Master Thesis

Heteroclinic connections in delay equations

Freie Universität Berlin

Alejandro López Nieto

July 2017

Abstract

Global attractors of scalar delay equation with monotone delay feedback have already been object of an intensive study. In this thesis we are interested in approaching the problem from an unconventional point of view in delay equations through the use of techniques borrowed from the field of Sturm attractors on S^1, a family of structures which arise in an apparently completely unrelated partial differential equation setting.

By taking advantage of features shared by both systems and particularities of our specific framework, we managed to prove transverse intersection of invariant manifolds of hyperbolic critical elements. This is a sufficient condition to endow the global attractor with a robust structure.

Contents

Acknowledgements

I must thank Professor Fiedler for introducing me to Dynamical Systems and also for taking care together with Isabelle Schneider of supervising this thesis. I would also like to thank everyone within the group of nonlinear dynamics for being always eager to share their knowledge, for their very enlightening comments and for providing a fantastic work environment.

This work was partially supported by SFB 910 "Control of Self-Organizing Nonlinear Systems: Theoretical Methods and Concepts of Application", project A4: "Spatio-temporal patterns: observation, control, and design" of the Deutsche Forschungsgemeinschaft.

Chapter 1

Introduction

1.1 Motivation of the work

The existence of a *discrete Lyapunov function* for systems of differential delay equations of the form:

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-1)), \\ vf(0,v) > 0 \text{ whenever } v \neq 0 \text{ and } f(0,0) = 0, \end{cases} \tag{1.1}$$

was first observed by Mallet-Paret for C^∞ nonlinearities in [13]. Moreover, he also proved the existence of a Morse decomposition of the global attractor defined precisely by this Lyapunov function together with existence of periodic orbits and even presented some results on connectivity [4].

The phase space for solutions of the class (1.1) is classically considered to be $C^0[-1,0]$. Within this set the previously mentioned Lyapunov function is, roughly speaking, something as simple as roughly speaking counting the number sign changes within the open interval $(-1,0)$, it is therefore just natural to call it *zero number* (see Section (2.3)). The reason to also refer it by the name Lyapunov function is the fact that if one considers a forward orbit of (1.1) defined by the solution semigroup $T(t)\phi$ for $t \geq 0$, $\phi \in C^0[-1,0]$, then it turns out that the zero number $z(T(t)\phi)$ is a non-increasing function in time (see Theorem (2.3.1)). In particular this very convenient feature of this type of systems allowed Mallet-Paret and Sell to formulate an analogous theorem to that of Poincaré-Bendixson in two dimensional ODE [16] which was later on used in [11] to describe the global attractor for a special class of nonlinearities.

As we will see in section (2.4) the zero number is very closely related to the unstable dimension of equilibria and periodic solutions of the system, which we will refer in the future as *critical elements* of the system since by the Poincaré-Bendixson theorem they will be our only candidates to α- and ω-limit sets of the system.

Very surprisingly the class of equations (1.1) shares its zero number property with solutions of the parabolic PDE on the circle:

$$w_t = w_{xx} + b(t,x)w_x + c(t,x)w \text{ for } t > 0,\ x \in \mathbb{R}/2\pi, \tag{1.2}$$

for $b(t,x)$ and $c(t,x)$ regular enough. Similarly to Section (2.3) if we count the number of sign changes of a x-profile $w(t,\cdot)$, it is also a nonincreasing function as time grows. Moreover, one can formulate a Poincaré-Bendixson theorem too [4].

The results from [5] and [6], show which global attractors (see Definition (2.1.1)) arise assuming hyperbolicity of equilibria ($\mathcal{E}$) and periodic orbits ($\mathcal{P}$). They can be seen as the set:

$$\mathcal{A} = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}, \tag{1.3}$$

where $\mathcal{H}$ are heteroclinics connecting the elements in $\mathcal{E}$ and $\mathcal{P}$.

Although historically both families of equations have received very different treatments, the scope of this thesis is to bring both settings a little closer. To do so we intend to study the behaviour of solutions approaching critical elements. By applying techniques already well-known in the field of PDE, we will prove the transverse intersection of the global unstable manifold and the local stable manifold of any two hyperbolic critical elements. This property, which plays a crucial role in the study of (1.2), will endow our system of a robust structure as indicated by [8].

1.2 Main results and overview

The main result of the thesis is the following theorem:

Theorem A. *Consider the scalar delay differential equation:*

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-1)), \text{ for } t > 0, \\ f_v(u,v) \lessgtr 0. \end{cases} \tag{1.4}$$

Where $f \in C^2(\mathbb{R}^2, \mathbb{R})$ generates a compact dissipative semiflow $T(t)$ on $C^0[-1,0]$. Given any two hyperbolic critical elements Φ^+, Φ^- of (1.4) i.e. they are either a hyperbolic equilibrium or periodic orbit, then their invariant manifolds (see Section (2.5)) intersect transversely, in other words:

Either

$$W^u(\Phi^+) \cap W^s_{loc}(\Phi^-) = \emptyset, \tag{1.5}$$

or in case there exists $\psi \in W^u(\Phi^+) \cap W^s_{loc}(\Phi^-)$ it follows that:

$$T_\psi W^u(\Phi^+) + T_\psi W^s_{loc}(\Phi^-) = C^0[-1,0]. \tag{1.6}$$

We will denote the transverse intersection by $W^u(\Phi^+) \pitchfork W^s_{loc}(\Phi^-)$.

The proof of this theorem is scattered along the thesis since it requires checking the result for the four possible cases treated in the corollaries (3.3.4), (4.2.2) and (4.2.3). However, the principles lying behind all of them follow always the same steps:

1. Characterize the tangent spaces in terms of the zero number of solutions lying in them, this is done through the iterations introduced in Section (3.1) and theorems (2.4.1) and (2.4.2).

2. Use this characterization to prove existence of subspaces that *add up* to the complete phase space, and which intersection must be trivial because of Lemma (2.3.1).

A result that comes as a by-product of the previous one is the following:

Theorem B. *Under the assumptions of Theorem A, assume there exists a heteroclinic x_t of (1.4) connecting Φ^- to Φ^+ so that:*

$$\Phi^- \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \Phi^+, \tag{1.7}$$

then $i(\Phi^-) > i(\Phi^+)$ where $i(\Phi)$ is the Morse index as defined in Section (2.5).

The proof of Theorem B is a result of combining lemmas (3.3.3), (4.1.3) and (4.1.4). In particular Theorem B prevents the existence of homoclinics to any critical element of the system.

The structure of the thesis consists of Chapter 2 introducing the most basic theory of differential delay equations, the zero number and invariant manifolds close to critical elements. All of these concepts are what the results obtained in Chapters 3 and 4 are built upon. Chapter 3 also introduces the iterations in Banach spaces that we use in order to characterize the tangent spaces to invariant manifolds of equilibria, after this, we proceed to prove transversality and Morse index ordering between equilibria in Section (3.3) following the steps of [2]. Chapter 4 can be regarded as an adaptation of the techniques developed in Chapter 3 so that they can be applied to the study of periodic orbits, in particular we acquire the point of view of [3] and show transversality and Morse index ordering for the couplings Periodic-Periodic, Equilibrium-Periodic and Periodic-Equilibrium. After finishing the proofs of theorems A and B we expose in Chapter 5 a few direct, but very useful conclusions that follow immediately from our results.

Chapter 2

Basics

2.1 The class of equations

We aim in this section to introduce the general theory required to study some more interesting properties of the system later on. We are interested in having a better understanding of the global dynamics of scalar differential delay equations (DDE) of the form:

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-1)), \\ f_v(u, v) \lesseqgtr 0. \end{cases} \tag{2.1}$$

Where our nonlinearity is regular enough so that $f \in C^2(\mathbb{R}^2, \mathbb{R})$ and generates a dissipative solution semigroup $T(t)$ over the space of continuous functions on $[-1, 0]$, $C^0 = (C^0[-1, 0], \| \cdot \|_\infty)$ as follows:

$$x_t = T(t)x_0 \in C^0, \tag{2.2}$$

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in [-1, 0], \tag{2.3}$$

without loss we will always assume that $T(t)$ is compact for $t \geq 1$. Particular examples of these nonlinearities have been researched in depth for instance in [12]. The condition $f_v(u, v) \lesseqgtr 0$ induces the existence of a discrete Lyapunov function [15] for the difference of any two solutions of (2.1). In particular this tool is the reason why a Poincaré-Bendixson-like result holds for this special class of equations [16].

Definition 2.1.1
Given a function $x_0 \in C^0$ we say that it defines a *global solution* whenever for any $t \geq 0$ there exists some $y \in C^0$ such that $T(t)y = x_0$, we will denote $y = T(-t)x_0 = x_{-t}$. A *global orbit* defined by x_0 is the set:

$$\gamma(x_0) = \{T(t)x_0 \,|\, t \in \mathbb{R}\}. \tag{2.4}$$

The *global attractor* $\mathcal{A}$ of the system (2.1) is the set of all bounded global solutions of the system:

$$\mathcal{A} = \left\{ x \in C^0 | T(t)x \text{ is defined for any } t \in \mathbb{R} \text{ and } \sup_{t\in\mathbb{R}} \|T(t)x\| < \infty \right\}, \tag{2.5}$$

the conditions of compactness and dissipativity of the semigroup are enough to ensure existence of $\mathcal{A}$.

The previously mentioned Poincaré-Bendixson Theorem vastly reduces the amount of possibilities that dynamics on $\mathcal{A}$ can show. Namely, assuming the set of equilibria ($\mathcal{E}$) and periodic orbits ($\mathcal{P}$) of (2.1) to be hyperbolic, then the global attractor is:

$$\mathcal{A} = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}, \tag{2.6}$$

where $\mathcal{H}$ denotes the set of heteroclinic orbits connecting the elements of $\mathcal{E}$ and $\mathcal{P}$.

2.2 General properties of the solution semigroup

In this section we will establish some very basic properties of the solution semigroup of (2.1) which turn out to be fundamental throughout the rest of the thesis. First of all we will define the variational equation along a difference of two global solutions of (2.1).

Definition 2.2.1
Let x_t^1 and x_t^2 for $t \in \mathbb{R}$ be two globally defined solutions for (2.1), then $y_t = x_t^1 - x_t^2$ fulfils the variational equation:

$$\dot{y}(t) = a(t)y(t) + b(t)y(t-1), \text{ for any } t \in \mathbb{R}, \tag{2.7}$$

where the linear nonautonomous coefficients $a(t)$ and $b(t)$ come defined by:

$$a(t) = \int_0^1 f_u\big(\theta x^1(t) + (1-\theta)x^2(t), x^1(t-1)\big)\mathrm{d}\theta, \tag{2.8}$$

$$b(t) = \int_0^1 f_v\big(x^2(t), \theta x^1(t-1) + (1-\theta)x^2(t-1)\big)\mathrm{d}\theta. \tag{2.9}$$

The continuity on t of the coefficients arises from the regularity conditions on f. In particular the condition on the derivative of f with respect to the second component gives:

$$b(t) \leqslant 0, \text{ for any } t \in \mathbb{R}. \tag{2.10}$$

The following lemma is a useful bound for solutions of the type of (2.7).

Lemma 2.2.1
Consider a continuous family of linear functionals $L(t) \in C^{0*}$, $t \in \mathbb{R}$ *of the type* $L(t)\phi = a(t)\phi(0) + b(t)\phi(-1)$, *and some function* $h \in C^0(\mathbb{R})$. *Define the following initial value problem:*

$$\begin{cases} \dot{x}(t) = L(t)x_t + h(t), & \textit{for } t \geq s, \\ x_s = \phi \in C^0. \end{cases} \tag{2.11}$$

Then the solution admits the following bound:

$$\|x_t\| \leq \|S(t,s)\|\|\phi\| + exp\left(\int_s^t \|L(\tau)\|_{C^{0*}}\, d\tau\right)\|h\|_{C^0[s,t]}, \textit{ for } t \geq s. \tag{2.12}$$

Where $S(t,s)$ *is the solution operator of the linear problem:*

$$\begin{cases} \dot{y}(t) = L(t)y_t, & \textit{for } t \geq s, \\ y_s = \phi \in C^0. \end{cases} \tag{2.13}$$

Proof. It follows directly from [9] Chapter 6, Corollary 2.1. □

The following result is widely known in functional analysis that yields some interesting properties of the solution of the variational problem (2.7).

Lemma 2.2.2
Let X *be a Banach space, for any* $T \in \mathcal{L}(X)$ *we can define the* adjoint operator $T^* \in L(X^*)$ *as follows: For any* $x \in X$ *and* $f \in X^*$:

$$T^*f(x) = \langle T^*f, x\rangle = \langle f, Tx\rangle = f(Tx).$$

Then it holds that:

$$T \textit{ has dense range} \iff T^* \textit{ is injective.}$$

Proof. Assume first that T has dense range and that there exist $f, g \in X^*$ such that $\langle T^*f, x\rangle = \langle T^*g, x\rangle$ for any $x \in X$. In particular this implies that f and g coincide on a dense set therefore they must be equal.

Now let T^* be injective and assume that $R(T)$ is not dense. Then there must exist some $x \in X$ and some open ball B_x around it such that $B_x \subset X \setminus R(T)$. By the Hahn-Banach Separation Theorem there exist a functional $f \in X^*$ and $k \in \mathbb{R}$ such that:

$$f(y_1) < k \leq f(y_2), \text{ for any } y_1 \in B_x \text{ and } y_2 \in R(T).$$

In particular this implies that $\langle T^*f, y\rangle = 0$ for any $y \in X$ since in case $\langle T^*f, y\rangle \neq 0$ one could choose some $\lambda \in \mathbb{R}$ for which $\langle T^*f, \lambda y\rangle < k$. By injectivity of T^* we get that $f = 0$, which is a contradiction. □

In particular for the solution semigroup of (2.1) and the solution family of a variational equation along a difference of global solutions (2.7) one has the following result:

Lemma 2.2.3
Let $T(t)$, $t \geq 0$ be the solution semigroup of (2.1) and $S(t,s)$, $t \geq s$ be the family of solution operators of variational equation along a difference of two global solutions (2.7), then the following holds:

1. *The semigroup $T(t)$ is injective for any $t \geq 0$.*

2. *The solution operator $S(t,s)$ is injective and has a dense range for any $t \geq s$.*

Proof. To prove part (1.) assume without loss of generality that there exists some $0 \neq \varphi \in X$ such that $T(t)\varphi = 0$ for $t \in (0,1)$. This implies that $f(0, \varphi(t-1)) = 0$ for $t \in (0,1)$ and contradicts the strict monotonoicity of f on its second component. Injectivity in (2.) follows analogously.

Proving the dense range in (2.) is based on (2.2.2). By Chapter 6, Theorem 4.1 in [7] in order to prove injectivity of the dual operator of $S(t,s)$, it is enough to prove that the solution operator of a DDE on C^{0^*} in backwards time direction is injective. Due to the very particular form of (2.7) such system admits an explicit expression, namely:

$$\dot{\tilde{y}}(s) = -a(s)\tilde{y}(s) - b(s+1)\tilde{y}(s+1) \text{ for } s \leq t-1 \tag{2.14}$$

Where $b(t)$ is precisely the same function as in (2.7). Injectivity of the solution operator of (2.14) is proved by the same argument we used in part (1.). □

2.3 The zero number

As we previously announced the key feature of the system that will allow us to learn more about its global dynamics is the *zero number*. The following results play a crucial role to prove both theorems A and B, we will consider the following differential delay equation:

$$\begin{cases} \dot{x}(t) = g(t, x(t), x(t-1)),\ t \geq t_0, \\ vg(t,0,v) \lesseqgtr 0 \text{ for } v \neq 0. \end{cases} \tag{2.15}$$

Definition 2.3.1
Given $\phi \in C^0$, we will consider the number of sign changes in the interior of its domain, formally:

$$sc(\phi) = \sup_{k \in \mathbb{N}} \left\{ \begin{array}{l} \text{There exists a sequence } -1 < t_0 < \cdots < t_k < 0 \\ \text{for which } \phi(t_{i-1})\phi(t_i) < 0,\ i = 1, \ldots, k \end{array} \right\}, \tag{2.16}$$

and a scalar DDE of the form (2.15), the *zero number* of ϕ is defined to be:

1. In case the feedback is positive (i.e. $vg(t, 0, v) > 0$ for $v \neq 0$):

$$z(\phi) = \begin{cases} sc(\phi) \text{ if } sc(\phi) \text{ is even,} \\ sc(\phi) + 1 \text{ if it is odd.} \end{cases} \tag{2.17}$$

2. Whenever the feedback is negative (i.e. $vg(t, 0, v) < 0$ for $v \neq 0$):

$$z(\phi) = \begin{cases} sc(\phi) \text{ if } sc(\phi) \text{ is odd,} \\ sc(\phi) + 1 \text{ if it is even.} \end{cases} \tag{2.18}$$

Then we have the following theorem:

Theorem 2.3.1 (Theorem 2.2, [15])
Let $x_t = S(t, t_0)x_{t_0}$ for $t \geq t_0$, where $S(t, s)$, $t \geq s \geq t_0$ is the family of solution operators of the DDE (2.15):

1. *$z(x_t)$, $t \geq t_0$ is a monotone nonincreasing function in time.*
2. *Moreover, if the function $x_{t_1} \neq 0$ presents a multiple zero i.e. there exists some $\theta \in (-1, 0)$ for which $x_{t_1}(\theta) = \dot{x}_{t_1}(\theta) = 0$, $t_1 - 3 \geq t_0$ then:*

$$z(x_{t_1}) > z(x_{t_1 - 3}) \text{ or } z(x_{t_1}) = \infty.$$

Remark 2.3.1. *In particular solutions of the system (2.7) satisfy the assumptions of the theorem. The result of the theorem is often called* zero dropping *and it is the reason why the zero number is often called discrete Lyapunov function.*

2.4 Spectra near critical elements

The following two theorems describe the spectra of the solution operators of the linearization of (2.1) around either an equilibrium or a periodic orbit. They will be key in upcoming sections since they provide the dimensionality of the eigenspaces and even characterize them through the *zero number*.

Theorem 2.4.1 (Corollary 3.3 in [15])
Consider the linearization of (2.1) around an equilibrium $\phi \in C^0$:

$$\dot{y}(t) = f_u(\phi, \phi)y(t) + f_v(\phi, \phi)y(t-1) = ay(t) + by(t-1), \text{ for } t \in \mathbb{R}, \tag{2.19}$$

it holds that $b \lesseqgtr 0$. For $T(t)$ solution semigroup of (2.1) it holds that $D_\phi T(1)$ time-1 solution operator of equation (2.19) fulfils:

1. *Assume* $b < 0$, *then* $\sigma(T(1))$ *is formed by an ordered family of of eigenvalues* $|\mu_1| \geq |\mu_2| \geq \ldots \xrightarrow[i\to+\infty]{} 0$ *so that:*

$$\mu_i = \overline{\mu_{i+1}} \text{ and } |\mu_{i+1}| > |\mu_{i+2}| \text{ for } i \text{ odd.} \tag{2.20}$$

We will denote by G_μ *the generalized eigenspace of* $\mu \in \sigma(S)$*:*

$$G_\mu = \bigcup_{k=0}^{\infty} Ker(\mu I - T)^k. \tag{2.21}$$

It is also useful to consider the real part of the generalized eigenspace spanned by each couple of eigenvalues, we shall denote it by $\mathcal{G}_i$*:*

$$\mathcal{G}_i = \operatorname{Re}\left(G_{\mu_i} \oplus G_{\mu_{i+1}}\right), \quad \mathcal{H}_i = \operatorname{Re}\bigoplus\nolimits_{j>i} G_{\mu_j}, \text{ for } i \text{ odd.} \tag{2.22}$$

Then it is also known that:

$$\dim \mathcal{G}_i = 2 \text{ for } i = 1, 3, 5, \ldots \text{ and } z(\phi) = i \text{ for } \phi \in \mathcal{G}_i. \tag{2.23}$$

2. *Assume now* $b(t) > 0$, *then* $\sigma(S_\omega)$ *is formed by a single real eigenvalue* $\mu_0 > 0$ *and a family* $\{\mu_i\}_{i\geq 1}$ *so that they are ordered* $|\mu_0| > |\mu_1| \geq |\mu_2| \geq \ldots \xrightarrow[i\to+\infty]{} 0$ *and the following holds:*

$$\mu_i = \overline{\mu_{i+1}} \text{ and } |\mu_{i+1}| > |\mu_{i+2}| \text{ for } i \text{ odd.} \tag{2.24}$$

With the notation previously defined and letting $\mathcal{G}_0 = G_{\mu_0}$ *we have that:*

$$\begin{aligned} &\dim \mathcal{G}_0 = 1 \text{ and } z(\phi) = 0 \text{ for any } \phi \in \mathcal{G}_0, \\ &\dim \mathcal{G}_i = 2 \text{ and } z(\phi) = i + 1 \text{ for any } \phi \in \mathcal{G}_i,\ i = 1, 3, 5, \ldots \end{aligned} \tag{2.25}$$

What follows is the counterpart for periodic solutions of the system. Unfortunately a little less is known for this next case, in particular the conjugated couples of eigenvalues from the previous result may have now turned into real eigenvalues with different modules. However, the *zero number characterization* of the eigenspaces remains essentially the same.

Theorem 2.4.2 (Theorem 5.1 and Proposition 5.6 in [14])
Consider the linearization of (2.1) around a periodic point with minimal period $\omega > 0$, $p \in C^0$*:*

$$\begin{aligned} \dot{y}(t) &= f_u(p(t), p(t-1))y(t) + f_v(p(t), p(t-1))y(t-1) \\ &= a(t)y(t) + b(t)y(t-1), \text{ for } t \in \mathbb{R}, \end{aligned} \tag{2.26}$$

it follows that $a(t)$ *and* $b(t)$ *are* ω*-periodic and continuous on* $[0, \omega]$*. Assuming without loss that* $T(\omega)$*, time-*ω *map of (2.1) is compact (otherwise consider an iteration), then the* monodromy operator $S_\omega = S(\omega, 0) = D_pT(\omega)$ *solving (2.26) is compact. The following holds:*

1. *Assume* $b(t) < 0$*, then* $\sigma(S_\omega)$ *is formed by an ordered family of of eigenvalues* $|\mu_1| \geq |\mu_2| \geq \ldots \xrightarrow[i \to +\infty]{} 0$ *so that:*

$$\mu_i \mu_{i+1} > 0 \text{ and } |\mu_{i+1}| > |\mu_{i+2}|, \text{ for } i \text{ odd.} \tag{2.27}$$

   In the notation of the previous theorem:

$$\mathcal{G}_i = \operatorname{Re}\left(G_{\mu_i} \oplus G_{\mu_{i+1}}\right), \quad \mathcal{H}_i = \operatorname{Re} \bigoplus\nolimits_{j>i} G_{\mu_j}, \text{ for } i \text{ odd.} \tag{2.28}$$

   Then it is also known that:

$$\dim \mathcal{G}_i = 2 \text{ for } i = 1, 3, 5, \ldots \text{ and } z(\phi) = i \text{ for } \phi \in \mathcal{G}_i. \tag{2.29}$$

2. *Assume now* $b(t) > 0$*, then* $\sigma(S_\omega)$ *is formed by a single real eigenvalue* $\mu_0 > 0$ *and a family* $\{\mu_i\}_{i \geq 1}$ *so that they are ordered* $|\mu_0| > |\mu_1| \geq |\mu_2| \geq \ldots \xrightarrow[i \to +\infty]{} 0$ *and the following spectral gap condition holds:*

$$\mu_i \mu_{i+1} > 0 \text{ and } |\mu_{i+1}| > |\mu_{i+2}| \text{ for } i \text{ odd,} \tag{2.30}$$

   and we know much about the eigenspaces:

$$\begin{aligned} &\dim \mathcal{G}_0 = 1 \text{ and } z(\phi) = 0 \text{ for any } \phi \in \mathcal{G}_0, \\ &\dim \mathcal{G}_i = 2 \text{ and } z(\phi) = i + 1 \text{ for any } \phi \in \mathcal{G}_i,\ i = 1, 3, 5, \ldots \end{aligned} \tag{2.31}$$

Remark 2.4.1. *This fully characterizes the spectrum of the linearization close to an equilibrium as well as the Floquet spectrum of our system close to a periodic orbit. In particular one can see that the smaller the multipliers are, the larger the zero number of the eigenfunctions gets.*

2.5 Invariant manifolds near critical elements

This section is devoted to the study the behaviour of solutions of equation (2.1) that approach equilibria or periodic orbits either in positive or negative time direction. In particular we are interested in the properties of stable and unstable manifolds of the semiflow of (2.1) and their tangent spaces.

In order to define them we will consider the following couple of complementary theorems on invariant manifolds of hyperbolic fixed points under a map iteration:

Theorem 2.5.1 (Theorem C.1. in [2])
Let $T : X \to X$ be a C^1 operator on a Banach space X. Assume that $\phi \in X$ is a fixed point of T and suppose that $\sigma(D_\phi T) \subset \sigma_u \cup \sigma_s$ spectral splitting dividing the phase space into two $D_\phi T-$invariant subspaces $X = X^u \oplus X^s$ related to the spectral sets σ_u and σ_s. Then there exists a neighborhood V_u of $0 \in X^s$ and a neighborhood V of $0 \in X$, and a C^1 map $h_u : V_u \to X$ such that:

$$W^u_{loc}(\phi) = \{x \in V | \lim_{n\to-\infty} T^n x = \phi\} = \{\phi + b + h_u(b) | b \in V_u\}. \tag{2.32}$$

Moreover, the following follows:

1. *X^u is the tangent space of $W^u_{loc}(\phi)$ at ϕ.*
2. *Consider a semiorbit $\{x_n\}_{n\leq 0} \subset W^u_{loc}(\phi)$ with $x_{n+1} = Tx_n$, if $y_0 \in T_{x_0}W^u_{loc}(\phi)$, there exists a semiorbit $\{y_n\}_{n\leq 0} \subset X$ given by $y_{n+1} = D_{x_n}Ty_n$ such that $y_n \in T_{x_n}W^u_{loc}(\phi)$ and:*
$$\limsup_{n\to-\infty} \|y_n\|^{\frac{1}{|n|}} \leq \frac{1}{\alpha_u}. \tag{2.33}$$
3. *If $\{x_n\}_{n\leq 0} \subset W^u_{loc}(\phi)$ with $x_{n+1} = Tx_n$, and the semiorbit $\{y_n\}_{n\leq 0} \subset X$ given by $y_{n+1} = D_{x_n}Ty_n$ fulfils:*
$$\limsup_{n\to-\infty} \|y_n\|^{\frac{1}{|n|}} < \frac{1}{\alpha_s}, \tag{2.34}$$
then $y_n \in T_{x_n}W^u_{loc}(\phi)$ and (2.33) holds.

Theorem 2.5.2 (Theorem C.2. in [2])
Let $T : X \to X$ be a C^1 operator on a Banach space X. Assume that $\phi \in X$ is a fixed point of T and suppose that $\sigma(D_\phi T) \subset \sigma_u \cup \sigma_s$ spectral splitting dividing the phase space into two $D_\phi T-$invariant subspaces $X = X^u \oplus X^s$ related to the spectral sets σ_u and σ_s. Then there exists a neighborhood V_s of $0 \in X^s$ and a neighborhood V of $0 \in X$, and a C^1 map $h_s : V_s \to X$ such that:

$$W^s_{loc}(\phi) = \{x \in V | \lim_{n\to\infty} T^n x = \phi\} = \{\phi + b + h_s(b) | b \in V_s\}. \tag{2.35}$$

Moreover, the following follows:

1. *X^s is the tangent space of $W^s_{loc}(\phi)$ at ϕ.*

2. *Consider a semiorbit $\{x_n\}_{n\geq 0} \subset W^s_{loc}(\phi)$ with $x_{n+1} = Tx_n$, if $y_0 \in T_{x_0}W^u_{loc}(\phi)$, there exists a semiorbit $\{y_n\}_{n\geq 0} \subset X$ given by $y_{n+1} = D_{x_n}Ty_n$ such that $y_n \in T_{x_n}W^s_{loc}(\phi)$ and:*
$$\limsup_{n\to\infty} \|y_n\|^{\frac{1}{n}} \leq \alpha_s. \tag{2.36}$$

3. *If $\{x_n\}_{n\geq 0} \subset W^s_{loc}(\phi)$ with $x_{n+1} = Tx_n$, and the semiorbit $\{y_n\}_{n\geq 0} \subset X$ given by $y_{n+1} = D_{x_n}Ty_n$ fulfils:*
$$\limsup_{n\to\infty} \|y_n\|^{\frac{1}{n}} < \alpha_u \tag{2.37}$$
, then $y_n \in T_{x_n}W^s_{loc}(\phi)$ and (2.36) holds.

Definition 2.5.1
Let $\phi \in X$ be a hyperbolic fixed point under $T(t)$ solution semigroup of (2.1). Consider the eigenvalues of $D_\phi T(1)$ ordered like in theorem (2.4.2) and let m be the index of the first eigenvalue with $|\mu_m| < 1$, then the *Morse index* of ϕ is:
$$i(\phi) = \dim \mathcal{H}_m.$$

Section 8 from [9] gives us a $D_\phi T(1)$ invariant splitting of $C^0 = X^u \oplus X^s$ where X^u is related to the *unstable* part of the spectrum ($\mathcal{G}_i \subset X^u$ for $i \leq i(\phi)$) and X^s to the *stable* one ($\mathcal{G}_i \subset X^s$ for $i > i(\phi)$). By applying the previous theorems (2.5.1) and (2.5.2) we can define respectively the local unstable and stable manifolds of $\phi \in X$ in a neighborhood V of ϕ under the semiflow $T(t)$:

$$W^u_{loc}(\phi) = \left\{ \begin{array}{c} x \in V | x = T(t)y_t \text{ for some } y_t \in X \\ \text{for every } t \geq 0 \text{ and } \lim_{t\to\infty} y_t = \phi \end{array} \right\} \tag{2.38}$$
$$= \{\phi + b + h_u(b) | b \in V_u\}, \tag{2.39}$$

$$W^s_{loc}(\phi) = \{x \in V | \lim_{t\to\infty} T(t)x = \phi\} = \{\phi + b + h_s(b) | b \in V_s\}, \tag{2.40}$$

where $V_u \subset X^u$ and $V_s \subset X^s$ are neighborhoods of 0 for a $D_\phi T(1)$-invariant splitting of the phase space $C^0 = X^u \oplus X^s$ like in the invariant manifold theorems and the h_u, h_s maps are precisely the same. We are also interested in the *global unstable set*:

$$W^u(\phi) = \left\{ \begin{array}{c} x \in X | x = T(t)y_t \text{ for some } y_t \in X \\ \text{for every } t \geq 0 \text{ and } \lim_{t\to\infty} y_t = \phi \end{array} \right\}. \tag{2.41}$$

Remark 2.5.1. *We already saw that $T(1)$ is injective, its Fréchet derivative at any point $x \in C^0$ is $S(1,0) = D_xT(1)$ solving:*

$$\dot{y}(t) = f_u(x(t), x(t-1))y(t) + f_v(x(t), x(t-1))y(t-1), \qquad (2.42)$$

which can be easily proved is injective, by Theorem 6.1.9 from [10] it follows that the global unstable set is actually an injectively immersed global manifold modeled over the unstable part of the phase space.

Given a periodic solution p of (2.1) with minimal period $\omega > 0$, it defines a periodic orbit Π of (2.1) in the following way:

$$\Pi = \{T(t)p \,|\, t \in [0, \omega]\}. \qquad (2.43)$$

For convenience from now on we will denote the elements $p_\alpha = T(\alpha)p$ for some periodic basepoint $p \in \Pi$.

Definition 2.5.2
The *monodromy operator* of (2.1) at $p \in \Pi$ is $D_pT(\omega)$. Some of its most important properties have already been summarized in Theorem (2.4.2). A periodic orbit Π is called *hyperbolic* whenever its monodromy operator has only one simple eigenvalue 1. The *Morse index* of a periodic orbit $i(\Pi)$ is the number of eigenvalues μ of its monodromy operator such that $|\mu| > 1$.

Section 8 from [9] gives a $D_pT(\omega)$-invariant splitting of the phase space $C^0 = X^u \oplus X^c \oplus X^s$ so that any eigenfunction with eigenvalue μ and $|\mu| > 1$ lies in X^u, $X^c = span\{\dot{p}\}$ and X^s contains the rest. This allows us to define invariant sets just like we did for equilibria. Given an open neighborhood V of Π one can define the *local unstable and stable sets of the periodic orbit*:

$$W^u_{loc}(\Pi) = \left\{ \begin{array}{c} x \in V | x = T(t)y_t \text{ for some } y_t \in X \\ \text{for every } t \geq 0 \text{ and } \lim_{t\to\infty} \mathrm{d}(y_t, \Pi) = 0 \end{array} \right\}, \qquad (2.44)$$

$$W^s_{loc}(\Pi) = \{x \in V | \mathrm{d}(x_t, \Pi)\}, \qquad (2.45)$$

In Theorem (2.5.3) we will see that for a hyperbolic periodic orbit Π, assuming enough regularity on the nonlinearity, the unstable and stable sets are indeed manifolds and are fibrated by strong manifolds of elements on the orbit.

Let $p_\alpha \in \Pi$ the *strong unstable manifold* of Π at p_α is

$$W^{su}_{loc}(p_\alpha) = \left\{ \begin{array}{c} x \in V | x = T(n\omega)y_n \text{ for some } y_n \in X \\ \text{for } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} y_n = p_\alpha \end{array} \right\}, \qquad (2.46)$$

analogously we can define the *strong stable manifold* at p_α

$$W^{ss}_{loc}(p_\alpha) = \{x \in V | \lim_{n\to\infty} T(n\omega)x = p_\alpha \text{ for } n \in \mathbb{N}\}. \qquad (2.47)$$

The following theorem summarizes all of the previous comments.

Theorem 2.5.3 (Section 3 in Chapter 10 of [9])
Consider equation (2.1) with $f \in C^2(\mathbb{R}^2, \mathbb{R})$ and let Π be a hyperbolic periodic orbit of (2.1), let $p \in \Pi$ and as before $p_\alpha = T(\alpha)p$. There is a neighborhood V of Π in which both $W^u_{loc}(\Pi)$ and $W^s_{loc}(\Pi)$ are diffeomorphic to $X^u \times [0, \omega)$ and $X^s \times [0, \omega)$ respectively. More precisely:

1. *There exists a neighborhood V of Π such that $W^u_{loc}(\Pi)$ is a C^2-submanifold of C^0 with $\dim W^u_{loc}(\Pi) = i(\Pi) + 1$ and:*

$$W^u_{loc}(\Pi) = \bigcup_{\alpha \in [0,\omega)} W^{su}_{loc}(p_\alpha), \tag{2.48}$$

where each of the strong unstable sets is a C^2-submanifold of C^0 so that for V_u a neighborhood of $0 \in X^u$:

$$W^{su}_{loc}(p_\alpha) = \{x \in V | p_\alpha + b + h^u(b, \alpha) | b \in V_u\} \tag{2.49}$$

h^u is a C^2-mapping from $X^u \times [0, \omega)$ to $X^s \oplus X^c$ with $h^u(0, \alpha) = 0$ and $Dh^u(0, \alpha) = 0$. For any $x \in W^u_{loc}(\Pi)$ there exists one single α such that $\lim_{n \to -\infty} T(n\omega)x = p_\alpha$.

2. *There exists a neighborhood V of Π such that $W^s_{loc}(\Pi)$ is a C^1-submanifold of C^0 with $\operatorname{codim} W^s_{loc}(\Pi) = i(\Pi)$ and:*

$$W^s_{loc}(\Pi) = \bigcup_{\alpha \in [0,\omega)} W^{ss}_{loc}(p_\alpha), \tag{2.50}$$

where each of the strong stable sets is a C^2-submanifold of C^0 so that for V_s a neighborhood of $0 \in X^s$:

$$W^{ss}_{loc}(p_\alpha) = \{x \in V | p_\alpha + b + h^s(b, \alpha) | b \in V_s\} \tag{2.51}$$

here h^s is a C^2-mapping from $X^s \times [0, \omega)$ to $X^c \oplus X^u$ with $h^s(0, \alpha) = 0$ and $Dh^s(0, \alpha) = 0$. For any $x \in W^s_{loc}(\Pi)$ there exists one single α such that $\lim_{n \to \infty} T(n\omega)x = p_\alpha$.

Once again, by backwards invariance of the local unstable manifold under $T(\omega)$, together with the injectivity of the solution operator and injectivity of its Fréchet derivative (see Remark (2.5.1)), one can just like in Remark (2.5.1) that the local unstable structure can be extended to a global unstable manifold $W^u(\Pi)$.

Remark 2.5.2 (The increase in regularity of delay equations)**.** *It follows immediately from the regularity assumption for f in (2.1) that after time 1 our solutions become C^1 for any initial condition in C^0. This has a series of immediate consequences, for instance any global solution (see section (2.1.1)) is automatically C^1, this includes examples very interesting for us such as heteroclinics or periodic orbits. In particular, eigenfunctions of the linearized equation around equilibria or periodic orbits will also be C^1.*

The following corollary relates the Floquet spectrum and the zero number of a periodic orbit.

Corollary 2.5.4
Consider a periodic solution $p \in C^0$ of (2.1) with minimal period $\omega > 0$, then:

- *p defines a periodic C^1 global solution $p(t)$, $t \in \mathbb{R}$ of (2.1). Let $\dot{p}_0 \in C^0$ defined by $\dot{p}_0(\theta) = f(p(\theta), p(\theta - 1))$, $\dot{p}_0$ is a periodic solution of:*

$$\dot{y}(t) = f_u(p(t), p(t-1))y(t) + f_v(p(t), p(t-1))y(t-1), \tag{2.52}$$

  with minimal period ω and constant zero number $z(\dot{p}) = z(\dot{p}_0)$.

- *The zero number of $\dot{p}$ is related to the unstable dimension of Π as follows:*

$$z(\dot{p}) \in \{i(\Pi), i(\Pi + 1)\}. \tag{2.53}$$

Proof. The first part follows easily from the increase in regularity of delay equations, the second part follows from the observation that $\dot{p}$ is a trivial eigenfunction of the monodromy operator $D_pT(\omega)$ associated to the trivial eigenvalue 1. By theorem (2.4.2) the result arises immediately. □

In particular the previous corollary does not exclude the existence of stable periodic solutions in the negative feedback case $f_v < 0$ (unlike its counterpart in the PDE setting). However, any periodic solution must be unstable in the case $f_v > 0$, since a stable periodic solution would have a derivative of constant sign (which cannot happen).

Chapter 3

Asymptotic behaviour near equilibria

In this chapter we will finally prove transversality between hyperbolic equilibria of (2.1). The first section introduces some rather general theorems on convergence of sequences in Banach spaces. This is the main tool that we use in the following parts in order to characterize the tangent spaces of invariant manifolds and, ultimately, prove transversality for equilibria. The following section follows essentially Appendix B from [2].

3.1 Iterations in general Banach spaces

Consider now a generic Banach space X and some bounded linear operator $T \in \mathcal{L}(X)$ the space of bounded linear operators on X. We will be interested in the asymptotic behavior of iterations of the form:

$$u_{n+1} = Tu_n + \xi_n \text{ for } n \geq 0, \tag{3.1}$$

where $\lim_{n\to\infty} \frac{\|\xi_n\|}{\|u_n\|} = 0$. Denote by Λ the set of modules of the eigenvalues of T:

$$\Lambda = \{|\mu| \mid \mu \in \sigma(T)\}. \tag{3.2}$$

Given any $a \in [0, \infty) \setminus \Lambda$, $P(a)$ will denote the projection of X associated to $\sigma(T) \cap \{|z| > a\}$ and $Q(a) = I - P(a)$.

Lemma 3.1.1
Assume the following:

1. *There exists* $\lambda \in \Lambda$, $\lambda > 0$ *so that*

$$\lim_{n\to\infty} \|u_n\|^{\frac{1}{n}} = \lambda. \tag{3.3}$$

2. *Either:*

   (a) *T has a couple of conjugated eigenvalues $\{\mu, \overline{\mu}\}$ for which $|\mu| = \lambda$ with generalized eigenspaces G_μ, $G_{\overline{\mu}}$ and:*

$$\mathcal{G} = \operatorname{Re}\left(G_\mu \oplus G_{\overline{\mu}}\right), \, \dim \mathcal{G} = 2, \tag{3.4}$$

   (b) *or T has a simple eigenvalue $\mu \in \mathbb{R}$ with $|\mu| = \lambda$ and*

$$\mathcal{G} = G_\mu, \, \dim \mathcal{G} = 1. \tag{3.5}$$

3. *There exists $\delta > 0$ for which:*

$$(\lambda - \delta, \lambda + \delta) \cap (\Lambda \setminus \{\lambda\}) = \emptyset, \tag{3.6}$$

then there exists some subsequence $\{n_k\}_{k=0}^{\infty}$ with $n_k \xrightarrow{k \to \infty} \infty$ for which:

$$\lim_{k\to\infty} \tfrac{u_{n_k}}{\|u_{n_k}\|} = \Psi, \quad \Psi \in \mathcal{G}. \tag{3.7}$$

Proof. This is a version of Theorem B.5. in [2], however, the proof does not fit the setting with complex conjugated eigenvalues. Our result is weaker since it does not prove convergence of the normalized sequence to an eigenfunction, but convergence of a subsequence to some object lying in the real generalized eigenspace $\mathcal{G}$.

From our assumptions we can choose some $\varepsilon < \delta$ in a way so that

$$\big(P(\lambda + \varepsilon) - P(\lambda - \varepsilon)\big)X = \mathcal{G}.$$

By Theorem B.4. the ω-limit $\omega\left(\left\{\frac{u_n}{\|u_n\|}\right\}_{n\geq 0}\right)$ is a nonempty compact subset on a finite dimensional sphere $\{x \in \mathcal{G} | \, \|x\| = 1\}$. Since $\left\{\frac{u_n}{\|u_n\|}\right\}_{n\geq 0}$ is a sequence of real functions it must hold that the ω-limit is real too and:

$$\omega\left(\left\{\frac{u_n}{\|u_n\|}\right\}_{n\geq 0}\right) \subset \mathcal{G}. \tag{3.8}$$

Then the result follows. $\square$

The previous theorem has its counterpart for negative semiorbits $\{u_n\}_{n\leq 0}$ with $n \to -\infty$ i.e. sequences satisfying:

$$u_{n+1} = Tu_n + \xi_n \text{ for } n \leq 0, \tag{3.9}$$

with the condition $\lim_{n\to-\infty} \frac{\|\xi_n\|}{\|u_n\|} = 0$, then the following lemma explains the asymptotic behaviour of solutions.

Lemma 3.1.2
Assume that:

1. *There exists* $\lambda \in \Lambda$, $\lambda > 0$ *so that*

$$\lim_{n\to-\infty} \|u_n\|^{\frac{1}{n}} = \lambda > 0 \tag{3.10}$$

2. *Either:*

   (a) *T has a couple of conjugated eigenvalues* $\{\mu, \overline{\mu}\}$ *for which* $|\mu| = \lambda$ *with generalized eigenspaces* G_μ, $G_{\overline{\mu}}$ *and:*

$$\mathcal{G} = \mathrm{Re}\,(G_\mu \oplus G_{\overline{\mu}})\,,\ \dim \mathcal{G} = 2, \tag{3.11}$$

   (b) *or T has a simple eigenvalue* $\mu \in \mathbb{R}$ *with* $|\mu| = \lambda$ *and*

$$\mathcal{G} = G_\mu,\ \dim \mathcal{G} = 1. \tag{3.12}$$

3. *There exists* $\delta > 0$ *for which:*

$$(\lambda - \delta, \lambda + \delta) \cap (\Lambda \setminus \{\lambda\}) = \emptyset, \tag{3.13}$$

then there exists some subsequence $\{n_k\}_{k=0}^{\infty}$ *with* $n_k \xrightarrow{k\to\infty} -\infty$ *for which:*

$$\lim_{k\to\infty} \tfrac{u_{n_k}}{\|u_{n_k}\|} = \Psi, \quad \Psi \in \mathcal{G}. \tag{3.14}$$

Proof. It follows the same lines as (3.1.1), but one has to consider the α-limit instead. □

3.2 Variational equations and iterations

Consider $\phi \in C^0$, a hyperbolic equilibrium of (2.1). Given a globally defined solution of the system with trajectory $x(t)$ for $t \in \mathbb{R}$, we consider the variational equation satisfied by its derivative $y(t) = \dot{x}(t)$:

$$\begin{cases} \dot{y}(t) = a(t)y(t) + b(t)y(t-1) = L(t)y_t, \\ y_s = \xi, \end{cases} \tag{3.15}$$

with

$$a(t) = f_u(x(t), x(t-1)), \tag{3.16}$$
$$b(t) = f_v(x(t), x(t-1)). \tag{3.17}$$

We will denote $S(t,s)$ with $t \geq s$ the family of solution operators. One can also consider the linearized equation around ϕ:

$$\begin{cases} \dot{z}(t) = az(t) + bz(t-1) = Lz_t, \\ z_0 = \xi. \end{cases} \tag{3.18}$$

With solution semigroup $D_\phi T(t)$, it also holds that given $T(t)$ the solution semigroup of (2.1), in fact $D_{x_0}T(1)\xi = S(1,0)\xi$.

If one considers now a global trajectory such that either $x_t \xrightarrow[C^0]{t\to\infty} \phi$ or $x_t \xrightarrow[C^0]{t\to-\infty} \phi$, by regularity of f in (2.1) it turns out that for $L(t)$ in (3.15) and L in (3.18):

$$\|L(t) - L\|_{C^{0*}} \xrightarrow{|t|\to\infty} 0,$$

the following lemma shows that actually the time-1 solution operators of (3.15) and (3.18) converge in $\mathcal{L}(C^0)$ for large times and the convergence actually happens in $\mathcal{L}(C^1)$.

Lemma 3.2.1
In the previous notation let $x(t)$ be a globally defined trajectory of (2.1) and ϕ a hyperbolic equilibrium. $S(t,s)$ is the family of solution operators of (3.15) and $D_\phi T(1)$ is the time-1 solution of (3.18), then:

- *If $\lim_{t\to\infty} x(t) = \phi$, then*

$$\lim_{n\to\infty} \|S(n+1,n) - D_\phi T(1)\|_{\mathcal{L}(C^1)} = 0. \tag{3.19}$$

- *If $\lim_{t\to-\infty} x(t) = \phi$, then*

$$\lim_{n\to-\infty} \|S(n+1,n) - D_\phi T(1)\|_{\mathcal{L}(C^1)} = 0 \tag{3.20}$$

Proof. The proof is split in two parts, first of all we prove convergence in $\mathcal{L}(C^0)$, convergence in $\mathcal{L}(C^1)$ follows from the regularity of the nonlinearity in (2.1).

Step 1. We will just prove convergence in $\mathcal{L}(C^0)$ for the positive time direction and the negative time direction follows analogously. Let us rename $v^n(s) := y(s+n)$ where $y(s)$ solves (3.15) with the initial condition $y_n = \xi$, $v^n(s)$ solves:

$$\begin{cases} \dot{v}^n(s) = a(s+n)v^n_s(0) + b(s+n)v^n_s(-1) = L(s+n)v^n_s, \text{ for } s \in (0,1], \\ v^n_0 = \xi. \end{cases} \tag{3.21}$$

Define now $w^n(s) = v^n(s) - z(s)$, $s \in [0,1]$ where $z(s)$ is a solution of (3.18), then $w^n(s)$ satisfies:

$$\begin{cases} \dot{w}^n(s) = Lw_s^n + (L(s+n) - L)v_s^n, \text{ for } s \in (0,1], \\ w_0^n = 0. \end{cases} \tag{3.22}$$

Consider the operator $D_\phi T(t)$ solving (3.18), $0 \le t \le 1$, it is clear that there is some constant $C > 0$ such that:

$$\|D_\phi T(t)\xi\|_{\mathcal{L}(C^0)} \le C\|\xi\|_{C^0}. \tag{3.23}$$

Since $f \in C^2$, the operator $L(t) \in C^{0^*}$ converges in norm to L. Thus there exists some $M > 0$ such that for n large enough $\|L(s+n)\|_{C^{0*}} \le M$. By Lemma (2.2.1), it follows that:

$$\|v_s^n\|_{C^0} \le CM\|\xi\|_{C^0} \text{ for any } s \in [0,1]. \tag{3.24}$$

The term to the right of equation (3.22) can be regarded as a forcing of a linear delay equation $F_n(t) = (L(t+n) - L)v_t^n$, $t \in (0,1]$, once again by (2.2.1):

$$\|w_s^n\|_{C^0} \le C \max_{\tau\in[0,1]} |F_n(\tau)| \le K \max_{\tau\in[0,1]} \|L(\tau+n) - L^+\|_{C^{0*}}\|v_\tau^n\|_{C^0}, \tag{3.25}$$

by convergence of the functionals we get that for any $\varepsilon > 0$ we can choose n large enough so that:

$$\|w_1^n\|_{C^0} \le \varepsilon\|\xi\|_{C^0}, \tag{3.26}$$

therefore:

$$\lim_{n\to-\infty} \|S(n+1,n) - D_\phi T(1)\|_{\mathcal{L}(C^0)} = 0. \tag{3.27}$$

Step 2: From (3.22) it follows that

$$|\dot{w}^n(s)| \le (|a| + |b|)\|w_s^n\|_{C^0} + \max_{\tau\in[0,1]} \|L(n+\tau) - L\|CM\|\xi\|_{C^0}, \tag{3.28}$$

$$\|\dot{w}_1^n\|_{C^0} \le (|a| + |b|)\|w_1^n\|_{C^0} + \max_{\tau\in[0,1]} \|L(n+\tau) - L\|CM\|\xi\|_{C^0}. \tag{3.29}$$

Convergence in C^1-norm is then immediate from Step 1. □

Remark 3.2.1.

1. *In order to prove the previous lemma we used little more than the special form of the operators $L(t)$ and L and convergence of the global solution to some critical element. We will refer to this sort of argument in the future for other approximations.*

2. *Provided that x_t converges to ϕ in forward or backward time, the previous lemma allows us to consider iterations of the forms (3.1) and (3.9) for y_t solving (2.7) as follows:*

$$y_{n+1} = S(n+1,n)y_n = D_\phi T(1)y_n + (S(n+1,n) - D_\phi T(1))y_n.$$

3.3 Tangent spaces to invariant manifolds of equilibria

The following lemma brings us closer to describing the tangent spaces of the invariant manifolds from Section (2.5) in terms of zero numbers of solutions.

Lemma 3.3.1
Consider $\phi \in C^0$ a hyperbolic equilibrium of (2.1) and $x(t)$ a global solution under such flow, $S(t,s)$, $t \geq s$ denotes the operator family solving (3.15) the variational equation along $x(t)$.

- *Let $x_0 \in W^s_{loc}(\phi)$, then it follows that:*

$$T_{x_0} W^s_{loc}(\phi) = \{\psi \in C^0 | \lim_{n\to\infty} \|S(n+1,0)\psi\|^{1/n} < 1\}. \tag{3.30}$$

- *Let $x_0 \in W^u(\phi)$ global unstable manifold of ϕ, then the tangent space can be described by:*

$$T_{x_0} W^u(\phi) = \left\{ \begin{array}{l} \psi \in C^0 | \text{ there exists } y_{n+1} = S(n+1,n)y_n \in C^0 \\ y_0 = \psi,\ n \leq 0 \text{ and } \lim_{n\to -\infty} \|y_n\|^{\frac{1}{n}} > 1 \end{array} \right\}. \tag{3.31}$$

Proof. Part 1. follows directly from the characterization given in the stable manifold theorem for map iterations (2.5.2) considering $x_n = T(1)x_0$ since $D_{x_n}T(1) = S(n+1,n)$.

Part 2. would follow just as easily for the local unstable manifold using Theorem (2.5.1). However, for the global set one has to consider that for m large enough $x_{-m} = T(-m)x_0$ belongs to the local unstable set. Therefore:

$$T_{x_m} W^u_{loc}(\phi) = \left\{ \begin{array}{l} \psi \in C^0 | \text{ there exists } y_{n+1} = S(n+1,n)y_n \in C^0 \\ y_m = \psi,\ n \leq m-1 \text{ and } \lim_{n\to -\infty} \|y_n\|^{\frac{1}{n}} > 1 \end{array} \right\}, \tag{3.32}$$

however, $S(m,0)T_{x_m} W^u_{loc}(\phi) = T_{x_0} W^u(\phi)$ and this ends the proof. □

Let now ϕ^-, $\phi^+ \in C^0$ be two hyperbolic equilibria of (2.1) and assume there exists a globally defined heteroclinic x_t for $t \in \mathbb{R}$ such that:

$$\phi^- \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \phi^+,$$

we will study the following couple of lemmas.

Lemma 3.3.2
For x_t the heteroclinic previously defined, assume without loss that $x_0 \in W^u(\phi^-) \cap W^s_{loc}(\phi^+)$, then it holds that:

- *$T_{x_0} W^s_{loc}(\phi^+)$ is a closed Banach space with codimension $i(\phi^+)$ and given $\psi \in T_{x_0} W^s_{loc}(\phi^+)$, $z(\psi) \geq i(\phi^+)$.*
- *$T_{x_0} W^u(\phi^-)$ is a closed Banach space with dimension $i(\phi^-)$ and given $\psi \in T_{x_0} W^u(\phi^+)$, $z(\psi) \leq i(\phi^-) - 1$.*

Proof. We show it for $T_{x_0} W^s_{loc}(\phi^+)$ and the case for the unstable manifold is similar. Consider the iteration:

$$y_{n+1} = S(n+1,n)y_n = D_{\phi^+}T(1)y_n + (S(n+1,n) - D_{\phi^+}T(1))y_n, \qquad (3.33)$$

where $S(t,s)$ solves (3.15) and the iteration takes place over C^0, $n \geq 0$ and $y_0 = \psi \in T_{x_0} W^s_{loc}(\phi^+)$. Define:

$$E(m) := \{\psi \in C^0 | \lim_{n\to\infty} \|S(n,m)\psi\|^{1/n} < 1, \text{ for } n > m\}. \qquad (3.34)$$

From lemma (3.2.1) together with theorems B.7. and B.8. from [2] we know that for m large enough there exists an isomorphism between $E(m)$ and $X^s = Q(1)C^0$, where $Q(1)$ denotes the projection associated to the eigenvalues of $D_{\phi^+}T(1)$ of module smaller or equal 1, therefore $E(m)$ is a closed Banach space of codimension $i(\phi^+)$ for m large enough. Since $S(m,0)$ is injective and has dense range, it follows that

$$T_{x_0} W^s_{loc}(\phi^+) = S(m,0)^{-1}E(m),$$

is just as we stated in the lemma. In order to see the property of the zero number one has to consider (3.33) in C^1, by (3.2.1) and Corollary B.3. from [2], there exists $\lambda \in \Lambda = \{|\mu| : \mu \in \sigma D_{\phi^+}T(1)\}$ such that $\lambda < 1$ and $\lim_{n\to\infty} \|y_n\|^{1/n} = \lambda$. In case $\lambda > 0$ we know from Lemma (3.1.1) that a normalized subsequence $\frac{y_{n_k}}{\|y_{n_k}\|}$ converges in C^1 to an eigenfunction of $D_{\phi^+}T(1)$ in the stable part of the spectrum, by the zero dropping lemma (2.3.1) and Theorem (2.4.1), it follows that $z(\psi) \geq z\left(\frac{y_{n_k}}{\|y_{n_k}\|}\right) \geq i(\phi^+)$. Convergence in C^1 is actually important since convergence in C^0 does not ensure that the zero numbers approach in finite time. Unlike in the PDE setting (1.2), it may happen that $\lambda = 0$; in such case the solution decays at a superexponential rate towards the equilibrium, however, these solutions are characterized by Theorem 2.8. in [1] implying that $z(\psi) = \infty$ and the claim still holds. □

Remark 3.3.1. *A superexponential approach to an equilibrium cannot happen in backwards time direction, this follows from the fact that the semiflow of (2.1) is a family of bounded operators.*

Lemma 3.3.3
Let again x_t denote the previously defined heteroclinic, $t \in \mathbb{R}$. Then, $i(\phi^-) > i(\phi^+)$.

Proof. Consider the linearized equation around $y(t) = x(t) - x(t-1)$:

$$\dot{y}(t) = a(t)y(t) + b(t)y(t-1) = L(t)y_t, \tag{3.35}$$
$$a(t) = \int_0^1 f_u\big(\theta x(t) + (1-\theta)x(t-1), x(t-1)\big)\mathrm{d}\theta, \tag{3.36}$$
$$b(t) = \int_0^1 f_v\big(x(t-1), \theta x(t-1) + (1-\theta)x(t-2)\big)\mathrm{d}\theta. \tag{3.37}$$

With solution family $S(t,s)$. By the regularity of f and the convergence of the heteroclinic it follows that given the linearized equations around the equilibria:

$$\dot{z}^{\pm}(t) = a^{\pm}z^{\pm}(t) + b^{\pm}z^{\pm}(t-1) = L^{\pm}z_t^{\pm}, \tag{3.38}$$
$$a^{\pm} = f_u(\phi^{\pm}, \phi^{\pm}), \tag{3.39}$$
$$b^{\pm} = f_v(\phi^{\pm}, \phi^{\pm}). \tag{3.40}$$

$\lim_{t\to\pm\infty} \|L(t) - L^{\pm}\|_{C^{0*}} = 0$, let $T^{\pm}(t) = D_{\phi^{\pm}}T(t)$ be the semigroups solving (3.38), one can then prove that $\|S(n+1,n) - T^{\pm}(1)\|_{\mathcal{L}(C^1)} \xrightarrow{n\to\pm\infty} 0$ applying the same reasoning as in the proof of Lemma (3.2.1). Then we consider the iterations given by:

$$y_{n+1} = S(n+1,n)y_n = T^+(1)y_n + (S(n+1,n) - T^+(1))y_n,\ n \geq 0, \tag{3.41}$$
$$y_{n+1} = S(n+1,n)y_n = T^-(1)y_n + (S(n+1,n) - T^-(1))y_n,\ n \leq 0. \tag{3.42}$$

Now we want to see that $\lim_{n\to\infty} \|y_n\|^{1/n} < 1$, and $\lim_{n\to-\infty} \|y_n\|^{1/n} > 1$. We will prove the first case and the second follows analogously, let $P^+(1)$ be the projection associated to the unstable eigenvalues of $T^+(1)$, $Q^+(1) = I - P^+(1)$. By Theorem B.2. in [2] we have to prove that:

$$\lim_{n\to\infty} \frac{\|P^+(1)y_n\|_{C^1}}{\|Q^+(1)y_n\|_{C^1}} = 0, \tag{3.43}$$

from the definition of the stable manifold, letting $x_n = \phi + b_n + h_s(b_n)$ for $b_n \in Q^+(1)C^0$. It follows that $y_n = b_n - b_{n-1} + h_s(b_n) - h_s(b_{n-1})$, $n \geq 1$, since $\dot{y}_n$ can be seen as a bounded perturbation of y_n for n large enough by (3.35), it is enough to prove:

$$\lim_{n\to\infty} \frac{\|P^+(1)y_n\|_{C^0}}{\|Q^+(1)y_n\|_{C^0}} = \lim_{n\to\infty} \frac{\|h_s(b_n) - h_s(b_{n-1})\|_{C^0}}{\|b_n - b_{n-1}\|_{C^0}} = 0, \tag{3.44}$$

and this follows immediately from $Dh_s(0) = 0 = h_s(0)$.

From these previous observations, arguing like in the proof of Lemma (3.3.2) and considering the zero dropping lemma (2.3.1), we conclude that $i(\phi^-) > z(y_0) \geq i(\phi^+)$. □

Corollary 3.3.4
Given two hyperbolic equilibria ϕ^+, ϕ^- in (2.1). Then their global unstable manifold and local stable manifold intersect transversely:

$$W^u(\phi^-) \pitchfork W^s_{loc}(\phi^+). \tag{3.45}$$

Proof. If they do not interesect at all we are done. Assume there exists some heteroclinic like the one we considered before:

$$\phi^- \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \phi^+,$$

Let without loss of generality $x_t = T(t)x_0$ for $t \in \mathbb{R}$ and for some $x_0 \in W^u(\phi^-) \cap W^s_{loc}(\phi^+)$. Then from the previous results we know that $T_{x_0}W^u(\phi^-)$ has dimension $i(\phi^-) > i(\phi^+) = \text{codim} T_{x_0}W^s_{loc}(\phi^+)$. If $i(\phi^+) = 0$ define $\lambda = |\mu_k|$ where μ_k is an eigenvalue of $D_{\phi^-}T(1)$ ordered like in theorem (2.4.2) so that $\dim \mathcal{H}_k = i(\phi^+)$, it is clear from the previous lemma that $|\lambda| > 1$ since $i(\phi^-) > i(\phi^+)$. Denote $S(t,s)$ the solution operator of (3.15) linear equation satisfied by $\dot{x}(t)$, we define:

$$F(m) := \left\{ \begin{array}{l} \psi \in C^0 | \text{ there exists } \psi_n \in C^0 \text{ so that } S(m,n)\psi_n = \psi, \\ \forall n \leq m, \lim_{n\to-\infty} \|\psi_n\|^{1/n} \geq |\lambda| \end{array} \right\}, \tag{3.46}$$

from theorems B.10, B.11 in [2], for m small enough this set is isomorphic to a closed Banach space of dimension $i(\phi^+)$, and arguing like in the proof of lemma (3.3.2) $z(\psi) < i(\phi^+)$ for $\psi \in F(m)$, $m \leq 0$. Once again by the injectivity and dense range of $S(0,m)$ it follows that $F(0) = S(0,m)F(m)$ and it holds that:

$$T_{x_0}W^u(\phi^-) + T_{x_0}W^s_{loc}(\phi^+) \supset F(0) + T_{x_0}W^s_{loc}(\phi^-) = X, \tag{3.47}$$

one can easily see this since the spaces have *complementary dimensions* and $F(0) \cap T_{x_0}W^s_{loc}(\phi^-) = \emptyset$ by (3.3.2).

In case $i(\phi^+) = 0$, $W^s_{loc}(\phi^+) = V$ where $V \subset C^0$ is an open neighborhood of ϕ^+ and consequently transversality is immediate. □

Chapter 4

Asymptotic behaviour near periodic orbits

The following sections follow the lines of [3] to study the behaviour of solutions close to periodic orbits. We already saw in Chapter 2 that solutions approaching a hyperbolic periodic orbit in either time direction do it along manifolds which are fibrated by strong invariant manifolds related to each of their elements, i.e. given a ω-periodic solution p of (2.1) defining a periodic orbit

$$\Pi = \{p_\alpha = T(\alpha)p \,|\, \alpha \in [0, \omega]\},$$

and a globally defined solution x_t such that $\lim_{t\to\infty} \mathrm{d}(x_t, \Pi) = 0$, by Theorem (2.5.3) there exists a unique α such that $\lim_{n\to\infty} T(n\omega)x_0 = p_\alpha$.

The first section of this chapter deals with the way in which solutions approach periodic orbits. The main tools used were already introduced in Section (3.1) and will allow us to obtain bounds for the zero numbers in lemmas (4.1.1) and (4.1.2). Later we combine this results in Lemma (4.1.3) using a different approach from the one used in the previous chapter, these results complete the proof of Theorem B. It is in the second section that we study the tangent spaces of the invariant manifolds, by using techniques already developed for equilibria, we will be able to finish proving Theorem A.

4.1 Estimates for the zero number of solutions approaching periodic orbits

The main content of this section is the following lemma together with its counterpart for backwards time direction.

Lemma 4.1.1
For the previously defined global solution of (2.1) x_t approaching the hyperbolic periodic orbit $\Pi = \{T(t)p \,|\, t \in \mathbb{R}\}$, so that $\lim_{n\to\infty} T(n\omega)x_0 = p_\alpha$. Then the following holds for $z(x_0 - p_\alpha)$:

1. *In case $f_v < 0$:*

$$
\begin{aligned}
z(x_0 - p_\alpha) &\geq i(\Pi) + 1 \text{ if } i(\Pi) = 2(N-1),\ N = 1, 2, \ldots && (4.1)\\
z(x_0 - p_\alpha) &\geq i(\Pi) + 2 \text{ if } i(\Pi) = 2N - 1,\ N = 1, 2, \ldots && (4.2)
\end{aligned}
$$

2. *In case $f_v > 0$:*

$$
\begin{aligned}
z(x_0 - p_\alpha) &\geq i(\Pi) + 2 \text{ if } i(\Pi) = 2(N-1),\ N = 1, 2, \ldots && (4.3)\\
z(x_0 - p_\alpha) &\geq i(\Pi) + 1 \text{ if } i(\Pi) = 2N - 1,\ N = 1, 2, \ldots && (4.4)
\end{aligned}
$$

Proof. Let $y(t) = x(t) - T(t)p_\alpha(0) = x(t) - p_\alpha(t)$, consider the linearized equation along the (global) trajectory $x(t)$ of (2.1):

$$
\begin{aligned}
\dot{y}(t) &= a(t)y(t) + b(t)y(t-1) = L(t)y_t, && (4.5)\\
a(t) &= \int_0^1 f_u\big(\theta x(t) + (1-\theta)p_\alpha(t), x(t-1)\big)\mathrm{d}\theta, && (4.6)\\
b(t) &= \int_0^1 f_v\big(p_\alpha(t), \theta x(t-1) + (1-\theta)p_\alpha(t-1)\big)\mathrm{d}\theta. && (4.7)
\end{aligned}
$$

We will denote the solution operator $S(t,s)$, consider now the linearization of (2.1) along the periodic trajectory $p_\alpha(t) = T(t)p_\alpha(0)$:

$$
\begin{aligned}
\dot{z^+}(t) &= a^+(t)z^+(t) + b^+(t)z^+(t-1) = L^+(t)y_t, && (4.8)\\
a^+(t) &= f_u(p_\alpha(t), p_\alpha(t-1)), && (4.9)\\
b^+(t) &= f_v(p_\alpha(t), p_\alpha(t-1)). && (4.10)
\end{aligned}
$$

Its solution operator $S^+(t,s)$ fulfils $S^+_\omega := S^+(\omega, 0) = D_{p_\alpha}T(\omega)$ and without loss we may assume it is compact. We are interested in the iteration:

$$u_{n+1} = y_{(n+1)\omega} = S((n+1)\omega, n\omega)y_{n\omega} = S^+_\omega u_n + (S((n+1)\omega, n\omega) - S^+_\omega), \quad (4.11)$$

$n \geq 0$, $u_0 = x_0 - p_\alpha$. Observe that $\lim_{t\to\infty} \|L(t) - L^+(t)\|_{(C^0)^*} = 0$, an argument similar to the one in Lemma (3.3.2) gives us that $\lim_{n\to\infty} \|S((n+1)\omega, n\omega) - S^+_\omega\|_{\mathcal{L}(C^1)} = 0$. Since we know that the phase space splits into three S^+_ω-invariant subspaces $C^0 = X^u \oplus X^c \oplus X^s$, using the theory from Chapter 3 we want to see if:

$$\limsup_{n\to\infty} \|u_n\|^{1/n} < 1.$$

Denote by $P = P(1-\varepsilon)$ for some $\varepsilon > 0$ small enough, the projection of the phase space onto $X^u \oplus X^c$, $Q = I - P$, by Theorem (2.5.3) it follows that for n large enough $u_n = b_n + h^s(b_n, \alpha)$ with $b_n \in X^s$, $h^s : X^s \times [0, \omega) \to X^u \oplus X^c$, $h^s(\cdot, \alpha)$ defined in a neighborhood of 0 in X^s, $h^s(0, \alpha) = Dh^s(0, \alpha) = 0$:

$$\lim_{n\to\infty} \frac{\|Pu_n\|_{C^0}}{\|Qu_n\|_{C^0}} = \lim_{n\to\infty} \frac{\|h^s(b_n, \alpha)\|_{C^0}}{\|b_n\|_{C^0}} = 0, \tag{4.12}$$

this implies by Theorem B.2. in [2] that $\limsup_{n\to\infty} \|u_n\|^{1/n} < 1$, let $\lambda := \lim_{n\to\infty} \|u_n\|^{1/n}$ by Theorem B.3. in [2], either $\lambda > 0$ is the module of an eigenvalue of S_ω^+ or $\lambda = 0$. In case $\lambda = 0$, once again we have superexponential convergence and $z(x_0 - p_\alpha) = \infty$. In case $\lambda > 0$, we can apply Lemma (3.1.1) and there must exist a subsequence such that $\left\{\frac{u_{n_k}}{\|u_{n_k}\|}\right\}$ converges to an eigenfunction Ψ related to an eigenvalue μ with $|\mu| = \lambda$, since eigenfunctions only have simple zeros, $u_{n_k} \in C^1$ and for large k, $\|\dot{u}_{n_k}\|$ can be bounded by $\|u_{n_k}\|$, there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$, $z\left(\frac{u_{n_k}}{\|u_{n_k}\|}\right) = z(\Psi)$. Theorem (2.4.2) together with the zero dropping lemma (2.3.1) yield the result. □

Lemma 4.1.2
For a global solution of (2.1) x_t approaching the hyperbolic periodic orbit $\Pi = \{T(t)p \,|\, t \in \mathbb{R}\}$ in backwards time direction, so that $\lim_{n\to-\infty} T(n\omega)x_0 = p_\alpha$. Then the following holds for $z(x_0 - p_\alpha)$:

1. *In case $f_v < 0$:*

$$\begin{aligned} z(x_0 - p_\alpha) &\leq i(\Pi) \text{ if } i(\Pi) = 2N-1,\ N = 1, 2, \dots && (4.13)\\ z(x_0 - p_\alpha) &\leq i(\Pi) - 1 \text{ if } i(\Pi) = 2N,\ N = 1, 2, \dots && (4.14)\end{aligned}$$

2. *In case $f_v > 0$:*

$$\begin{aligned} z(x_0 - p_\alpha) &\leq i(\Pi) - 1 \text{ if } i(\Pi) = 2N-1,\ N = 1, 2, \dots && (4.15)\\ z(x_0 - p_\alpha) &\leq i(\Pi) \text{ if } i(\Pi) = 2N,\ N = 1, 2, \dots && (4.16)\end{aligned}$$

Proof. It is completely analogous to the proof of (4.1.1). □

As we will see now these two previous lemmas allow us to characterize possible heteroclinic connections on the attractor. The next result is a version of Lemma (3.3.3) for periodic orbits. Its proof relies essentially on the C^1 convergence of heteroclinics to periodic orbits, and the compactness of the periodic orbits and follows the lines of Section 7 in [3].

Lemma 4.1.3
Let x_t be a globally defined solution so that:

$$\Pi^- \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \Pi^+, \tag{4.17}$$

for two periodic orbits $\Pi^\pm = \{T(t)p^\pm | t \in \mathbb{R}\}$ with minimal periods $\omega^\pm$¿0. Then it follows that:

$$i(\Pi^-) > i(\Pi^+). \tag{4.18}$$

In particular, there cannot exist homoclinics to a periodic orbit of (2.1).

Proof. Assume without loss that $p^\pm$ are the only elements in $\Pi^\pm$ respectively such that $\lim_{t\to\pm\infty} T(t)(x_0 - p^\pm) = 0$, the main idea of the proof is that when the expression $\|T(t)(x_0 - p^\pm)\|_{C^1}$ is small enough, its contribution to $z(p^+ - p^-)$ is negligible. By Lemmas 7.1. and 7.2. in [3], there exists some $\varepsilon > 0$ for which one can find a finite cover $\bigcup_{i=1}^n B_{\varepsilon_i}(v_{(t_i,s_i)})$ in C^1 of the set

$$\{v_{(t,s)} = T(t)p^+ - T(s)p^- \,|\, t, s \in \mathbb{R}\},$$

so that the zero number is constant inside the cover ($z(T(t)p^+ - T(s)p^-)$ in particular is constant) and so that $\varepsilon_i > \varepsilon$ for $i = 1, \dots, n$. Choosing $|t|$ large enough so that $\|x_{\pm t} - T(\pm t)p^\pm\| < \varepsilon/2$. Then the following holds:

$$\begin{aligned} i(\Pi^-) &\geq z(x_0 - p^-) \geq z(x_t - T(t)p^+ + T(t)p^+ - T(t)p^-) \\ &= z(T(t)p^+ - T(t)p^-) = z(T(-t)p^- - T(-t)p^+) \\ &= z(x_{-t} - T(-t)p^- + T(-t)p^- - T(t)p^-). \\ &\geq z(x_0 - p^+) \geq i(\Pi^+) + 1. \end{aligned}$$

This combined with the previous lemmas yields the result. □

Remark 4.1.1. *Although we announced this result as an analogous of Lemma (3.3.3), the way in which we proved it follows a completely different philosophy. It should be possible to prove the result with the arguments of Lemma (3.3.3), but this method shows better some of the common characteristics that the Sturm PDE treated in [3] shares with (2.1). Knowing all of this one can prove rather easily a similar lemma for the pairing equilibrium-periodic orbit and vice versa.*

Lemma 4.1.4
Let Π be a hyperbolic periodic orbit of (2.1) and ϕ be a hyperbolic equilibrium, then the following hold:

- *Let x_t be a heteroclinic so that:*

$$\phi \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \Pi, \tag{4.19}$$

then $i(\phi) > i(\Pi)$.

- *Let x_t be a heteroclinic so that:*

$$\Pi \overset{t\to-\infty}{\longleftarrow} x_t \overset{t\to\infty}{\longrightarrow} \phi, \tag{4.20}$$

then $i(\Pi) > i(\phi)$.

Proof. It is enough to prove the first case, let $p \in \Pi$ be the element of the periodic orbit such that $T(t)x_0 \overset{C^1}{\longrightarrow} T(t)p$. Like in (4.1.3), there exists some $\varepsilon > 0$ some finite cover $\bigcup_{i=1}^{n} B_{\varepsilon_i}(v_{t_i})$ in C^1 of the set $\{v_t = T(t)p - \phi \,|\, t, s \in \mathbb{R}\}$ with constant zero number in the covering and $\varepsilon \leq \varepsilon_i$ for $i = 1, \ldots, n$. □

This ends the proof of the previously announced Theorem B.

4.2 Tangent spaces to invariant manifolds of periodic orbits

Lemma 4.2.1
Consider $p \in C^0$ a periodic solution of (2.1) with minimal period $\omega > 0$ and let $\Pi = \{T(t)p \,|\, t \in \mathbb{R}\}$ be hyperbolic. Assume without loss that the semigroup $T(t)$ is compact for $t \geq \omega$ and for $x(t)$ a global trajectory of (2.1), define the family of operators $S(t,s)$, $t \geq s$ solving:

$$\begin{aligned} \dot{y}(t) &= a(t)y(t) + b(t)y(t-1) = L(t)y_t, && (4.21)\\ y_s &= \xi, \text{ with} && (4.22)\\ a(t) &= f_u(x(t), x(t-1)), && (4.23)\\ b(t) &= f_v(x(t), x(t-1)). && (4.24) \end{aligned}$$

- *If $x_0 \in W^s_{loc}(\Pi)$, then it follows that:*

$$T_{x_0}W^s_{loc}(\phi) = \{\psi \in C^0 \,|\, \lim_{n\to\infty} \|S((n+1)\omega, 0)\psi\|^{1/n} \leq 1\}. \tag{4.25}$$

- *If $x_0 \in W^u(\Pi)$ global unstable manifold of Π, then:*

$$T_{x_0}W^u(\phi) = \left\{ \begin{array}{l} \psi \in C^0 \,|\, \exists\, u_{n+1} = S((n+1)\omega, n\omega)u_n \in C^0, \\ n \leq 0,\ u_0 = \psi \text{ and } \lim_{n\to-\infty} \|u_n\|^{\frac{1}{n}} \geq 1 \end{array} \right\}. \tag{4.26}$$

Proof. From (2.5.3) we know that, assuming $x_0 \in W^{ss}_{loc}(p_\alpha)$:

$$\begin{aligned} x_0 &= p_\alpha + b_0 + h^s(b_0, \alpha), && (4.27)\\ T_{x_0}W^s(\Pi) &= \{\beta \dot{p}_\alpha + \psi + Dh^s(b_0,\alpha)(\psi,\beta) \,|\, (\psi,\beta) \in X^s \times \mathbb{R}\}. && (4.28) \end{aligned}$$

On the one hand we have that for n large enough $T_{x_{n\omega}}W^s_{loc}(\Pi)$ is isomorphic to $X^c \oplus X^s$, since h^s is C^1 and $Dh(0,\alpha) = 0$. On the other hand we have that $\|S((n+1)\omega, n\omega) - D_{p_\alpha}T(\omega)\|_{\mathcal{L}(C^1)} \xrightarrow{n\to\infty} 0$. Therefore by theorems B.7. and B.8. in [2], the set:

$$E(m) := \{\psi \in C^0 | \lim_{n\to\infty} \|S(n\omega, m\omega)\psi\|^{1/n} \leq 1, \text{ for } n > m\},$$

is isomorphic to $X^c \oplus X^s$, a closed Banach space of codimension $i(\Pi)$ for m large enough. Taking into account that $S(m\omega, 0)$ is an injective operator with dense range by the arguments from Lemma (2.2.3), the following holds for m big enough:

$$\begin{aligned} T_{x_0}W^s_{loc}(\Pi) &= S(m\omega,0)^{-1}T_{x_{m\omega}}W^s_{loc}(\Pi) \cong X^c \oplus X^s & (4.29)\\ &\cong E(m) = S(m\omega,0)E(0). & (4.30)\end{aligned}$$

However, it is easy to see that $T_{x_0}W^s_{loc}(\Pi) \subset E(0)$ since given $\xi \in T_{x_0}W^s_{loc}(\Pi)$, one can write $x_{n\omega} = p_\alpha + h^s(b_n, \alpha)$ and $y_n = S(n,0)\xi$ comes given by

$$y_n = \beta_n \dot{p}_\alpha + \psi_n + Dh^s(b_n,\alpha)(\psi_n, \beta_n).$$

With this parameterization and considering that $X^s \supset b_n \xrightarrow{n\to\infty} 0$ and $h^s(0,\alpha) = Dh^s(0,\alpha) = 0$ one can see that:

$$\lim_{n\to\infty} \frac{\|P(1)y_n\|}{\|Q(1)y_n\|} = 0, \tag{4.31}$$

this implies once again by Theorem B.2. in [2] that $\xi \in E(0)$ and completes that $T_{x_0}W^s_{loc}(\Pi) = E(0)$. Proving the result for the tangent space of the unstable manifold follows the exact same steps. □

The last step left to prove transversality for hyperbolic periodic orbits follows our usual strategy. However, one has to be very careful since the spectrum of the monodromy operator may (and does) present simple real eigenvalues. Nevertheless, the proof still works as we show below.

Corollary 4.2.2
Given two hyperbolic periodic orbits $\Pi^\pm$ of (2.1), their global unstable manifold and local stable manifold intersect transversely:

$$W^u(\Pi^-) \pitchfork W^s_{loc}(\Pi^+). \tag{4.32}$$

Proof. If they do not interesect we are done. Assume there exists some heteroclinic like the one we considered before:

$$\Pi^- \xleftarrow{t\to-\infty} x_t \xrightarrow{t\to\infty} \Pi^+.$$

Let without loss of generality $x_t = T(t)x_0$ for $t \in \mathbb{R}$ and for some $x_0 \in W^{su}(p^-) \cap W^{ss}_{loc}(p^+)$. Then from the previous results we know that:

$$\dim T_{x_0} W^u(\Pi^-) = i(\Pi^-) + 1 \geq i(\Pi^+) + 2 = \text{codim } T_{x_0} W^s_{loc}(\Pi^+) + 2.$$

Let $\mu_j^\pm$ be the eigenvalues of the monodromy operators around $\Pi^\pm$ respectively ordered like in Theorem (2.4.2), in particular $\mu^-_{i(\Pi^-)+1} = \mu^+_{i(\Pi^+)+1} = 1$. One must consider the following cases, which are quite delicate and fix the parity of the unstable dimensions and the sign of f_v in (2.1):

- In case $f_v < 0$ and $i(\Pi^-)$ is even, if we consider the family of operators $S(t,s)$ solving (4.21) then, arguing like in the proof (3.3.4) we have that the subspace:

$$\begin{aligned} F &:= \left\{ \begin{array}{l} \psi \in C^0 \mid \text{there is } u_{n+1} = S((n+1)\omega, n\omega)u_n \in C^0, \\ n \leq 0,\ u_0 = \psi \text{ and } \lim_{n\to-\infty} \|u_n\|^{\frac{1}{n}} > 1 \end{array} \right\} && (4.33) \\ &\subset T_{x_0} W^u(\Pi^-), && (4.34) \end{aligned}$$

  is a Banach space with dimension $i(\Pi^-)$, applying an argument like in the proof of Lemma (3.3.2) we can show that for any $\psi \in F$, $z(\psi) \leq i(\Pi^-) - 1$. In a same fashion the subset:

$$\begin{aligned} E &:= \left\{ \psi \in C^0 \mid \lim_{n\to\infty} \|S(n\omega, 0)\psi\|^{1/n} < |\mu^+_{i(\Pi^-)}| \right\} && (4.35) \\ &\subset T_{x_0} W^s_{loc}(\Pi^+), && (4.36) \end{aligned}$$

  is a Banach space with codimension $i(\Pi^-)$ so that given $\psi \in E$, $z(\psi) \geq i(\Pi^-) + 1$, thus:

$$F + E = C^0. \tag{4.37}$$

- In case $f_v < 0$ and $i(\Pi^-)$ is odd, we have that the subspace:

$$\begin{aligned} E &:= \left\{ \psi \in C^0 \mid \lim_{n\to\infty} \|S(n\omega, 0)\psi\|^{1/n} < |\mu^+_{i(\Pi^-)+1}| \right\} && (4.38) \\ &\subset T_{x_0} W^s_{loc}(\Pi^+), && (4.39) \end{aligned}$$

  is a Banach space with codimension $i(\Pi^-) + 1$ so that given $\psi \in E$, $z(\psi) \geq i(\Pi^-) + 2$. Since we know from Lemma (4.2.1) that for $\psi \in T_{x_0} W^u(\Pi^-)$, $z(\psi) \leq i(\Pi^-)$, it follows that:

$$T_{x_0} W^u(\Pi^-) + E = C^0. \tag{4.40}$$

The case for positive feedback $f_v > 0$ is exactly the same since the first real eigenvalue $\mu_0^{\pm}$ just shifts the spectrum and does not play any special role. □

Corollary 4.2.3
Given a hyperbolic equilibrium ϕ and a hyperbolic periodic orbit Π of (2.1), their invariant manifolds intersect transversely:

$$W^u(\phi) \pitchfork W^s_{loc}(\Pi), \text{ and} \tag{4.41}$$
$$W^u(\Pi) \pitchfork W^s_{loc}(\phi). \tag{4.42}$$

Proof. It is essentially the same as in the previous corollary considering the equilibrium to be a periodic orbit which unstable dimension has a fixed parity (even if $f_v < 0$ and odd otherwise). □

This concludes the proof of Theorem A.

Chapter 5

Conclusions

The results we obtained showcase the strength of the *zero number*. This particular feature is the only thing ensuring that the invariant manifolds of critical elements intersect transversely. Apart from it, some specific properties of the special setting provided by (2.1) that were used are the compact dissipative semiflow (quite a common requirement when one studies global attractors), the injectivity and dense range of the derivative of the semiflow and the increase in regularity of solutions.

Theorem A immediately brings our equation (2.1) into the family of Morse-Smale semiflows, for which a large theory has already been developed in the past. One of the most interesting consequences is the fact that we can apply now rather general structural stability theorems (see for instance [8]). In particular our results come in handy when one considers that many of the already studied attractors arise from nonlinearities of the form:

$$f(x(t), x(t-1)) = -\mu x(t) + g(x(t-1)), \tag{5.1}$$

where g regular enough fulfils some monotonicity condition. In particular, under hyperbolicity assumption on the critical elements of (2.1), Theorem A extends substantially the family of nonlinearities for which the structure of the global attractor is known.

Concerning Theorem B, it is unknown to us to which extent a result of this type was already proved in the delay equation setting, but it is widely used in the study of Sturm attractors on the circle (1.2) and we thought that stating it in a similar way here would help to highlight even more how closely related the global dynamics of both systems seem to be.

We expect to be able in the future to use the results here presented and the way in which they were obtained to throw some more light on the general structure of global attractors of (2.1). We also hope to find some day an explanation to why two completely different settings (equations (2.1) and (1.2)) show so many similarities and if there is any reasonable general principle governing both dynamics.

Bibliography

[1] Yulin Cao,
The discrete Lyapunov function for scalar differential delay equations, Journal of Differential Equations, Volume 87, Issue 2, October 1990, Pages 365-390.

[2] Mingxiang Chen, Xu-Yan Chen and Jack K. Hale,
Structural stability for time-periodic one-dimensional parabolic equations, Journal of Differential Equations Volume 96, Issue 2, April 1992, Pages 355-418.

[3] R. Czaja and C. Rocha,
Transversality in scalar reaction-diffusion equations on a circle, Journal of Differential Equations, Volume 245, Issue 3, August 2008, Pages 692-721.

[4] Bernold Fiedler and John Mallet-Paret,
A Poincaré-Bendixson theorem for scalar reaction diffusion equations, Archive for Rational Mechanics and Analysis, Volume 107, Issue 4, September 1989, Pages 325–345.

[5] Bernold Fiedler, Carlos Rocha and Matthias Wolfrum,
Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle, Journal of Differential Equations, Volume 201, Issue 1, June 2004, Pages 99-138.

[6] Bernold Fiedler, Carlos Rocha and Matthias Wolfrum,
Sturm global attractors for S^1-equivariant parabolic equations, Networks and Heterogeneous Media Volume 7, 2012, Pages 617-659.

[7] Jack K. Hale,
Theory of functional differential equations, Springer-Verlag, New York, 1977.

[8] Jack K. Hale and Xiao-Biao Lin,
Heteroclinic orbits for retarded functional differential equations, Journal of Differential Equations Volume 65, Issue 2, November 1986, Pages 175-202.

[9] Jack K. Hale, Sjoerd M. Verduyn Lunel,
Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[10] Daniel Henry,
Geometric theory of semilinear parabolic equations, Springer-Verlar, Heidelberg, 1981.

[11] Tibor Krisztin, Hans-Otto Walther and Jianhong Wu,
Shape, smothness and invariant stratification of an attracting set for delayed monotone positive feedback, Fields Institute Monographs, Vol. 11, Amer. Math. Soc., Providence, RI, 1999.

[12] Tibor Krisztin and Hans-Otto Walther,
Unique periodic orbits for delayed positive feedback and the global attractor, Journal of Dynamics and Differential Equations, Volume 13, Issue 1, January 2001 Pages 1–57.

[13] John Mallet-Paret,
Morse Decompositions for delay differential equations, Journal of Differential Equations Volume 72, Issue 2, April 1988, Pages 270-315.

[14] John Mallet-Paret and Roger D. Nussbaum,
Tensor products, positive linear operators, and delay-differential equations, Journal of Dynamics and Differential Equations, Volume 25, Issue 4, December 2013, Pages 843–905. .

[15] John Mallet-Paret and George Sell,
Systems of differential delay equations: Floquet multipliers and discrete Lyapunov functions, Journal of Differential Equations Volume 125, Issue 2, March 1996, Pages 385-440.

[16] John Mallet-Paret and George Sell,
The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay, Journal of Differential Equations Volume 125, Issue 2, March 1996, Pages 441-489.

Selbständigkeitserklärung

Hiermit bestätige ich, Alejandro López Nieto, dass ich die vorgelegte Masterarbeit mit dem Thema *Heteroclinic connections in delay equations*, selbstständig angefertigt und nur die erwähnten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

Berlin, den