt{6}c\right)e_3 + \left(-b + \sqrt{\frac{2}{3}}c\right)e_4 + 0e_5. \qquad (2.2.1)\end{aligned}$$

Here $a, b, c \in \mathbb{C}$ denote the corresponding coefficients from the normal form of [IoRo]. One gets this by comparison of the normal form of [IoRo] restricted to the subspace with the general equation. Define coefficients $\alpha, \beta, \gamma \in \mathbb{C}$:

$$\begin{aligned} \alpha &= a - \tfrac{1}{2}b - \sqrt{\tfrac{3}{2}}c, & a &= \gamma, \\ \beta &= a - \sqrt{\tfrac{8}{3}}c, & b &= -2\alpha + \tfrac{3}{2}\beta + \tfrac{1}{2}\gamma, \\ \gamma &= a, & c &= \sqrt{\tfrac{3}{8}}(\gamma - \beta). \end{aligned}$$

Then the vector field has the form

$$\begin{aligned} \dot{z} &= (\lambda + \mathrm{i}\omega)(e_1 + e_2) + \alpha\pi_1 e_1 + \beta(\pi_1 e_2 + \pi_2 e_1) + \gamma\pi_2 e_2 + \\ &\quad 2(\alpha - \gamma)e_3 + 2(\alpha - \beta)e_4 \\ &= \Big((\lambda + \mathrm{i}\omega) + \alpha\pi_1 + \beta\pi_2\Big)e_1 + \Big((\lambda + \mathrm{i}\omega) + \beta\pi_1 + \gamma\pi_2\Big)e_2 + \\ &\quad 2(\alpha - \gamma)e_3 + 2(\alpha - \beta)e_4 \end{aligned} \tag{2.2.2}$$

with $\lambda, \omega \in \mathbb{R}$.

Let $\dot{x} = f(x)$ be a differential equation on a vector space X. Let the mapping f be equivariant with respect to the representation of the compact Lie-group G on X. Since

$$\dot{(gx)} = g\dot{x} = gf(x) = f(gx), \ \forall g \in G,$$

$gx(t), \ g \in G$, is a solution if $x(t)$ is a solution. This means one has to deal with group orbits Gx of solutions. Let G_x denote the isotropy of a point x. Then we have

$$\frac{G}{G_x} \cong Gx.$$

Here $\frac{G}{G_x}$ and Gx are compact manifolds and we have (cf. [Di])

$$\mathrm{Dim}\, Gx = \mathrm{Dim} G - \mathrm{Dim} G_x.$$

In order to get rid of the redundancy in our system induced by the group G one studies the orbit space $\frac{X}{G}$. Here points lying on a group orbit are identified:

$$x \simeq y \iff x = gy \text{ with } x, y \in X \text{ and } g \in G.$$

The orbit space is homeomorphic to the image of the Hilbert-map $\Pi(X)$

$$\begin{array}{rccc} \Pi & : & X & \to & \mathbb{R}^k \\ & & x & \to & (\pi_i(x)) \end{array}$$

(cf. [La2], [Bi]). Here k denotes the minimal number of generators of the ring of G-invariant polynomials $P : X \to \mathbb{R}$ and let $\pi_i, \ i = 1, \ldots, k$, be such a system of generators.
The original differential equation is reduced to a differential equation on $\Pi(X)$ of the form

$$\dot{\pi} = g(\pi) \text{ with } \pi = (\pi_1, \ldots, \pi_k).$$

The reduced equation can be calculated as follows:

$$\dot{\pi_i} = \nabla_x \pi_i \, \dot{x} = \nabla_x \pi_i \, f(x), \ i = 1, \ldots, k.$$

The advantage of this reduction lies in the fact that the dimension of the reduced problem is in general smaller than the original one. Furthermore e.g. symmetry induced periodic solutions in the original system correspond to fixed points in the reduced system and can be dealt with more easily analytically. The disadvantage is that the orbit space is in general no vector space but a stratified space.

In our case the differential equation up to third order (equation (2.2.2)) is given in the form

$$\dot{z} = \sum_{j=1}^{5} q_j \, e_j \, .$$

Here

$$q_j : \mathbb{R}^6 \to \mathbb{C},\ \ j = 1, \ldots, 5,$$

are invariant polynomials. So one gets

$$\begin{aligned} \dot{\pi_i} &= \nabla_z \pi_i \dot{z} + \nabla_{\overline{z}} \pi_i \dot{\overline{z}} \\ &= \overline{e_i} \dot{z} + e_i \dot{\overline{z}} \\ &= 2\mathrm{Re}\ (\overline{e_i} \dot{z}) \\ &= 2\mathrm{Re} \left(\sum_{j=1}^{5} q_j \overline{e_i} e_j \right). \end{aligned}$$

The products $\overline{e_i} e_j,\ i \leq j \in \{1, \ldots, 5\}$, are

$$\begin{array}{ll} \overline{e_1} e_1 = \pi_1 & \overline{e_2} e_2 = \pi_2 \\ \overline{e_1} e_2 = 0 & \overline{e_2} e_3 = 2\pi_3 \\ \overline{e_1} e_3 = 0 & \overline{e_2} e_4 = \pi_4 + \mathrm{i}\pi_5 \\ \overline{e_1} e_4 = \pi_4 - \mathrm{i}\pi_5 & \overline{e_2} e_5 = -\mathrm{i}\pi_4 + \pi_5 \\ \overline{e_1} e_5 = \mathrm{i}\pi_4 + \pi_5 & \end{array}$$

$$\begin{array}{lll} \overline{e_3} e_3 = \pi_2 \pi_3 & \overline{e_4} e_4 = \frac{1}{4}\pi_1^2 \pi_2 + \pi_1 \pi_3 & \overline{e_5} e_5 = \frac{1}{4}\pi_1^2 \pi_2 + \pi_1 \pi_3. \\ \overline{e_3} e_4 = \frac{1}{2}\pi_2(\pi_4 + \mathrm{i}\pi_5) & \overline{e_4} e_5 = -\frac{i}{4}\pi_1^2 \pi_2 + \mathrm{i}\pi_1 \pi_3 & \\ \overline{e_3} e_5 = \frac{1}{2}\pi_2(-\mathrm{i}\pi_4 + \pi_5). & & \end{array}$$

For $i > j \in \{1, \ldots, 5\}$ we have

$$\overline{e_i} e_j \quad = \quad \overline{\overline{e_j} e_i}.$$

So the following lemma is proved.

Lemma 2.2.7: *The vector field (2.2.2) yields the following reduced vector field on the orbit space*

$$\begin{aligned} \dot{\pi_1} &= 2(\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4\Big((\alpha - \beta)^r \pi_4 + (\alpha - \beta)^i \pi_5\Big) \\ \dot{\pi_2} &= 2(\lambda + \beta^r \pi_1 + \gamma^r \pi_2)\pi_2 + 8(\alpha - \gamma)^r \pi_3 + 4\Big((\alpha - \beta)^r \pi_4 - (\alpha - \beta)^i \pi_5\Big) \\ \dot{\pi_3} &= 4(\lambda + \beta^r \pi_1 + \alpha^r \pi_2)\pi_3 + 2\pi_2\Big((\alpha - \beta)^r \pi_4 - (\alpha - \beta)^i \pi_5\Big) \\ \dot{\pi_4} &= 2\Big(2\lambda + (\alpha + \beta)^r (\pi_1 + \pi_2)\Big)\pi_4 + 2(\alpha - \beta)^i (-\pi_1 + \pi_2)\pi_5 + (\alpha - \beta)^r \pi_1 (\pi_1 \pi_2 + 4\pi_3) \\ \dot{\pi_5} &= 2\Big(2\lambda + (\alpha + \beta)^r (\pi_1 + \pi_2)\Big)\pi_5 + 2(\alpha - \beta)^i (\pi_1 - \pi_2)\pi_4 + (\alpha - \beta)^i \pi_1 (-\pi_1 \pi_2 + 4\pi_3). \end{aligned}$$

Here $\alpha^r, \beta^r, \gamma^r$ resp. $\alpha^i, \beta^i, \gamma^i$ denote the real resp. imaginary parts of α, β, γ.

2.2.3 Lattice of isotropy subgroups

All isotropy subgroups $G \subsetneq \mathrm{O}(2) \times \mathrm{S}^1$ can be written as twisted subgroups in the form

$$G = H^{\Theta} = \{(h, \Theta(h)) \in \mathrm{O}(2) \times \mathrm{S}^1 \,|\, h \in H\}$$

(cf. [GoSt], [GoStSch]). Here $H \subset \mathrm{O}(2)$ denotes a closed subgroup of $\mathrm{O}(2)$ and

$$\Theta \quad : \quad \mathrm{O}(2) \quad \to \quad \mathrm{S}^1$$

is a group homomorphism. For a closed subgroup $H \subset \mathrm{O}(2)$ let

$$H' = \left\langle g^{-1} h^{-1} g h \,|\, g, h \in H \right\rangle$$

denote the the commutator of H and

$$H^{ab} = \frac{H}{H'}$$

the abelianisation of H. Since $\Theta(H) \subset \mathrm{S}^1$ is abelian the possible twist typs $\Theta(H)$ of H can be concluded from the abelianisation H^{ab}. One gets the following table.

H	H'	H^{ab}	$\Theta(H)$
$\mathrm{O}(2)$	$\mathrm{SO}(2)$	$\mathbb{Z}_2$	$1, \mathbb{Z}_2$
$\mathrm{SO}(2)$	1	$\mathrm{SO}(2)$	$1, \mathrm{S}^1$
$\mathrm{D_n}$	$\mathbb{Z}_{\frac{\mathrm{n}}{2}}$, n even $\mathbb{Z}_{\mathrm{n}}$, n odd	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$, n even $\mathbb{Z}_2$, n odd	$1, \mathbb{Z}_2$
$\mathbb{Z}_{\mathrm{n}}$	1	$\mathbb{Z}_{\mathrm{n}}$	$1, \mathbb{Z}_{\mathrm{d}}$, $d \vert n$

Lemma 2.2.8: *For our representation of the group* $\mathrm{O}(2) \times \mathrm{S}^1$ *on the space* $\mathbb{R}^6$ *one gets the following lattice of isotropy subgroups.*

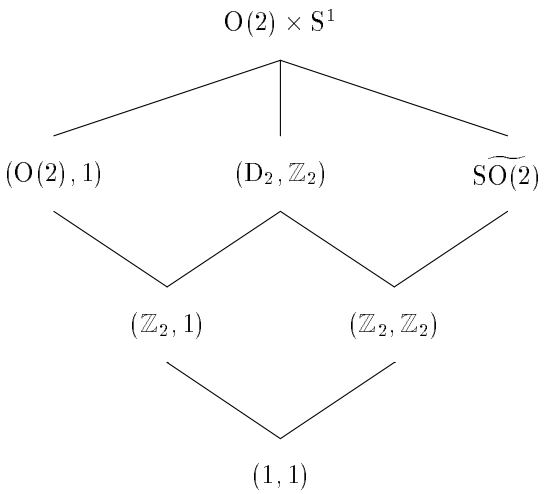


Figure 2: Lattice of isotropy subgroups of $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{R}^6$

The following table contains generating elements, representatives and the dimension of the associated fixed point space for every group H^{Θ}.

H^{Θ}	generators	representative	$\mathrm{Dim}\,\mathrm{Fix}(H^{\Theta})$
$\mathrm{O}(2) \times \mathrm{S}^1$	$\mathrm{O}(2) \times \mathrm{S}^1$	$(0,0,0)$	0
$(\mathrm{O}(2), 1)$	$(\mathrm{O}(2), 1)$	$(0, z_0, 0)$	2
$\widetilde{\mathrm{SO}(2)}$	$\langle(\phi, \phi),\, \phi \in \mathrm{S}^1\rangle$	$(z_{-2}, 0, 0)$	2
$(\mathrm{D}_2, \mathbb{Z}_2)$	$\langle(\kappa, 1), (\pi, \pi)\rangle$	$(z_2, 0, z_2)$	2
$(\mathbb{Z}_2, \mathbb{Z}_2)$	$\langle(\pi, \pi)\rangle$	$(z_{-2}, 0, z_2)$	4
$(\mathbb{Z}_2, 1)$	$\langle(\kappa, 1)\rangle$	(z_2, z_0, z_2)	4
$(1, 1)$	$\{(1, 1)\}$	(z_{-2}, z_0, z_2)	6

Proof: The dimension of the fixed point space of a potential isotropy subgroup

$$H^{\Theta} \quad \subset \quad \mathrm{O}(2) \times \mathrm{S}^1$$

is given by the trace formula (cf. [GoSt], [GoStSch])

$$\mathrm{Dim}\,\mathrm{Fix}\, H^{\Theta} \quad = \quad \int_{H^{\Theta}} \mathrm{Tr}(h, \theta(h))\, \mathrm{d}h.$$

The values of $\mathrm{Tr}(h,\theta(h))$, $h \in \mathrm{O}(2)$, $\Theta(h) \in \mathrm{S}^1$, are known by section 2.2.1. Since we use the diagonal representation of the group $\mathrm{O}(2) \times \mathrm{S}^1$ on $\mathbb{C}^3 \oplus \mathbb{C}^3 \supset \mathbb{R}^6$ the formula yields the real dimension of the fixed point space.
a. Let $\Theta(H) = 1$. Then

$$\begin{aligned}
\mathrm{Dim\,Fix}\,(\mathrm{O}(2),1) &= \frac{1}{2}\left(\frac{1}{2\pi}\int_{\delta=0}^{2\pi} 2(1+2\cos\delta)\,\mathrm{d}\delta + \int_{\delta=0}^{2\pi} 2\,\mathrm{d}\delta\right) = 2,\\
\mathrm{Dim\,Fix}\,(\mathrm{SO}(2),1) &= \frac{1}{2\pi}\int_{\phi=0}^{2\pi} 2(1+2\cos\phi)\,\mathrm{d}\phi = 2,\\
\mathrm{Dim\,Fix}\,(\mathrm{D_n},1) &= \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2\right) = \begin{cases} 4 & n=1,\\ 2 & n\geq 2,\end{cases}\\
\mathrm{Dim\,Fix}\,(\mathbb{Z}_n,1) &= \frac{1}{n}\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right) = 2.
\end{aligned}$$

The subspaces $\{(0,z_0,0)\}$ resp. $\{(z_2,z_0,z_2)\}$ have isotropy $(\mathrm{O}(2),1)$ resp. $(\mathbb{Z}_2,1)$ and consequently $(\mathrm{O}(2),1)$ resp. $(\mathbb{Z}_2,1)$ are isotropy subgroups with two resp. four dimensional fixed point spaces. Let $\mathbb{Z}_2 = \mathrm{D}_1$ denote the $\mathbb{Z}_2$ generated by κ. The other groups with trivial twist are no isotropy subgroups.
b. Let $\Theta(H) = \mathrm{S}^1$. Possible twists are

$$\begin{array}{rcccc} \Theta_k & : & \mathrm{SO}(2) & \to & \mathrm{S}^1 \\ & & \phi & \to & k\phi \end{array}$$

with $k \in \mathbb{N}$. Then we have

$$\mathrm{Dim\,Fix}\,\widetilde{\mathrm{SO}(2)}^{\,k} = \frac{1}{2\pi}\int_{\phi=0}^{2\pi} 2(1+2\cos\phi)\cos k\phi\,\mathrm{d}\phi = \begin{cases} 2 & k=1,\\ 0 & k>1.\end{cases}$$

The subspace $\{(z_{-2},0,0)\}$ has isotropy $\widetilde{\mathrm{SO}(2)}$ and therefore $\widetilde{\mathrm{SO}(2)}$ is an isotropy group with two dimensional fixed point space.
c. Let $\Theta(H) = \mathbb{Z}_2$. Then

$$\mathrm{Dim\,Fix}\,(\mathrm{O}(2),\mathbb{Z}_2) = \frac{1}{2}\left(\frac{1}{2\pi}\int_{\delta=0}^{2\pi} 2(1+2\cos\delta)\,\mathrm{d}\delta - \int_{\delta=0}^{2\pi} 2\,\mathrm{d}\delta\right) = 0.$$

In the case $(\mathrm{D_n},\mathbb{Z}_2)$ there are several possibilities. Let first n be even. Here we have three possible twists. To begin with let

$$H^{\,\Theta_{1,n}} = \left\langle \left(\frac{2\pi}{n},\pi\right),(\kappa,1)\right\rangle.$$

Then

$$\mathrm{Dim\,Fix}\,H^{\,\Theta_{1,n}} = \frac{1}{2n}\left(\sum_{j=1}^{n} 2(-1)^j\left(1+2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2(-1)^j\right) = \begin{cases} 2 & n=2,\\ 0 & n\geq 4.\end{cases}$$

Defining

$$H^{\,\Theta_{2,n}} = \left\langle \left(\frac{2\pi}{n},\pi\right),(\kappa,\pi)\right\rangle$$

we have

$$\mathrm{Dim\,Fix}\,H^{\,\Theta_{2,n}} = \frac{1}{2n}\left(\sum_{j=1}^{n} 2(-1)^j\left(1+2\cos\frac{2\pi}{n}j\right) + \sum_{j=1}^{n} 2(-1)^{j+1}\right) = \begin{cases} 2 & n=2,\\ 0 & n\geq 4.\end{cases}$$

Finally let

$$H^{\,\Theta_{3,n}} = \left\langle \left(\frac{2\pi}{n},1\right),(\kappa,\pi)\right\rangle.$$

Then

$$\mathrm{Dim}\,\mathrm{Fix}\,H^{\Theta_{3,n}} \;=\; \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right)+\sum_{j=1}^{n}-2\right) \;=\; 0.$$

Setting

$$(\mathrm{D}_2,\mathbb{Z}_2) \;=\; \langle(\pi,\pi),(\kappa,1)\rangle \;=\; H^{\Theta_{1,2}}$$

we have

$$\left(-\frac{\pi}{2},1\right)H^{\Theta_{2,2}}\left(\frac{\pi}{2},1\right) \;=\; H^{\Theta_{1,2}}.$$

Therefore both groups are conjugated.
The subspace $\{(z_2,0,z_2)\}$ has isotropy $(\mathrm{D}_2,\mathbb{Z}_2)$ and consequently $(\mathrm{D}_2,\mathbb{Z}_2)$ is an isotropy group with two dimensional fixed point space.
Now let n be odd. Then

$$\mathrm{Dim}\,\mathrm{Fix}\,(\mathrm{D}_n,\mathbb{Z}_2) \;=\; \frac{1}{2n}\left(\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right)+\sum_{j=1}^{n}-2\right) \;=\; \begin{cases} 2 & n=1,\\ 0 & n\geq 3.\end{cases}$$

$(\mathrm{D}_1,\mathbb{Z}_2)=\langle(\kappa,\pi)\rangle$ is extended by $H^{\Theta_{2,2}}$ and consequently is no isotropy group.
In the case $(\mathbb{Z}_n,\mathbb{Z}_2)$, in particular n has to be even, we have

$$\mathrm{Dim}\,\mathrm{Fix}\,(\mathbb{Z}_n,\mathbb{Z}_2) \;=\; \frac{1}{n}\sum_{j=1}^{n} 2(-1)^j\left(1+2\cos\frac{2\pi}{n}j\right) \;=\; \begin{cases} 4 & n=2,\\ 0 & n\geq 4.\end{cases}$$

The subspace $\{(z_{-2},0,z_2)\}$ has isotropy $(\mathbb{Z}_2,\mathbb{Z}_2)=\langle(\pi,\pi)\rangle$ and therefore $(\mathbb{Z}_2,\mathbb{Z}_2)$ is an isotropy group with four dimensional fixed point space.
d. Finally we have to study the case $(\mathbb{Z}_n,\mathbb{Z}_d)$ with $d|n$ and $n\geq 2$. Possible nontrivial twists for $\mathbb{Z}_n$ are

$$\begin{array}{rccc}\Theta_k & : & \mathbb{Z}_n & \to & \mathrm{S}^1\\ & & \frac{2\pi}{n}j & \to & \frac{2\pi}{n}jk\end{array}$$

with $1\leq k<n$. This gives

$$\begin{aligned}\mathrm{Dim}\,\mathrm{Fix}\,(\mathbb{Z}_n,\Theta_k(\mathbb{Z}_n)) &= \frac{1}{n}\sum_{j=1}^{n} 2\left(1+2\cos\frac{2\pi}{n}j\right)\cos\frac{2\pi}{n}jk\\ &= \begin{cases} 4 & n=2,\,k=1\\ 2 & n\geq 3,\,k\in\{1,n-1\}\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Studying the representations T_k of $\mathrm{D}_n=\langle\sigma,\kappa\rangle$ on $\mathbb{C}^2$

$$\begin{aligned}T_k(\sigma) &= \begin{pmatrix} e^{-i\frac{2\pi}{n}k} & 0\\ 0 & e^{i\frac{2\pi}{n}k}\end{pmatrix} \text{ and}\\ T_k(\kappa) &= \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.\end{aligned}$$

the last equality follows. The representations T_k are irreducible for $n\geq 3$. The representations T_1 and T_{n-1} are conjugated since

$$\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{2\pi}{n}} & 0\\ 0 & e^{i\frac{2\pi}{n}}\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \;=\; \begin{pmatrix} e^{i\frac{2\pi}{n}} & 0\\ 0 & e^{-i\frac{2\pi}{n}}\end{pmatrix}.$$

Orthogonality relations for this representations (cf. [La2]) yield the equality.
The case $(\mathbb{Z}_2,\mathbb{Z}_2)$ has been dealt with in part **c** of the proof, the other cases correspond to conjugated twists of typ

$$\begin{array}{rccc}\Theta_k & : & \mathbb{Z}_n & \to & \mathrm{S}^1\\ & & \frac{2\pi}{n}j & \to & \pm\frac{2\pi}{n}j.\end{array}$$

These are extended by the isotropy group $\widetilde{\mathrm{SO}(2)}$. ⋈

Lemma 2.2.9: *For the isotropy groups $H^\Theta \subset \mathrm{SO}(3) \times \mathrm{S}^1$ introduced in the first chapter we have*

H^Θ	$\frac{H^\Theta \cap \mathrm{N}(\Sigma)}{\Sigma}$
$(\mathbb{Z}_2, 1)$	$(1, 1)$
$(\mathbb{Z}_4, \mathbb{Z}_2)$	$(\mathbb{Z}_2, \mathbb{Z}_2)$
$(\mathrm{D}_2, 1)$	$(\mathbb{Z}_2, 1)$
$(\mathrm{O}(2), 1)$	$(\mathrm{O}(2), 1)$
$(\mathrm{D}_4, \mathbb{Z}_2)$	$(\mathrm{D}_2, \mathbb{Z}_2)$
$\widetilde{\mathrm{SO}(2)}^2$	$\widetilde{\mathrm{SO}(2)}$
$(\mathbb{T}, \mathbb{Z}_3)$	$(\mathbb{Z}_2, 1)$.

Note that

$$H^\Theta \quad \subset \quad N(\Sigma)$$

for all isotropy groups H^Θ except for the group $(\mathbb{T}, \mathbb{Z}_3)$. The group $(\mathbb{T}, \mathbb{Z}_3)$ does not correspond to a special isotropy typ in the $\mathrm{O}(2) \times \mathrm{S}^1$-equivariant system. But the restricted vector field (2.2.2) leaves the corresponding two dimensional fixed point space lying in $\mathrm{Fix}\,(\Sigma)$ invariant.

Lemma 2.2.10:

$$\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3) \quad = \quad \left\{ \left(\frac{i}{\sqrt{2}} z_0, z_0, \frac{i}{\sqrt{2}} z_0 \right),\ z_0 \in \mathbb{C} \right\}.$$

Proof: Using the representation of $\mathrm{SO}(3)$ on the space $V_2 \oplus \mathrm{i}V_2$ introduced in the first chapter one gets the following representation of the group

$$\mathbb{T} \ = \ \langle \pi, \tau \rangle \ \subset \ \mathrm{SO}(3)$$

with

$$\tau \quad = \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

on the subspace $\{(z_2, z_0, z_2)\} \subset \mathbb{R}^6$:

$$\begin{aligned} \pi(z_2, z_0, z_2) &= (z_2, z_0, z_2) \\ \tau(z_2, z_0, z_2) &= \left(-\frac{1}{2} z_2 - \frac{1}{2} \sqrt{\frac{3}{2}} z_0, \sqrt{\frac{3}{2}} z_2 - \frac{1}{2} z_0, -\frac{1}{2} z_2 - \frac{1}{2} \sqrt{\frac{3}{2}} z_0 \right). \end{aligned}$$

Exactly if an element has the form

$$\left\{ \left(\frac{i}{\sqrt{2}} z_0, z_0, \frac{i}{\sqrt{2}} z_0 \right),\ z_0 \in \mathbb{C} \right\}$$

then

$$\left(\tau, e^{i \frac{2\pi}{3}} \right) (z_2, z_0, z_2) \quad = \quad (z_2, z_0, z_2). \qquad \bowtie$$

2.2.4 Critical points of the reduced vector field

Lemma 2.2.11: *The image of the Hilbert-map $\Pi(\mathbb{R}^6)$ is sketched in figure 3.*

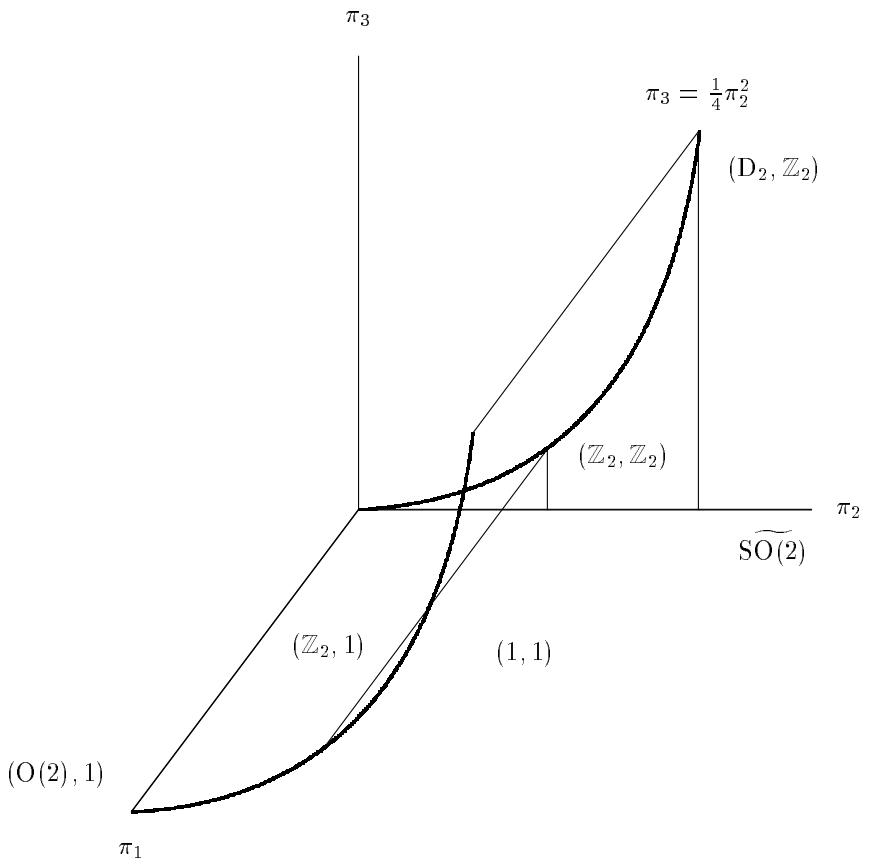


Figure 3: Image of the Hilbert-map

One has to imagine circles of radius

$$\pi_4^2 + \pi_5^2 = \pi_1^2 \pi_3$$

attached to points of the sketch. We have the following assignment

$(\pi_1, \ldots, \pi_5) \in \Pi(\mathbb{R}^6)$	*isotropy typ*
π_1-*axis*	$(\mathrm{O}(2), 1)$
π_2-*axis*	$\widetilde{\mathrm{SO}(2)}$
$\pi_1 = 0,\ \pi_3 = \frac{1}{4}\pi_2^2$	$(\mathrm{D}_2, \mathbb{Z}_2)$
$\pi_1 = 0,\ 0 < \pi_3 < \frac{1}{4}\pi_2^2$	$(\mathbb{Z}_2, \mathbb{Z}_2)$
$\pi_1 > 0,\ \pi_3 = \frac{1}{4}\pi_2^2$	$(\mathbb{Z}_2, 1)$
$\pi_1 > 0,\ 0 \leq \pi_3 < \frac{1}{4}\pi_2^2$	$(1, 1)$.

Remark 2.2.12: *The image of the Hilbert-map $\Pi(\mathbb{R}^6)$ will be denoted Hilbert-set in the following. Since the invariants π_1, π_2 and π_3 mean radii by definition only nonnegative values are possible. In (π_1, π_2, π_3)-space the Hilbert-set is a wedge (cf. figure 3) limited at the top by the surface $\pi_3 = \frac{1}{4}\pi_2^2$, at the bottom by the surface $\pi_3 = 0$ and at the back by the surface $\pi_1 = 0$.*

Proof: By definition of the invariants in lemma 2.2.4 we have

$$\pi_1,\ \pi_2,\ \pi_3 \geq 0.$$

A calculation using Lagrange-multipliers yields the possible values of π_3

$$0 \leq \pi_3 \leq \frac{1}{4}\pi_2^2.$$

The relation

$$\pi_4^2 + \pi_5^2 = \pi_1^2 \pi_3$$

has to be satisfied by lemma 2.2.4. ⋈

Remark 2.2.13: *Points with isotropy* $(\mathrm{O}(2),1)$ *and* $(\mathrm{D}_2,\mathbb{Z}_2)$ *and images of points with isotropy* $(\mathbb{T},\mathbb{Z}_3)$ *in the original system (cf. lemma 2.2.15) under the Hilbert-map satisfy the relation*

$$\Delta \ = \ \frac{1}{4}\pi_2^2 - \pi_3 \ = \ 0.$$

In the following we will study the reduced vector field (cf. lemma 2.2.7) on the Hilbert-set $\Pi(\mathbb{R}^6)$.

Lemma 2.2.14: *Let*

$$\Delta \ = \ \frac{1}{4}\pi_2^2 - \pi_3.$$

Then

$$\dot{\Delta} \ = \ 4\,\Delta\,(\lambda + \beta^r \pi_1 + \gamma^r \pi_2)\,.$$

Proof: The stratum

$$\Delta \ = \ 0$$

corresponds to points with a certain isotropy and is therefore flow invariant. Thus we have $\dot{\Delta} = 0$ for $\Delta = 0$ and there exists a relation of the form

$$\dot{\Delta} \ = \ \Delta\, r(\pi_1, \ldots, \pi_5).$$

A simple calculation gives the precise relation. ⋈

Lemma 2.2.15: *The orbit space reduction maps* $\mathrm{Fix}\,(\mathbb{T},\mathbb{Z}_3)$ *to the invariant curve*

$$\left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, -\frac{1}{2}\pi_1^2, 0\right) \subset \Pi(\mathbb{R}^6),\ \ \pi_1 > 0,$$

that is located on the stratum $\Delta = 0$.

Proof: The proof follows directly from the lemmata 2.2.4 and 2.2.10. ⋈

In the following let the parameter of the Hopf-bifurcation λ be positive:

$$\lambda \ > \ 0.$$

We are only interested in supercritical bifurcations.

The restriction of the reduced vector field (cf. lemma 2.2.7) to the statum $\Delta = 0$ is

$$\dot{\pi_1} = 2(\lambda + \alpha^r \pi_1 + \beta^r \pi_2)\pi_1 + 4\Big((\alpha-\beta)^r \pi_4 + (\alpha-\beta)^i \pi_5\Big) \tag{2.2.3}$$

$$\dot{\pi_2} = 2(\lambda + \beta^r \pi_1 + \alpha^r \pi_2)\pi_2 + 4\Big((\alpha-\beta)^r \pi_4 - (\alpha-\beta)^i \pi_5\Big) \tag{2.2.4}$$

$$\dot{\pi_3} = \frac{1}{2}\pi_2 \dot{\pi_2} \tag{2.2.5}$$

$$\begin{aligned}\dot{\pi_4} &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_4 + 2(\alpha-\beta)^i(-\pi_1+\pi_2)\pi_5 \\ &\quad +(\alpha-\beta)^r \pi_1\pi_2(\pi_1+\pi_2)\end{aligned} \tag{2.2.6}$$

$$\begin{aligned}\dot{\pi_5} &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5 + 2(\alpha-\beta)^i(\pi_1-\pi_2)\pi_4 \\ &\quad +(\alpha-\beta)^i \pi_1\pi_2(-\pi_1+\pi_2).\end{aligned} \tag{2.2.7}$$

Here α^r, β^r resp. α^i, β^i denote the real resp. imaginary parts of α, β.

Lemma 2.2.16: *Let $\alpha^r,\beta^r<0$ and $\alpha^r\neq\beta^r$. Then the set of critical points of the equations 2.2.3 to 2.2.7 on the stratum $\Delta=0$ is given by a curve*

$$g(\pi_1)=\left(\pi_1,\pi_2=-\left(\pi_1+\frac{\lambda}{\alpha^r}\right),\frac{1}{4}\pi_2^2,\frac{1}{2}\pi_1\pi_2,0\right),\ 0\leq\pi_1\leq-\frac{\lambda}{\alpha^r},$$

parametrised by π_1, and

$$h(\pi_1)=\left(\pi_1,\pi_1,\frac{1}{4}\pi_1^2,-\frac{1}{2}\pi_1^2,0\right),\ \pi_1=-\frac{\lambda}{2\beta^r}.$$

The curve $g(\pi_1)$, $0\leq\pi_1\leq-\frac{\lambda}{\alpha^r}$, connects a critical point with isotropy $(\mathrm{O}(2),1)$,

$$g\left(-\frac{\lambda}{\alpha^r}\right) \quad = \quad \left(-\frac{\lambda}{\alpha^r},0,0,0,0\right),$$

with a critical point with isotropy $(\mathrm{D}_2,\mathbb{Z}_2)$,

$$g(0) \quad = \quad \left(0,\pi_2=-\frac{\lambda}{\alpha^r},\frac{1}{4}\pi_2^2,0,0\right).$$

The critical point $h(\pi_1)$, $\pi_1=-\frac{\lambda}{2\beta^r}$, lies in $\Pi(\mathrm{Fix}\,(\mathbb{T},\mathbb{Z}_3))$, the image of points with isotropy $(\mathbb{T},\mathbb{Z}_3)$ in the original system under the Hilbert-map.

Proof: By addition resp. subtraction of equations 2.2.3 and 2.2.4 one gets the following equations

$$\begin{aligned}0 &= \lambda(\pi_1+\pi_2)+\alpha^r(\pi_1^2+\pi_2^2)+2\beta^r\pi_1\pi_2+4(\alpha-\beta)^r\pi_4, &(2.2.8)\\ 0 &= \lambda(\pi_1-\pi_2)+\alpha^r(\pi_1^2-\pi_2^2)+4(\alpha-\beta)^i\pi_5. &(2.2.9)\end{aligned}$$

Let $(\alpha-\beta)^i\neq0$ then

$$\begin{aligned}\pi_4 &= -\frac{\lambda(\pi_1+\pi_2)+\alpha^r(\pi_1^2+\pi_2^2)+2\beta^r\pi_1\pi_2}{4(\alpha-\beta)^r},\\ \pi_5 &= -\frac{\lambda(\pi_1-\pi_2)+\alpha^r(\pi_1^2-\pi_2^2)}{4(\alpha-\beta)^i}.\end{aligned}$$

Inserting this in equations 2.2.6 and 2.2.7 gives

$$\begin{aligned}0 &= -\frac{(\pi_1+\pi_2)\left(\lambda+\alpha^r(\pi_1+\pi_2)\right)\left(\lambda+\beta^r(\pi_1+\pi_2)\right)}{(\alpha-\beta)^r},\\ 0 &= -\frac{(\pi_1-\pi_2)\left(\lambda+\alpha^r(\pi_1+\pi_2)\right)}{2(\alpha-\beta)^r(\alpha-\beta)^i}\cdot\\ &\quad\left(2\lambda(\alpha-\beta)^r+(\pi_1+\pi_2)(\alpha^{r2}-\beta^{r2}+(\alpha-\beta)^{i2}\right). &(2.2.10)\end{aligned}$$

Therefore looking for nontrivial critical points one has to study two cases.
First let $\pi_1+\pi_2=-\frac{\lambda}{\alpha^r}$. Since we assume $\lambda>0$ only the choice $\alpha^r<0$ gives solutions that lie in $\Pi(\mathbb{R}^6)$. By insertion one gets the curve

$$g(\pi_1)=\left(\pi_1,\pi_2=-\left(\pi_1+\frac{\lambda}{\alpha^r}\right),\frac{1}{4}\pi_2^2,\frac{1}{2}\pi_1\pi_2,0\right),\ 0\leq\pi_1\leq-\frac{\lambda}{\alpha^r},$$

of critical points. Lemma 2.2.11 gives the associated orbit types.
Now let $\pi_1+\pi_2=-\frac{\lambda}{\beta^r}$. Only the choice $\beta^r<0$ gives solutions that lie in $\Pi(\mathbb{R}^6)$ as in the first case. By insertion in equation 2.2.10 one gets the condition

$$0 \quad = \quad \frac{\left((\alpha-\beta)^{r2}+(\alpha-\beta)^{i2}\right)\lambda^2\left(\lambda+2\beta^r\pi_2\right)}{2\beta^{r3}(\alpha-\beta)^i}.$$

In order to get critical points one has to choose

$$\pi_1=\pi_2=-\frac{\lambda}{2\beta^r}.$$

By insertion one has the critical point

$$h(\pi_1) = \left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, -\frac{1}{2}\pi_1^2, 0\right),\ \pi_1 = -\frac{\lambda}{2\beta^r},$$

that lies in $\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3))$ (cf. lemma2.2.15). It will be shown that there are no other critical points with radius

$$\pi_1 + \pi_2 \quad = \quad -\frac{\lambda}{\beta^r}.$$

Because of this the group orbit of periodic orbits with isotropy $(\mathbb{T}, \mathbb{Z}_3)$ in the original system can only intersect the stratified space in the curve given in lemma 2.2.15.
Now let $(\alpha - \beta)^i = 0$. Equations 2.2.8 and 2.2.9 yield

$$\begin{aligned} 0 &= \lambda(\pi_1 + \pi_2) + \alpha^r(\pi_1^2 + \pi_2^2) + 2\beta^r\pi_1\pi_2 + 4(\alpha - \beta)^r\pi_4, \\ 0 &= (\pi_1 - \pi_2)\Big(\lambda + \alpha^r(\pi_1 + \pi_2)\Big). \end{aligned}$$

Consequently we have to study two cases.
First let $\pi_1 = \pi_2$. Then

$$\pi_4 \quad = \quad -\frac{\Big(\lambda + (\alpha + \beta)^r\pi_1\Big)\pi_1}{2(\alpha - \beta)^r}.$$

By insertion in equation 2.2.6 one gets

$$0 \quad = \quad -\frac{\pi_1\,(\lambda + 2\beta^r\pi_1)(\lambda + 2\alpha^r\pi_1)}{(\alpha - \beta)^r}.$$

The choice $\pi_1 = \pi_2 = -\frac{\lambda}{2\alpha^r}$ together with the relation

$$\pi_4^2 + \pi_5^2 = \pi_1^2\pi_3 = \frac{1}{4}\pi_1^4$$

gives the critical point

$$\left(\pi_1, \pi_1, \frac{1}{4}\pi_1^2, \frac{1}{2}\pi_1^2, 0\right),\ \pi_1 = -\frac{\lambda}{2\alpha^r},$$

that lies on the curve $g(\pi_1)$.
The case $\pi_1 = \pi_2 = -\frac{\lambda}{2\beta^r}$ yields again the solution $h(\pi_1),\ \pi_1 = -\frac{\lambda}{2\beta^r}$.
Finally we have to study the case $\pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r}$. We get

$$\begin{aligned} \pi_4 &= -\frac{\pi_1}{2\alpha^r}(\lambda + \alpha^r\pi_1) \\ &= \frac{1}{2}\pi_1\pi_2. \end{aligned}$$

The relation

$$\pi_4^2 + \pi_5^2 \quad = \quad \frac{1}{4}\pi_1^2\pi_2^2$$

yields $\pi_5 = 0$. So we get the curve $g(\pi_1)$ again. ⋈

Lemma 2.2.17:

$$\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3)) \cap g(\pi_1) = \emptyset,\ 0 \le \pi_1 \le -\frac{\lambda}{\alpha^r}.$$

The critical point $h(\pi_1),\ \pi_1 = -\frac{\lambda}{2\beta^r}$, (cf. lemma 2.2.16) that lies in $\Pi(\mathrm{Fix}(\mathbb{T}, \mathbb{Z}_3))$ is isolated in the Hilbert-set $\Pi(\mathbb{R}^6)$.

Proof: For points lying on the curve $g(\pi_1)$ we have $\pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r}$. Points in $\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3))$ satisfy the condition $\pi_1 = \pi_2$ (cf. lemma 2.2.15). For a potential intersection this means $\pi_1 = \pi_2 = -\frac{\lambda}{2\alpha^r}$. We have

$$g\left(-\frac{\lambda}{2\alpha^r}\right) \quad = \quad \left(-\frac{\lambda}{2\alpha^r}, -\frac{\lambda}{2\alpha^r}, \frac{1}{16}\frac{\lambda^2}{\alpha^{r2}}, +\frac{1}{8}\frac{\lambda^2}{\alpha^{r2}}, 0\right),$$

whereas

$$\Pi(\mathrm{Fix}\,(\mathbb{T}, \mathbb{Z}_3)) \cap \left(\pi_1 = -\frac{\lambda}{2\alpha^r}\right) \quad = \quad \left(-\frac{\lambda}{2\alpha^r}, -\frac{\lambda}{2\alpha^r}, \frac{1}{16}\frac{\lambda^2}{\alpha^{r2}}, -\frac{1}{8}\frac{\lambda^2}{\alpha^{r2}}, 0\right).$$

On the stratum $\Delta = 0$ the critical point $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$, (cf. lemma 2.2.16) that lies on $\Pi(\mathrm{Fix}(\mathbb{T}, \mathbb{Z}_3))$ is therefore isolated. We will show in lemma 2.2.18 that there are no further critical points in the Hilbert-set in the region $\Delta \neq 0$ near $h(\pi_1)$, $\pi_1 = -\frac{\lambda}{2\beta^r}$. ⋈

Now we are looking for critical points of the reduced vector field (cf. lemma 2.2.7) in $\Pi(\mathbb{R}^6)$ that do not lie on the stratum $\Delta = 0$. Such a critical points has to meet the condition (cf. lemma 2.2.14)

$$\dot{\Delta} = 4\,\Delta\,(\lambda + \beta^r\pi_1 + \gamma^r\pi_2) = 0.$$

Since we assumed $\Delta \neq 0$ this means

$$\lambda + \beta^r\pi_1 + \gamma^r\pi_2 \quad = \quad 0. \tag{2.2.11}$$

So we get the following equations:

$$\begin{aligned}
0 &= 2(\lambda + \alpha^r\pi_1 + \beta^r\pi_2)\pi_1 + 4\Big((\alpha-\beta)^r\pi_4 + (\alpha-\beta)^i\pi_5\Big) && (2.2.12)\\
0 &= 8(\alpha-\gamma)^r\pi_3 + 4\Big((\alpha-\beta)^r\pi_4 - (\alpha-\beta)^i\pi_5\Big) && (2.2.13)\\
0 &= \frac{1}{2}\pi_2\dot{\pi}_2 && (2.2.14)\\
0 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_4 + 2(\alpha-\beta)^i(-\pi_1+\pi_2)\pi_5 \\
&\quad +(\alpha-\beta)^r\pi_1(\pi_1\pi_2 + 4\pi_3) && (2.2.15)\\
0 &= 2\Big(2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)\Big)\pi_5 + 2(\alpha-\beta)^i(\pi_1-\pi_2)\pi_4 \\
&\quad +(\alpha-\beta)^i\pi_1(-\pi_1\pi_2 + 4\pi_3) && (2.2.16)\\
\pi_2 &= -\frac{\lambda + \beta^r\pi_1}{\gamma^r}. && (2.2.17)
\end{aligned}$$

Here $\alpha^r, \beta^r, \gamma^r$ resp. α^i, β^i denote the real resp. imaginary parts of α, β, γ.
In the following we will assume

$$\beta^r < \alpha^r < \gamma^r < 0.$$

In lemma 2.2.20 we will show that only for this choice of the coefficients the solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $\widetilde{\mathrm{SO}(2)}$ can be stable simultaneously. Investigations using the topological Conlex-index suggested to study this case. In the following lemma the solution with isotropy $\widetilde{\mathrm{SO}(2)}$ is being described. For our choice of the coefficients we have the following lemma for the critical points of the reduced vector field in $\Delta \neq 0$.

Lemma 2.2.18: *Let* $\beta^r < \alpha^r < \gamma^r < 0$. *Then*

$$\left(0, -\frac{\lambda}{\gamma^r}, 0, 0, 0\right)$$

is the only critical point of the reduced vector field in $\Pi(\mathbb{R}^6)$ *with* $\Delta \neq 0$. *This solution has isotropy* $\widetilde{\mathrm{SO}(2)}$.

Proof: First let $(\alpha-\beta)^i \neq 0$. By addition resp. subtraction of equations 2.2.12 and 2.2.13 we get

$$\begin{aligned}
0 &= (\lambda + \alpha^r\pi_1 + \beta^r\pi_2)\pi_1 + 4(\alpha-\beta)^r\pi_4 + 4(\alpha-\gamma)^r\pi_3,\\
0 &= (\lambda + \alpha^r\pi_1 + \beta^r\pi_2)\pi_1 + 4(\alpha-\beta)^i\pi_5 - 4(\alpha-\gamma)^r\pi_3.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\pi_4 &= -\frac{(\lambda + \alpha^r\pi_1 + \beta^r\pi_2)\pi_1 + 4(\alpha-\gamma)^r\pi_3}{4(\alpha-\beta)^r},\\
\pi_5 &= -\frac{(\lambda + \alpha^r\pi_1 + \beta^r\pi_2)\pi_1 - 4(\alpha-\gamma)^r\pi_3}{4(\alpha-\beta)^i}.
\end{aligned}$$

Insertion in equation 2.2.15 yields

$$\frac{(\beta-\gamma)^r\left(\lambda(\alpha-\gamma)^r+\alpha^r\pi_1(\beta-\gamma)^r\right)(-\lambda\pi_1-\beta^r\pi_1^2+4\gamma^r\pi_3)}{(\alpha-\beta)^r\gamma^{r2}} = 0.$$

First let

$$\pi_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r} > 0.$$

Using equation 2.2.17 we get

$$\pi_2 = \frac{\lambda(\alpha-\beta)^r}{\alpha^r(\beta-\gamma)^r} > 0.$$

Together with equation 2.2.16 this yields

$$0 = \frac{(\alpha-\gamma)^r\left((\alpha-\beta)^{r2}+(\alpha-\beta)^{i2}\right)\lambda\left(-\lambda^2(\alpha-\beta)^{r2}+4\alpha^{r2}\,\pi_3\,(\beta-\gamma)^{r2}\right)}{2\alpha^{r3}(\alpha-\beta)^r(\alpha-\beta)^i(\beta-\gamma)^{r2}}.$$

So we have

$$\pi_3 = \frac{\lambda^2(\alpha-\beta)^{r2}}{4\alpha^{r2}(\beta-\gamma)^{r2}} = \frac{1}{4}\pi_2^2.$$

This solution lies on the stratum $\Delta = 0$.
Now let

$$\pi_3 = \frac{\pi_1(\lambda+\beta^r\pi_1)}{4\gamma^r} = -\frac{1}{4}\pi_1\pi_2.$$

Insertion in equation 2.2.16 yields

$$\begin{aligned} 0 &= -2(\alpha-\beta)^{i2}\pi_1^2\pi_2 - \frac{(\alpha-\beta)^{i2}\pi_1\left(\lambda+(\beta+\gamma)^r\pi_1\right)^2}{2\gamma^{r2}} \\ &\quad -\frac{\pi_1\left(\lambda(\alpha+\beta-2\gamma)^r+\pi_1(\alpha+\beta)^r(\beta-\gamma)^r\right)^2}{2\gamma^{r2}}. \end{aligned}$$

Since all elements of the sum are nonpositive in $\Pi(\mathbb{R}^6)$ the sum can only be zero if all elements are zero. This is only possible if $\pi_1 = 0$. This yields

$$\left(0, -\frac{\lambda}{\gamma^r}, 0, 0, 0\right),$$

the solution with isotropy $\widetilde{\mathrm{SO}(2)}$.
Now let $(\alpha-\beta)^i = 0$. Again by addition resp. subtraction of the equations 2.2.12 and 2.2.13 we get

$$\begin{aligned} \pi_3 &= \frac{(\lambda+\alpha^r\pi_1+\beta^r\pi_2)\pi_1}{4(\alpha-\gamma)^r}, \\ \pi_4 &= -\frac{2(\alpha-\gamma)^r\pi_3}{(\alpha-\beta)^r}. \end{aligned} \tag{2.2.18}$$

Furthermore we have

$$\pi_2 = -\frac{\lambda+\beta^r\pi_1}{\gamma^r}.$$

Insertion in equation 2.2.16 yields

$$0 = \left(2\lambda+(\alpha+\beta)^r(\pi_1+\pi_2)\right)\pi_5.$$

In order to solve this equation we have to look at several cases.
First let $\pi_5 = 0$. Then

$$\pi_4^2+\pi_5^2 = \pi_1^2\pi_3$$

and equation 2.2.18 yields

$$\frac{4(\alpha-\gamma)^{r2}}{(\alpha-\beta)^{r2}}\pi_3^2 \quad = \quad \pi_1^2\pi_3.$$

For $\pi_3 \neq 0$ we get

$$\pi_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r} > 0,\ \pi_2 = \frac{\lambda(\alpha-\beta)^r}{\alpha^r(\beta-\gamma)^r} > 0,\ \pi_3 = \frac{1}{4}\pi_2^2.$$

Therefore the solution lies on the stratum $\Delta = 0$.
The choice $\pi_3 = 0$ together with equation 2.2.16 yields the solution

$$\left(0, -\frac{\lambda}{\gamma^r}, 0, 0, 0\right)$$

with isotropy $\widetilde{\mathrm{SO}(2)}$.
For $\pi_5 \neq 0$ and

$$0 \quad = \quad 2\lambda + (\alpha+\beta)^r(\pi_1+\pi_2)$$

equation 2.2.15 gives

$$0 \quad = \quad \pi_1(\pi_1\pi_2 + 4\pi_3).$$

Choosing $\pi_1 = 0$ produces again the solution with isotropy $\widetilde{\mathrm{SO}(2)}$.
For $\pi_1 \neq 0$ one gets the solution

$$\begin{aligned}
\pi_1 &= -\frac{\lambda(\alpha+\beta-2\gamma)^r}{(\alpha+\beta)^r(\beta-\gamma)^r} > 0,\\
\pi_2 &= \frac{\lambda(\alpha-\beta)^r}{(\alpha+\beta)^r(\beta-\gamma)^r} > 0,\\
\pi_3 &= \frac{\lambda^2(\alpha-\beta)^r(\alpha+\beta-2\gamma)^r}{4(\alpha+\beta)^{r2}(\beta-\gamma)^{r2}} < 0,
\end{aligned}$$

that does not lie in $\Pi(\mathbb{R}^6)$. ⋈

Figure 4 sketches the position of the critical points of the reduced vector field (cf. lemma 2.2.7) in the Hilbert-set $\Pi(\mathbb{R}^6)$ known by lemmata 2.2.16 and 2.2.18 under the assumption

$$\beta^r < \alpha^r < \gamma^r < 0.$$

We now study the curve

$$g(\pi_1) = \left(\pi_1, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \frac{1}{4}\pi_2^2, \frac{1}{2}\pi_1\pi_2, 0\right), \quad 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

of critical points of the equations (2.2.3) to (2.2.7) (cf. lemma 2.2.16).

Lemma 2.2.19: *The preimage of a point $g(\pi_1), \pi_1 \in \left(0, -\frac{\lambda}{\alpha^r}\right)$, in $\mathbb{R}^6$ is a 2−torus. It is fibered with periodic solutions.*

Proof: The curve g of critical points lies on the statum

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3 = 0.$$

Introducing polar coordinates in the form

$$z_j = r_j\, e^{i\phi_j}, \quad j \in \{-2, 0, 2\},$$

yields

$$r_{-2} \quad = \quad r_2.$$

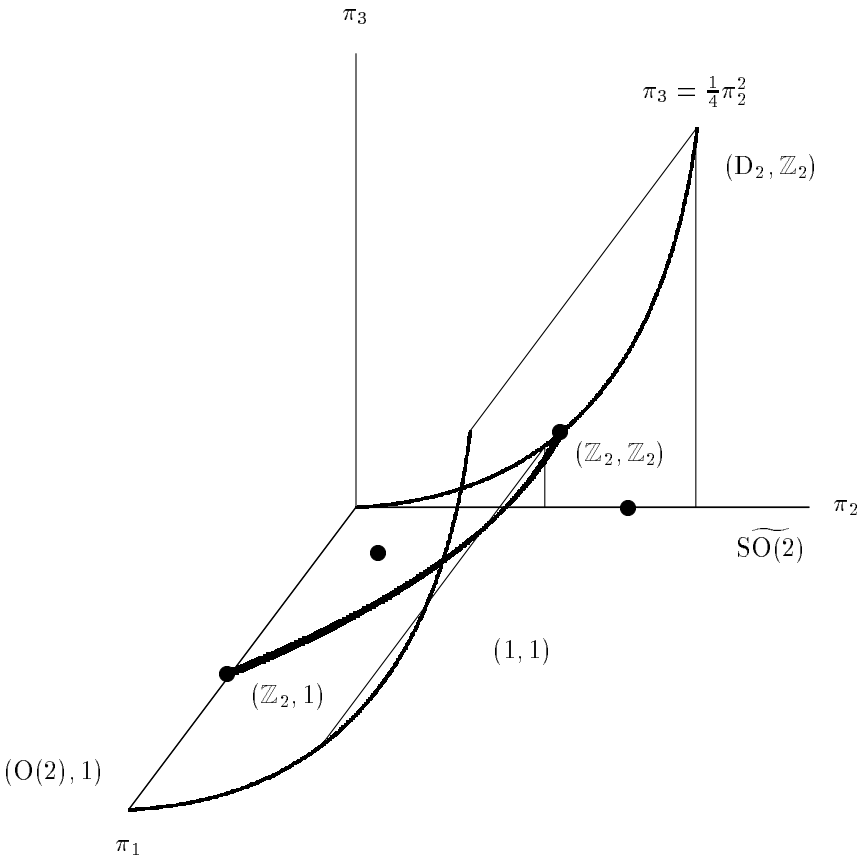


Figure 4: Critical points of the reduced vector field in the Hilbert-set

The the choice of

$$\pi_1 = r_0^2 \in \left(0, -\frac{\lambda}{\alpha^r}\right)$$

and the condition

$$\pi_1 + \pi_2 \quad = \quad -\frac{\lambda}{\alpha^r}$$

determine the radii. The conditions for π_4 resp. π_5 yield in polar coordinates the phase relations

$$\cos\theta = 1, \quad \sin\theta = 0$$

and thus

$$\theta \quad = \quad 0 \bmod 2\pi,$$

with

$$\theta \quad = \quad 2\phi_0 - \phi_{-2} - \phi_2.$$

So one angle is determined two are still available, the preimage is a 2-torus. Points on the surface $\Delta = 0$ have the (conjugated) isotropy $(\mathbb{Z}_2, 1)$. So it is possible to look only at points of the form (z_2, z_0, z_2) in order to determine the resulting flow on the preimage of a point on the curve of fixed points. Thus we have the additional condition

$$\phi_{-2} \quad = \quad \phi_2.$$

Using $\theta = 0 \bmod 2\pi$ one sees that

$$\phi_0 \quad = \quad \phi_2 \bmod \pi.$$

Inserting this into the differential equation yields

$$\dot{\phi}_0 = \omega_0 = \omega + \alpha^i \left(r_0^2 + 2r_2^2\right).$$

Thus the 2−torus is fibered with periodic solutions of period near $\frac{2\pi}{\omega}$. ⋈

2.2.5 Stability of the critical points of the reduced vector field

In lemmata 2.2.16 and 2.2.18 we have shown that in the case of supercritical bifurcation ($\lambda > 0$) the coefficients $\alpha^r, \beta^r, \gamma^r$ have to be negative in order that the corresponding solutions lie in the image of the Hilbert-map $\Pi(\mathbb{R}^6)$. The following lemma gives a condition on the choice of the coefficients relative to each other.

Lemma 2.2.20: *Only by choosing the coefficients*

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right),$$

the critical points with isotropy $(\mathrm{O}(2), 1)$ *resp.* $\widetilde{\mathrm{SO}(2)}$ *of the reduced vector field (cf. lemma 2.2.7) can be stable simultaneously. The stability of the solution with isotropy* $(\mathrm{O}(2), 1)$ *is determined by higher order terms because of the existence of a curve of critical points (cf. lemma 2.2.16).*

Proof: The calculations of [IoRo] yield (using our parameters) up to third order the following conditions for the stability of the periodic solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $\widetilde{\mathrm{SO}(2)}$ in the original, ten dimensional system:

isotropy	nontrivial Floquet-exponents
$(\mathrm{O}(2), 1)$	$-2\lambda < 0, -\frac{2\lambda}{\alpha^r}(\alpha - \beta)^r < 0, -\frac{2\lambda}{\alpha^r}(-4\alpha + \beta + 3\gamma)^r < 0$
$\widetilde{\mathrm{SO}(2)}$	$-2\lambda < 0, -\frac{2\lambda}{\gamma^r}(\alpha - \gamma), cc, \frac{\lambda}{\gamma^r}(\gamma - \beta), cc, \frac{3\lambda}{2\gamma^r}(\gamma - \beta), cc.$

Here *cc* denotes the complex conjugate of the preceding number.
So we get the conditions

$$\beta^r < \alpha^r < \gamma^r < 0$$

and

$$\beta^r + 3\gamma^r < 4\alpha^r.$$

The ansatz

$$\alpha^r \quad = \quad t\,\beta^r + (1-t)\,\gamma^r, \quad t \in (0,1),$$

yields

$$(\beta - \gamma)^r (1 - 4t) \quad < \quad 0,$$

and therefore we have

$$t \quad \in \quad \left(0, \frac{1}{4}\right).$$

This means

$$\alpha^r \quad \in \quad \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right).$$

Especially

$$\frac{(\alpha - \gamma)^r}{(\beta - \gamma)^r} \quad \in \quad \left(0, \frac{1}{4}\right).$$

⋈

Now we want to determine the linearisation of the reduced vector field (cf. lemma 2.2.7) along the curve $g(\pi_1)$ of critical points (cf. lemma 2.2.16). As the general linearisation L one gets

$$L = \begin{pmatrix} 2\lambda + 4\pi_1\alpha^r + 2\pi_2\beta^r \\ 2\pi_2\beta^r \\ 4\pi_3\beta^r \\ 2\pi_4(\alpha+\beta)^r - 2\pi_5(\alpha-\beta)^i + (2\pi_1\pi_2 + 4\pi_3)(\alpha-\beta)^r \\ 2\pi_5(\alpha+\beta)^r + 2\pi_4(\alpha-\beta)^i + (-2\pi_1\pi_2 + 4\pi_3)(\alpha-\beta)^i \end{pmatrix}$$

$$\begin{pmatrix} 2\pi_1\beta^r \\ 2\lambda + 2\pi_1\beta^r + 4\pi_2\gamma^r \\ 4\pi_3\alpha^r + 2\big(\pi_4(\alpha-\beta)^r - \pi_5(\alpha-\beta)^i\big) \\ 2\pi_4(\alpha+\beta)^r + 2\pi_5(\alpha-\beta)^i + \pi_1^2(\alpha-\beta)^r \\ 2\pi_5(\alpha+\beta)^r - 2\pi_4(\alpha-\beta)^i - \pi_1^2(\alpha-\beta)^i \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4(\alpha-\beta)^r & 4(\alpha-\beta)^i \\ 8(\alpha-\gamma)^r & 4(\alpha-\beta)^r & -4(\alpha-\beta)^i \\ 4(\lambda+\beta^r\pi_1+\alpha^r\pi_2) & 2\pi_2(\alpha-\beta)^r & -2\pi_2(\alpha-\beta)^i \\ 4\pi_1(\alpha-\beta)^r & 4\lambda + 2(\pi_1+\pi_2)(\alpha+\beta)^r & 2(-\pi_1+\pi_2)(\alpha-\beta)^i \\ 4\pi_1(\alpha-\beta)^i & 2(\pi_1-\pi_2)(\alpha-\beta)^i & 4\lambda + 2(\pi_1+\pi_2)(\alpha+\beta)^r \end{pmatrix}.$$

We are interested in the eigenvalues of L along the curve $g(\pi_1)$ with reference to $\Pi(\mathbb{R}^6) \subset \mathbb{R}^5$. Thus we have to determine the tangent space at points of the curve in $\Pi(\mathbb{R}^6)$. It is given by the relation

$$\pi_4^2 + \pi_5^2 = \pi_1^2\pi_3.$$

The curve itself lies on the stratum

$$\Delta = \frac{1}{4}\pi_2^2 - \pi_3 = 0.$$

So we get the following lemma.

Lemma 2.2.21: *The tangent space at the stratum $\Delta = 0$ along the curve*

$$g(\pi_1),\ 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

is spanned by the vectors

$$\begin{aligned} t_1 &= \left(1, -1, -\frac{1}{2}\pi_2, \frac{1}{2}(\pi_2 - \pi_1), 0\right), \\ t_2 &= \left(\pi_1, \pi_2, \frac{1}{2}\pi_2^2, \pi_1\pi_2, 0\right), \\ t_3 &= \big(2\alpha^r(\alpha-\beta)^i, -2\alpha^r(\alpha-\beta)^i, -\pi_2\alpha^r(\alpha-\beta)^i, \alpha^r(\pi_2-\pi_1)(\alpha-\beta)^i, \\ &\quad -\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r\big). \end{aligned}$$

The vectors t_1, t_2, t_3 are eigenvectors of L to the eigenvalues

$$\begin{aligned} ew_1 &= 0, \\ ew_2 &= -2\lambda \\ &= 2\alpha^r(\pi_1+\pi_2), \\ ew_3 &= \frac{2(\alpha-\beta)^r\lambda}{\alpha^r} \\ &= -2(\alpha-\beta)^r(\pi_1+\pi_2). \end{aligned}$$

By definition of the curve $g(\pi_1)$ we have

$$\pi_1 + \pi_2 = -\frac{\lambda}{\alpha^r}.$$

Therefore on the stratum $\Delta = 0$ the curve $g(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, is stable.

Proof: The relations $\pi_4^2 + \pi_5^2 = \pi_1^2\pi_3$ and $\Delta = 0$ yield the following vectors normal to the tangent space at the surface $\Delta = 0$ in $\Pi(\mathbb{R}^6) \subset \mathbb{R}^5$:

$$\begin{aligned} n_1 &= \left(-2\pi_1\pi_3, 0, -\pi_1^2, 2\pi_4, 2\pi_5\right), \\ n_2 &= \left(0, \frac{1}{2}\pi_2, -1, 0, 0\right). \end{aligned}$$

The orthogonal complement to $Span(n_1, n_2)$ is spanned by the vectors t_1, t_2, t_3. A simple calculation shows that this vectors are eigenvectors to the given eigenvalues. The eigenvector t_1 points along the curve of critical points. Therefore the associated eigenvalue is zero. ⋈

Now we want to determine the linearisation of the reduced vector field along the curve $g(\pi_1)$ of fixed points in the direction of the principal stratum. We will show that there exists a point $\tilde{\pi}_1$ on the curve $g(\pi_1)$ in which the stability of the curve in the direction of the principal stratum changes from stable to unstable. In this point the linearisation L of the vector field of the reduced equation has a nontrivial two dimensional Jordan-block to the eigenvalue zero.
Let

$$t = (0, 1, 0, 0, 0).$$

Then $n_1 t = 0$ and $n_2 t \neq 0$ for $\pi_2 \neq 0$. Thus the vectors t_1, t_2, t_3, t span the tangent space at the Hilbert-set $\Pi(\mathbb{R}^6)$ along the curve $g(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$. One gets

$$L\,t = a\,t_1 + b\,t_2 + c\,t_3 + d\,t$$

with

$$\begin{aligned} a &= -2\pi_1 \frac{(\alpha-\beta)^{r2} + (\alpha-\beta)^{i2}}{(\alpha-\beta)^r}, \\ b &= 2\alpha^r, \\ c &= \pi_1 \frac{(\alpha-\beta)^i}{\alpha^r(\alpha-\beta)^r}, \\ d &= 4\frac{\lambda(\alpha-\gamma)^r + \pi_1\,\alpha^r(\beta-\gamma)^r}{\alpha^r}. \end{aligned}$$

Restricted to the tangent space at the curve $g(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, according to our choice of the vectors t_1, t_2, t_3, t L has the form $\tilde{L}$:

$$\tilde{L} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & -2\lambda & 0 & b \\ 0 & 0 & \frac{2(\alpha-\beta)^r\lambda}{\alpha^r} & c \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Especially the fourth eigenvalue is

$$\begin{aligned} ew_4 &= d \\ &= 4\frac{\lambda(\alpha-\gamma)^r + \pi_1\alpha^r(\beta-\gamma)^r}{\alpha^r} \\ &= -4\Big(\pi_1(\alpha-\beta)^r + \pi_2(\alpha-\gamma)^r\Big). \end{aligned}$$

For

$$\tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$$

we have

$$ew_4(\tilde{\pi}_1) = 0.$$

Choosing the coefficients according to lemma 2.2.20 yields

$$0 < \tilde{\pi}_1 < -\frac{\lambda}{4\alpha^r}.$$

The point $g(\tilde{\pi}_1)$ is exactly the intersection point of the curve $g(\pi_1)$, $0 \leq \pi_1 \leq -\frac{\lambda}{\alpha^r}$, with the surface

$$\lambda + \beta^r \pi_1 + \gamma^r \pi_2 = 0$$

(cf. lemma 2.2.14). Only on this surface we can have critical points of the reduced vector field (cf. lemma 2.2.7) outside the stratum $\Delta = 0$ (cf. lemma 2.2.16).
For

$$h(\tilde{\pi}_1) \quad = \quad \begin{pmatrix} -\frac{(\alpha-\gamma)^r \lambda \left((\alpha-\beta)^{r2} - (\alpha-\beta)^{i2}\right)}{(\alpha-\beta)^{r2}(\beta-\gamma)^r} \\ \frac{(\alpha-\gamma)^r \lambda \left((\alpha-\beta)^{r2} - (\alpha-\beta)^{i2}\right)}{(\alpha-\beta)^{r2}(\beta-\gamma)^r} \\ \frac{\lambda^2 \left((\alpha-\beta)^{r3} - (\alpha-\gamma)^r (\alpha-\beta)^{i2}\right)}{2\alpha^r (\alpha-\beta)^r (\beta-\gamma)^{r2}} \\ -\frac{(\alpha-\gamma)^r \lambda^2 \left(2(\alpha-\beta)^{r3} + (-2\alpha+\beta+\gamma)^r (\alpha-\beta)^{i2}\right)}{2\alpha^r (\alpha-\beta)^{r2}(\beta-\gamma)^{r2}} \\ \frac{(\alpha-\gamma)^r (\alpha-\beta)^i \lambda^2}{2\alpha^r (\alpha-\beta)^r (\beta-\gamma)^r} \end{pmatrix}$$

we have

$$h(\tilde{\pi}_1) \quad = \quad \alpha^r t_2(\tilde{\pi}_1) - \frac{(\alpha-\beta)^i \tilde{\pi}_1}{2(\alpha-\beta)^{r2}} t_3(\tilde{\pi}_1) + \lambda t.$$

Consequently

$$h(\tilde{\pi}_1) \quad \in \quad \text{Span}\left(t_1(\tilde{\pi}_1), t_2(\tilde{\pi}_1), t_3(\tilde{\pi}_1), t\right),$$

and one sees that

$$L\, h(\tilde{\pi}_1) \quad = \quad j\, t_1(\tilde{\pi}_1)$$

with

$$j \;=\; \frac{2\lambda^2(\alpha-\gamma)^r \left((\alpha-\beta)^{r2} + (\alpha-\beta)^{i2}\right)}{\alpha^r (\alpha-\beta)^r (\beta-\gamma)^r} \;<\; 0.$$

So we have shown the following lemma.

Lemma 2.2.22: *In the point $g(\tilde{\pi}_1)$, $\tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$, the linearisation L of the vector field of the reduced equation (cf. lemma 2.2.7) has a nontrivial, two dimensional Jordan-block to the eigenvalue zero.*

Up to now we have studied the reduced vector field resulting from the normal form up to third order (cf. [IoRo]). It has been shown in this section that this vector field is degenerate. In the next section we will use fifth order terms to investigate this degeneracy.

2.2.6 Fifth order terms

Restricted to $\text{Fix}(\mathbb{Z}_2, 1)$ the normal form (cf. [IoRo]) yields the following fifth order terms (FOT). Proceed as in chapter 2.2.2 to get this terms.

$$\begin{aligned} \text{FOT} \quad = \quad & (\delta_1 \pi_1^2 + \delta_2 \pi_1 \pi_2 + \delta_3 \pi_2^2 + \delta_4 \pi_3 + \delta_5 \pi_4)\, e_1 \\ & + (\delta_6 \pi_1^2 + \delta_7 \pi_1 \pi_2 + \delta_8 \pi_2^2 + \delta_9 \pi_3 + \delta_{10} \pi_4 + \delta_{11} i \pi_5)\, e_2 \\ & + (\delta_{12} \pi_1 + \delta_{13} \pi_2)\, e_3 + (\delta_{14} \pi_1 + \delta_{15} \pi_2)\, e_4 - \frac{1}{2} \delta_{11} \pi_2 i e_5 . \end{aligned}$$

The coefficients $\delta_1, \ldots, \delta_{15} \in \mathbb{C}$ result from a transformation of the coefficients $d_1, \ldots, d_9 \in \mathbb{C}$ of the normalform (cf. [IoRo])

$$\begin{aligned} d_3 &\to \sqrt{6}\, d_3 \\ d_4 &\to -d_4 \\ d_5 &\to -\sqrt{6}\, d_5 \\ d_6 &\to \sqrt{\tfrac{3}{2}}\, d_6 \\ d_7 &\to \sqrt{\tfrac{3}{2}}\, d_7 \\ d_9 &\to \tfrac{3}{8}\, d_9 \end{aligned}$$

as follows

$$\begin{aligned} \delta_1 &= d_1 + \frac{1}{4}d_2 - 3d_3 + \frac{1}{2}d_4 - \frac{3}{2}d_5 + d_6 + d_7 - d_8 \\ \delta_2 &= 2d_1 - 7d_3 + \frac{1}{2}d_4 - 2d_5 - 2d_7 + 3d_8 \\ \delta_3 &= d_1 - 4d_3 - 2d_8 + d_9 \\ \delta_4 &= d_2 + 2d_5 + 12d_6 + 4d_7 - 4d_9 \\ \delta_5 &= d_2 + 2d_5 - 4d_6 - 4d_7 \\ \delta_6 &= d_1 + \frac{1}{4}d_2 - 4d_3 + \frac{1}{2}d_5 - 2d_7 + d_8 \\ \delta_7 &= 2d_1 - 4d_3 + 4d_7 - 2d_8 \\ \delta_8 &= d_1 \\ \delta_9 &= d_2 - 6d_5 \\ \delta_{10} &= d_2 - 2d_5 + 4d_7 - 2d_8 \\ \delta_{11} &= -4d_5 + 4d_7 + 2d_8 \\ \delta_{12} &= -6d_3 + d_4 - 4d_5 + 12d_6 - 2d_8 \\ \delta_{13} &= -6d_3 + d_4 \\ \delta_{14} &= 2d_3 + d_4 - 4d_5 - 4d_6 + 2d_8 \\ \delta_{15} &= 2d_3 + d_4 - 2d_5 + 2d_7 - 3d_8. \end{aligned}$$

In the following we want to study the vector field perturbed in fifth order of the form

$$\dot{\pi} = f(\pi) + \varepsilon\, \mathrm{RFOT}(\pi),\ \varepsilon \ll 1.$$

By reduction of the fifth order terms (FOT) to the orbit space one gets the perturbation RFOT (reduced fifth order terms) with components RFOT1, ..., RFOT5.

$$\begin{aligned} \mathrm{RFOT1} &= 2\pi_1(\delta_1^r \pi_1^2 + \delta_2^r \pi_1\pi_2 + \delta_3^r \pi_2^2 + \delta_4^r \pi_3) \\ &\quad + \pi_4\Big(2(\delta_5 + \delta_{14})^r \pi_1 + (\delta_{11} + 2\delta_{15})^r \pi_2\Big) \\ &\quad + \pi_5\Big(2\delta_{14}^i \pi_1 + (\delta_{11} + 2\delta_{15})^i \pi_2\Big) \\ \mathrm{RFOT2} &= 4\delta_{12}^r \pi_1\pi_3 + 2\pi_2\Big(\delta_6^r \pi_1^2 + \delta_7^r \pi_1\pi_2 + \delta_8^r \pi_2^2 + (\delta_9 + 2\delta_{13})^r \pi_3\Big) \\ &\quad + \pi_4\Big(2\delta_{14}^r \pi_1 + (2\delta_{10} - \delta_{11} + 2\delta_{15})^r \pi_2\Big) \\ &\quad - \pi_5\Big(2\delta_{14}^i \pi_1 + (\delta_{11} + 2\delta_{15})^i \pi_2\Big) \\ \mathrm{RFOT3} &= 2\pi_3\Big(2\delta_6^r \pi_1^2 + (2\delta_7 + \delta_{12})^r \pi_1\pi_2 + (2\delta_8 + \delta_{13})^r \pi_2^2 + 2\delta_9^r \pi_3\Big) \\ &\quad + \pi_4\left(\pi_2\left(\delta_{14}^r \pi_1 - \left(\frac{1}{2}\delta_{11} - \delta_{15}\right)^r \pi_2\right) + 4\delta_{10}^r \pi_3\right) \\ &\quad - \pi_5\left(\pi_2\left(\delta_{14}^i \pi_1 - \left(\frac{1}{2}\delta_{11} - \delta_{15}\right)^i \pi_2\right) + 4\delta_{11}^i \pi_3\right) \end{aligned}$$

$$
\begin{aligned}
\mathrm{RFOT4} &= 2\pi_4\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2+\left(\delta_3+\delta_8+\frac{1}{2}\delta_{13}\right)^r\pi_2^2\right.\\
&\quad\left.+(\delta_4+\delta_9)^r\pi_3+(\delta_5+\delta_{10})^r\pi_4\right)\\
&\quad-2\pi_5\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2+\left(\delta_3-\delta_8-\frac{1}{2}\delta_{13}\right)^i\pi_2^2\right.\\
&\quad\left.+(\delta_4-\delta_9)^i\pi_3-\delta_{11}^r\pi_5\right)\\
&\quad-2\pi_4\pi_5(\delta_5-\delta_{10}+\delta_{11})^i\\
&\quad+\pi_1\left(\frac{1}{2}\delta_{14}^r\pi_1^2\pi_2-\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{15}\right)^r\pi_1\pi_2^2+2\delta_{14}^r\pi_1\pi_3+(\delta_{11}+2\delta_{15})^r\pi_2\pi_3\right)\\
\mathrm{RFOT5} &= 2\pi_4\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2+\left(\delta_3-\delta_8-\frac{1}{2}\delta_{13}\right)^i\pi_2^2\right.\\
&\quad\left.+(\delta_4-\delta_9)^i\pi_3+(\delta_5-\delta_{10})^i\pi_4\right)\\
&\quad+2\pi_5\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2+\left(\delta_3+\delta_8+\frac{1}{2}\delta_{13}\right)^r\pi_2^2\right.\\
&\quad\left.+(\delta_4+\delta_9)^r\pi_3-\delta_{11}^i\pi_5\right)\\
&\quad+2\pi_4\pi_5(\delta_5+\delta_{10}-\delta_{11})^r\\
&\quad+\pi_1\left(-\frac{1}{2}\delta_{14}^i\pi_1^2\pi_2+\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{15}\right)^i\pi_1\pi_2^2+2\delta_{14}^i\pi_1\pi_3+(\delta_{11}+2\delta_{15})^i\pi_2\pi_3\right).
\end{aligned}
$$

Lemma 2.2.23: *Restriction to the stratum* $\Delta=0$ *yields*

$$
\begin{aligned}
\Delta\mathrm{RFOT1} &= 2\pi_1\left(\delta_1^r\pi_1^2+\delta_2^r\pi_1\pi_2+\left(\delta_3+\frac{1}{4}\delta_4\right)^r\pi_2^2\right)\\
&\quad+\pi_4\left(2(\delta_5+\delta_{14})^r\pi_1+(\delta_{11}+2\delta_{15})^r\pi_2\right)\\
&\quad+\pi_5\left(2\delta_{14}^i\pi_1+(\delta_{11}+2\delta_{15})^i\pi_2\right)\\
\Delta\mathrm{RFOT2} &= 2\pi_2\left(\delta_6^r\pi_1^2+\left(\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2+\left(\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2\right)\\
&\quad+\pi_4\left(2\delta_{14}^r\pi_1+(2\delta_{10}-\delta_{11}+2\delta_{15})^r\pi_2\right)\\
&\quad-\pi_5\left(2\delta_{14}^i\pi_1+(\delta_{11}+2\delta_{15})^i\pi_2\right)\\
\Delta\mathrm{RFOT3} &= \frac{1}{2}\pi_2\,\Delta\mathrm{RFOT2}\\
\Delta\mathrm{RFOT4} &= 2\pi_4\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3+\frac{1}{4}\delta_4+\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2+(\delta_5+\delta_{10})^r\pi_4\right)\\
&\quad-2\pi_5\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3+\frac{1}{4}\delta_4-\delta_8-\frac{1}{4}\delta_9-\frac{1}{2}\delta_{13}\right)^i\pi_2^2-\delta_{11}^r\pi_5\right)\\
&\quad-2\pi_4\pi_5(\delta_5-\delta_{10}+\delta_{11})^i\\
&\quad+\pi_1\pi_2\left(\frac{1}{2}\delta_{14}^r\pi_1^2-\left(\frac{1}{4}\delta_{11}-\frac{1}{2}\delta_{14}-\frac{1}{2}\delta_{15}\right)^r\pi_1\pi_2+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{15}\right)^r\pi_2^2\right)\\
\Delta\mathrm{RFOT5} &= 2\pi_4\left((\delta_1-\delta_6)^i\pi_1^2+\left(\delta_2-\delta_7-\frac{1}{2}\delta_{12}\right)^i\pi_1\pi_2\right.\\
&\quad\left.+\left(\delta_3+\frac{1}{4}\delta_4-\delta_8-\frac{1}{4}\delta_9-\frac{1}{2}\delta_{13}\right)^i\pi_2^2+(\delta_5-\delta_{10})^i\pi_4\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\pi_5\left((\delta_1+\delta_6)^r\pi_1^2+\left(\delta_2+\delta_7+\frac{1}{2}\delta_{12}\right)^r\pi_1\pi_2\right.\\
&\left.+\left(\delta_3+\frac{1}{4}\delta_4+\delta_8+\frac{1}{4}\delta_9+\frac{1}{2}\delta_{13}\right)^r\pi_2^2-\delta_{11}^i\pi_5\right)\\
&+2\pi_4\pi_5(\delta_5+\delta_{10}-\delta_{11})^r\\
&+\pi_1\pi_2\left(-\frac{1}{2}\delta_{14}^i\pi_1^2+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{14}-\frac{1}{2}\delta_{15}\right)^i\pi_1\pi_2+\left(\frac{1}{4}\delta_{11}+\frac{1}{2}\delta_{15}\right)^i\pi_2^2\right).
\end{aligned}
$$

Here ΔRFOT1, ..., ΔRFOT5 *denote the components of the reduced fifth order terms (RFOT) restricted to the statum* $\Delta = 0$.

2.2.7 Singular perturbation theory

For the moment we want to restrict our considerations to the stratum $\Delta = 0$. Here lies (cf. lemma 2.2.16) the curve $g(\pi_1)$, $0 \le \pi_1 \le -\frac{\lambda}{\alpha^r}$, of critical points of the reduced vector field $\dot{\pi} = f(\pi)$ (cf. equations 2.2.3 to 2.2.7). According to lemma 2.2.21 this curve is asmyptotically stable for our choice of the coefficients

$$\beta^r < \alpha^r < \gamma^r < 0.$$

Now we want to study the perturbed vector field (cf. lemma 2.2.23)

$$\dot{\pi} \quad = \quad f^\varepsilon(\pi) \;=\; f(\pi) + \varepsilon\,\Delta\mathrm{RFOT}(\pi),\ \varepsilon \ll 1. \tag{2.2.19}$$

We have the following propostion.

Proposition 2.2.24: *For the perturbed vector field 2.2.19 there persists for* $0 < |\varepsilon| < \varepsilon_0$ *an invariant curve* g_ε *near* g *on the stratum* $\Delta = 0$. *This curve* g_ε *is parametrised over* π_1. *The vector field on* g_ε *has the form*

$$r(\pi_1) \quad = \quad 2\frac{\pi_1\pi_2}{\pi_1+\pi_2}\left(16\pi_1^2+16\frac{\lambda}{\alpha^r}\pi_1+3\frac{\lambda^2}{\alpha^{r2}}\right)d$$

with

$$0 < \pi_1 < -\frac{\lambda}{\alpha^r},\ \pi_1+\pi_2 = -\frac{\lambda}{\alpha^r},$$

$$d = \left((d_6+d_7-d_8)^r+\frac{(\alpha-\beta)^i}{(\alpha-\beta)^r}(d_6+d_7-d_8)^i\right).$$

Remark 2.2.25: *In the following let*

$$d \quad \neq \quad 0.$$

Proof: In lemma 2.2.21 we showed that the curve $g(\pi_1)$, $0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, is normally hyperbolic. Thus an invariant curve g_ε near g is conserved under small pertubations. But the curve g_ε will no longer consist of critical points but there will be a resulting flow on g_ε. This flow is determined to the lowest order by projection of the perturbation onto the curve g.
Let

$$E \quad = \quad \left\{g(\pi_1);\, 0 < \pi_1 < -\frac{\lambda}{\alpha^r}\right\}$$

be the curve of critical points of the vector field $f^0(\pi)$ on the stratum

$$F = \left\{\pi \in \Pi(\mathbb{R}^6) \,|\, \Delta(\pi) = 0\right\}.$$

For a point $\pi \in E$ let

$$Tf^0(\pi) \quad : \quad T_\pi F \quad \to \quad T_\pi F$$

denote the linearisation of f^0 in π. By construcion $T_\pi E$ lies in the kernel of $Tf^0(\pi)$. So a linear map

$$Qf^0(\pi) \quad : \quad \frac{T_\pi F}{T_\pi E} \quad \to \quad \frac{T_\pi F}{T_\pi E}$$

is induced on the quotient space. The eigenvalues of $Qf^0(\pi)$ have been determined in lemma 2.2.21 and are both negative. Thus $T_\pi E$ has for every $\pi \in E$ a unique complement N_π that is invariant under $Tf^0(\pi)$. Let P^E denote the projection onto TE defined by the splitting

$$TF_{|E} \quad = \quad TE \oplus N$$

On E we define the vector field

$$f_R(\pi) \quad = \quad P^E \frac{\partial}{\partial \varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}\,.$$

Now we have the extended vector field

$$f^\varepsilon(\pi) \times \{0\} \text{ on } F \times (-\varepsilon_0, \varepsilon_0).$$

For this system, according to [Fe], it exists for small $\varepsilon_0 \ll 1$ a two dimensional center manifold C. The second dimension has its origin in the extension of the systemn in ε-direction.
On C near $E \times \{0\}$ a smooth vector field

$$f_C \quad = \quad \begin{cases} \frac{1}{\varepsilon} f^\varepsilon(\pi) \times 0, & \varepsilon \neq 0 \\ f_R(\pi) \times 0, & \varepsilon = 0 \end{cases}$$

is defined. The center manifold is fibered in ε-direction with invariant curves g_ε. The flow on g_ε has the form

$$\dot{\pi} \quad = \quad \varepsilon\, f_R(\pi) + O(\varepsilon^2).$$

Now we want to explicitly calculate the vector field

$$f_R(\pi) \quad = \quad P^E \frac{\partial}{\partial \varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}\,.$$

According to lemma 2.2.21 the vectors t_1, t_2, t_3 that span the tangent space to the stratum F along the curve g are known. The vector t_1 is the tangent vector along the curve g. Now we want to write the terms of higher order ΔRFOT along the curve g in the form

$$\Delta\text{RFOT}(\pi) \quad = \quad a(\pi)t_1 + b(\pi)t_2 + c(\pi)t_3.$$

This gives the projection P^E we are looking for and we have

$$f_R(\pi) \quad = \quad a(\pi).$$

As the restriction of the vector field $\frac{\partial}{\partial \varepsilon} f^\varepsilon(\pi)_{|\varepsilon=0}$ along the curve g one gets

$$\begin{aligned}
r_1 &= \pi_1 \left(2\delta_1^r \pi_1^2 + (2\delta_2 + \delta_5 + \delta_{14})^r \pi_1 \pi_2 + \left(2\delta_3 + \frac{1}{2}\delta_4 + \frac{1}{2}\delta_{11} + \delta_{15} \right)^r \pi_2^2 \right) \\
r_2 &= \pi_2 \left((2\delta_6 + \delta_{14})^r \pi_1^2 + \left(2\delta_7 + \delta_{10} - \frac{1}{2}\delta_{11} + \delta_{12} + \delta_{15} \right)^r \pi_1 \pi_2 \right. \\
&\quad \left. + \left(2\delta_8 + \frac{1}{2}\delta_9 + \delta_{13} \right)^r \pi_2^2 \right) \\
r_3 &= \frac{1}{2}\pi_2\, r_2 \\
r_4 &= \pi_1 \pi_2 \left(\left(\delta_1 + \delta_6 + \frac{1}{2}\delta_{14} \right)^r \pi_1^2 \right. \\
&\quad + \left(\delta_2 + \frac{1}{2}\delta_5 + \delta_7 + \frac{1}{2}\delta_{10} - \frac{1}{4}\delta_{11} + \frac{1}{2}\delta_{12} + \frac{1}{2}\delta_{14} + \frac{1}{2}\delta_{15} \right)^r \pi_1 \pi_2 \\
&\quad \left. + \left(\delta_3 + \frac{1}{4}\delta_4 + \delta_8 + \frac{1}{4}\delta_9 + \frac{1}{4}\delta_{11} + \frac{1}{2}\delta_{13} + \frac{1}{2}\delta_{15} \right)^r \pi_2^2 \right) \\
r_5 &= \pi_1 \pi_2 \left(\left(\delta_1 - \delta_6 - \frac{1}{2}\delta_{14} \right)^i \pi_1^2 \right. \\
&\quad + \left(\delta_2 + \frac{1}{2}\delta_5 - \delta_7 - \frac{1}{2}\delta_{10} + \frac{1}{4}\delta_{11} - \frac{1}{2}\delta_{12} + \frac{1}{2}\delta_{14} - \frac{1}{2}\delta_{15} \right)^i \pi_1 \pi_2 \\
&\quad \left. + \left(\delta_3 + \frac{1}{4}\delta_4 - \delta_8 - \frac{1}{4}\delta_9 + \frac{1}{4}\delta_{11} - \frac{1}{2}\delta_{13} + \frac{1}{2}\delta_{15} \right)^i \pi_2^2 \right).
\end{aligned}$$

Always we have

$$\pi_1 + \pi_2 \quad = \quad -\frac{\lambda}{\alpha^r}.$$

Written in components we get the following equations

$$\begin{aligned} r_1 &= a + b\pi_1 + 2c\alpha^r(\alpha-\beta)^i && (2.2.20)\\ r_2 &= -a + b\pi_2 - 2c\alpha^r(\alpha-\beta)^i && (2.2.21)\\ r_4 &= \frac{1}{2}a(\pi_2-\pi_1) + b\pi_1\pi_2 + c\alpha^r(\pi_2-\pi_1)(\alpha-\beta)^i && (2.2.22)\\ r_5 &= -c\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r. && (2.2.23) \end{aligned}$$

Thus

$$c \quad = \quad -\frac{r_5}{\alpha^r(\pi_1+\pi_2)(\alpha-\beta)^r},$$

$$r_1 + r_2 \quad = \quad b(\pi_1+\pi_2)$$

and

$$r_1 - r_2 \quad = \quad 2a + b(\pi_1-\pi_2) + 4c\alpha^r(\alpha-\beta)^i.$$

Finally we get

$$b \quad = \quad \frac{r_1+r_2}{\pi_1+\pi_2}$$

and

$$\begin{aligned} a &= \frac{1}{2}(r_1-r_2) - \frac{1}{2}b(\pi_1-\pi_2) - 2c\alpha^r(\alpha-\beta)^i\\ &= \frac{r_1\pi_2 - r_2\pi_1}{\pi_1+\pi_2} - 2c\alpha^r(\alpha-\beta)^i. \end{aligned}$$

Insertion of r_1, r_2, c and retranslation of the coefficients $\delta_1, \ldots, \delta_{15}$ into the coefficients $d_1, \ldots, d_9$ finishes the proof. ⋈

Proposition 2.2.26: *On the invariant curve g_ε (cf. propostion 2.2.24) exactly two critical points persist for $0 < |\varepsilon| < \tilde{\varepsilon}_0 < \varepsilon_0$ under the perturbed vector field 2.2.19. In the entire ten dimensionel system these critical points have isotropy* $(\mathrm{O}(2), 1)$ *resp.* $(\mathrm{D}_4, \mathbb{Z}_2)$. *The latter corresponds to the isotropy* $(\mathrm{D}_2, \mathbb{Z}_2)$ *in the reduced system. Their stability in $\mathbb{R}^6$ is determined by the sign of*

$$d \quad = \quad \left((d_6+d_7-d_8)^r + \frac{(\alpha-\beta)^i}{(\alpha-\beta)^r}(d_6+d_7-d_8)^i\right).$$

Especially there persists for small ε a connection in $\mathbb{R}^6$ between the group orbits of solutions with isotropy $(\mathrm{O}(2), 1)$ *resp.* $(\mathrm{D}_4, \mathbb{Z}_2)$.
The position of the critical points, their isotropy in the entire system und the direction of the resulting flow on g_ε is given in figure 5.

Proof: On the persisting curve $g_\varepsilon(\pi_1), 0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, near g there are two critical points of the Fenichel vector field $r(\pi_1)$ (cf. proposition 2.2.24) for

$$\pi_1 \quad \in \quad \left\{-\frac{\lambda}{4\alpha^r}, -\frac{3\lambda}{4\alpha^r}\right\}.$$

Linearisation of the vector field $r(\pi_1)$ in this critical points yields

π_1	$\frac{\mathrm{d}r}{\mathrm{d}\pi_1}$
$-\frac{\lambda}{4\alpha^r}$	$-3d\frac{\lambda^2}{\alpha^{r2}}$
$-\frac{3\lambda}{4\alpha^r}$	$3d\frac{\lambda^2}{\alpha^{r2}}$,

thus they are hyperbolic. Here d is defined as in proposition 2.2.24. Therefore these critical points persist for $|\varepsilon| < \tilde{\varepsilon}_0 < \varepsilon_0$ in the perturbed vector field 2.2.19.
Now we will show that the persisting critical points lie on the group orbits of solutions with isotropy $(\mathrm{O}(2), 1)$ resp. $(\mathrm{D}_4, \mathbb{Z}_2)$ with reference to the entire system.
First let

$$\tilde{\pi}_1 = -\frac{\lambda}{4\alpha^r}.$$

Using the representation of the group elements

$$\tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

introduced in lemma 2.2.10 and

$$\pi \in \mathrm{S}^1$$

we have

$$\begin{aligned} \tau\,\pi\,(0,z,0) &= \tau\,(0,-z,0) \\ &= \left(\frac{1}{2}\sqrt{\frac{3}{2}}z, \frac{1}{2}z, \frac{1}{2}\sqrt{\frac{3}{2}}z\right). \end{aligned}$$

Points of the form $(0,z,0)$ with

$$|z|^2 = -\frac{\lambda}{\alpha^r}$$

are mapped to the critical point of isotropy $(\mathrm{O}(2), 1)$ in the reduced system by the Hilbert-map. Thus

$$\Pi(\tau\pi\,(0,z,0)) = \left(\pi_1 = -\frac{\lambda}{4\alpha^r}, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \pi_3 = \frac{1}{4}\pi_2^2, \pi_4 = \frac{1}{2}\pi_1\pi_2, \pi_5 = 0\right).$$

Therefore

$$\Pi\big(\tau\,\pi\,(0,z,0)\big) = g(\tilde{\pi}_1).$$

Now let

$$\tilde{\pi}_1 = -\frac{3\lambda}{4\alpha^r}.$$

Correspondingly the Hilbert-map maps points of the form $(z,0,z)$ with

$$|z|^2 = -\frac{\lambda}{2\alpha^r}$$

to the critical point of isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ in the reduced system. We have with $r_\pi \in \mathrm{O}(2)$

$$\begin{aligned} r_\pi\,\tau\,(z,0,z) &= r_\pi\left(-\frac{1}{2}z, \sqrt{\frac{3}{2}}z, -\frac{1}{2}z\right) \\ &= \left(\frac{1}{2}z, \sqrt{\frac{3}{2}}z, \frac{1}{2}z\right). \end{aligned}$$

Thus

$$\begin{aligned} \Pi(r_\pi\,\tau\,(z,0,z)) &= \left(\pi_1 = -\frac{3\lambda}{4\alpha^r}, \pi_2 = -\left(\pi_1 + \frac{\lambda}{\alpha^r}\right), \pi_3 = \frac{1}{4}\pi_2^2, \pi_4 = \frac{1}{2}\pi_1\pi_2, \pi_5 = 0\right) \\ &= g(\tilde{\pi}_1) \end{aligned}$$

Since the perturbation respects the symmetry the critical points persisting for small ε on the curve have the same isotropies.

Besides this two critical points there are no critical points on g_ε for small $\varepsilon > 0$. Since the two critical points are hyperbolic, in a neighbourhood of these points no other critical points exists by the implicit function theorem. If there were critical points (x_n, ε_n) in the remaining part of g_ε, for a sequence $(\varepsilon_n) \to 0$ the accumulation point $(\bar{x}, 0)$ had to be a critical point of the resulting vector field in contradiction to proposition 2.2.24. ⋈

Figure 5 shows the resulting flow on the invariant curve $g_\varepsilon(\pi_1),\ 0 < \pi_1 < -\frac{\lambda}{\alpha^r}$, in a schematic way for $d > 0$ and small $\varepsilon > 0$. Choosing $d < 0$ will change the direction of the arrows. The isotropies of the solutions in the entire ten dimensional system are indicated in the scetch.
For $\varepsilon = 0$ (i.e. $g_\varepsilon = g$) $g(0)$ resp. $g(-\frac{\lambda}{\alpha^r})$ are fixed points of isotropy $(\mathrm{D}_2, \mathbb{Z}_2)$ resp. $(\mathrm{O}(2), 1)$ (cf. lemma 2.2.16). The curve itself consists of fixed points.

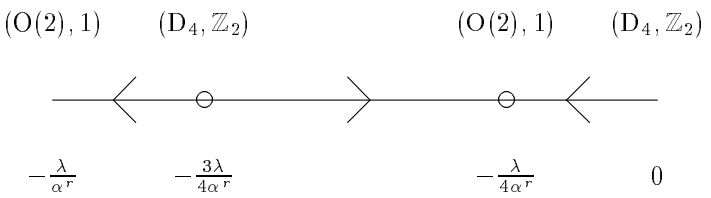


Figure 5: Resulting flow on g_ε

2.2.8 Invariant tori

In this section we want to show that for small $\varepsilon > 0$ a fixed point bifurcates from the critical point $\tilde{\pi}_1$ that lies on the curve g on the stratum $\Delta = 0$ in the direction of the principal stratum. According to lemma 2.2.22 the linearisation of the vector field of the reduced equation (cf. lemma 2.2.7) has a nontrivial Jordan-block to the eigenvalue zero in the point

$$g(\tilde{\pi}_1),\ \tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}.$$

The position of this point on the invariant curve

$$g(\pi_1),\ 0 < \pi_1 < -\frac{\lambda}{\alpha^r},$$

depends on the relative choice of the coefficients

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta^r + 3\gamma^r), \gamma^r\right),$$

according to figure 6. Making the ansatz

$$\alpha^r \ = \ t\,\beta^r + (1-t)\,\gamma^r, \quad t \in \left(0, \frac{1}{4}\right)$$

this follows as in the proof of lemma 2.2.20. Thus

$$\tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r} = -\frac{\lambda}{\alpha^r}t, \quad t \in \left(0, \frac{1}{4}\right).$$

Now we want to determine the form of the resulting vector field on the lokal two dimensional center manifold W^c_{loc} near the point $g(\tilde{\pi}_1)$. The center manifold W^c_{loc} is tangential to Span (t_1, h) (cf. lemma 2.2.22) and intersects the stratum $\Delta = 0$ in a part of the invariant curve $g(\pi_1)$ near $g(\tilde{\pi}_1)$. Here let t_1 be the tangent vector in the direction of the curve $g(\pi_1)$ and h be the hauptvector associated to the

Figure 6: Possible region of the point $\tilde{\pi}_1$

Jordan-block of the linearisation. By definition of the vectors t_1, t, h in lemma 2.2.22 h points in the direction of the principal stratum.
We introduce x-coordinates in the direction of $(-t_1)$ along the invariant curve $g(\pi_1)$ and y-coordinates in the direction of $(-h)$ with origin in $g(\tilde{\pi}_1)$. Therefore the vector field on W^c_{loc} has the form

$$\begin{aligned} \dot{x} &= -y + H(x,y) \\ \dot{y} &= y\,G(x,y). \end{aligned} \tag{2.2.24}$$

We are only interested in the region $y \leq 0$ that describes a part of the Hilbert-set $\Pi(\mathbb{R}^6)$ according to our choice of the coordinates. The $(-y)$-term in the x-equation models the Jordan-block, the minus sign follows from the equation

$$Lh = j\,t_1$$

with $j < 0$ according to lemma 2.2.22. The y-term in the y-equation models the flow invariance of the curve $y = 0$, i.e. of the stratums $\Delta = 0$.
The function $H(x,y)$ has the following properties

$$\begin{aligned} H(x,y) &= O(x^2, xy, y^2), \\ H(x,0) &\equiv 0, \\ \frac{\partial H}{\partial x}(x,0) &\equiv 0. \end{aligned}$$

The last two properties come from the fact that points of the form $(x,0)$ are critical points of the system 2.2.24 by construction. The linearisation of the vector field 2.2.24 in such a point $(x,0)$ yields

$$A = \begin{pmatrix} 0 & -1 + \frac{\partial H}{\partial y}(x,0) \\ 0 & G(x,0) \end{pmatrix}.$$

Consequently the eigenvalues are zero in the direction of the curve of fixed points and $G(x,0)$ in the direction of the principal stratum. This eigenvalue has been calculated in lemma 2.2.22 and has the form

$$e_4 = 4\frac{\lambda(\alpha-\gamma)^r + \pi_1\,\alpha^r\,(\beta-\gamma)^r}{\alpha^r}.$$

Therefore in our coordinates we have

$$G(x,0) = a\,x + O(x^2)$$

with $a > 0$. The invariant curve changes the stability in the direction of the principal stratum in the first order from stable to unstable in the point $(0,0)$ (transversality condition).
Now let's look at the extended system

$$\begin{aligned} \dot{\pi} &= f(\pi) + \varepsilon\,\mathrm{RFOT}(\pi) \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{2.2.25}$$

Here near the point

$$(g(\tilde{\pi}_1), 0)$$

there exists a local center manifold. This manifold is fibered in ε-direction with two dimensional invariant manifolds $W^c_{loc,\varepsilon}$. For $\varepsilon = 0$ the manifold $W^c_{loc,0}$, tangential to Span $\{t_1, h\}$, intersects the stratum $\Delta = 0$ in a part of the curve g near $g(\tilde{\pi}_1)$ transversally. This property is preserved for small

$$\varepsilon < \overline{\varepsilon} < \tilde{\varepsilon}.$$

On the two dimensional center manifolds $W^c_{loc,\varepsilon}$ again we introduce, now ε-dependend, coordinates x_ε in the direction of g_ε and y_ε in the direction of the principal stratum. We will continue writing x resp. y for x_ε resp. y_ε.
Now let $\tilde{\pi}_1$ be tuned in a way such that the Fenichel-drift in $g_\varepsilon(\tilde{\pi}_1)$, $|\varepsilon| < \overline{\varepsilon}$, is not zero. Then the flow on the corresponding center manifold $W^c_{loc,\varepsilon}$ has the form

$$\begin{aligned} \dot{x} &= -y + \varepsilon + H(x, y, \varepsilon) \\ \dot{y} &= y\, G(x, y, \varepsilon). \end{aligned} \tag{2.2.26}$$

The sign of ε depends on the direction of the resulting Fenichel-drift. We want to assume the solution of isotropy (O(2), 1) to be stable. Therefore according to proposition 2.2.26 we have to choose $d > 0$ and the resulting Fenichel-drift has the form indicated in figure 5. For the parameters

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right),$$

we have

$$\tilde{\pi}_1 \in \left(0, -\frac{\lambda}{4\alpha^r}\right)$$

and thus we have to choose $\varepsilon < 0$.
The functions $G(x, y, \varepsilon)$ resp. $H(x, y, \varepsilon)$ have the following properties

$$\begin{aligned} G(x, y, \varepsilon) &= O(x, y, \varepsilon), \\ G(x, y, 0) &= G(x, y) \end{aligned}$$

resp.

$$\begin{aligned} H(x, y, \varepsilon) &= O(x^2, xy, y^2, \varepsilon x, \varepsilon y, \varepsilon^2), \\ H(x, y, 0) &= H(x, y). \end{aligned}$$

Proposition 2.2.27: *Let*

$$\beta^r < \alpha^r < \gamma^r < 0, \quad \alpha^r \in \left(\frac{1}{4}(\beta + 3\gamma)^r, \gamma^r\right).$$

Then there exists $\overline{\varepsilon} > 0$ and a unique curve

$$\Big(x(\varepsilon), y(\varepsilon) \le 0\Big), \quad -\overline{\varepsilon} < \varepsilon \le 0,$$

of critical points of the flow on the center manifold $W^c_{loc,\varepsilon} \cap \Pi(\mathbb{R}^6)$ with

$$\Big(x(0), y(0)\Big) = (0, 0).$$

The critical points are saddles.

Proof: We are looking for critical points of the vector field 2.2.26. Therefore we first solve the equation

$$P(x, y, \varepsilon) = -y + \varepsilon + H(x, y, \varepsilon) = 0.$$

We have

$$P(0, 0, 0) = 0$$

and

$$\frac{\partial P}{\partial y}(0, 0, 0) = -1,$$

since $H(x,y,\varepsilon)$ is of second order. Using the implicit function theorem locally near $(x,\varepsilon)=(0,0)$ one gets a unique surface $y=y(x,\varepsilon)$ with $y(0,0)=0$ and $P(x,y(x,\varepsilon),\varepsilon)=0$. Furthermore

$$y(x,\varepsilon) \quad = \quad \varepsilon + O(x^2, x\varepsilon, \varepsilon^2)$$

and

$$\frac{\partial y}{\partial x}(0,0) \quad = \quad 0.$$

Now we want to solve the equation

$$G(x,y(x,\varepsilon),\varepsilon) \quad = \quad 0.$$

We have

$$G(0,0,0) \quad = \quad 0$$

and

$$\frac{\partial G}{\partial x}(0,0,0) \quad = \quad a > 0$$

because of the transversality property of G and the condition $\frac{\partial y}{\partial x}(0,0)=0$. Therefore, again by the implicit function theorem, there exists a unique curve

$$(x(\varepsilon),y(\varepsilon)),\ 0 \le |\varepsilon| < \overline{\varepsilon},\ \varepsilon \le 0,$$

of critical points of the vector field 2.2.26. Furthermore

$$x \quad = \quad O(\varepsilon).$$

Thus the curve $y(\varepsilon)$ has the form

$$y(\varepsilon) \quad = \quad \varepsilon + O(\varepsilon^2).$$

The sign of $y(\varepsilon)$ is determined by the sign of ε for small ε. Here we have $\varepsilon < 0$ and therefore $y(\varepsilon) < 0$. Consequently the curve lies in the Hilbert-set $\Pi(\mathbb{R}^6)$.
The linear stability of the critical point $(x(\varepsilon),y(\varepsilon)),\ -\overline{\varepsilon} < \varepsilon \le 0$, is to be determined. The linearisation of the vector field 2.2.26 in the point $(x(\varepsilon),y(\varepsilon))$ yields

$$\begin{aligned} D \quad &= \quad \begin{pmatrix} \frac{\partial H}{\partial x}(x(\varepsilon),y(\varepsilon),\varepsilon) & -1+\frac{\partial H}{\partial y}(x(\varepsilon),y(\varepsilon),\varepsilon) \\ y(\varepsilon)\,\frac{\partial G}{\partial x}(x(\varepsilon),y(\varepsilon),\varepsilon) & G(x(\varepsilon),y(\varepsilon),\varepsilon)+y(\varepsilon)\,\frac{\partial G}{\partial y}(x(\varepsilon),y(\varepsilon),\varepsilon) \end{pmatrix} \\ &= \quad \begin{pmatrix} O(\varepsilon) & -1+O(\varepsilon) \\ \varepsilon\,a(\varepsilon)+O(\varepsilon^2) & O(\varepsilon) \end{pmatrix}. \end{aligned}$$

We have $\frac{\partial G}{\partial x}(0,0,0)=a>0$. Thus

$$\frac{\partial G}{\partial x}(x(\varepsilon),y(\varepsilon),\varepsilon) \quad = \quad a(\varepsilon) \ > \ 0$$

with $a(0)=a$ for small ε. So we get the following eigenvalues of D

$$\rho_{1,2} \quad = \quad O(\varepsilon) \pm \sqrt{O(\varepsilon^2) - \varepsilon\,a(\varepsilon)}$$

with $\varepsilon<0$ and $a(\varepsilon)>0$. For small ε the $\sqrt{-\varepsilon}$-term is dominating, the critical point is a saddle. $\bowtie$
The bifurcating critical point lies in the principal stratum. The preimages are two 2-tori. Since there are no additional, symmetry given phase relations (cf. lemma 2.2.19) in general we have quasiperiodic solutions.

2.2.9 Stability of the invariant tori

In order to apply Conley-index theory later on we have to know the stability of the group orbit of the quasiperiodic solutions (cf. proposition 2.2.27) in the entire ten dimensional system. We want to determine the Floquet-exponents of the periodic solutions that correspond to the critical points on the curve

$$g(\pi_1), \quad 0 < \pi_1 < -\frac{\lambda}{\alpha^r}.$$

According to our choice of the coefficients only the interval

$$0 < \pi_1 < -\frac{\lambda}{4\alpha^r}$$

is of interest. Here, in dependence on the relative choice of the coefficients, critical points of the reduced system bifurcate (cf. proposition 2.2.27).
The periodic solutions are rotating waves. In a rotating coordinate system one gets a static problem which is accessible more easily. We make the ansatz

$$\begin{aligned} z_0 &= (r_0 + \rho_0)\, e^{i(\omega_0 t + \phi_0)} \\ z_{\pm 2} &= (r_2 + \rho_{\pm 2})\, e^{i(\omega_0 t + \phi_{\pm 2})} \\ z_{\pm 1} &= y_{\pm 1}\, e^{i\omega_0 t} \end{aligned}$$

with

$$\omega_0 = \omega + \alpha^i \left(r_0^2 + 2r_2^2\right)$$

and

$$r_0^2 + 2r_2^2 = -\frac{\lambda}{\alpha^r}.$$

In the lowest order one gets the following systems which decouple for symmetry reasons:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ \overline{y}_1 \\ y_{-1} \\ \overline{y}_{-1} \end{pmatrix} = \begin{pmatrix} s & t & t & s \\ \overline{t} & \overline{s} & \overline{s} & \overline{t} \\ t & s & s & t \\ \overline{s} & \overline{t} & \overline{t} & \overline{s} \end{pmatrix} \begin{pmatrix} y_1 \\ \overline{y}_1 \\ y_{-1} \\ \overline{y}_{-1} \end{pmatrix}$$

with

$$\begin{aligned} s &= r_0^2 \left(-\alpha + \frac{1}{4}\beta + \frac{3}{4}\gamma\right) + r_2^2 \left(-2\alpha + \frac{3}{2}\beta + \frac{1}{2}\gamma\right), \\ t &= 2\sqrt{\frac{3}{8}}(\gamma - \beta) r_0 r_2 \end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \rho_{-2} \\ \rho_0 \\ \rho_2 \\ \phi_{-2} \\ \phi_0 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 2r_2^2\gamma^r - r_0^2(\alpha-\beta)^r & 2r_0 r_2 \alpha^r \\ 2r_0 r_2 \alpha^r & 2r_0^2 \alpha^r \\ -2r_2^2\gamma^r + r_0^2(\alpha-\beta)^r + 4r_2^2\alpha^r & 2r_0 r_2 \alpha^r \\ 2r_2\gamma^i - \frac{r_0^2}{r_2}(\alpha-\beta)^i & 2r_0\alpha^i \\ 2r_2\alpha^i & 2r_0\alpha^i \\ -2r_2\gamma^i + \frac{r_0^2}{r_2}(\alpha-\beta)^i + 4r_2\alpha^i & 2r_0\alpha^i \end{pmatrix}$$

$$\begin{pmatrix} -2r_2^2\gamma^r + r_0^2(\alpha-\beta)^r + 4r_2^2\alpha^r & r_0^2 r_2 (\alpha-\beta)^i \\ 2r_0 r_2 \alpha^r & -2r_0 r_2^2 (\alpha-\beta)^i \\ 2r_2^2\gamma^r - r_0^2(\alpha-\beta)^r & r_0^2 r_2 (\alpha-\beta)^i \\ -2r_2\gamma^i + \frac{r_0^2}{r_2}(\alpha-\beta)^i + 4r_2\alpha^i & -(\alpha-\beta)^r r_0^2 \\ 2r_2\alpha^i & 2(\alpha-\beta)^r r_2^2 \\ 2r_2\gamma^i - \frac{r_0^2}{r_2}(\alpha-\beta)^i & -(\alpha-\beta)^r r_0^2 \\ -2r_0^2 r_2 (\alpha-\beta)^i & r_0^2 r_2 (\alpha-\beta)^i \\ 4r_0 r_2^2 (\alpha-\beta)^i & -2r_0 r_2^2 (\alpha-\beta)^i \\ -2r_0^2 r_2 (\alpha-\beta)^i & r_0^2 r_2 (\alpha-\beta)^i \\ 2(\alpha-\beta)^r r_0^2 & -(\alpha-\beta)^r r_0^2 \\ -4(\alpha-\beta)^r r_2^2 & 2(\alpha-\beta)^r r_2^2 \\ 2(\alpha-\beta)^r r_0^2 & -(\alpha-\beta)^r r_0^2 \end{pmatrix} \begin{pmatrix} \rho_{-2} \\ \rho_0 \\ \rho_2 \\ \phi_{-2} \\ \phi_0 \\ \phi_2 \end{pmatrix}.$$

One gets the following eigenvalues

$$\begin{aligned} \mu_{1,2} &= 0, \\ \mu_3 &= 2(s+t)^r, \\ \mu_4 &= 2(s-t)^r. \end{aligned}$$

Our choice of coordinates yields

$$\begin{aligned} \left(-\alpha + \frac{1}{4}\beta + \frac{3}{4}\gamma\right)^r &< 0, \\ \left(-\alpha + \frac{3}{4}\beta + \frac{1}{4}\gamma\right)^r &< 0, \\ 2\sqrt{\frac{3}{8}}(\gamma-\beta)^r &> 0, \end{aligned}$$

and therefore

$$\mu_4 < 0.$$

Finally we want to show

$$\mu_3 < 0.$$

By insertion on gets

$$\mu_3 = 2\left((\gamma-\beta)^r\left(\frac{1}{2}r_0^2 + 2\sqrt{\frac{3}{8}}r_0\sqrt{-\frac{\lambda}{2\alpha^r} - \frac{r_0^2}{2}}\right) - \frac{\lambda}{2\alpha^r}\left(-2\alpha + \frac{3}{2}\beta + \frac{1}{2}\gamma\right)^r\right).$$

The ansatz

$$r_0^2 = -t\frac{\lambda}{\alpha^r}, \quad t \in \left(0, \frac{1}{4}\right),$$

yields

$$\mu_3 = -2\frac{\lambda}{\alpha^r}\left((\gamma-\beta)^r\left(\frac{1}{2}t + 2\sqrt{\frac{3}{8}}\sqrt{\frac{t(1-t)}{2}}\right) + \frac{1}{2}\left(-2\alpha + \frac{3}{2}\beta + \frac{1}{2}\gamma\right)^r\right).$$

In the admissible region we have

$$\mu_3 < 0.$$

The eigenvalues of the second system are (cf. lemmata 2.2.21 and 2.2.22),

$$\begin{aligned} \mu_{1,2,3} &= 0, \\ \mu_4 &= -2\lambda < 0, \\ \mu_5 &= 2\frac{\lambda}{\alpha^r}(\alpha-\beta)^r < 0, \\ \mu_6 &= -2\left(r_0^2(\alpha-\beta)^r + 2r_2^2(\alpha-\gamma)^r\right). \end{aligned}$$

Therefore in the bifurcation point

$$\tilde{\pi}_1 = -\frac{\lambda(\alpha-\gamma)^r}{\alpha^r(\beta-\gamma)^r}$$

there are six trivial and four negative Floquet-exponents. In the entire system the solution has isotropy $(D_2, 1)$ in the bifurcation point, thus the group orbit is four dimensional. Therefore four trivial exponents are symmetry given. The sign of the Floquet-exponents of the periodic solution corresponds to the sign of the eigenvalues of the associated fixed point in the stratified space. Dealing with fixed point bifurcation in the stratified space the group orbit of the bifurcating solution inherits the stability of the bifurcation point. The double zero eigenvalue at the bifurcation point becomes one positive and one negative eigenvalue (cf. proposition 2.2.27). Therefore the bifurcating fixed point is hyperbolic. In the entire system the bifurcating fixed point has isotropy $(\mathbb{Z}_2, 1)$.
The following lemma is shown.

Lemma 2.2.28: *The bifurcating group orbit (cf. proposition 2.2.27) has the unstable dimension one.*

3 Topological methods

3.1 Homotopy and homology index

Definition 3.1.1: *$S \subset X = \mathbb{R}^{10}$ is called an isolated invariant set if:*
(a) S is invariant.
(b) It exists a compact set N with $S \subset \mathrm{Int}(N)$.
(c) S is the maximal invariant set in N.
Such a set N is called isolating neighbourhood.

Definition 3.1.2: *An index pair of an isolated invariant set S is a pair of compact sets $N_0 \subset N_1$ with*

(a) $\mathrm{Cl}(N_1 \backslash N_0)$ is an isolating neighbourhood of S and $S \subset \mathrm{Int}(N_1 \backslash N_0)$.

(b) N_0 is positively invariant in N_1, i.e. if $x \in N_0$ and $\phi_s x \in N_1$ for all $s \in [0,t]$ with $t \geq 0$ then $\phi_s x \in N_0$ for all $s \in [0,t]$.

(c) N_0 is the exit set of N_1, i.e. if for $x \in N_1$ there exists $t_1 > 0$ such that $\phi_{t_1} x \notin N_1$ then there exists $t_0 \in [0,t_1)$ such that $\phi_s x \in N_1$ for all $s \in [0,t_0)$ and $\phi_{t_0} x \in N_0$.

For a given isolated invariant set S und an associated index pair (N_1, N_0) the homotopy index (resp. Conley-index) $h(S)$ is the homotopy typ of the pointed space $(N_1/N_0, [N_0])$ that one gets by identifying N_0 to a distinguished point. We will here use the homology index that can be calculated more easily

$$\mathrm{CH}(S) \quad = \quad \mathrm{H}(N_1/N_0, [N_0]; \mathbb{Z}_2).$$

These definitions are independend of the special choice of the index pair (cf. [Co]). For a given isolated invariant set S and an associated isolating neighbourhood N there always exists an index pair (N_1, N_0) in N and therefore the homotopy resp. homology index is defined. Given two disjoint isolated invariant sets S and T

$$C(S,T) \quad = \quad \{x \in X \,|\, \omega(x) \subset T \text{ und } \alpha(x) \subset S\}$$

denotes the set of connecting orbits from S to T.
The occuring group orbits

$$M \quad = \quad \left(\mathrm{SO}(3) \times \mathrm{S}^1\right) x, \quad x \in X,$$

are compact manifolds

$$M \quad \cong \quad \frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Omega},$$

x having the isotropiy Ω. The dimension of M is

$$m \quad = \quad \mathrm{Dim}(M) \quad = \quad \mathrm{Dim}(\mathrm{SO}(3) \times \mathrm{S}^1) - \mathrm{Dim}(\Omega) \quad = \quad 4 - \mathrm{Dim}(\Omega).$$

The homology index of the group orbit M is

$$\mathrm{CH}_{n+k}(M) \quad = \quad \mathrm{H}_n(M, \mathbb{Z}_2)$$

where k denotes the unstable dimension of M (cf. [FiMi]). Thus the determination of the homology indices $\mathrm{CH}(M)$ is reduced to the calculation of the homology groups $\mathrm{H}_n(M, \mathbb{Z}_2)$. This will be done in the next chapter.

3.2 Calculation of the homology groups

First we want to calculate the homology groups of the manifolds of periodic solutions. This is done in the following proposition.

Proposition 3.2.1: *The homology groups of the manifolds of periodic solutions are*

	$H_n\left(\frac{SO(3)\times S^1}{\Omega}, \mathbb{Z}_2\right)$				
Ω	H_0	H_1	H_2	H_3	H_4
$(O(2),1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	0
$SO(2)^1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$SO(2)^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$(D_4, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
$(\mathbb{T}, \mathbb{Z}_3)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Here Ω denotes the isotropy of the respective manifold.

The proof is done in the following lemmata.

Lemma 3.2.2: $\frac{SO(3)\times S^1}{(O(2),1)} \approx \mathbb{R}R^2 \times S^1$ *and we get homology groups according to proposition 3.2.1.*

Proof: In this case the twist is trivial. Thus $\frac{SO(3)\times S^1}{(O(2),1)}$ is homeomorphic to $\frac{SO(3)}{O(2)} \times S^1$. The manifold $\frac{SO(3)}{O(2)}$ exists and is homeomorphic to $SO(3)x$, x having isotropy $O(2)$. The group $O(2) \subset SO(3)$ is generated by a planar rotation and a reflection in this plane and in the direction orthogonal to this plane. Thus $SO(3)x \approx \mathbb{R}P^2$. The homology groups of $\mathbb{R}P^2 \times S^1$ are calculated using the universal coefficient theorem and the Künneth-formula (cf. [StZi]):

	H_0	H_1	H_2	H_3
$H(\mathbb{R}P^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0
$H(\mathbb{R}P^2, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$H(S^1)$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$H(S^1, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
$H(\mathbb{R}P^2 \times S^1, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$

⋈

Lemma 3.2.3: $\frac{SO(3)\times S^1}{SO(2)^1} \approx SO(3) \approx \mathbb{R}P^3$ *and we get homology groups according to proposition 3.2.1.*

Proof: Let

$$H = SO(2)^1 = \{(\psi, \psi); \psi \in SO(2)\}.$$

Then the map

$$\begin{array}{rccl} h & : & \frac{SO(3)\times S^1}{SO(2)^1} & \to & SO(3) \\ & & (g,\theta)H & \to & g\theta^{-1} \end{array}$$

is a homeomorphism. The homology groups of $SO(3) \approx \mathbb{R}P^3$ are:

	H_0	H_1	H_2	H_3
$H(\mathbb{R}P^3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$H(\mathbb{R}P^3, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

⋈

Lemma 3.2.4: $\frac{SO(3)\times S^1}{SO(2)^2} \approx \frac{SO(3)}{\mathbb{Z}_2}$ *and we get homology groups according to proposition 3.2.1..*

Proof: Let

$$H = \mathrm{SO}(2)^2 = \{(\psi, \psi^2); \psi \in \mathrm{SO}(2)\}.$$

For $\mathbb{Z}_2 = \{1, e^{i\pi}\}$ the map

$$\begin{array}{rccc} h & : & \frac{\mathrm{SO}(3)\times \mathrm{S}^1}{\mathrm{SO}(2)^2} & \to & \frac{\mathrm{SO}(3)}{\mathbb{Z}_2} \\ & & (g,\theta)H & \to & g\theta^{-\frac{1}{2}}\mathbb{Z}_2 \end{array}$$

is a homeomorphism. Here $e^{i\pi} \in \mathrm{SO}(2)$ denotes the rotation of 180 degrees around an axis.
The group SO(3) and therefore the three dimensional differentiable manifold $\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}$ is pathwise connected. Thus

$$\mathrm{H}_0\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}\right) = \mathbb{Z}.$$

Using the universal coefficient theorem (cf. [StZi]) one gets

$$\mathrm{H}_0\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right) = \mathbb{Z}_2.$$

The Poincaré-duality (vgl. [StZi]) yields

$$\mathrm{H}_0\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right) = \mathrm{H}_3\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right) = \mathbb{Z}_2.$$

The quotient spaces $\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}$ and $\frac{\mathrm{S}^3}{\mathbb{Z}_4}$ are homeomorphic (cf. remark 3.2.8). This follows from the fact that S^3 is a double covering of SO(3). Here the $\mathbb{Z}_4$ is as a complex multiplication properly discontinuous on the simply connected space S^3, the unit quaternions. So we get the fundamental group (cf. [StZi])

$$\pi_1\left(\frac{\mathrm{S}^3}{\mathbb{Z}_4}\right) = \mathbb{Z}_4.$$

Since $\mathbb{Z}_4$ is abelian we have (vgl. [Sch])

$$\mathrm{H}_1\left(\frac{\mathrm{S}^3}{\mathbb{Z}_4}\right) = \mathbb{Z}_4.$$

Using the universal coefficient theorem one gets

$$\mathrm{H}_1\left(\frac{\mathrm{S}^3}{\mathbb{Z}_4}, \mathbb{Z}_2\right) = \mathbb{Z}_2.$$

The Poincaré-duality again yields

$$\mathrm{H}_1\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right) = \mathrm{H}_2\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right) = \mathbb{Z}_2.$$

Thus we have shown that

	H_0	H_1	H_2	H_3
$\mathrm{H}\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

.

Now we want to calculate the homology groups of the compact, simply connected manifolds $\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma}$ with $\Sigma = (H, \Theta)$ a discrete twisted subgroup of $\mathrm{SO}(3) \times \mathrm{S}^1$. Thus the manifolds are four dimensional and as above

$$\mathrm{H}_0(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma}, \mathbb{Z}_2) \cong \mathrm{H}_4(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma}, \mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{3.2.27}$$

Define the following map

$$\begin{array}{rccc} F & : & \mathrm{S}^3 \times \mathbb{R} & \to & \mathrm{SO}(3) \times \mathrm{S}^1 \\ & & (x,t) & \to & (f_x, q(t)). \end{array}$$

Here

$$\begin{array}{rcccc} f & : & \mathrm{S}^3 & \to & \mathrm{SO}(3) \\ & & x & \to & f_x \end{array}$$

is the covering map defined in remark 3.2.8 and

$$\begin{array}{rcccc} q & : & \mathbb{R} & \to & \mathrm{S}^1 \\ & & t & \to & e^{\mathrm{i}2\pi t} \end{array}$$

is the universal covering of S^1 by $\mathbb{R}$. Let H be a subgroup of $\mathrm{SO}(3)$ and define

$$\overline{H} = f^{-1}(H).$$

Let

$$\begin{array}{rcccc} \overline{\Theta} & : & \overline{H} & \to & \mathrm{S}^1 \\ & & \overline{h} & \to & \Theta(f_{\overline{h}}) \end{array}$$

and the group

$$\overline{\Sigma} = \Big\{ (\,\overline{h}, \overline{\Theta}(\overline{h}) + n\,) \,|\, \overline{h} \in \overline{H}, n \in \mathbb{Z} \Big\}.$$

Thus

$$\overline{\Sigma} = F^{-1}(\Sigma).$$

Lemma 3.2.5:

$$\mathrm{H}_1(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma}) \cong \Big\{ \big([\overline{h}], \overline{\Theta}(\overline{h}) + n\big) \mid \overline{h} \in \overline{H}_{ab}, n \in \mathbb{Z} \Big\}.$$

Proof: The map F is by construction a continuous surjective and open homomorphism, $\mathrm{SO}(3) \times \mathrm{S}^1$ is compact. Thus

$$B = \frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma} \approx \frac{\mathrm{S}^3 \times \mathbb{R}}{\overline{\Sigma}}.$$

Since $\overline{\Sigma}$ is as a discrete group properly discontinuous on the simply connected space $\mathrm{S}^3 \times \mathbb{R}$

$$\pi_1(B) \cong \pi_1(\frac{\mathrm{S}^3 \times \mathbb{R}}{\overline{\Sigma}}) \cong \overline{\Sigma}.$$

Hence $\mathrm{H}_1(B)$ is the abelianisation of $\overline{\Sigma}$.
For a group G let $[G,G]$ denote the commutator of G and G_{ab} the abelianisation of G. Since

$$(\overline{h}, \overline{\Theta}(\overline{h}) + n)(\overline{k}, \overline{\Theta}(\overline{k}) + m)(\overline{h}^{-1}, -\overline{\Theta}(\overline{h}) - n)(\overline{k}^{-1}, -\overline{\Theta}(\overline{k}) - m) = (\overline{h}\,\overline{k}\,\overline{h}^{-1}\overline{k}^{-1}, 0),$$

we have

$$\frac{\overline{\Sigma}}{\overline{[\Sigma, \Sigma]}} = \frac{\overline{\Sigma}}{[\overline{H}, \overline{H}] \times \{0\}} = \Big\{ \big([\overline{h}], \overline{\Theta}(\overline{h}) + n\big) \mid \overline{h} \in \overline{H}_{ab}, n \in \mathbb{Z} \Big\}.$$

⋈

Now we look at the group

$$\Sigma_1 = (\mathrm{D}_4, \Theta(\mathrm{D}_4)).$$

The group

$$\mathrm{D}_4 = \{x, y \,|\, x^2 = y^4 = (xy)^2 = 1\}$$

is generated by a rotation x of degree π around the x-axis

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ with } \Theta(x) = 1 \in \mathrm{S}^1$$

and a rotation y of degree $\frac{\pi}{2}$ around the z-axis

$$y = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \Theta(y) = e^{\mathrm{i}\pi} \in \mathrm{S}^1.$$

The preimages $\overline{x}_{1,2}$ and $\overline{y}_{1,2}$ of x and y under the covering map f are (cf. remark 3.2.8)

$$\overline{x}_{1,2} = \pm \mathrm{i} \text{ and } \overline{y}_{1,2} = \pm(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{k}).$$

Thus the preimage of D_4 is the group

$$\overline{\mathrm{D}_4} = \{\overline{x}, \overline{y} \mid \overline{x}^2 = \overline{y}^{\,4} = (\overline{x}\,\overline{y})^2;\ \overline{x}^{\,4} = 1\}$$

with

$$\overline{x} = \mathrm{i} \text{ and } \overline{y} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{k}.$$

Let $[\overline{x}]$ and $[\overline{y}]$ be the factors of $\overline{x}$ and $\overline{y}$ in the abelianisation $\widetilde{\mathrm{D}_4}$ of $\overline{\mathrm{D}_4}$. Then we have

$$\begin{aligned} [\overline{x}]^4 = 1 \quad &\Rightarrow \quad [\overline{y}]^4 = [\overline{y}]^4[\overline{x}]^4 = 1 \\ \Rightarrow \quad [\overline{x}]^2 = 1 \quad &\Rightarrow \quad [\overline{y}]^2 = [\overline{y}]^2[\overline{x}]^2 = 1 \end{aligned}$$

and thus

$$\widetilde{\mathrm{D}_4} = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

According to lemma 3.2.5 we get

$$\begin{aligned} \mathrm{H}_1(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_1}) &\cong \left\{ \left([\overline{h}], \overline{\Theta}(\overline{h}) + n\right) \mid \overline{h} \in \widetilde{\mathrm{D}_4}, n \in \mathbb{Z} \right\} = \\ &= \left\{ \left(1, n\right), \left([\overline{x}], n\right), \left([\overline{y}], n + \tfrac{1}{2}\right), \left([\overline{x}][\overline{y}], n + \tfrac{1}{2}\right) \Big| \ n \in \mathbb{Z} \right\}. \end{aligned}$$

The map

$$\begin{array}{rccc} T_1 & : & \mathrm{H}_1(\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_1}) & \rightarrow & \mathbb{Z} \oplus \mathbb{Z}_2 \\ & & (1, n) & \rightarrow & (2n, 0) \\ & & ([\overline{x}], n) & \rightarrow & (2n, 1) \\ & & ([\overline{y}], n) & \rightarrow & (2n + 1, 0) \\ & & ([\overline{x}][\overline{y}], n) & \rightarrow & (2n + 1, 1) \end{array}$$

is a group homomorphism and thus the universal coefficient theorem and the Poincaré-duality yield

$$\mathrm{H}_1(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_1}, \mathbb{Z}_2) \cong \mathrm{H}_3(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_1}, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{3.2.28}$$

The case

$$\Sigma_2 = (\mathbb{T}, \Theta(\mathbb{T}))$$

is similar. The group

$$\mathbb{T} = \{x, y \mid x^2 = y^3 = (xy)^3 = 1\}$$

is generated by a rotation x of degree π around the z-axis

$$x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \Theta(x) = 1 \in \mathrm{S}^1$$

and a rotation y of degree $\frac{2\pi}{3}$ around the diagonal $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ with } \Theta(y) = e^{\mathrm{i}\frac{2\pi}{3}} \in \mathrm{S}^1.$$

The preimages $\overline{x}_{1,2}$ and $\overline{y}_{1,2}$ of x and y under the covering map f are (cf. remark 3.2.8)

$$\overline{x}_{1,2} = \pm \mathrm{k} \text{ and } \overline{y}_{1,2} = \pm\frac{1}{2}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k}).$$

Thus the preimage of $\mathbb{T}$ is the group

$$\overline{\mathbb{T}} = \{\overline{x}, \overline{y} \mid \overline{x}^2 = \overline{y}^3;\ (\overline{x}\,\overline{y})^3 = \overline{x}^4 = 1\}$$

with

$$\overline{x} = \mathrm{k} \text{ and } \overline{y} = \frac{1}{2}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k}).$$

Let $[\overline{x}]$ and $[\overline{y}]$ denote the factors of $\overline{x}$ and $\overline{y}$ in the abelianisation $\widetilde{\mathbb{T}}$ of $\overline{\mathbb{T}}$. Then we have

$$\begin{aligned} [\overline{x}]^2 = [\overline{y}]^3 \quad &\Rightarrow \quad [\overline{x}]^3[\overline{y}]^3 = [\overline{x}]^5 = 1 \\ [\overline{x}]^4 = 1 \quad &\Rightarrow \quad [\overline{x}] = 1 \\ &\Rightarrow \quad [\overline{y}]^3 = 1 \end{aligned}$$

and thus

$$\widetilde{\mathbb{T}} = \mathbb{Z}_3.$$

According to lemma 3.2.8 we get

$$\begin{aligned} \mathrm{H}_1(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_2}) &\cong \left\{ \left([\overline{h}], \overline{\Theta}(\overline{h}) + n\right) \mid \overline{h} \in \widetilde{\mathbb{T}}, n \in \mathbb{Z} \right\} = \\ &= \left\{ \left(1, n\right), \left([\overline{y}], n + \tfrac{1}{3}\right), \left([\overline{y}]^2, n + \tfrac{2}{3}\right) \middle|\ n \in \mathbb{Z} \right\}. \end{aligned}$$

The map

$$\begin{array}{rccc} T_2 & : & \mathrm{H}_1(\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_2}) & \to & \mathbb{Z} \\ & & (1, n) & \to & 3n \\ & & ([\overline{y}], n + \frac{1}{3}) & \to & 3n + 1 \\ & & ([\overline{y}]^2, n + \frac{2}{3}) & \to & 3n + 2 \end{array}$$

is a group homomorphism and thus the universal coefficient theorem and the Poincaré-duality yield

$$\mathrm{H}_1(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_1}, \mathbb{Z}_2) \cong \mathrm{H}_3(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_1}, \mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{3.2.29}$$

The compact manifolds

$$B_{1,2} = \frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_{1,2}}$$

are finite CW-spaces (cf. [Mil]). As discrete finite groups $\Sigma_{1,2}$ are properly discontinuous on $\mathrm{SO}(3) \times \mathrm{S}^1$ and therefore the projection

$$\pi \quad : \quad \mathrm{SO}(3) \times \mathrm{S}^1 \quad \to \quad \tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_{1,2}}$$

is a covering, where the number of sheets equals the cardinal number $\sigma_{1,2}$ of $\Sigma_{1,2}$ (cf. [StZi]). According to [StZi] one can represent $\mathrm{SO}(3) \times \mathrm{S}^1$ as a CW-space in such a way that the projection π maps the cells of $\mathrm{SO}(3) \times \mathrm{S}^1$ to cells of $B_{1,2}$ in a homeomorphic way. Dealing with CW-spaces B we have for the Euler-characteristic

$$\chi(B) = \sum_{i=0}^{\infty} (-1)^i \alpha_i$$

where α_i denotes the number of i-cells in B. This yields

$$\chi(\mathrm{SO}(3) \times \mathrm{S}^1) = \sigma_{1,2}\chi(B_{1,2}).$$

Since

$$\chi(\mathrm{SO}(3) \times \mathrm{S}^1) = 0 \text{ and } \sigma_{1,2} \neq 0$$

we have

$$\chi(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_{1,2}}) = 0.$$

In our case the Euler-characteristic is calculated in the following way

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}_i(X, \mathbb{Z}_2).$$

Thus

$$0 = \chi(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_{1,2}}) = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}_i(\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{\Sigma_{1,2}}, \mathbb{Z}_2). \tag{3.2.30}$$

Therefore equations (3.2.27), (3.2.28), (3.2.29) and (3.2.30) yield

$$\begin{aligned} \dim \mathrm{H}_2(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_1}) &= 2 \\ \dim \mathrm{H}_2(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_2}) &= 0. \end{aligned}$$

This means

$$\begin{aligned} \mathrm{H}_2(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_1}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathrm{H}_2(\tfrac{\mathrm{SO}(3)\times \mathrm{S}^1}{\Sigma_2}) &\cong 0. \end{aligned}$$

So we get the following lemma.

Lemma 3.2.6: *The homology groups of* $\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{(\mathrm{D}_4, \Theta(\mathrm{D}_4))}$ *and* $\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{(\mathbb{T}, \Theta(\mathbb{T}))}$ *are as listed in proposition 3.2.1.*

Now we want to calculate the homology groups of the manifold of quasiperiodic solutions. The existence of such solutions has been proven in the previous chapter.

Proposition 3.2.7: *The homology groups of the quotient space*

$$\frac{\mathrm{SO}(3) \times \mathrm{S}^1}{(\mathbb{Z}_2, 1)}$$

are

H_0	H_1	H_2	H_3	H_4
$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$

.

Proof: Here the twist is trivial. Thus $\frac{\mathrm{SO}(3)\times \mathrm{S}^1}{(\mathbb{Z}_2,1)}$ is homeomorphic to $\frac{\mathrm{SO}(3)}{\mathbb{Z}_2} \times \mathrm{S}^1$. Using the results of lemma 3.2.4 we have

	H_0	H_1	H_2	H_3
$\mathrm{H}\left(\frac{\mathrm{SO}(3)}{\mathbb{Z}_2}, \mathbb{Z}_2\right)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathrm{H}(\mathrm{S}^1, \mathbb{Z}_2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0

.

Applying the Künneth-formula (cf. [StZi]) one gets the result. ⋈

In order to calculate the homology groups we have the following useful remark (cf. [StZi] and [DuetAl]).

Remark 3.2.8: *Let*

$$\mathbb{H} = \{x = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \mid a, b, c, d \in \mathbb{R}\}$$

denote the quaternions,

$$\mathbb{H}_1 = \{x \in \mathbb{H} \mid |x| = 1\} = \mathrm{S}^3$$

the unit quaternions and

$$\mathbb{H}_0 = \{x \in \mathbb{H} \mid \mathrm{Re}(x) = 0\} = \mathbb{R}^3$$

the quaternions with zero real part.
Then the map

$$\begin{array}{rccc} f & : & \mathrm{S}^3 & \to & \mathrm{SO}(3) \\ & & x & \to & f_x \end{array}$$

with

$$f_x y = xyx^{-1},\ y \in \mathbb{R}^3,$$

is a double covering.

3.3 Morse-decompositions and connection graphs

Let S be an isolated invariant set and $(P,>)$ a finite set with a partial order such that for $i,j,k \in P$

$$i \not> i$$

and

$$i > j,\, j > k \quad \Rightarrow \quad i > k.$$

Two elementes $i,j \in P$ with $i > j$ are called neighbouring under $>$ if there is no $k \in P$ with $i > k > j$. Given two partial orders $>_0$ and $>_1$ $>_1$ is called an extension of $>_0$ if

$$i >_0 j \quad \Rightarrow \quad i >_1 j.$$

Definition 3.3.1: *A Morse-decomposition of S*

$$M(S) \quad = \quad \{M(i) \,|\, i \in (P,>)\}$$

is a family of mutual disjoint isolated invariant subsets of S indexed by P. For $x \in S$ either

$$x \in M(i), \quad i \in P,$$

or

$$\exists\, i,j \in P \quad \text{with } i > j \text{ and } x \in C\big(M(i), M(j)\big).$$

The sets $M(i)$ are called Morse-sets. This means that $x \in S$ either lies in one of the Morse-sets or on a connecting orbit between two different Morse-sets. Identifying the Morse-sets to (different) points one gets a gradient like flow whose critical points are exactly the Morse-sets. For a family $\{M(i)\}$ of Morse-sets of S the flow defined order $>_F$ is defined in the following way:

$$\begin{aligned} i >_F j \quad \Leftrightarrow \quad & \exists \text{ sequence } (p_k) \subset P,\ 1 \le k \le K, \\ & \text{with } p_1 = i,\ p_K = j \\ & \text{and } C\big(M(p_k), M(p_{(k+1)})\big) \neq \emptyset \quad \forall\ k = 1, \ldots, K-1. \end{aligned}$$

The knowledge of the order $>_F$ is equivalent to the knowledge of the connecting orbits between neighbouring Morse-sets.
Now let

$$M(S) \quad = \quad \{M(i) \,|\, i \in (P,>_F)\}$$

be a Morse-decomposition of S and

$$\{e_k^n(i) \,|\, n = 1, \ldots, N_k(i)\}$$

be basis of $\mathrm{CH_k}\big(M(i)\big)$.

Definition 3.3.2: *A connection graph (C-graph) is a directed graph with the following properties:*

1. *$\{e_k^n(i) \,|\, i \in P,\ n = 1, \ldots, N_k(i),\ k = 0, 1, 2, \ldots\}$ is the set of vertices.*
2. *Every vertex has at most one edge.*
3. *Every edge goes from $e_k^n(i)$ to $e_{(k-1)}^m(j)$ with $i \neq j$.*
4. *The set of vertices without edges is a basis of $\mathrm{CH}(S)$.*
5. *If $M(i)$ is an attractor then there are no edges of the form $e_k^n(i) \to e_{(k-1)}^m(j)$.*
6. *If $M(j)$ is a repellor then there are no edges of the form $e_k^n(i) \to e_{(k-1)}^m(j)$.*
7. *There are no R-cycles (see below).*

A vertex without edge is called free.

Definition 3.3.3: *A R(ecurrence)-cycle is a family of edges*

$$\left\{ e^{n_q}_{k_q}(i_q) \to e^{m_q}_{(k_q-1)}(j_q) \,|\, q = 1, \ldots Q \right\}$$

with $j_q = i_{(q+1)}$ *and* $j_Q = i_1$.

Since R-cycles do not occur a C-graph defines a order $>_G$ on $\{M(i)\}$ with

$$\begin{aligned} i >_G j \quad \Leftrightarrow \quad & \exists \text{ sequence } \{q_\alpha \,|\, \alpha = 1, \ldots, L\} \subset P \\ & \text{and edges } e^{n_{q_\alpha}}_{k_{q_\alpha}}(q_\alpha) \to e^{m_{q_{(\alpha+1)}}}_{(k_{q_\alpha}-1)}(q_{(\alpha+1)}) \\ & \text{with } \alpha = 1, \ldots, L-1,\ q_1 = i \text{ and } q_L = j. \end{aligned}$$

We have the following proposition.

Proposition 3.3.4: *For a given Morse-decomposition*

$$M(S) \quad = \quad \{M(i) \,|\, i \in (P, >_F)\}$$

the set of possible C-graphs $\{G_k\}$ *is not empty. Let* $>_k$ *be the order defined by the C-graph* G_k. *Then there exists* k_0 *such that* $>_F$ *is an extension of* $>_{k_0}$.

Proof: [FiMi] ⋈

Corollary 3.3.5: *Let* $\{M(i)\}$ *be a family of isolated invariant sets with* $M(i) \subset S$ *for all* i. *If there is no C-graph then* $\{M(i)\}$ *is no Morse-decomposition of* S.

3.4 Topological results

In [IoRo] it has been shown that there exists a region in parameter space where the group orbits of the solutions of isotropy $(\mathrm{O}(2), 1)$ resp. $\mathrm{SO}(2)^2$ are stable simultaneously. The unstable dimensions of the group orbits of the periodic orbits in this region in parameter space are given in the next table (cf. [IoRo]).

$(\mathrm{O}(2), 1)$	0
$\mathrm{SO}(2)^1$	4
$\mathrm{SO}(2)^2$	0
$(\mathrm{D}_4, \mathbb{Z}_2)$	2
$(\mathbb{T}, \mathbb{Z}_3)$	5

Firstly we will prove that in this case the group orbits of periodic solutions together with the trivial solution are no Morse-decomposition. In the region in parameter space in question the existence of another isolated invariant set (the group orbit of quasiperiodic solutions) has been estabished by proposition 2.2.27. This set contributes to the Conley-index. Actually the results of this topological investigations were the starting point of analytical efforts to find other solutions. Secondly we want to show that it is possible to fill in a C-graph together with this "new" solutions and thirdly we will draw conclusions from the C-graphs for the existence of heteroclinic orbits between different group orbits of solutions.

So we will start in the situation where the existence of quasiperiodic solutions was unknown.

All solutions bifurcate supercritically. In the following we want to assume that:

A1: It exists a compact neighbourhood $N \subset \mathbb{R}^{10}$ in which for all λ near zero besides the trivial solution and the known group orbits of periodic solutions and possible connecting orbits between them there are no other bounded solutions.
For λ near zero

$$S_\lambda \quad \subset \quad \mathrm{Int}(N)$$

denotes the maximal invariant set in N. If λ is small enough then all bifurcating solutions lie in $\mathrm{Int}(N)$. Thus N is the isolating neighbourhood associated to S_λ.

A2: For all λ near zero the trivial solution and the periodic solutions establish a Morse-decomposition of S_λ. In the bifurcation point $\lambda = 0$ S_0 consists of the trivial solution only and has a homology index. Because of the continuation property of the homotopy and the homology index this index coincides with the homology indices of S_+ resp. S_-. Here $S_\pm$ denotes the maximal invariant set S_λ for λ slightly bigger resp. smaller then zero. S_- consists of the stable trivial solution only. Thus

$$\mathrm{CH}(S_-) \quad = \quad \begin{cases} \mathbb{Z}_2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If it is now possible to fill in C-graphs for the Morse-decomposition of S_+ according to the rules of the definition 3.3.2 in such a way that there remains exactly one free vertex on level zero then the flow defined order $>_F$ is an extension of the graph order (cf. proposition 3.3.4) and it is possible to draw conclusions on connecting orbits between different Morse-sets. If this not possible our assumptions are not valid. Other invariant objects have to exist or there is recurrent motion between different group orbits of solutions.
We have the following vertices of the C-graph

$\mathrm{O}(3)$								10
								$\downarrow$
$\mathbb{T}$	5	6					8	9
							$\downarrow$	
$\mathrm{SO}(2)^1$	4	5				6	7	
						$\downarrow$		
D_4	2	3	3	4	4	5	5	6
$\mathrm{SO}(2)^2$	0	1	2	3				
$\mathrm{O}(2)$	0	1	1	2	2	3		

Here $\mathrm{O}(3) \times \mathrm{S}^1$ denotes the trivial solution $x \equiv 0$, which is unstable for $\lambda > 0$. As the homotopy index we get a pointed 10-sphere and therefore as the homology index a $\mathbb{Z}_2$ on the tenth level. In the rows the bases of the homology indices of the Morse-sets are listed. An entry n denotes a $\mathbb{Z}_2$ on the n-th homology level. Some necessary edges are already registered in the C-graph.
In order to get a valid C-graph we have to fill in the graph in such a way that exactly one free vertex on level zero remains. Since we have two vertices on level zero that cannot be connected with vertices on level one (all vertices on level one belong to attractors and therefore cannot connect to any other vertices by rule five in definition 3.3.2) this is impossible. Therefore we get the following proposition.

Proposition 3.4.1: *It is not possible to construct a C-graph in our case. Therefore the group orbits of periodic solutions together with the trivial solution are no Morse-decomposition. Other bifurcation phenomena (other solutions or recurrent motion) have to occur.*

Now we want to take the additional quasiperiodic solutions (cf. proposition 2.2.27) into account. Thus we have to slightly modify our assumptions.

A1A: It exists a compact neighbourhood $N \subset \mathbb{R}^{10}$ in which for all λ near zero besides the trivial solution, the known group orbits of periodic and quasiperiodic solutions and possible connecting orbits between them there are no other bounded solutions.

A2A: For all λ near zero the trivial solution and the periodic resp. quasiperiodic solutions establish a Morse-decomposition of S_λ.
If the "new" quasiperiodic solutions complete the C-graph a possible explanation of the phenomenon found in proposition 3.4.1 is given.
We have the following vertices of the C-graph

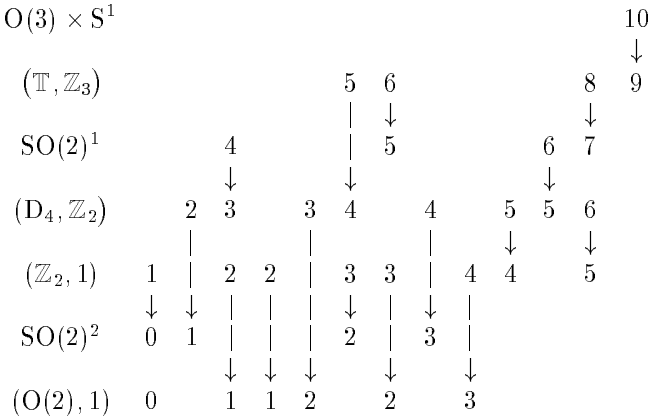
$\mathrm{O}(3) \times \mathrm{S}^1$	10							
$(\mathbb{T}, \mathbb{Z}_3)$	5	6	8	9				
$\mathrm{SO}(2)^1$	4	5	6	7				
$(\mathrm{D}_4, \mathbb{Z}_2)$	2	3	3	4	4	5	5	6
$(\mathbb{Z}_2, 1)$	1	2	2	3	3	4	4	5
$\mathrm{SO}(2)^2$	0	1	2	3				
$(\mathrm{O}(2), 1)$	0	1	1	2	2	3.		

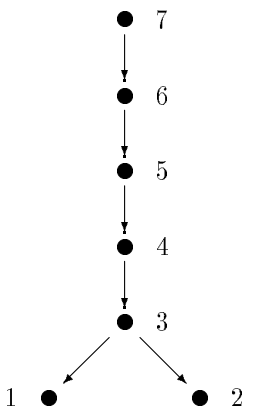
It turns out that there are many different possibilities to fill in C-graphs according to the rules of definition 3.3.2. Three examples are listed. With the assignment

$O(3) \times S^1$	7
$(\mathbb{T}, \mathbb{Z}_3)$	6
$SO(2)^1$	5
$(D_4, \mathbb{Z}_2)$	4
$(\mathbb{Z}_2, 1)$	3
$SO(2)^2$	2
$(O(2), 1)$	1

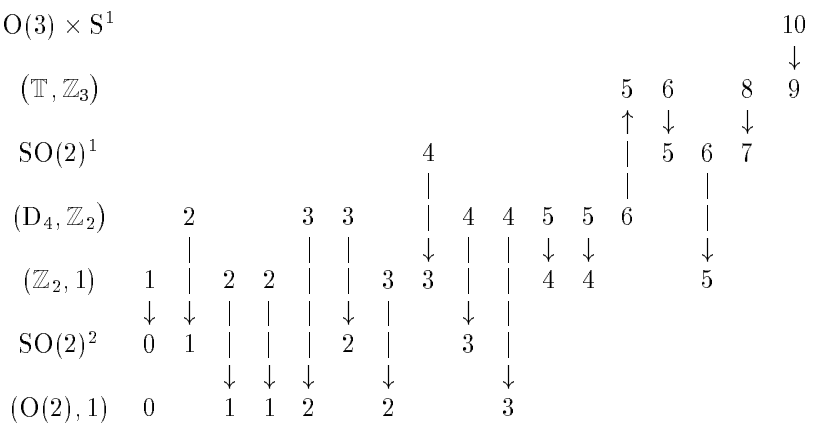
we list different C-graphs and the corresponding graph defined orders $>_G$.

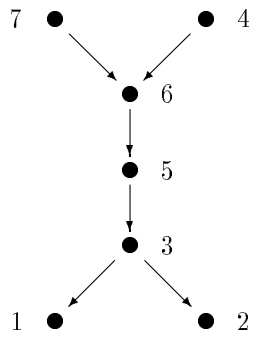
Example 1:



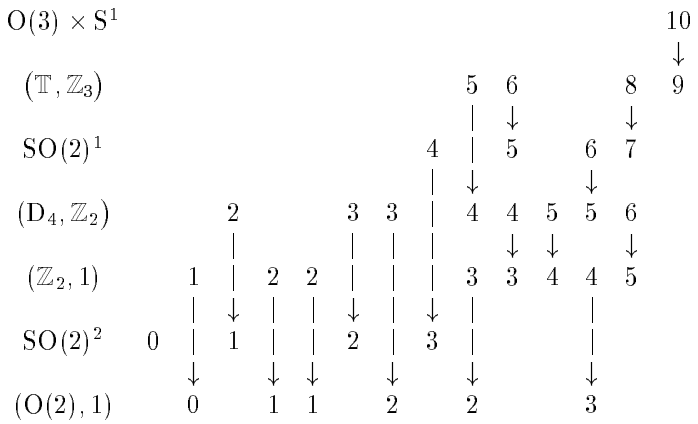


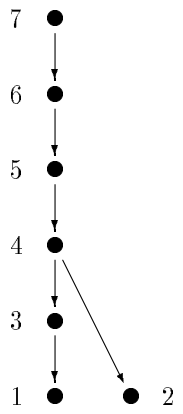
Example 2:



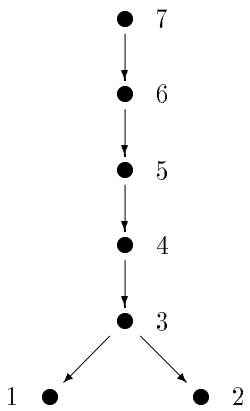


Example 3:





Studying all possibilities of C-graphs one sees that there are eight different types of graph defined orders.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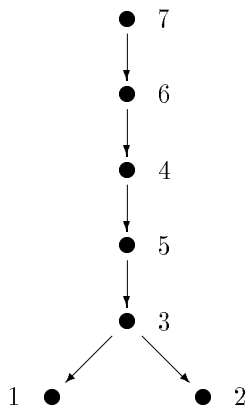
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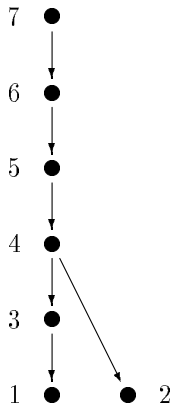
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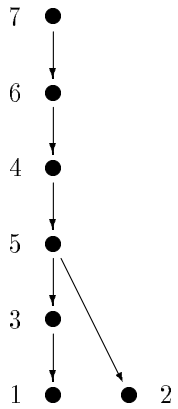
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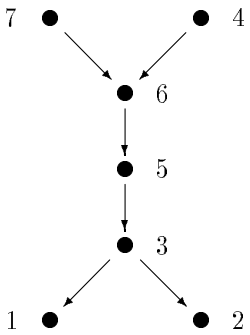
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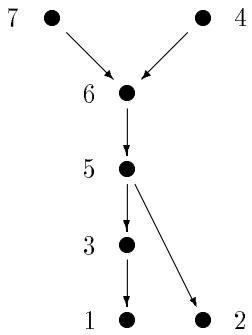
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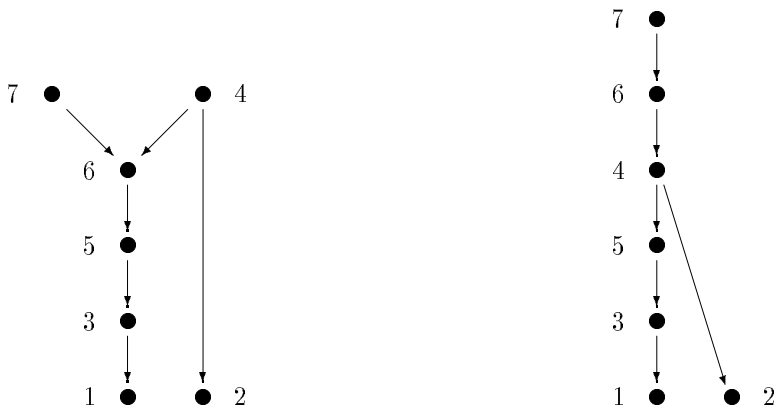

For all this orders we have

$$\{7,6,5,4\} > 3 > 1.$$

Every extension of the graph defined orders und therefore the unknown flow defined order preserves this relations. Since the solution of isotropy $\mathrm{SO}(2)^2$, here denoted by (2), is an attactor the solutions of isotropy types $(\mathbb{Z}_2, 1)$, i.e. (3), resp. $(\mathrm{O}(2), 1)$, i.e. (1), are neighbouring in the flow defined order too. Therefore it exists a connecting orbit. So we have shown the following propositions.

Proposition 3.4.2: *Under the assumptions (A1A, A2A) it is possible to build C-graphs.*

Remark 3.4.3: *The quasiperiodic solutions of isotropy* $(\mathbb{Z}_2, 1)$ *are a possible explanation of the phenomenon found in proposition 3.4.1.*

Proposition 3.4.4: *Under the assumptions (A1A, A2A) it exists a connecting orbit from the group orbit of quasiperiodic orbits of isotropy* $(\mathbb{Z}_2, 1)$ *to the group orbit of periodic orbits of isotropy* $(\mathrm{O}(2), 1)$.

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