Finite dimensional approximations of dynamical systems generated by delay equations

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Master Thesis in Mathematics

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Contents

Introduction

In this thesis, we study three different approximation methods. First, we study the pseudospectral approximation method, where we approximate eigenvalues of delay equations. Moreover, we look at the parametrisation method, which we can use to approximate invariant manifolds of both finite and infinite dimensional dynamical systems. Lastly, we study Trotter-Kato approximation methods, where we approximate flows and orbits of delay equations.

Delay differential equations

Let X be a Banach space and let $A : X \to X$ be a bounded linear operator; on X, let us study the differential equation

$$\begin{cases} \dot{x}(t) & = Ax(t), \quad t \geq 0, \\ x(0) & = x_0 \in X. \end{cases} \tag{1}$$

Associated to this abstract differential equation is a uniformly continuous semigroup of operators, whose generator is given by A. In fact, the semigroup is given by a shift along the solution and can concretely be represented as

$$T(t)x_0 = e^{At}x_0, \quad t \geq 0, \ x_0 \in X.$$

The stability of the zero solution of (1) is determined by the eigenvalues of the generator A: if all the eigenvalues of A are in the left half of the complex plane, the solution $x \equiv 0$ of (1) is stable. If one of the eigenvalues of (1) is in the right half of the complex plane, the solution $x \equiv 0$ of (1) is unstable.

Now, let us turn towards the (linear) delay differential equation

$$\dot{x}(t) = Bx(t) + Cx(t - \tau), \quad t \geq 0 \tag{2}$$

with $x(t) \in \mathbb{R}^d$ and B, C both $d \times d$ matrices. The state space of (2) is chosen to be $X = C\left([-\tau, 0], \mathbb{R}^d\right)$ and (2) induces a strongly continuous semigroup of operators $\{T(t)\}_{t \geq 0}$ on X, which is defined by a shift along the solution of (2). The generator of this strongly continuous semigroup is an unbounded operator given by

$$\mathcal{D}(A) = \left\{ \phi \in X \mid \in C^1\left([-\tau, 0], \mathbb{R}^d\right), \ \dot{\phi}(0) = B\phi(0) + C\phi(-\tau) \right\}, \quad A\phi = \dot{\phi}. \tag{3}$$

We note that the action of the generator is differentiation, which reflects that the semigroup $\{T(t)\}_{t \geq 0}$ describes a shift. The fact that the semigroup shifts according to some prescribed rule (defined by the delay equation) is reflected in the domain condition.

In fact, the delay equation (2) is equivalent to the abstract ordinary differential equation

$$\dot{x}(t) = Ax(t), \quad t \geq 0 \tag{4}$$

on X, with the unbounded operator A given by the generator (3).

The pseudospectral method

Similar to the problem (1), the stability of the zero solution in (4) – and hence in (2) – is determined by the spectrum of the generator A. The spectrum of A can be characterised as roots of a transcendental characteristic equation.

In the pseudospectral method, we discretise the unbounded operator A given in (3) to obtain finite dimensional linear maps $A_n : \mathbb{R}^{(n+1)\times d} \to \mathbb{R}^{(n+1)\times d}$ whose eigenvalues can be viewed as an approximation of the eigenvalues of A. It turns out that the characteristic equation of A_n has a natural interpretation as a discretisation of the characteristic equation associated to the problem (2).

We note that the delay equation (2) and the pseudospectral approximation maps $A_n : \mathbb{R}^{(n+1)\times d} \to \mathbb{R}^{(n+1)\times d}$ are linear. If we study a nonlinear delay equation, then one can write down a family of nonlinear pseudospectral approximation maps $A_n : \mathbb{R}^{(n+1)\times d} \to \mathbb{R}^{(n+1)\times d}$. These nonlinear pseudospectral approximation maps have the property that there is a clear correspondence between the nonlinear terms in the pseudospectral approximation and the nonlinear terms in the original delay equation. Moreover, the eigenvalues of the original delay equation are well approximated by the eigenvalues of the pseudospectral approximation. This combination of features hints that the following two properties are approximated in the pseudospectral method:

- Invariant manifolds;

- Bifurcation behaviour.

The parametrisation method

To study the relation between invariant manifolds of delay equations and of their pseudospectral approximation, we use the parametrisation method.

In the parametrisation method, we let $\mathcal{M}$ be an invariant at the origin of either a delay equation or an ordinary differential equation. Let $y(t)$ be the restricted flow on the invariant manifold and denote by X_0 the tangent space to $\mathcal{M}$ at the origin. In the parametrisation method, we conjugate $y(t)$ to a 'simpler' flow $u(t)$ on X_0 via a conjugation map P. We want to choose $u(t)$ in such a way that i) a conjugation between $y(t)$ and $u(t)$ is possible, i.e. the conjugation map P exists and ii) the conjugation map P gives a local description of the invariant manifold near zero. Having chosen $u(t)$ in such a way, we then algorithmically compute the coefficients of P.

In the field of rigorous computations, where one combines numerical methods with analytical estimates to give rigorous proofs of existence results, the parametrisation method is much used. This is mainly because it is very suitable for a posteriori error analysis. In the context of pseudospectral approximation, the method is attractive because it provides a general framework for the study of invariant manifolds in both delay equations and ordinary differential equations. This allows us to jump back and forth easily between the invariant manifolds of delay equations and their pseudospectal approximation.

Trotter-Kato Theorem

In the pseudospectral method, we discretised the generator of delay equations in order to approximate characteristic properties of the system (2) such as eigenvalues and invariant manifolds. To study the approximation of actual orbits of delay equations, we look at the Trotter-Kato theorem, in which we discretise the semigroup associated to a delay equation. We study the functional analytic framework to set up an approximation of the semigroup, and apply this to the cases of spline approximation and approximation using Legendre polynomials.

Organisation of the thesis

This thesis is divided in three parts. Part A deals with the pseudospectral method and the parametrisation method. In Chapter 1, we study the pseudospectral method and the approximation of eigenfunctions. In Chapter 2, we study the parametrisation method for ordinary differential equations and study its relation with normal form theory. In Chapter 3, we turn our attention towards the parametrisation method for delay equations.

In Part B, we combine the methods introduced in part A to study approximation of invariant manifolds in the pseudospectral method. In Chapter 4, we give a characterisation of the eigenvectors of the pseudospectral matrices A_n. Using this characterisation, we then study the approximation of center manifolds and unstable manifolds in Chapter 5.

In Part C, we study the framework of Trotter-Kato approximation and, using different schemes for function approximation, we then apply this to delay equations.

In the Appendix, we give an overview of various schemes for function approximation used in this thesis. Moreover, we provide some background on numerical methods for solving ordinary differential equations.

Part C can be read independently of Part A and Part B. The interdependence of Part A and Part B is represented in the following diagram:
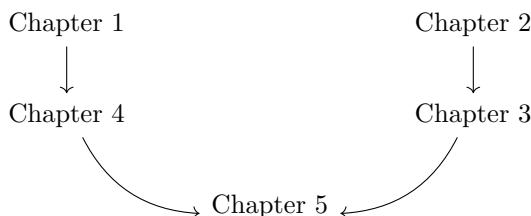
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Part A

Chapter 1

Pseudospectral approximation

1.1   Introduction

In this chapter, we study the pseudospectral approximation method for delay equations. Consider the delay differential equation

$$\begin{cases} \dot{x}(t) &= Lx_t, \quad t > 0, \\ x_0 &= \phi, \end{cases} \tag{1.1}$$

where $X = C\left([-\tau, 0], \mathbb{C}^d\right)$ for some $\tau > 0$ and $d \in \mathbb{N}$, $L : X \to \mathbb{C}^d$ is a bounded linear map, $\phi \in X$ and for $t \geq 0$ the function $x_t \in X$ is defined via $x_t(\theta) = x(t + \theta)$. Since we are interested in approximating the spectrum of delay equations, we will work with complex Banach spaces; hence our choice for the state space as the complex Banach space $X = C\left([-\tau, 0], \mathbb{C}^d\right)$. To the initial value problem (1.1) we can associate a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of solution operators. The generator A of this semigroup is given by

$$\mathcal{D}(A) = \{\phi \in X \mid \phi \in C^1\left([-\tau, 0], \mathbb{C}^d\right), \ \dot{\phi}(0) = L\phi\}, \quad A\phi = \dot{\phi}.$$

In fact, the initial value problem (1.1) is equivalent to the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= Au(t), \quad t > 0 \\ u(0) &= \phi \end{cases} \tag{1.2}$$

with $\phi \in X$ [11].

In the pseudospectral approximation method, we approximate the generator A by approximating continuous functions – which make up the state space X of the problem – by polynomials. In this way, we obtain finite-dimensional maps $A_n : \mathbb{C}^{d \times (n+1)} \to \mathbb{C}^{d \times (n+1)}$. The maps A_n in turn induce ODEs on $\mathbb{C}^{d \times (n+1)}$, which approximate the abstract ODE (1.2). Of course, we have to make the meaning of 'approximate' in this last statement more precise. As it turns out, the eigenvalues of A are approximated by the eigenvalues of A_n.

Pseudospectral discretisation is a well-known method in the study of eigenvalues of partial differential equations, see for example [16]. In [7], Breda et al. introduced the pseudospectral method for linear differential delay equations, proving that every eigenvalue of the delay equation is approached by eigenvalues of the approximating problem. In [5], the pseudospectral discretisation method was extended to nonlinear differential delay equations and other classes of delay equations, such as renewal equations. It was proven that linearisation and pseudospectral approximation commute. This makes that it is interesting to compare the bifurcation behaviour of the approximating ODE with the bifurcation behaviour of the original delay

equation. This topic was further explored for a specific example of a renewal equation in [6].

This chapter is structured as follows: in Section 1.2, we define the pseudospectral approximation maps A_n and give motivation for this definition. Section 1.3 provides an interlude on finite-difference schemes and rational approximations. This we will then use in Section 1.4 to give a characterisation of the eigenvalues of A_n; and in Section 1.5 to prove results on convergence of eigenvalues in the pseudospectral approximation. In Section 1.6, we study how we can use pseudospectral approximation in the context of nonlinear delay equations and bifurcation problems. Throughout Section 1.2, 1.4–1.5, we will follow [7]; Section 1.3 is based on [2] and in Section 1.6, we will study results from [5] and [6].

1.2 Definition of pseudospectral approximation for delay equations

In pseudospectral approximation, we approximate elements of the state space of continuous functions X by (interpolating) polynomials. In this way, we obtain a map $A_n : \mathbb{C}^{d \times (n+1)} \to \mathbb{C}^{d \times (n+1)}$, which can be viewed as a discretisation of the generator A.

To obtain the map A_n for $n \in \mathbb{N}$, we let $-\tau \leq \theta_{n,n} < \ldots < \theta_{n,0} \leq 0$ be a mesh on the interval $[-\tau, 0]$. Moreover, we let $x = (x_0, \ldots, x_n) \in \mathbb{C}^{d \times (n+1)}$ and let $\mathcal{L}_n(x)$ be the interpolating polynomial through x with respect to the chosen mesh. This means that $\mathcal{L}_n(x) : [-\tau, 0] \to \mathbb{C}^d$ is the unique polynomial of degree n satisfying $\mathcal{L}_n(x)(\theta_{n,i}) = x_i$ for all $0 \leq i \leq n$; see also Section A.3 in the appendix. We now define the discretised infinitesimal generator $A_n : \mathbb{C}^{d \times (n+1)} \to \mathbb{C}^{d \times (n+1)}$ as

$$A_n : \mathbb{C}^{d \times (n+1)} \to \mathbb{C}^{d \times (n+1)},$$

$$A_n(x) = \left(L\mathcal{L}_n(x), \left. \frac{d}{d\theta} \right|_{\theta = \theta_{n,1}} \mathcal{L}_n(x)(\theta), \ldots, \left. \frac{d}{d\theta} \right|_{\theta = \theta_{n,n}} \mathcal{L}_n(x)(\theta) \right). \tag{1.3}$$

We note that the first component of (1.3) discretises the domain condition of the generator, and the other component discretises the action of the generator.

Let us now study the case where $L : X \to \mathbb{C}^d$ is of the form $L\phi = A\phi(0) + B\phi(-\tau)$ with A, B both $d \times d$–matrices. Using the Lagrange-form of the interpolating polynomial $\mathcal{L}_n(x)$ (see Section A.3), we find that the operator A_n as defined in (1.3) has the matrix-representation

$$A_n = \begin{pmatrix} A & 0 & \ldots & \ldots & B \\ a_{10} & a_{11} & \ldots & a_{1(n-1)} & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n0} & a_{n1} & \ldots & a_{n(n-1)} & a_{nn} \end{pmatrix}$$

with $a_{ij} = \ell'_{n,j}(\theta_{n,i})I_d$ (where $\ell_{n,j}$ is the Lagrangian base polynomial as in Section A.3) and I_d is the identity operator on $\mathbb{C}^d$.

1.3 Interlude: finite-difference schemes and rational approximation

In this interlude, we turn our attention to finite-difference schemes and rational approximations, where we approach semigroups by leaving the generator intact, but approximating the exponential relation between the semigroup and the generator by a rational function. In particular, we will see that all collocation methods to numerically solve ODEs can be viewed as rational approximations. Collocation methods will then return in Section 1.4 and 1.5 in the study of eigenvalues in the pseudospectral approximation scheme.

Throughout this section, we follow [2].

Let X be a Banach space and let $A : \mathcal{D}(A) \to X$ be the generator of a $\mathcal{C}_0$-semigroup $\{T(t)\}_{t\geq 0}$. Let us consider the following abstract Cauchy problem on X:

$$\begin{cases} \dot{u}(t) = Au(t), \quad t \geq 0 \\ u(0) = u_0. \end{cases}$$

We make the following definition:

Definition 1.3.1. Let $F : [0, \infty) \to \mathcal{L}(X)$ be such that $F(0) = I$ and F is strongly continuous.

a) Suppose that $D \subseteq \mathcal{D}(A)$ is a dense subset of X such that

$$\lim_{h \to 0} \frac{F(h)T(t)x - T(t+h)x}{h} = 0$$

for all $x \in D$, with the limit uniform for t in bounded intervals. Then we say that F is a *consistent finite different scheme*.

b) A consistent finite difference scheme is called *stable* if, for all $t_0 > 0$, there exists a $M \geq 1$ such that

$$\|F(h)^n\| \leq M$$

for all $h > 0$ and $n \in \mathbb{N}$ with $hn \leq t_0$.

c) A consistent finite difference scheme is called *convergent* if for all $t > 0$, $h_k \to 0$, $n_k \to \infty$ with $h_k n_k \in [0, t]$ for all $n \in \mathbb{N}$ and $h_k n_k \to t$, we have that

$$\lim_{k \to \infty} F(h_k)^{n_k} f = T(t)f$$

for all $f \in X$.

Example 1.3.1. If we set $F(h) = T(h)$, then F is a consistent finite difference scheme that is both stable and convergent.

We have the following theorem relating stability and convergence [2, Theorem 4.6]:

Theorem 1.3.1 (Lax Equivalence Theorem). *For a consistent finite difference scheme, stability is equivalent to convergence.*

We now look at finite difference schemes $F : [0, \infty) \to \mathcal{L}(X)$ that are of the form $F(h) = r(hA)$, where r is a rational function.

Definition 1.3.2. Let $r : \mathbb{C} \to \mathbb{C}$ be a rational function, then we call r a *rational approximation (of the exponential) of order p* if there exists a $C, \delta > 0$ such that

$$|r(z) - e^z| \leq C \, |z|^{p+1}$$

for all $z \in \mathbb{C}$ with $|z| \leq \delta$.

Let us now study the ODE

$$\begin{cases} \dot{y}(t) = \lambda y(t), \quad t \geq 0 \\ y(0) = y_0 \end{cases} \tag{1.4}$$

for some $\lambda, y_0 \in \mathbb{C}$. We recall that a Runge–Kuta method is defined via the equations (B.4), (B.5). If we apply the Runge–Kutta method (B.4), (B.5) to the ODE (1.4), we find (by rescaling $k_{s,i}$ by a factor $\frac{1}{\lambda}$) that

$$
\begin{aligned}
k_{s,i} &= y_s + h\lambda \sum_{j=1}^n a_{ij} k_{s,j} \\
y_{s+1} &= y_s + h\lambda \sum_{j=1}^n b_j k_{s,j}
\end{aligned}
\tag{1.5}
$$

If we set $\mathbf{k}_s = (k_{s,1}, \ldots, k_{s,n})^T$, $\mathbf{1} = (1, \ldots, 1)^T$, $b = (b_1, \ldots, b_n)^T$ and define the matrix $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,n}$ then (1.5) implies that

$$
\begin{aligned}
y_{s+1} &= y_s + z\mathbf{b}^T \mathbf{k}_s \\
\mathbf{k}_s &= (1 - zA)^{-1} y_s
\end{aligned}
$$

where we have set $z = \lambda h$. Thus, we find that y_{s+1} is given by

$$
y_{s+1} = \left(\mathbf{1} + z\mathbf{b}^T (1 - zA)^{-1} \right) y_s
$$

We conclude that all Runge–Kuta methods can in fact be viewed as rational finite difference schemes.

1.4 Characterisation of the eigenvalues

In this section, we give a characterisation of the eigenvalues of A_n.

Let $\lambda \in \sigma(A_n)$ and let $x = (x_0, \ldots, x_n) \in \mathbb{C}^{d \times (n+1)}$, $x \neq 0$, be the associated eigenvector. Using the definition of A_n as in (1.3), we find that $A_n x = \lambda x$ implies that

$$
L(\mathcal{L}_n(x)) = \lambda x_0 \tag{1.6}
$$

$$
(\mathcal{L}_n(x))' (\theta_{n,i}) = \lambda x_i, \quad 1 \le i \le n. \tag{1.7}
$$

Let us denote by $p_n(\lambda, u)$ the collocation solution of the ODE

$$
\begin{cases}
\dot{y}(\theta) &= \lambda y(\theta), \quad \theta \in [-\tau, 0] \\
y(0) &= u
\end{cases}
\tag{1.8}
$$

with respect to the chosen mesh (see Section 1.2).

We prove the following lemma:

Lemma 1.4.1. *Let* $x = (x_0, \ldots, x_n) \in \mathbb{C}^{(n+1) \times d}$. *Then* x *satisfies* (1.6)–(1.7) *if and only if*

$$
x_0 = L(p_n(\lambda, x_0)) \tag{1.9}
$$

and $x_i = p_n(\lambda, x_0)(\theta_{n,i})$ *for* $1 \le i \le n$.

Proof. We first show that (1.7) holds if and only if $\mathcal{L}_n x = p_n(\lambda, x_0)$. Let us assume that (1.7) holds. Since $(\mathcal{L}_n x)(0) = x_0$ and $\mathcal{L}_n x$ is a polynomial of degree n, we find by definition of the collocation solution that $\mathcal{L}_n x$ is a collocation solution of (1.8).

Now suppose that $\mathcal{L}_n x = p_n(\lambda, x_0)$, where $p_n(\lambda, x_0)$ is a collocation solution of (1.8). Then we find that

$$
\begin{aligned}
(\mathcal{L}_n x)' (\theta_{n,i}) &= p_n(\lambda, x_0)'(\theta_{n,i}) \\
&= \lambda p_n(\lambda, x_0)(\theta_{n,i}) \\
&= \lambda (\mathcal{L}_n x) (\theta_{n,i}) \\
&= \lambda x_i
\end{aligned}
$$

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for all $1 \leq i \leq n$, i.e. (1.7) holds.

Now, let $x \in \mathbb{C}^{(n+1) \times d}$ be such that (1.6)–(1.7) hold, then by (1.7) $\mathcal{L}_n x = p_n(\lambda, x_0)$, so $x_i = p_n(\lambda, x_0)(\theta_{n,i})$ for $1 \leq i \leq n$ and by (1.6) we find that $\lambda x_0 = L(p_n(\lambda, x_0))$. If we now suppose that $x_i = p_n(\lambda, x_0)(\theta_{n,i})$ for $1 \leq i \leq n$ and $\lambda x_0 = L(p_n(\lambda, x_0))$, then we find that $\mathcal{L}_n x = p_n(\lambda, x_0)$, so (1.7) holds, and $L(p_n(\lambda, x_0)) = L(\mathcal{L}_n x) = \lambda x_0$. This proves the lemma. $\qquad \square$

We now make the following definition:

Definition 1.4.1. For $n \in \mathbb{N}$, let us define

$$\sigma_n = \{\lambda \in \mathbb{C} \mid \text{there exists a } x_0 \in \mathbb{C}^d, \ x_0 \neq 0 \text{ such that } \lambda x_0 = L(p_n(\lambda, x_0))$$

We note that $\sigma_n \subseteq \sigma(A_n)$: for $\lambda \in \sigma_n$, let $x_0 \in \mathbb{C}^d$, $x_0 \neq 0$ be such that $\lambda x_0 = L(p_n(\lambda, x_0))$. If we define $x_i = p_n(\lambda, x_0)(\theta_{n,i})$ for $1 \leq i \leq n$, then $x = (x_0, \ldots, x_n) \neq 0$ (since $x_0 \neq 0$) and by Lemma 1.4.1, we find that $A_n x = \lambda x$.

We recall that $\lambda \in \sigma(A)$ if and only if there exist a $u \in \mathbb{C}^d, u \neq 0$ such that

$$\lambda u = L(e_\lambda u), \tag{1.10}$$

where $e_\lambda \in X$ defined by $e_\lambda(\theta) = e^{\lambda \theta}$ is the actual solution of (1.8). Therefore, the condition (1.9) can be viewed as the discretised counterpart of (1.10), where we have replaced the actual solution of (1.8) by the collocation solution to (1.8).

1.5 Convergence analysis

In this section, we state results from [7] on the convergence of the eigenvalues of A_n to the eigenvalues of A.

Although we gave a characterisation of eigenvalues of A_n in terms collocation solutions of (1.8), we have not studied the existence and uniqueness of the collocation polynomial. We now discuss this topic for a specific choice of mesh points, namely the Chebyshev-nodes on $[-\tau, 0]$, which are defined as

$$\theta_{n,i} = \frac{\tau}{2} \left(\cos \left(\frac{i\pi}{n} \right) - 1 \right), \quad 0 \leq i \leq n. \tag{1.11}$$

We have the following result:

Lemma 1.5.1. *Let $\lambda_0 \in \mathbb{C}$ and let $\rho_0 > 0$. Let us choose as mesh on the interval $[-\tau, 0]$ the Chebyshev nodes (1.11). There exists a $N_0 \in \mathbb{N}$ such that for $n \geq N_0$, for $\lambda \in B(\lambda_0, \rho_0)$, and for all $u \in \mathbb{C}^d$, we have that the collocation solution $p_n(\lambda, 0)$ to (1.8) exists and is unique. Moreover, we obtain the estimate*

$$\max_{\theta \in [-\tau, 0]} \left| p_n(\lambda, u)(\theta) - e^{\lambda \theta} u \right| \leq \frac{C_0}{\sqrt{n}} \left(\frac{C_1}{n} \right)^n |u| \tag{1.12}$$

where C_0, C_1 are constants that depend on λ_0 and ρ_0 but not on n. Furthermore, for $u \in \mathbb{C}^d$ and $n \geq N_0$, the map

$$B(\lambda_0, \rho_0) \to X, \quad \lambda \mapsto p_n(\lambda, u)$$

is holomorphic.

Proof. For $\lambda \in \mathbb{C}$, let us define the Volterra operator

$$K_\lambda : X \to X$$

$$(K_\lambda(\phi))(\theta) = \lambda \int_0^\theta \phi(s) ds.$$

Moreover, for $n \in \mathbb{N}$, let us define the Langrange interpolation operator

$$L_n : X \to X$$
$$L_n(f) = \mathcal{L}_{n-1}(f(\theta_{n,1}), \ldots, f(\theta_{n,n}))$$

The operator L_n is linear and bounded, see [27].

By integration, we see that y is a solution of (1.8) if and only if

$$y = u + K_\lambda y \tag{1.13}$$

By definition, we have that $p_n(\lambda, u)$ is a collocation solution of (1.8) if and only if $p_n(\lambda, u)' = \lambda L_n p_n$, which gives that $p_n(\lambda, u)$ is a collocation solution of (1.8) if and only if

$$p_n = u + K_\lambda L_n p_n \tag{1.14}$$

Let us now write $e_n = p_n - y$ and $r_n = L_n y - y$. Then e_n is the error between the collocation solution and the actual solution of (1.8) and r_n is the error in the polynomial interpolation of y. Substracting (1.13) from (1.14) gives that

$$
\begin{aligned}
e_n = p_n - y &= K_\lambda L_n p_n - K_\lambda y \\
&= K_\lambda (L_n(p_n - y)) + K_\lambda(L_n y - y) \\
&= K_\lambda L_n e_n + K_\lambda r_n
\end{aligned}
$$

i.e. we have that

$$e_n = K_\lambda L_n e_n + K_\lambda r_n \tag{1.15}$$

We make the following claim:

Claim 1.5.2. *e_n is a solution of* (1.15) *if and only if $e_n = K_\lambda \hat{e}_n$, where $\hat{e}_n$ satisfies*

$$\hat{e}_n = L_n K_\lambda \hat{e}_n + r_n \tag{1.16}$$

We prove the claim at the end of the proof of Lemma 1.5.1.

We know that the Volterra operator K_λ is compact and has no point spectrum; therefore, we have that $\sigma(K_\lambda) \subseteq \{0\}$, which implies that $I - K_\lambda$ is invertible. Moreover, since $K_\lambda X \subseteq C^1\left([-\tau, 0], \mathbb{C}^d\right)$, we have that $\lim_{n \to \infty} \|L_n K_\lambda - K_\lambda\| = 0$. Using Neumann series, we find that there exists a $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, the operator $I - L_n K_\lambda$ is invertible and that

$$\left\|(I - L_n K_\lambda)^{-1}\right\| \leq 2 \left\|(I - K_\lambda)^{-1}\right\|$$

Thus, the equation (1.16) has a unique solution $\hat{e}_n$ for $n \geq N_0$; and by Claim 1.5.2, we find that (1.15) has a unique solution e_n for $n \geq N_0$. Moreover, we obtain the estimate

$$
\begin{aligned}
\|e_n\| &\leq \|K_\lambda\| \|\hat{e}_n\| \\
&\leq 2 \|K_\lambda\| \left\|(I - K_\lambda)^{-1}\right\| \|r_n\|
\end{aligned}
$$

Since $r_n = L_n y - y$ is the interpolation error in $y = e^{\lambda \cdot} u$, we find by Lemma A.3.2 that

$$\|r_n\| \leq \max\{1, e^{-\mathrm{Re}\,(\lambda)\tau}\} \frac{(\tau |\lambda|)^n}{n!} |u|$$

Using the Stirling formula $n! \geq \sqrt{2\pi n}(n/e)^n$, the estimate (1.12) follows, up to the proof of Claim 1.5.2.

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To see that, for fixed $u \in \mathbb{C}^d, n \geq N_0$, the map

$$B(\lambda_0, \rho_0) \to X, \quad \lambda \mapsto p_n(\lambda, u)$$

is holomorphic, we write

$$p_n = y + e_n = y + K_\lambda \hat{e}_n.$$

But since $\hat{e}_n$ is given by

$$\hat{e}_n = (I - L_n K_\lambda)^{-1} r_n = (I - L_n K_\lambda)^{-1} (L_n - I) y$$

we find that

$$p_n(\lambda, u) = y + K_\lambda (I - L_n K_\lambda)^{-1} (L_n - I) y$$

which yields the analyticity of $\lambda \mapsto p_n(\lambda, u)$. This proves the lemma up to the proof of Claim 1.5.2.

Proof. (of Claim 1.5.2)
The claim easily follows in the case $\lambda = 0$, therefore we restrict ourselves to the case $\lambda \neq 0$. First, let $\hat{e}_n$ be a solution of (1.16), and set $e_n = K_\lambda \hat{e}_n$, then

$$\begin{aligned} e_n &= K_\lambda L_n K_\lambda \hat{e}_n + K_\lambda r_n \\ &= K_\lambda L_n e_n + K_\lambda r_n \end{aligned}$$

so $e_n = K_\lambda \hat{e}_n$ solves (1.15). Now, suppose that e_n solves (1.15), then in particular e_n is differentiable. Therefore, we can set $\hat{e}_n = \frac{1}{\lambda} e_n'$. Because (1.15) implies that $e_n(0) = 0$, we have that $K_\lambda \hat{e}_n = e_n$. Moreover, by (1.15), we have that

$$\begin{aligned} \hat{e}_n &= \frac{1}{\lambda} e_n' = L_n e_n + r_n \\ &= L_n K_\lambda \hat{e}_n + r_n \end{aligned}$$

i.e. $\hat{e}_n$ satisfies (1.16). This proves Claim 1.5.2. $\qquad\square$

We now state two results from [7] without proof. Using Rouché's Theorem and Lemma 1.5.1, the following result is proven in [7]:

Theorem 1.5.3. *Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of A with multiplicity ν. Then there exists $C_1 = C_1(\lambda_0), C_2 = C_2(\lambda_0), C_3 = C_3(\lambda_0)$ such that for*

$$\rho_n = \left(\frac{C_1}{C_3} \right)^{1/\nu} \left(\frac{1}{\sqrt{n}} \left(\frac{C_1}{n} \right)^n \right)^{1/m}$$

and n large enough, the set σ_n has exactly ν elements (counting multiplicities) $\lambda_1, \ldots, \lambda_\nu$ such that

$$\max_{1 \leq i \leq \nu} |\lambda_0 - \lambda_i| \leq \rho_n.$$

Moreover, it is proven in [7] that the pseudospectral method cannot produce any 'ghost solutions':

Lemma 1.5.4. *For $n \in \mathbb{N}$, let $\lambda_n \in \sigma_n$. Assume that $\lim_{n \to \infty} \lambda_n = \lambda$ for some $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma(A)$.*

We note that Theorem 1.5.3 is a *local* result: it tells us that when the eigenvalue $\lambda_0 \in \sigma(A)$ has multiplicity ν, then in a neighbourhood of λ_0 we find exactly ν eigenvalues $\lambda_{n,1}, \ldots, \lambda_{n,\nu}$ of A_n, and that $\lim_{n \to \infty} \lambda_{n,i} = \lambda_0$ for all $1 \leq i \leq \nu$. Together with Lemma 1.5.4, this makes the pseudospectral method very suited to the numerical approximation of eigenvalues. Theorem 1.5.3 gives us, however, no information on the *global* behaviour of the eigenvalues of A_n. For example, Theorem 1.5.3 and Lemma 1.5.4 do not rule out the existence of a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \in \sigma_n$ such that $\lim_{n \to \infty} \lambda_n = \infty$.

1.6 Application to nonlinear problems

In this section, we define the pseudsospectral approximation of a nonlinear delay equation, following [5]. We show that for a nonlinear delay equation, linearisation around an equilibrium and pseudospectral approximation commute. We then proceed by discussing the possible applications of pseudospectral approximation to bifurcation analysis and invariant manifolds of delay equations.

Let us study the delay equation:

$$\begin{cases} \dot{x}(t) & = Lx_t + G(x_t), \quad t > 0 \\ x_0 & = \phi \end{cases} \tag{1.17}$$

with $L : X \to \mathbb{C}^d$ a linear operator and $G \in C^1(X, \mathbb{C}^d)$ a function satisfying $G(0) = 0, DG(0) = 0$, and $\phi \in X$. To this equation we can associate a strongly continuous semigroup of (nonlinear) solution operators $\{T(t)\}_{t \geq 0}$, whose generator is given by

$$\mathcal{D}(A) = \{\phi \in X \mid \phi \in C^1\left([-\tau, 0], \mathbb{C}^d\right), \; \dot{\phi}(0) = L\phi + G(\phi)\}, \quad A\phi = \dot{\phi}.$$

For $n \in \mathbb{N}$, we define in the spirit of Section 1.2 the discretised infinitesimal generator A_n associated to the equation (1.17) as:

$$A_n : \mathbb{C}^{d \times (n+1)} \to \mathbb{C}^{d \times (n+1)}$$

$$A_n(x) = \left(L\mathcal{L}_n x + G\left(\mathcal{L}_n x\right), \frac{d}{d\theta}\bigg|_{\theta=\theta_{n,1}} (\mathcal{L}_n x)(\theta), \ldots, \frac{d}{d\theta}\bigg|_{\theta=\theta_{n,n}} (\mathcal{L}_n x)(\theta) \right). \tag{1.18}$$

We note that the nonlinearity of A_n is contained in the first component of the map, which captures the domain condition of the generator A.

Let us now study the nonlinear ODE

$$\begin{cases} \dot{y}(t) & = A_n y(t) \\ y(0) & = y_0 \end{cases} \tag{1.19}$$

with $y_0 \in \mathbb{C}^{d \times (n+1)}$. We first give a characterisation of the equilibria of (1.19).

Lemma 1.6.1. *Let $x_0 \in \mathbb{C}^d$ be an equilibrium of the delay equation (1.17). Then $\overline{x} = (x_0, \ldots, x_0) \in \mathbb{C}^{d \times (n+1)}$ is an equilibrium of (1.19). Vice versa, if $x = (x_0, \ldots, x_n) \in \mathbb{C}^{d \times (n+1)}$ is an equilibrium of (1.19), then $x_i = x_0$ for all $0 \leq i \leq n$ and x_0 is an equilibrium of (1.17).*

Proof. Let us first assume that $x_0 \in \mathbb{C}^d$ is an equilibrium of the delay equation (1.17). Then we have that $Lx_0 + G(x_0) = 0$ (where with $Lx_0, G(x_0)$ we mean the operator L, G applied to the constant function x_0). If we now set $\overline{x} = (x_0, \ldots, x_0) \in \mathbb{C}^{d \times (n+1)}$, then $\mathcal{L}_n \overline{x} \equiv x_0$. This gives that

$$A_n(x) = \left(L\mathcal{L}_n(x) + G(\mathcal{L}_n x), \frac{d}{d\theta}\bigg|_{\theta=\theta_{n,1}} (\mathcal{L}_n x)(\theta), \ldots, \frac{d}{d\theta}\bigg|_{\theta=\theta_{n,n}} (\mathcal{L}_n x)(\theta) \right)$$

$$= (Lx_0 + G(x_0), 0, \ldots, 0)$$

$$= (0, \ldots, 0)$$

which proves that $\overline{x}$ is an equilibrium of (1.19).

Now, assume that $x \in \mathbb{C}^{d \times (n+1)}$ is an equilibrium of (1.19). Then we have that $A_n x = 0$, so in particular $(\mathcal{L}_n x)'(\theta_{n,i}) = 0$ for $1 \leq i \leq n$. Since $(\mathcal{L}_n x)'$ is a polynomial of degree $n-1$, the fact that it has n zeros $\theta_{n,1}, \ldots, \theta_{n,n}$ implies that $(\mathcal{L}_n x)' \equiv 0$. Thus, $\mathcal{L}_n x$ is constant and by $(\mathcal{L}_n x)(0) = x_0$ we find that $\mathcal{L}_n x \equiv x_0$. Since $A_n x = 0$, we also see that

$$L\mathcal{L}_n(x) + G(\mathcal{L}_n(x)) = Lx_0 + G(x_0) = 0$$

This shows that x_0 is an equilibrium of (1.17). $\qquad\square$

In particular, we see that there is a one-to-one correspondence between equilibria of (1.17) and (1.19). Using this correspondence, we can prove the following:

Theorem 1.6.2. *Linearisation around an equilibrium and approximation commute in the sense of Figure 1.1.*

Proof. Without loss of generality, let us study the equilibrium $x \equiv 0 \in \mathbb{C}^d$. Then the linearisation of (1.17) around this equilibrium is given by

$$\begin{cases} \dot{x}(t) & = Lx_t, \quad t > 0 \\ x_0 & = \phi \end{cases}$$

The pseudospectral approximation of this system is given by

$$A_n x = \left(L\mathcal{L}_n x, (\mathcal{L}_n x)' (\theta_{n,1}), \ldots, (\mathcal{L}_n x)' (\theta_{n,n}) \right)$$

If we first approximate the system (1.17) using pseudospectral approximation, we obtain the map

$$\tilde{A}_n x = \left(L\mathcal{L}_n(x) + G(\mathcal{L}_n x), (\mathcal{L}_n x) (\theta_{n,1}), \ldots, (\mathcal{L}_n x) (\theta_{n,n}) \right)$$

From Lemma 1.6.1 we know that if $0 \in \mathbb{C}^d$ is an equilibrium of (1.17), then $\bar{x} = (0, \ldots, 0) \in \mathbb{C}^{d \times (n+1)}$ is the corresponding equilibrium of (1.19). Let us denote the linearisation of $\tilde{A}_n$ around 0 by B_n, then we have that

$$B_n x = \left(L\mathcal{L}_n x, (\mathcal{L}_n x)' (\theta_{n,1}), \ldots, (\mathcal{L}_n x)' (\theta_{n,n}) \right)$$

i.e. $B_n = A_n$. This proves the lemma. □

Combining Theorem 1.5.3 and Theorem 1.6.2, we find that we can use the stability of the approximating ODEs corresponding to (1.17) to obtain information on the (local) stability of the equilibria of the DDE (1.17).

In general, the bifurcation behaviour of the delay equation (1.17) is determined by

(i) the behaviour of the eigenvalues of the linearisation of (1.17) and

(ii) the nonlinear terms of the equation (1.17).

By Theorem 1.5.3 and Theorem 1.6.2, the eigenvalues of the linearisation of (1.17) are well approximated by the eigenvalues of the linearisation of the pseudospectral approximation of (1.17). Moreover, we see that the nonlinear terms of the pseudospectral approximation (1.18) correspond to the nonlinear terms of the delay equation (1.17). This motivates us to study whether the bifurcation behaviour and invariant manifolds of the delay equation (1.17) are well approximated by the bifurcation behaviour of the ODEs (1.19). For a further exploration of the approximation of bifurcation behaviour, see [6], and for a further discussion of approximations of invariant manifolds, see Chapter 5.

Nonlinear DDE $\xrightarrow{\text{Linearisation around } x_0 \in \mathbb{C}^d}$ Linear DDE

Pseudospectral $|$ Approximation $\qquad$ Pseudospectral $|$ Approximation

Nonlinear ODE $\xrightarrow[\text{Linearisation around } (x_0, \ldots, x_0) \in \mathbb{C}^{d \times (n+1)}]{}$ Linear ODE
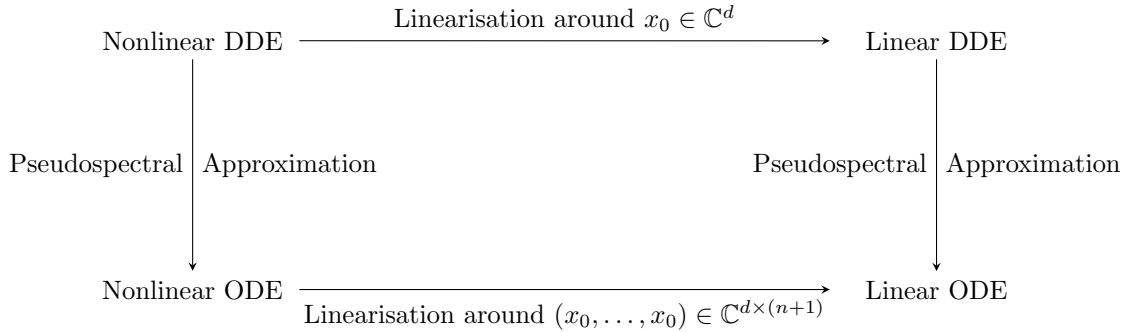
Figure 1.1: Approximation and linearisation commute.

1.7 Example: Wright's equation

In this section, we give an numerical example of approximation of eigenvalues by way of the pseudospectral method.

Consider Wright's equation

$$\dot{x}(t) = \alpha x(t-1)\left(x(t)+1\right), \quad t \geq 0 \tag{1.20}$$

with $x(t) \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ a parameter. Wright's equation was introduced in [31] and was one of the first nonlinear delay differential equations which was intensively studied [11, Page 387].

We note that equation (1.20) has an equilibrium at $x \equiv 0$. The linearisation of (1.20) around this equilibrium is given by

$$\dot{x}(t) = \alpha x(t-1), \quad t \geq 0. \tag{1.21}$$

The characteristic equation of (1.21) is given by

$$\Delta(\lambda) = \lambda - \alpha e^{-\lambda}. \tag{1.22}$$

For $\alpha = -\frac{\pi}{2}$, the equation (1.21) has exactly two eigenvalues $\pm i\frac{\pi}{2}$ on the imaginary axis. Figure 1.2 shows the error between the eigenvalue $i\frac{\pi}{2}$ and the pseudospectral approximation to this eigenvalue. In the computation of the approximate eigenvalues, the pseudospectral matrices A_n were computed in MATLAB following the implementation as described in [8, Chapter 7.2]. Figure 1.3 shows the spectrum of (1.21) for $\alpha = -\frac{\pi}{2}$ as computed using DDEBiftool [12] and the eigenvalues of the pseudospectral matrix A_n for (1.21) for $n = 10$.

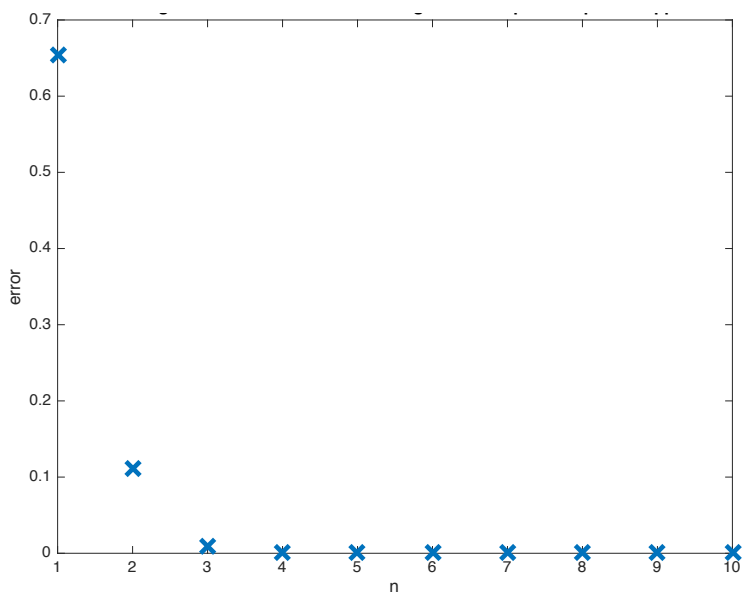Figure 1.2: Error between the eigenvalues $i\frac{\pi}{2}$ of (1.21) for $\alpha = -\frac{\pi}{2}$ and the pseudospectral approximation to this eigenvalue.
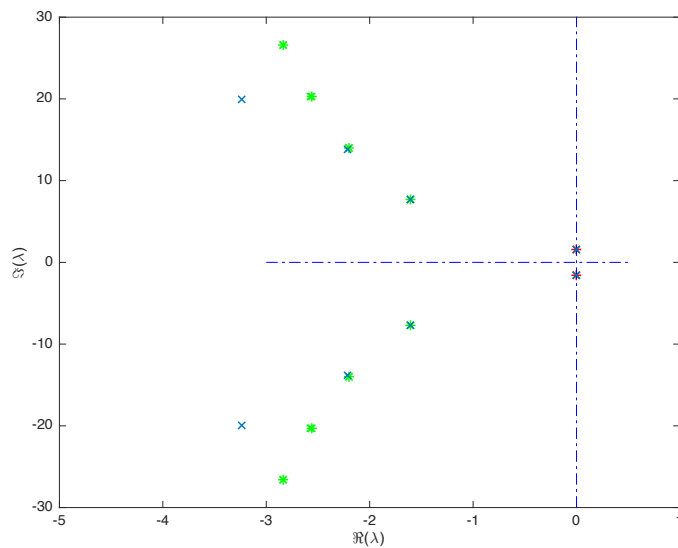

Figure 1.3: Spectrum of (1.21) for $\alpha = -\frac{\pi}{2}$ as computed using DDEBiftool (green and red stars) and the eigenvalues of the pseudospectral matrix A_n for (1.21) with $\alpha = -\frac{\pi}{2}$ for $n = 10$ (blue crosses).

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Chapter 2

The parametrisation method and normal form theory for ordinary differential equations

The parametrisation method, introduced in [9], provides a method to obtain numerical approximations for invariant manifolds at the origin of dynamical systems. In the parametrisation method, we make a conjugation between the dynamics on the invariant manifold and simpler dynamics on the tangent space of the invariant manifold at the origin. Assuming that the conjugation map is analytic, we obtain equations for the coefficients of its series expansion; solving those equations up to a certain order then gives an approximate parametrisation of the invariant manifold.

The set-up of the parametrisation method suits itself to error analysis of the approximation: this approach was for example taken in [28], where the parametrisation method was discussed in the framework of rigorous computations. Here, we will turn our attention towards the relation between the parametrisation method and normal form theory. We see that the two are closely connected, and that normal form theory can tell us how to choose the flow on the tangent space.

This chapter is structured as follows: we give an introduction to the parametrisation method and normal form theory in Section 2.1. Throughout Sections 2.2–2.4, we discuss the relation between normal form theory and the parametrisation method in the context of (un)stable manifolds. In Section 2.5, we turn our attention towards the center manifold and the Hopf bifurcation.

2.1 Introduction to the parametrisation method and normal form theory

Let $n \in \mathbb{N}$ and let us study the ordinary differential equation

$$\dot{x}(t) = g(x(t)) \tag{2.1}$$

where $g : \mathbb{C}^n \to \mathbb{C}^n$ is such that $g(0) = 0$. We discuss the parametrisation method for unstable manifolds; the parametrisation of the stable manifold can be treated in a similar fashion by sending $g \mapsto -g$.

Let us assume that $Dg(0)$ has d eigenvalues in the right half of the complex plane (counting multiplicities) and let us write $\{\lambda_1, \ldots, \lambda_d\} = \sigma(Dg(0)) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Denote by $\xi_i, 1 \leq i \leq d$ the associated (generalised) eigenvectors. In the parametrisation method, we look for an analytic map that conjugates the flow on the unstable manifold with a flow on the unstable generalised eigenspace, which we can identify with $\mathbb{C}^d$.

Let us choose a map $h : \mathbb{C}^d \to \mathbb{C}^d$, let $U \subseteq \mathbb{C}^d$ be a neighbourhood of the origin and let $P : U \to \mathbb{C}^n$. If P conjugates the flow associated to

$$\dot{\theta} = h(\theta) \tag{2.2}$$

to the flow associated to (2.1), then differentiating $x = P(\theta)$ gives that

$$g(P(\theta)) = DP(\theta)h(\theta), \quad \theta \in U. \tag{2.3}$$

Conversely, if P satisfies (2.3) then using the uniqueness of the initial value problems associated to (2.2), (2.1) one can show that the coordinate transformation $x = P(\theta)$ conjugates the flow associated to (2.2) into the flow associated to (2.1) (see also the notes). Additionally, we prescribe the constraints

$$P(0) = 0, \quad \partial_i P(0) = \xi_i \text{ for } 1 \leq i \leq d \tag{2.4}$$

which make sure that we obtain a parametrisation of the local invariant manifold near the origin in all directions.

Given a map $h : \mathbb{C}^d \to \mathbb{C}^d$, it is not clear that a map P satisfying (2.3), (2.4) exists, let alone that it is analytic. Therefore, one of the challenges of the parametrisation method lies in making the 'right' choice of h, i.e. in choosing $h : \mathbb{C}^d \to \mathbb{C}^d$ in such a way that there exists an analytic P satisfying (2.3), (2.4). It is at this point that normal form theory can give us an idea of how to choose h.

Suppose that the system (2.1) has a d-dimensional invariant manifold. In normal form theory, we look for a coordinate transformation that locally transforms the system (2.1) on the invariant manifold into a system

$$x'(t) = f(x(t)) \tag{2.5}$$

on $\mathbb{C}^d$, where f is such that it is easier to read of the local dynamics of the system. Based on the eigenvalues of the linearisation and their resonances (see Definition 2.2.1 below), normal form theory provides an algorithm to find a formal power series f such that the flow induced by (2.5) is formally conjugate to the flow induced by (2.1); see [3], [10]. For many classes of resonances of eigenvalues, the actual convergence of the normal form and conjugating map are classical results, see for example [3].

Thus, normal form theory provides us with an algorithm to find a h such there exists a formal power series P satisfying (2.3), (2.4); if we choose h as the normal form of (2.1), then we are sure that a formal power series P satisfying (2.3), (2.4) actually exists and we can compute its coefficients using the parametrisation method.

If we are in a situation where i) the convergence of the conjugating map is a 'classical result' from normal form theory and ii) the conjugating map between the original flow and its normal form is unique, then we are sure that the conjugation map we compute in the parametrisation method is actually the conjugating map from normal form theory, and the power series is convergent.

However, in general, the conjugating map need not be unique, a situation that we will further discuss in Section 2.3. If the conjugating map is not unique, but the existence of a convergent conjugation map is a result from normal form theory, we can do the following: suppose that the coefficients of the conjugating map are unique up to order k, then we can use the parametrisation method to compute those coefficients. Normal form theory then guarantees the existence of higher order coefficients (that we do *not* explicitly compute in the parametrisation method) of a convergent conjugating map. Although limiting ourselves to computing only the first k coefficients of the conjugating map may sound restrictive, this can already give us interesting and relevant information, as we will see in Section 2.5.

2.2 Example: the non-resonance case

In this section, we will apply the approach introduced in Section 2.1 to the case where the eigenvalues are non-resonant (see Definition 2.2.1 below). In this section, we no longer assume that the eigenvalues $\lambda_1, \ldots, \lambda_d$

are in the right half of the complex plane and in the following definition, the complex numbers $\lambda_1, \ldots, \lambda_d$ are not related to the complex numbers $\lambda_1, \ldots, \lambda_d$ from the previous section.

We make the following definition:

Definition 2.2.1. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and set $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then we say that the numbers $\{\lambda_1, \ldots, \lambda_n\}$ are non-resonant if

$$\lambda \cdot \alpha \neq \lambda_i$$

for all $1 \leq i \leq n$ and all multi-indices $\alpha \in \mathbb{N}^n$ with $\alpha \neq e_i$. If $\alpha \in \mathbb{N}^n$ is such that $\lambda \cdot \alpha \in \{\lambda_1, \ldots, \lambda_n\}$, then we call $|\alpha|$ the order of the resonance.

We state the following lemma from normal form theory:

Lemma 2.2.1. *Consider the system* (2.1) *and assume that $Dg(0)$ is diagonalisable; and that its eigenvalues are non–resonant. Then for any formal power series $f : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $f(0) = 0, Df(0) = 0$, there exists a formal coordinate transformation $x = P(\theta)$ that transforms the system*

$$\dot{\theta} = Dg(0)\theta + f(\theta)$$

into the complexification of the system (2.1) *and satisfies $\partial_i P(0) = e_i$ for $1 \leq i \leq n$, where e_i is the i-th basis vector. Moreover, the coordinate transformation $x = P(\theta)$ with these properties is unique.*

For a proof in the language of normal forms, see [3, Theorem 2.1]. We can also prove this result using the parametrisation method:

Proof. (of Lemma 2.2.1). We will prove that for any formal power series $f : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $f(0) = 0, Df(0) = 0$, there exists a formal coordinate transformation $x = P(\theta)$ that transforms the system

$$\dot{\theta} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} + f(\theta) \tag{2.6}$$

into the system (2.1) and satisfies (2.4); moreover, we will prove that the coordinate transformation $x = P(\theta)$ with these properties is unique. A (unique) linear coordinate transformation then leads to the result.

Let us fix a formal power series $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f(0) = 0, Df(0) = 0$. Let us assume that the formal coordinate transformation $P : \mathbb{C}^n \to \mathbb{C}^n$ is such that it brings (2.6) over into (2.1); then P should formally satisfy

$$g(P(\theta)) = DP(\theta)h(\theta)$$

for all $\theta \in \mathbb{C}^n$, where h is given by

$$h(\theta) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} + f(\theta) \tag{2.7}$$

Let us write the formal series expansion of P as

$$P(\theta) = \sum_{|\alpha|=0}^{\infty} P_\alpha \theta^\alpha$$

where we have used multi-index notation.

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We note that

$$DP(\theta)h(\theta) = \left(\sum_{|\alpha|=0}^{\infty} \alpha_1 P_\alpha \theta^{\alpha-e_1}, \dots, \sum_{|\alpha|=0}^{\infty} \alpha_n P_\alpha \theta^{\alpha-e_n} \right) \left(\begin{pmatrix} \lambda_1 \theta_1 \\ \vdots \\ \lambda_n \theta_n \end{pmatrix} + \begin{pmatrix} f_1(\theta) \\ \vdots \\ f_n(\theta) \end{pmatrix} \right) \tag{2.8}$$

Let us set $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda \cdot \alpha = \lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n$ for multi-indices $\alpha \in \mathbb{N}^n$. Then we can rewrite (2.8) as

$$DP(\theta)h(\theta) = \sum_{|\alpha|=0}^{\infty} (\lambda \cdot \alpha) P_\alpha \theta^\alpha + \sum_{|\alpha|=0}^{\infty} \sum_{i=1}^{n} \alpha_i P_\alpha \theta^{\alpha-e_i} f_i(\theta)$$

Because $f_i(\theta)$ satisfies $f_i(0) = 0, f_i'(0) = 0$ for $1 \le i \le n$ (i.e. f has a formal series expansion starting at second order), we can rewrite this as

$$\sum_{|\alpha|=0}^{\infty} (\lambda \cdot \alpha) P_\alpha \theta^\alpha + \sum_{|\alpha|=0}^{\infty} \sum_{i=1}^{n} \alpha_i P_\alpha \theta^{\alpha-e_i} f_i(\theta) = \sum_{|\alpha|=0}^{\infty} (\lambda \cdot \alpha) P_\alpha \theta^\alpha + r_\alpha \theta^\alpha$$

where r_α depends on P_β with $|\beta| < |\alpha|$.

We can expand $g(P(\theta))$ as

$$g(P(\theta)) = \sum_{|\alpha|=0}^{\infty} (Dg(0)P_\alpha + q_\alpha)\, \theta^\alpha$$

where q_α depends on P_β with $|\beta| < |\alpha|$; see [17]. Thus, $g(P(\theta)) = DP(\theta)h(\theta)$ gives that

$$\sum_{|\alpha|=0}^{\infty} (Dg(0)P_\alpha + q_\alpha)\, \theta^\alpha = \sum_{|\alpha|=0}^{\infty} (\lambda \cdot \alpha) P_\alpha \theta^\alpha + r_\alpha \theta^\alpha$$

This implies that

$$Dg(0)P_\alpha + q_\alpha = (\lambda \cdot \alpha)P_\alpha + r_\alpha$$

for all multi-indices $\alpha \in \mathbb{N}^n$; this, we can rewrite as

$$(\lambda \cdot \alpha - Dg(0))\, P_\alpha = r_\alpha - q_\alpha \tag{2.9}$$

for all $n \in \mathbb{N}^n$.

Since for $|\alpha| = 0, 1$, we have that $r_\alpha = q_\alpha = 0$, we note that the equation (2.9) does not conflict with the constraint (2.4). By our assumption that the eigenvalues of $Dg(0)$ are non-resonant, we find that $\lambda \cdot \alpha \notin \sigma(Dg(0))$ for $|\alpha| \ge 2$, and thus that (2.9) has a unique solution for $|\alpha| \ge 2$. This proves the claim. $\square$

2.3 Uniqueness of the conjugating map

In the previous section, we studied a situation where the conjugating map between the original system and its normal form is unique. In general this is, however, not the case. As in [3], we make the following definitions:

Definition 2.3.1. Let us consider the system (2.1) and let us denote by $\lambda = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of $Dg(0)$. Moreover, let us consider a formal power series $f : \mathbb{C}^n \to \mathbb{C}^n$, satisfying $f(0) = 0, Df(0) = 0$ and let us write $f(x) = \sum_{|\alpha|=2}^{\infty} f_\alpha x^\alpha$. Then we say that, corresponding to system (2.1), the coefficient f_α is *non-resonant* if $\lambda \cdot \alpha \notin \sigma(Dg(0))$; we say that the coefficient f_α is *resonant* if $\lambda \cdot \alpha \in \sigma(Dg(0))$.

With this terminology, we can state the following lemma:

Lemma 2.3.1. *The complexification of system* (2.1) *is formally conjugate to a system*

$$\dot{x} = Dg(0)x + \sum_{|\alpha|=2}^{\infty} f_\alpha x^\alpha \tag{2.10}$$

where the non-resonant coefficients of $f(x) = \sum_{|\alpha|=2}^{\infty} f_\alpha x^\alpha$ *can be chosen arbitrarily, but the resonant terms are uniquely determined given that the non-resonant terms are fixed. If we denote by* P *the map conjugating* (2.10) *to* (2.1) *and satisfying* $\partial_i P(0) = e_i$ *for* $1 \le i \le n$, *the resonant coefficients of* P *are arbitrary and the non-resonant terms are uniquely determined provided the resonant terms are fixed.*

For a proof, see [3, p. 10-11]. Following [3], we also introduce the following terminology:

Definition 2.3.2. System (2.10) where all the non-resonant terms in f are equal to zero is called the *normal form* of (2.1).

Using the parametrisation method, we can prove part of Lemma 2.3.1.

Proof. (of part of Lemma 2.3.1). Let the power series $f(x) = \sum_{|\alpha|=2}^{\infty} f_\alpha x^\alpha$ be such that (2.10) is formally conjugate to the system (2.1); denote by $P(x) = \sum_{|\alpha|=0}^{\infty} P_\alpha x^\alpha$ the conjugating map transforming (2.10) into (2.1) and satisfying (2.4). We prove that the resonant terms of P are not unique, but that the non-resonant terms of P are unique once the resonant ones are fixed.

As in the proof of Lemma 2.2.1, we find that the coefficients P_α should satisfy

$$\left((\lambda \cdot \alpha) - Dg(0)\right) P_\alpha = r_\alpha - q_\alpha \tag{2.11}$$

where r_α, q_α depend on P_β with $|\beta| < |\alpha|$.

For the resonant terms, it holds that $\lambda \cdot \alpha \in \sigma(Dg(0))$. Thus, we can either have that $r_\alpha - q_\alpha$ does not lie in the range of $(\lambda \cdot \alpha) - Dg(0)$; or that $r_\alpha - q_\alpha$ lies in the range of $(\lambda \cdot \alpha) - Dg(0)$ but that the system (2.11) does not have a unique solution (because $(\lambda \cdot \alpha) - Dg(0)$ has a non-trivial kernel). If we assume that (2.10) is formally conjugate to (2.1), we assume that a power series $P(x) = \sum_{|\alpha|=0}^{\infty} P_\alpha x^\alpha$ satisfying (2.11) for all $\alpha \in \mathbb{N}^n$ exists. Therefore, we cannot have that for resonant terms (2.11) has no solution, i.e. we cannot have that $r_\alpha - q_\alpha \notin \mathcal{R}((\lambda \cdot \alpha) - Dg(0))$. By the previous remarks, this then implies that (2.11) does not have a unique solution. Thus, we see that the resonant terms of $P(x) = \sum_{|\alpha|=0}^{\infty} P_\alpha x^\alpha$ are not uniquely determined.

For the non-resonant terms, we have that $\lambda \cdot \alpha \notin \sigma(Dg(0))$, thus the equation (2.11) has a unique solution P_α. This gives that the non-resonant terms of P are unique given that the non-resonant terms of P_α are fixed. $\qquad \square$

2.4 Normal forms on invariant manifolds

In our discussion of normal form theory so far, we have discussed conjugations between the normal form on $\mathbb{C}^n$ and the original flow (2.1) on all of $\mathbb{C}^n$. In the parametrisation method, we make a conjugation between an invariant manifold and $\mathbb{C}^d$, where $d \le n$ is the dimension of the invariant manifold. To study the connection between this and normal form theory, we turn our attention towards normal forms on invariant manifolds, where we conjugate the flow on an invariant manifold with a simpler flow on $\mathbb{C}^d$, where d is the dimension of the manifold. We see that the coordinate transform will also give us a (local) description of the invariant manifold.

We state the following lemma from [10] on normal form theory on invariant manifolds:

Lemma 2.4.1. *Let us study the system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} E(x,y) \\ F(x,y) \end{pmatrix} \tag{2.12}$$

where $x \in \mathbb{C}^d$, $y \in \mathbb{C}^{n-d}$, A is a $d \times d$ matrix, B is a $(n-d) \times (n-d)$ matrix and the maps E, F satisfy $E(0,0) = F(0,0) = 0$ and $DE(0,0) = DF(0,0) = 0$. Denote by $\mu = (\mu_1, \ldots, \mu_n)$ the vector consisting of all the eigenvalues of A and denote by $\{\nu_1, \ldots, \nu_{n-d}\}$ the eigenvalues of B. Let us assume that

$$\mu \cdot \alpha \neq \nu_k \quad \text{for all } \alpha \in \mathbb{N}^n, \ k = 1, \ldots, n-d. \tag{2.13}$$

Let us set

$$D_1 = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_d \end{pmatrix}, \quad D_2 = \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_{n-d} \end{pmatrix}$$

Then there exists

- *maps $\overline{E} = \overline{E}(\overline{x}, \overline{y}), \overline{F} = \overline{F}(\overline{x}, \overline{y})$ such that $\overline{F}(\overline{x}, 0) = 0$ and the system $\dot{\overline{x}} = D_1 \overline{x} + \overline{E}(\overline{x}, 0)$ is in normal form*

- *a formal change of variables $(x, y) = Q(\overline{x}, \overline{y}) = (\overline{x} + Q_1(\overline{x}), \overline{y} + Q_2(\overline{x}))$ with $Q_1(0) = Q_2(0) = 0, DQ_1(0) = DQ_2(0) = 0$*

such that the coordinate transform $(x, y) = Q(\overline{x}, \overline{y})$ formally transforms the system

$$\begin{pmatrix} \dot{\overline{x}} \\ \dot{\overline{y}} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} + \begin{pmatrix} \overline{E}(\overline{x}, \overline{y}) \\ \overline{F}(\overline{x}, \overline{y}) \end{pmatrix} \tag{2.14}$$

into the system (2.12).

We note that since the $\overline{F}(\overline{x}, 0) = 0$, the set $\overline{y} = 0$ is an invariant manifold for the flow (2.14); if we set $P : \mathbb{C}^d \to \mathbb{C}^n$, $P(\overline{x}) = (\overline{x} + Q_1(\overline{x}), Q_2(\overline{x}))$, then the set $\{P(\overline{x}) \mid \overline{x} \in \mathbb{C}^d\}$ is a invariant set for the flow (2.12). Thus, the conjugating map gives us a description of the local invariant manifold. See also Figure 2.1.
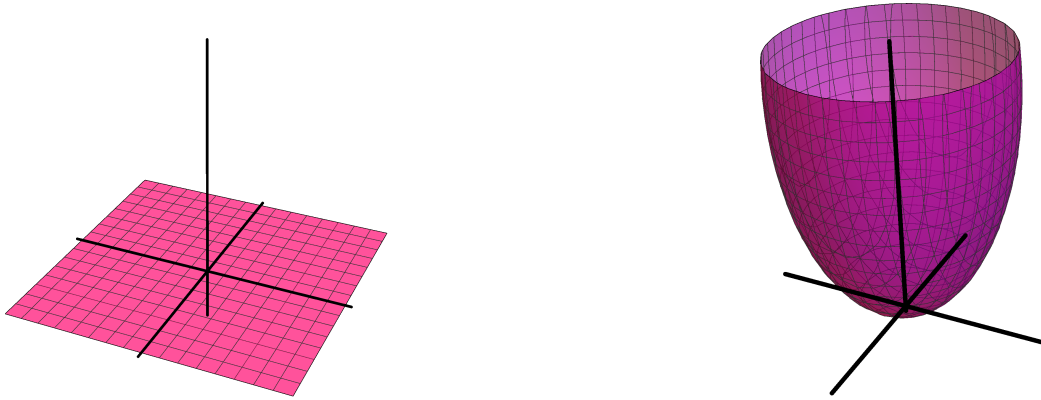

Figure 2.1: Invariant plane $\overline{y} = 0$ of system (2.14) (left) and corresponding invariant manifold of (2.1) (right).

2.5 Center manifolds

In this section, we apply the discussion from Section 2.4 to a specific situation, namely the situation where we have exactly two eigenvalues on the imaginary axis.

Let $n \geq 3$ and let us study (2.1) with $g : \mathbb{C}^n \to \mathbb{C}^n$. Let us assume that $Dg(0)$ has eigenvalues $\{\lambda_1 = i\omega, \lambda_2 = -i\omega, \lambda_3, \ldots, \lambda_n\}$ where $\operatorname{Re} \lambda_i \neq 0$ for $i \geq 3$. By a coordinate transformation, we can bring (2.1) in the following form:

$$
\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} E(x_1, x_2) \\ F(x_1, x_2) \end{pmatrix}
$$

where $x_1 \in \mathbb{C}^2, x_2 \in \mathbb{C}^{n-2}$, A has eigenvalues $\pm i\omega$ and B has eigenvalues $\lambda_3, \ldots, \lambda_n$ and E, F satisfy $E(0,0) = F(0,0) = 0$, $DE(0,0) = 0 = DF(0,0) = 0$ for all $x_1 \in \mathbb{C}^2, x_2 \in \mathbb{C}^{n-2}$. We note that the condition (2.13) is satisfied. Let us write

$$
D_1 = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \quad D_2 = \begin{pmatrix} \lambda_3 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}
$$

Lemma 2.4.1 implies that there exist

- maps $\overline{E} = \overline{E}(\overline{x}_1, \overline{x}_2), \overline{F} = \overline{F}(\overline{x}_1, \overline{x}_2)$ such that $\overline{F}(\overline{x}_1, 0) = 0$ and the system $\dot{\overline{x}}_1 = D_1 \overline{x}_1 + \overline{E}(\overline{x}_1, 0)$ is in normal form

- a formal change of variables $(x, y) = Q(\overline{x}, \overline{y}) = (\overline{x} + Q_1(\overline{x}), \overline{y} + Q_2(\overline{x}))$ with $Q_1(0) = Q_2(0) = 0, DQ_1(0) = DQ_2(0) = 0$

transforming system (2.1) into the system (2.14).

Let us write $\mu = (i\omega, -i\omega)$, then we find that μ has non-trivial resonances of the form $\alpha_n^+ = (n, n-1)$ and $\alpha_n^- = (n-1, n)$ for $n \geq 2$, since

$$
\mu \cdot \alpha_n^+ = ni\omega - (n-1)i\omega = i\omega
$$
$$
\mu \cdot \alpha_n^- = (n-1)i\omega - ni\omega = -i\omega
$$

Moreover, all non-trivial resonances of μ are of this form. Since $|\alpha_n| = 2n - 1$ is always odd, i.e. the order of the resonances is always odd, an argument similar to Lemma 2.3.1, but then for normal forms on invariant manifolds, tells us that in the normal form $\dot{\overline{x}}_1 = D_1 \overline{x}_1 + \overline{E}(\overline{x}_1, 0)$ the map $\overline{E}(\overline{x}_1, 0)$ has all even terms equal to zero (see [3, Theorem 3.1]).

Now let us write $P : \mathbb{C}^2 \to \mathbb{C}^n$ for the coordinate transform that transforms the system $\dot{\overline{x}}_1 = D_1 \overline{x}_1 + \overline{E}(\overline{x}_1, 0)$ into the system (2.1) $P(0) = 0, \partial_i P(0) = e_i$ for $i = 1, 2$. This map exists as a formal power series (but is not necessarily unique!) because $x \mapsto (x + Q_1(x), Q_2(x))$ transforms system $\dot{\overline{x}}_1 = D_1 \overline{x}_1 + \overline{E}(\overline{x}_1, 0)$ into system (2.1).

To compute the coefficients of the series expansion of P, we note that $x = P(\theta)$ satisfying (2.4) transforms the system $\theta = D_1 \theta + \overline{E}(\theta, 0)$ into system (2.1) if and only if $P(0) = 0, \partial_i P(0) = e_i$ for $i = 1, 2$ is satisfied and $DP(\theta)h(\theta) = g(P(\theta))$, where $h(\theta) = D_1 \theta + \overline{E}(\theta, 0)$. Let us write $P(\theta) = \sum_{|\alpha|=0}^{\infty} P_\alpha \theta^\alpha$, then a similar computation as in the proof of Lemma 2.3.1 gives that

$$
\sum_{|\alpha|=0}^{\infty} (\mu \cdot \alpha) P_\alpha \theta^\alpha + \sum_{|\alpha|=0}^{\infty} \alpha_1 P_\alpha \overline{E}_1(\theta, 0)\theta^{\alpha - e_1} + \sum_{|\alpha|=0}^{\infty} \alpha_2 P_\alpha \overline{E}_2(\theta, 0)\theta^{\alpha - e_2}
$$
$$
= \sum_{|\alpha|=0}^{\infty} (Dg(0)P_\alpha + q_\alpha)\theta^\alpha
$$

where q_α depends on P_β with $|\beta| < |\alpha|$.

Since $\overline{E}(\theta, 0)$ has a series expansion starting at order three, we find that that up to and including order 2 the equations for P_α are given by

$$
(\mu \cdot \alpha) P_\alpha = Dg(0)P_\alpha + q_\alpha, \quad |\alpha| \leq 2 \tag{2.15}
$$

In particular, since the non-trivial resonances have order at least three, we see that for $|\alpha| = 2$ we have that $\mu \cdot \alpha \notin \sigma(Dg(0))$, hence the equation (2.15) has a unique solution for $|\alpha| = 2$ (the coefficients of order 0 and 1 are determined by the constraint $P(0) = 0, \partial_i P(0) = e_i$ for $i = 1, 2$).

Thus, for $|\alpha| \leq 2$, we can compute the unique coefficients P_α of the coordinate transformation P. For $|\alpha| > 2$, the existence of coefficients P_α such that P transforms system $\dot{\theta} = D_1\theta + \overline{E}(\theta, 0)$ into system (2.1) and such that P is a convergent power series, is guaranteed by normal form theory.

The computation of the coefficients of P up to and including order two is interesting for the following reason: suppose a Hopf bifurcation occurs in an ordinary differential equation. To determine the direction of this Hopf bifurcation, we can use a description of the center manifold up to and including second order combined with the eigenvectors of the adjoint generator of the linearised problem; see [11]. Since P gives a local parametrisation of the center manifold, we can use the coefficients P_α, $|\alpha| \leq 2$, combined with the eigenvectors of the adjoint problem, to compute the direction of the Hopf bifurcation.

Chapter 3

The parametrisation method and normal form theory for delay differential equations

In this chapter, we study the parametrisation method for delay differential equations. In Chapter 2, we saw that there is a close connection between the parametrisation method and normal form theory. Therefore, we start this chapter with a very short introduction to normal form theory for delay equations in Section 3.1. In Section 3.2, we then discuss the parametrisation method for delay equations.

3.1   Normal form theory for delay equations

Consider the delay equation

$$\dot{x}(t) = Lx_t + G(x_t), \quad t \geq 0 \tag{3.1}$$

where $L : X \to \mathbb{C}^d$ is a bounded linear map and $G \in C^1(X, \mathbb{C}^d)$ satisfies $G(0) = 0, DG(0) = 0$.

To study the parametrisation method for delay differential equations, we will rewrite the DDE (3.1) as an abstract ODE on a Banach space, and then try to adapt the methods introduced in Chaper 2 to the case where the state space is infinite dimensional. A natural abstract ODE to study is the ODE

$$\dot{u}(t) = Au(t), \quad t \geq 0 \tag{3.2}$$

on the state space X, where A is the nonlinear generator associated to (3.1). However, the abstract ODE (3.2) is not very suitable to the parametrisation method. As we saw in the last chapter, there is a close connection between the parametrisation method and normal form theory. Therefore, one would like to consider an abstract ODE that is in some sense suitable to normal form theory. Since in the abstract ODE (3.2) the dependence on the DDE (3.1) appears in the domain condition, this abstract ODE is not very suitable for normal form theory – and therefore for the parametrisation method.

The way to circumvent this problem is to enlarge the state space X and to extend the operator A to an operator on the larger state space in such way that in the extension of A, the dependence on (3.1) appears in the action of the operator and not in the domain. This approach is taken in sun-star calculus, a functional analytic framework as described in [11].

In sun-star calculus, we define the space $X^\odot \subset X^\star$ as the largest space on which the adjoint semigroup $\{T(t)^*\}_{t \geq 0}$ is strongly continuous. Then, we embed the state space X into the space $X^{\odot*}$, and on $X^{\odot*}$ study the unbounded operator $A^{\odot*}$, where the dependence of the DDE appears in the action and not in the domain. For a detailed study of sun-star calculus, see [11]; an overview of the main results can for example be found in [23] and [4].

If we restrict the flow of (3.1) to a finite dimensional invariant manifold, the restricted flow satisfies a (finite dimensional) ordinary differential equation, that we can explicitly write down in the sun-star framework [11, Chapter IX.8]. This allows us to 'lift' normal form theory from ODEs to DDEs using the sun-star framework. For a detailed account of normal form theory for delay equations in light of sun-star calculus, see [23] and [29].

3.2 Parametrisation method for delay equations

Using the sun-star framework as introduced in the previous section, we now turn our attention towards the parametrisation method for delay equations.

Let us assume that the following hypothesis holds:

Hypothesis 1. Let $\{\lambda_1, \ldots, \lambda_m\} \subseteq \sigma(A)$ be m simple and distinct eigenvalues of the generator A; for $1 \leq i \leq m$, let us denote by $\xi_i \in \mathbb{C}^d$ the vector such that

$$\Delta(\lambda_i)\xi_i = \lambda_i \xi_i - L(e_{\lambda_i}\xi_i) = 0 \tag{3.3}$$

and such that $\|\xi_i\| = 1$. Let $X_0 = \text{span}\{\xi_1, \ldots, \xi_m\}$ and let us write $\lambda = (\lambda_1, \ldots, \lambda_m)$.
We assume that there exists a $k \in \mathbb{N}, k \geq 2$ such that

(i) There exists a locally invariant manifold $W_{loc} \subseteq X$ near the origin for (3.1), such that W_{loc} is locally given by a C^{k+1}-graph over X_0 (i.e. there exists a $\Psi \in C^{k+1}(X_0, X)$ and an open set $V \subseteq X_0, 0 \in V$ such that $\Psi(V) = W_{loc}$). Moreover, if $\phi \in W_{loc}$, then the equation (3.1) with initial condition $x_0 = \phi$ has a backward solution for all time.

(ii) For multi-indices $\alpha \in \mathbb{N}^m$ with $2 \leq |\alpha| \leq k$, we have that $\lambda \cdot \alpha \notin \sigma(A)$.

In particular, one should have the following two situations in mind where Hypothesis 1 is satisfied:

Situation 1. In the case where $\sigma(A)$ has exactly two (counting multiplicities) eigenvalues $\pm i\omega_0 \neq 0$ on the imaginary axis, we have that Hypothesis 1 is satisfied with $m = 2$ and $k = 2$. In this case, the locally invariant manifold W_{loc} is the center manifold (see [11, Corollary IX.7.8]).

Situation 2. In the case where A has exactly m simple eigenvalues $\lambda_1, \ldots, \lambda_m$ in the right half of the complex plane such that $\lambda \cdot \alpha \notin \sigma(A)$ for $2 \leq |\alpha| \leq k$, and the non-linearity G in (3.1) is C^{k+1}, Hypothesis 1 is satisfied with W_{loc} the local unstable manifold [11, Corollary VIII.4.11].

We state the following lemma on the parametrisation method for delay equations:

Lemma 3.2.1. *Assume that Hypothesis 1 holds. Denote by* $\Lambda : \mathbb{C}^m \to \mathbb{C}^m$ *the operator*

$$\Lambda\theta = \Lambda(\theta_1 + \ldots + \theta_m) = \lambda_1\theta_1 + \ldots + \lambda_m\theta_m.$$

Then there exists a power series

$$P : X_0 \to X, \quad P(\theta) = \sum_{|\alpha|=0}^{k} P_\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}) \tag{3.4}$$

such that the coordinate transformation $u = P(\theta)$ *locally brings the system*

$$\dot{\theta} = \Lambda\theta + \mathcal{O}(|\theta|^{k+1}) \tag{3.5}$$

into the flow of (3.1) restricted to the invariant manifold W_{loc}. *In particular, for* $0 \leq |\alpha| \leq k$, *the coefficients* P_α *are of the form*

$$P_\alpha = P_\alpha(0)e_{(\lambda \cdot \alpha)}.$$

Here, $e_{(\lambda \cdot \alpha)} \in X$ is defined as $e_{(\lambda \cdot \alpha)}(s) = e^{(\lambda \cdot \alpha)s}$ for $s \in [-\tau, 0]$ and $P_\alpha(0)$ satisfies

$$(\lambda \cdot \alpha)P_\alpha(0) = L(P_\alpha(0)e_{(\lambda \cdot \alpha)}) + q_\alpha.$$

Here, q_α is such that

$$G(P(\theta)) = \sum_{|\alpha|=0}^{k} q_\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}). \tag{3.6}$$

Proof. Let $u(t)$ be a solution of (3.1) on the invariant manifold W_{loc}; since the solution $u(t)$ exists for all $t \in \mathbb{R}$ by assumption (i) in Hypothesis 1, $u(t)$ satisfies

$$\dot{u}(t) = A^{\odot *}u(t) + R(u(t)) \tag{3.7}$$

Here, the unbounded linear operator $A^{\odot *} : \mathcal{D}(A^{\odot *}) \subseteq X^{\odot *} \to X^{\odot *} \simeq \mathbb{C}^d \times L^\infty \left([0, \tau], \mathbb{C}^d\right)$ is the sun-star operator associated to the linear equation $\dot{x}(t) = Lx_t$ and $R : X \to X^{\odot *}$ is given by $R(\phi) = (G(\phi), 0)$ [23, Page 25].

The coordinate transformation $z = P(\theta)$ transforms the system (3.5) into the system (3.7) if P satisfies

$$jDP(\theta)\Lambda z(\theta) + \mathcal{O}\left(|z|^{k+1}\right) = A^{\odot *}P(\theta) + R(P(\theta)). \tag{3.8}$$

Here,

$$j : X \to X^{\odot *}, \quad j(\phi) = (\phi(0), \phi')$$

denotes the canonical embedding operator. Expanding (3.8) as its power series gives that

$$\sum_{|\alpha|=0}^{k} (\lambda \cdot \alpha)j(P_\alpha)\theta^\alpha + \mathcal{O}\left(|\theta|^{k+1}\right) = \sum_{|\alpha|=0}^{k} A^{\odot *}j(P_\alpha)\theta^\alpha + (q_\alpha, 0)\theta^\alpha + \mathcal{O}\left(|\theta|^{k+1}\right),$$

where $q_\alpha, 0 \leq |\alpha| \leq k$, is such that

$$G(P(\theta)) = \sum_{|\alpha|=0}^{k} q_\alpha \theta^\alpha + \mathcal{O}\left(|\theta|^{k+1}\right).$$

We note that, since $G(0) = 0, DG(0) = 0$, we have that q_α depends only on P_β with $|\beta| < |\alpha|$.

If $jP_\alpha \in \mathcal{D}(A^{\odot *})$, then $A^{\odot *}jP_\alpha = (LP_\alpha, P'_\alpha)$. Thus, for $0 \leq |\alpha| \leq k$, the coefficients P_α should satisfy

$$(\lambda \cdot \alpha)(P_\alpha(0), P_\alpha) = (LP_\alpha, P'_\alpha) + (q_\alpha, 0). \tag{3.9}$$

The second component of (3.9), $P'_\alpha = \lambda \cdot P_\alpha$, implies that $P_\alpha(s) = P_\alpha(0)e^{(\lambda \cdot \alpha)s}, s \in [-\tau, 0]$. With this choice of P_α, the condition $P_\alpha \in \mathcal{D}(A^{\odot *})$ is also satisfied.

The first component of (3.9) now implies that

$$\Delta(\lambda \cdot \alpha)P_\alpha(0) = (\lambda \cdot \alpha)P_\alpha(0) - L(e_{(\lambda \cdot \alpha)}P_\alpha(0)) = q_\alpha \tag{3.10}$$

We note that for $|\alpha| = 0, 1$, we have that $q_\alpha = 0$. Thus, for $|\alpha| = 0$, equation (3.10) is satisfied for $P_\alpha = 0$. For $|\alpha| = 0$, i.e. $\alpha = e_i$ for some $1 \leq i \leq m$, (3.10) is satisfied with $P_{e_i}(0) = \xi_i$. For $2 \leq |\alpha| \leq k$, we have by Hypothesis 1 that $\lambda \cdot \alpha \notin \sigma(A)$, i.e. the equation (3.10) has a unique solution. $\square$

Part B

Chapter 4

Convergence of eigenvectors in the pseudospectral method

In this chapter, we study the eigenvectors of the pseudospectral approximation. The result obtained in Lemma 4.0.2 will be useful in Chapter 5 when studying the approximation of invariant manifolds in the pseudospectral method.

Let us consider the linear delay equation

$$\dot{x}(t) = Lx_t, \quad t \geq 0 \tag{4.1}$$

with $L : X \to \mathbb{C}^d$ a bounded linear map. Denote by A_n the pseudospectral approximation to the generator A that is associated to (4.1). Moreover, for $\lambda \in \mathbb{C}$, $y_0 \in \mathbb{C}^d$, let us denote by $p_n(\lambda, y_0)$ the n-th order collocation solution to the initial value problem

$$\begin{cases} \dot{y}(t) & = \lambda y(t), \ t \in [-\tau, 0], \\ y(0) & = y_0. \end{cases} \tag{4.2}$$

We recall from Lemma 1.4.1 that $\lambda \in \sigma_n \subseteq \sigma(A_n)$ if and only if there exists a $x_0 \in \mathbb{C}^d, x_0 \neq 0$ such that $x_0 = L(p_n(\lambda, x_0))$. Moreover, if $x \in \mathbb{C}^{(n+1) \times d}$ is an eigenvector of A_n to the eigenvalue λ, then the components of $x_i \in \mathbb{C}^d$ of x are given by

$$x_i = p_n(\lambda, x_0)(\theta_{n,i}), \quad 0 \leq i \leq n.$$

To study the convergence of the eigenvectors of A_n, we first prove the following technical result:

Lemma 4.0.1. *Let $(\lambda_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ be such that $\lim_{n \to \infty} \lambda_n = \lambda \in \mathbb{C}$; let $(u_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}^d$ be such that $\lim_{n \to \infty} u_n = u \in \mathbb{C}^d$. Then*

$$\lim_{n \to \infty} \|p_n(\lambda_n, u_n) - e_\lambda u\|_\infty = 0$$

where $\|.\|_\infty$ denotes the supremum-norm on $X = C([-\tau, 0], \mathbb{C}^d)$ and $e_\lambda \in X$ is defined as $e_\lambda(\theta) = e^{\lambda \theta}$ for $\theta \in [-\tau, 0]$.

Proof. we let $\rho = \sup_{n \in \mathbb{N}} |\lambda_n - \lambda|$, $\tilde{\rho} = \sup_{n \in \mathbb{N}} \|u_n - u\|_{\mathbb{C}^d}$. Then by Lemma 1.5.1, we can find a $N \in \mathbb{N}$ such that

$$\left\| p_n(\tilde{\lambda}, \tilde{u}) - p_m(\tilde{\lambda}, \tilde{u}) \right\|_\infty < \frac{\epsilon}{3}, \quad \text{for all } n, m \geq N, \ \tilde{\lambda} \in B(\lambda, \rho), \ \tilde{u} \in B(u, \tilde{\rho}).$$

Moreover, for this $N \in \mathbb{N}$ we also have that

$$\|p_N(\lambda, u) - e_\lambda u\|_\infty < \frac{\epsilon}{3}.$$

Thus, we find for $n \geq N$ that

$$
\begin{aligned}
\|p_n(\lambda_n, u_n) - e_\lambda u\|_\infty &\leq \|p_N(\lambda_n, u_n) - e_\lambda u\|_\infty + \|p_N(\lambda_n, u_n) - p_n(\lambda_n, u_n)\|_\infty \\
&\leq \|p_N(\lambda_n, u_n) - p_N(\lambda, u)\|_\infty + \|p_N(\lambda, u) - e_\lambda u\|_\infty + \|p_N(\lambda_n, u_n) - p_n(\lambda_n, u_n)\|_\infty \\
&< \|p_N(\lambda_n, u_n) - p_N(\lambda, u)\|_\infty + \frac{\epsilon}{3} + \frac{\epsilon}{3}.
\end{aligned}
$$

For fixed $v \in \mathbb{C}^d$, we find by Lemma 1.5.1 that the map

$$B(\lambda, \rho) \to X, \quad \mu \mapsto p_N(\mu, v)$$

is holomorphic, so in particular continuous. For fixed $\mu \in B(\lambda, \rho)$, the map

$$\mathbb{C}^d \to X, \quad v \mapsto p_N(\mu, v)$$

is linear, hence continuous. Thus, we find that $\lim_{n \to \infty} \|p_N(\lambda_n, u_n) - p_N(\lambda, u)\|_\infty = 0$. This proves the lemma. $\qquad \square$

Using Lemma 4.0.1, we now find the following result on the convergence of eigenvectors in the pseudospectral method:

Lemma 4.0.2. *For $n \in \mathbb{N}$, let $\lambda_n \in \sigma_n$ and let $\lambda \in \sigma(A)$ be a simple eigenvalue such that $\lim_{n \to \infty} \lambda_n = \lambda$. Let $x_n \in \mathbb{C}^{(n+1) \times d}$ be such that*

$$A_n x_n = \lambda_n x_n$$

and $\|(x_n)_0\|_{\mathbb{C}^d} = 1$. Let $x \in \mathbb{C}^d$ be such that

$$\Delta(\lambda)x = \lambda x - L(e^{\lambda \cdot} x) = 0$$

and $\|x\|_{\mathbb{C}^d} = 1$. Then

$$\lim_{n \to \infty} \|\mathcal{L}_n x_n - e_\lambda x\|_\infty = 0, \tag{4.3}$$

where $e_\lambda \in X$ is defined as $e_\lambda(\theta) = e^{\lambda \theta}$ for $\theta \in [-\tau, 0]$.

Proof. Let us define $u_n = (x_n)_0 \in \mathbb{C}^d$ for $n \in \mathbb{N}$. We first prove that

$$\lim_{n \to \infty} u_n = x. \tag{4.4}$$

By Lemma 1.4.1 and Definition 1.4.1 we have that $\lambda_n u_n = L(p_n(\lambda_n, u_n))$. Since $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{C}^d$, we can extract a converging subsequence $(u_{n_k})_{k \in \mathbb{N}}$ with limit $\bar{x}$, for which we have that

$$\lambda_{n_k} u_{n_k} = L(p_{n_k}(\lambda_{n_k}, u_{n_k})). \tag{4.5}$$

Taking the limit on both sides in (4.5) and using Lemma 4.0.1, we find that $\lambda \bar{x} = L(e^{\lambda \cdot} \bar{x})$, i.e. $\bar{x}$ is an eigenvector associated to the eigenvalue $\lambda \in \sigma(A)$. But since λ is a simple eigenvalue, and since $\|\bar{x}\| = \|x\| = 1$, we conclude that $x = \bar{x}$.

Thus, we find that the limit of any converging subsequence of $(u_n)_{n \in \mathbb{N}}$ is given by x. By Urysohn's Lemma, this implies that (4.4) holds.

To prove the lemma, we recall that by Lemma 1.4.1

$$\mathcal{L}_n x = p_n(\lambda_n, (x_n)_0)$$

Since $\lim_{n \to \infty} \lambda_n = \lambda$, $\lim_{n \to \infty} (x_n)_0 = x$, the identity (4.3) follows using Lemma 4.0.1. $\qquad \square$

Chapter 5

Approximation of invariant manifolds using the pseudospectral method

In this chapter, we study the approximation of invariant manifolds in the pseudospectral method, using the parametrisation method as introduced in Chapters 2–3. The strength of the parametrisation method is that the conjugation map whose coefficients we compute in the parametrisation method, at once describes both the invariant manifold and the dynamics restricted to this manifold. This property we will exploit in studying approximation of invariant manifolds in the pseudospectral method: in fact, we will not study the 'approximate manifold' directly, but rather study the approximate conjugation map.

Our approach to constructing the approximate conjugation map is the following: in the parametrisation method, we make a conjugation between the flow in the invariant manifold and a 'simpler' flow on the tangent space to the invariant manifold at the origin (which we will denote by X_0). The 'simpler' flow is chosen in such a way that the conjugation map locally parametrises the invariant manifold. To construct the approximate conjugation map, we approximate the flow on the invariant manifold in the original delay equation using the pseudospectral method. The 'simpler' flow on X_0 we approximate using the approximation of eigenvalues in the pseudospectral method. We then show that we can make a conjugation between the 'approximate flow on the invariant manifold' and the 'approximate simpler flow' on X_0; resulting in the approximate conjugation map. See Figure 5.1.

This chapter is structured as follows: In Section 5.1, we define the approximate conjugation map and show that (for n large enough), this map exists. In Section 5.2, we prove that approximate conjugation map is actually the right terminology, in the sense that the coefficients of the approximate conjugation map converge to the coefficients of the original parametrisation map in the delay equation. Throughout this chapter, we assume that Hypothesis 1 from Chapter 3 holds.

5.1   Definition of approximate conjugation map

Consider the delay equation

$$\dot{x}(t) = Lx_t + G(x_t), \quad t \geq 0 \tag{5.1}$$

with $X = C\left([-\tau, 0], \mathbb{C}^d\right)$, $L : X \to \mathbb{C}^d$ a bounded linear operator and $G \in C^1(X, \mathbb{C}^d)$ satisfying $G(0) = 0$ and $DG(0) = 0$. Let us assume that Hypothesis 1 holds. For $n \in \mathbb{N}$, let A_n^L be the nonlinear pseudospectral approximation associated to (5.1) as defined in equation (1.18). Let us write $A_n^L = DA_n^L(0)$ (with the superscript 'L' for 'linear') and $A_n^N = A_n - A_n^L$ (with the superscript 'N' for 'nonlinear'). For $n \in \mathbb{N}$ and $1 \leq i \leq m$ (with m as in Hypothesis 1), denote by λ_n^i the element of σ_n (as in Definition 1.4.1) closest to λ_i. Moreover, let us write

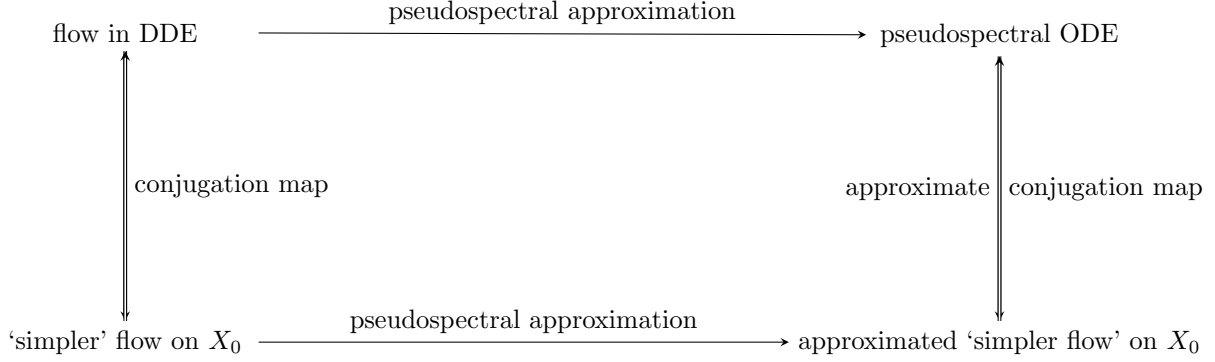$$\lambda_n = (\lambda_n^1, \ldots, \lambda_n^m)$$

Figure 5.1: Schematic representation of definition of approximate conjugation map.

and

$$\Lambda_n = \begin{pmatrix} \lambda_n^1 & 0 & \ldots & 0 \\ 0 & \lambda_n^2 & \ldots & 0 \\ \ddots & \vdots & \ddots & \ddots \\ 0 & 0 & \ddots & \lambda_n^m \end{pmatrix}.$$

We make the following definition:

Definition 5.1.1. Let W_{loc} be as in Hypothesis 1. Then $P_n : \mathbb{C}^m \to \mathbb{C}^{(n+1)\times d}$ is an approximate conjugation map to the invariant manifold W_{loc} if

- $P_n(0) = 0$,

- $DP_n(0)e_i = \xi_n^i$, where $\xi_n^i \in \mathbb{C}^{(n+1)\times d}$ is such that

$$A_n^L \xi_n^i = \lambda_n^i \xi_n^i, \quad \left\| (\xi_n^i)_0 \right\|_{\mathbb{C}^d} = 1. \tag{5.2}$$

- For x in a neighbourhood of $0 \in \mathbb{C}^m$, we have that the coordinate transform $u = P_n(x)$ transforms the system

$$\dot{\theta}(t) = \Lambda_n \theta(t) + \mathcal{O}\left(|\theta|^{k+1}\right) \tag{5.3}$$

on $\mathbb{C}^m$ into the system

$$\dot{x}(t) = A_n^L x(t) + A_n^N(x(t)). \tag{5.4}$$

on $\mathbb{C}^{(n+1)\times d}$.

To prove the existence of the approximate conjugation map, we first prove the following lemma concerning resonances in the approximate system:

Lemma 5.1.1. *Let us assume that Hypothesis 1 holds. Let $\alpha \in \mathbb{N}^m$ be a multi-index such that $\lambda \cdot \alpha \notin \sigma(A)$. Then there exists a $N = N(\alpha) \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$, we have that $\lambda_n \cdot \alpha \notin \sigma(A_n)$.*

Proof. Suppose the statement of the lemma is not true. Then there exists a sequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ satisfying $\lambda_{n_k} \cdot \alpha \in \sigma(A_{n_k})$ for all $k \in \mathbb{N}$. Because for all $k \in \mathbb{N}$, the sequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ is bounded, Lemma 9.0.1 implies that for each $k \in \mathbb{N}$ there exists a $l = l(k) \geq k$ such that $\lambda_{n_{l(k)}} \in \sigma_{l(k)}$. We note that, using Theorem 1.5.3 and the definition of λ_n^i, $1 \leq i \leq m$, that $\lim_{n\to\infty} \lambda_n^i = \lambda_i$, and thus we have that $\lim_{k\to\infty} \lambda_{n_{l(k)}} \cdot \alpha = \lambda \cdot \alpha$. By Lemma 1.5.4 (the pseudospectral approximation scheme for eigenvalues has no 'ghost solutions'), this implies that $\lambda \cdot \alpha \in \sigma(A)$, which gives a contradiction with the assumption that $\lambda \cdot \alpha \notin \sigma(A)$. This proves the lemma. $\qquad\square$

We can now prove the following:

Lemma 5.1.2. *Assume that Hypothesis 1 holds. Then there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, there exists a Taylor series*

$$P_n : \mathbb{C}^m \to \mathbb{C}^{(n+1) \times d}, \quad P_n(\theta) = \sum_{|\alpha|=0}^{k} P_n^\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}) \tag{5.5}$$

that satisfies Definition 5.1.1 up to and including order k.

Proof. The coordinate transformation $x = P(\theta)$ transforms system (5.3) into system (5.4) if

$$DP_n(\theta) \left(\Lambda_n \theta + \mathcal{O}(|\theta|^{k+1}) \right) = A_n P(\theta) = A_n^L P_n(\theta) + A_n^N(P_n(\theta)). \tag{5.6}$$

If P_n satisfies (5.5), then we can expand (5.6) as

$$\sum_{|\alpha|=0}^{k} (\lambda_n \cdot \alpha) P_n^\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}) = \sum_{|\alpha|=0}^{k} (A_n^L P_n^\alpha) \theta^\alpha + \overline{q}_n^\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}).$$

Here, $\overline{q}_n^\alpha$, $0 \leq |\alpha| \leq k$ is such that

$$A_n^N(P_n(\theta)) = \sum_{|\alpha|=0}^{k} \overline{q}_n^\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1}). \tag{5.7}$$

Since $A_n^N(0) = 0, DA_n^N(0) = 0$, we find that $\overline{q}_n^\alpha$ only depends on P_n^β with $|\beta| < |\alpha|$.

Thus, for $0 \leq |\alpha| \leq k$ the coefficients P_n^α should satisfy

$$(\lambda_n \cdot \alpha) P_n^\alpha - A_n^L P_n^\alpha = \overline{q}_n^\alpha. \tag{5.8}$$

Since for $|\alpha| = 0, 1$ it holds that $\overline{q}_n^\alpha = 0$, we have that for $|\alpha| = 0$ equation (5.8) is satisfied for $P_n^\alpha = 0$; for $|\alpha| = 1$, i.e. $\alpha = e_i$ for some $1 \leq i \leq k$, equation (5.8) is satisfied if we choose $P_n^{e_i} = \xi_n^i$ as in (5.2). We note that with this choice of P_n^α for $|\alpha| = 0, 1$, the first two criteria of Definition 5.1.1 are satisfied. For $2 \leq |\alpha| \leq k$, we have by assumption that $\lambda \cdot \alpha \notin \sigma(A)$. Thus, by Lemma 5.1.1, we can find a $N \in \mathbb{N}$ such that for $n \geq N$, $\lambda_n \cdot \alpha \notin \sigma(A_n)$ for $2 \leq |\alpha| \leq k$. Thus, for $n \geq N$ and $2 \leq |\alpha| \leq k$, the equation (5.8) has a unique solution P_n^α. $\square$

5.2 Convergence of approximate conjugation map

In this section, we show that the coefficients of the approximate conjugation map converge – when rightly embedded in the state space X – to the coefficients of the conjugation map of the original invariant manifold of the delay equation (5.1). More formally, we state the following lemma:

Lemma 5.2.1. *Let us assume that Hypothesis 1 holds. Let*

$$P : X_0 \to X, \quad P(\theta) = \sum_{|\alpha|=0}^{k} P_\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1})$$

be as in Lemma 3.2.1. Let

$$P_n : \mathbb{C}^k \to \mathbb{C}^{(n+1) \times d}, \quad P_n(\theta) = \sum_{|\alpha|=0}^{k} P_n^\alpha \theta^\alpha + \mathcal{O}(|\theta|^{k+1})$$

be the approximate conjugation map to the invariant manifold W_{loc} as in Definition 5.1.1. Then for $0 \leq |\alpha| \leq k$, we have that

$$\lim_{n \to \infty} \|\mathcal{L}_n(P_n^\alpha) - P_\alpha\|_\infty = 0, \tag{5.9}$$

where $\|.\|_\infty$ denotes the supremum-norm on X.

Proof. We recall from the proof of Lemma 5.1.2 that the coefficients P_n^α, $0 \leq |\alpha| \leq k$ satisfy (5.8), where $\bar{q}_n^\alpha$, $0 \leq |\alpha| \leq k$ is such that (5.7) holds. Since

$$A_n^N(x) = (G(\mathcal{L}_n x), 0, \ldots, 0),$$

we see that $\bar{q}_n^\alpha$ is of the form

$$\bar{q}_n^\alpha = (q_n^\alpha, 0, \ldots, 0),$$

with $q_n^\alpha \in \mathbb{C}^k$. In fact, for q_α as in (3.6) we can write

$$q_\alpha = h_\alpha \left(\{P_\beta\}_{|\beta| < |\alpha|} \right)$$

for some function h_α; and q_n^α is given by

$$q_n^\alpha = h_\alpha \left(\{\mathcal{L}_n(P_n^\beta)\}_{|\beta| < |\alpha|} \right)$$

for $0 \leq |\alpha| \leq k$.

Using the definition of the matrix A_n^L, we can rewrite (5.8) as

$$\begin{cases} L(\mathcal{L}_n(P_n^\alpha)) = (\lambda_n \cdot \alpha)(P_n^\alpha)_0 + q_n^\alpha, \\ \mathcal{L}_n(P_n^\alpha)'(\theta_{n,i}) = (\lambda_n \cdot \alpha)(P_n^\alpha)_i, \quad 1 \leq i \leq n. \end{cases} \tag{5.10}$$

As in Lemma 1.4.1, we see that the second equation of (5.10) holds if and only if $\mathcal{L}_n(P_n^\alpha) = p_n(\lambda_n \cdot \alpha, (P_n^\alpha)_0)$. Thus, we see that (5.10) holds if and only if

$$(\lambda_n \cdot \alpha)(P_n^\alpha)_0 + q_n^\alpha = L(p_n(\lambda_n \cdot \alpha, (P_n^\alpha)_0)) \tag{5.11}$$

and $x_i = p_n(\lambda_n \cdot \alpha, (P_n^\alpha)_0)(\theta_{n,i})$ for $1 \leq i \leq n$.

We recall that the coefficients P_α of $P : X_0 \to X$ satisfy (3.10); we note that (5.11) can be viewed as a discretised version of (3.10).

Let us introduce the operators

$$T : \mathbb{C}^2 \to \mathbb{C}^2, \quad T(x) = (\lambda \cdot \alpha)x - L(e_{(\lambda \cdot \alpha)}x),$$

$$T_n : \mathbb{C}^2 \to \mathbb{C}^2, \quad T_n(x) = (\lambda_n \cdot \alpha)x - L(p_n(\lambda_n \cdot \alpha, x)).$$

We note that if $\lambda \cdot \alpha \notin \sigma(A)$, then the operator T is invertible. Moreover, if $\lambda \cdot \alpha \notin \sigma(A)$, then by Lemma 5.1.1 and the characterisation of the eigenvalues of A_n^L as in Lemma 1.4.1, the operators T_n are invertible for n large enough. We make the following claim:

Claim 5.2.2. *Let $\alpha \in \mathbb{C}^2$ be such that $\lambda \cdot \alpha \notin \sigma(A)$, then we have that*

$$\lim_{n \to \infty} T_n^{-1} = T^{-1},$$

where the convergence is in operator norm.

We prove the claim after proving the rest of the lemma.

We now prove the result of the lemma by induction. For $|\alpha| = 0$, we have that $P_n^\alpha(0) = P_\alpha(0) = 0$, so we directly see that (5.9) holds. For $|\alpha| = 1$, i.e. $\alpha = e_i$ for $1 \le i \le m$, we have that $P_n^{e_i} = \xi_n^i$ with ξ_n^i as in (5.2). The statement (5.9) then follows by Lemma 4.0.2.

Now, let us assume that (5.9) holds for all multi-indices with norm smaller or equal than l for some $l \le k - 1$; let α be such that $|\alpha| = l + 1$. Since h_α is continuous and the fact that (5.9) holds for $|\alpha| \le l$, we have that $\lim_{n \to \infty} q_n^\alpha = q_\alpha$ for $|\alpha| = l + 1$. We note that $P_\alpha(0) = T^{-1}(q_\alpha)$ and $(P_n^\alpha)(0) = T_n^{-1}(q_n^\alpha)$. Thus, we find that

$$\|P_\alpha(0) - (P_n^\alpha)_0\| \le \left\|T_n^{-1}(q_n^\alpha - q_\alpha)\right\| + \left\|(T_n^{-1} - T^{-1})q_\alpha\right\|.$$

Using Claim 5.2.2, we then find that $\lim_{n \to \infty} \|P_\alpha(0) - (P_n^\alpha)_0\| = 0$, i.e. $\lim_{n \to \infty}(P_n^\alpha)_0 = P_\alpha(0)$. Then, since $\mathcal{L}_n(P_n^\alpha) = p_n(\lambda_n \cdot \alpha, (P_n^\alpha)_0)$, we find that

$$\lim_{n \to \infty} \left\|\mathcal{L}_n(P_n^\alpha) - P_\alpha(0)e_{(\lambda \cdot \alpha)}\right\|_\infty = 0,$$

i.e. (5.9) also holds for $|\alpha| = l + 1$. This proves the lemma up to the proof of Claim 5.2.2.

To prove Claim 5.2.2, we recall that if $\lambda \cdot \alpha \notin \sigma(A)$, the operator T is invertible and the operators T_n are invertible for n large enough. Moreover, we see that $\lim_{n \to \infty} T_n = T$ in operator norm.

As already remarked, we know that T_n is invertible for n large enough; by the Neumann series, we then know that

$$\left\|T_n^{-1}\right\| \le \frac{\left\|T^{-1}\right\|^{-1}}{1 - \|T - T_n\| \|T^{-1}\|^{-1}}, \tag{5.12}$$

where the norms denote the operator norm. Moreover, we note that

$$T_n^{-1} - T^{-1} = -T_n^{-1}(T_n - T)T^{-1}$$

so

$$\left\|T_n^{-1} - T^{-1}\right\| \le \left\|T_n^{-1}\right\| \left\|T^{-1}\right\| \|T_n - T\|.$$

By (5.12), the sequence $(\left\|T_n^{-1}\right\|)_{n \in \mathbb{N}}$ is bounded. Since also $\|T_n - T\| \to 0$ as $n \to \infty$, this gives that $\lim_{n \to \infty} \left\|T_n^{-1} - T^{-1}\right\|$, which proves the claim. $\qquad\square$

5.3 Example: Wright's equation

As a numerical example, let us consider Wright's equation

$$\dot{x}(t) = \alpha x(t - 1)(x(t) + 1), \quad t \ge 0 \tag{5.13}$$

with $x(t) \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ a parameter. We recall from Section 1.7 that for $\alpha = -\frac{\pi}{2}$ the linearisation of (5.13) around $x = 0$ has two eigenvalues $\pm i\frac{\pi}{2}$ on the imaginary axis. Thus, system (5.13) has a local center manifold [11, Corollary IX.7.8] and Hypothesis 1 is satisfied with $k = 2$ (in particular, we are in Situation 1 from page 28).

Let $P : \mathbb{C}^2 \to X$ be as in Lemma 3.2.1 and let $P_n : \mathbb{C}^2 \to \mathbb{C}^{(n+1) \times d}$ be as in Lemma 5.1.2. In Figure 5.2, we plotted the coefficients P_α and the approximate coefficients $\mathcal{L}_n P_n^\alpha$ for $1 \le |\alpha| \le 2$ and $n = 5$.

Instead of studying invariant manifolds of the system (5.13), one can also consider the delay equation

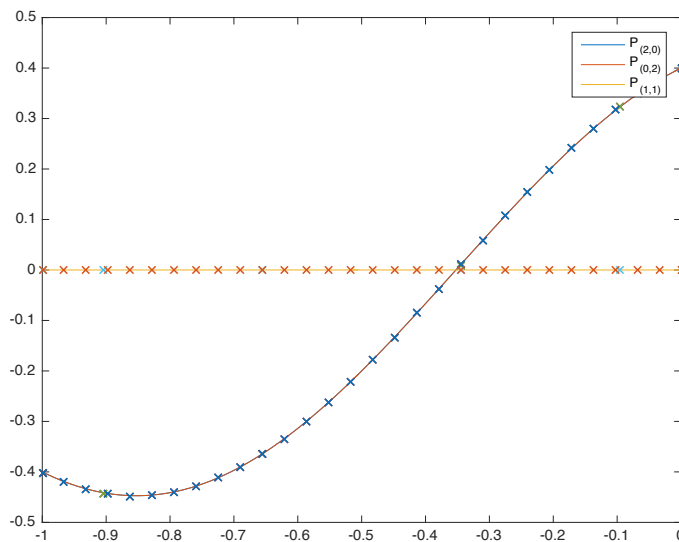$$\begin{cases} \dot{x}(t) &= \alpha(t)x(t - 1)(1 + x(t)), \\ \dot{\alpha}(t) &= 0, \end{cases} \tag{5.14}$$

Figure 5.2: For the center manifold of (5.13) for $\alpha = -\frac{\pi}{2}$, this plot shows the coefficients P_α (lines) and the approximate coefficients $\mathcal{L}_n P_n^\alpha$ (crosses) for $1 \leq |\alpha| \leq 2$ and $n = 5$.

where the second equation reflects the fact that $\alpha \in \mathbb{R}$ is a constant parameter. Since for any $\alpha_0 \in \mathbb{R}$, the second equation in (5.14) results in a zero eigenvalue for the linearisation of (5.14) around $(x, \alpha) = (0, \alpha_0)$, the equation (5.14) will have an invariant center manifold, which results in a parameter dependent invariant manifold in the system (5.13) [23]. The (approximation of) this parameter dependent center manifold can give us much insight into the (approximation of) the bifurcation behaviour of the system (5.13); see also [4] for a discussion of parameter dependent manifolds and bifurcation behaviour for delay equations.

Part C

Chapter 6

Trotter-Kato approximation of linear delay equations

In Part A and Part B of this thesis, we used the pseudospectral method and the parametrisation method to study approximation of eigenvalues and invariant manifolds of delay equations. In Part C, we now turn our attention towards the approximation of orbits of delay equations using the Trotter-Kato theorem.

Suppose we have a Banach space X with a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of linear operators; moreover, suppose we have a family of strongly continuous semigroup of linear operators $\{T_n(t)\}_{t \geq 0}$ on finite dimensional subspaces $X_n \subseteq X$. The Trotter-Kato Theorem (see Appendix C) gives necessary and sufficient conditions for pointwise convergence of the semigroups $\{T_n(t)\}_{t \geq 0}$ towards the semigroup $\{T(t)\}_{t \geq 0}$.

When we apply the Trotter-Kato theorem to approximation of semigroups associated to linear delay equations, there is a wide range of possibilities to define the approximating semigroups $\{T_n(t)\}_{t \geq 0}$ and in this chapter, we specifically focus on a systematic approach to the definition of the approaching semigroups $\{T_n(t)\}_{t \geq 0}$. This approach is based upon using embedding and projection operators associated approximation schemes for the functions in the state space. In this way, we can replace the condition the conditions in the Trotter-Kato theorem by conditions on the function approximation scheme used.

The rest of this chapter is structured as follows. In Section 6.1 and 6.2 we follow [1]; we introduce a systematic approach to the definition of the approximating generator in Section 6.1 and apply this to the case of spline approximation in Section 6.2. In Section 6.3, we then apply the approximation scheme introduced in Section 6.1 to the case of Legendre approximation.

6.1 Definition and convergence of approximating semigroups

Whereas in Part A and B of this thesis, we used $X = C\left([-\tau, 0], \mathbb{R}^d\right)$ as state space, we now choose for the state space $Z = \mathbb{R}^d \times L^2\left([-\tau, 0], \mathbb{R}^d\right)$, mainly for technical reasons: if we equip Z with the inner product $\langle ., . \rangle$ given by

$$\langle (\eta, \phi), (\zeta, \psi) \rangle = \langle \eta, \zeta \rangle_{\mathbb{R}^d} \times \langle \phi, \psi \rangle_{L^2([-\tau,0],\mathbb{R}^d)}, \quad (\eta, \phi), (\zeta, \psi) \in Z.$$

then Z has the advantage of being a Hilbert space. We denote the induced norm by $\|.\|$.

On Z we study the initial value problem

$$\begin{cases} \dot{x}(t) = L(\phi) & \text{for } t \geq 0, \\ x(0) = \eta, \\ x(\theta) = \phi(\theta) & \text{a.e. on } [-\tau, 0]. \end{cases} \tag{6.1}$$

with $(\eta, \phi) \in Z$ and $L : \mathcal{D}(L) \subseteq Z \to \mathbb{R}^d$ a linear, but possibly unbounded, operator.

The delay equation (6.1) generates a strongly continuous semigroup on the Banach space $(Z, \|.\|)$, which we will denote by $\{T(t)\}_{t\geq 0}$. The generator A of this semigroup is given by

$$\mathcal{D}(A) = \{(\eta, \phi) \in Z \mid \phi \in H^1\left([-\tau, 0], \mathbb{R}^d\right), \ \eta = \phi(0)\},$$
$$A(\eta, \phi) = \left(L\phi, \dot{\phi}\right). \tag{6.2}$$

In fact, one has that the abstract ODE

$$\frac{d}{dt}v(t) = Av(t), \quad t \geq 0$$
$$v(0) = (\eta, \phi) \tag{6.3}$$

is equivalent to the delay equation (6.1).

Throughout the rest of this chapter, we will use the following notation: we write $\langle \phi, \psi \rangle_{L^2}$ as shorthand notation for $\langle \phi, \psi \rangle_{L^2([-\tau, 0], \mathbb{R}^d)}$ and we write $\|\phi\|_{L^2}$ as shorthand for $\|\phi\|_{L^2([-\tau, 0], \mathbb{R}^d)}$. Moreover, for $k \in \mathbb{N}$, we write C^k for $C^k\left([-\tau, 0], \mathbb{R}^d\right)$, which is the space of k times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^d$. Furthermore, we will write

$$\mathcal{C}^k = \{(\phi(0), \phi) \mid \phi \in C^k\}.$$

If $\hat{\phi} \in \mathcal{C}^k$, then we denote by ϕ the element in C^k such that $\hat{\phi} = (\phi(0), \phi)$. Conversely, for $\phi \in C^k$, we denote by $\hat{\phi}$ the element $(\phi(0), \phi) \in \mathcal{C}^k$.

With this notation, we state the following theorem.

Theorem 6.1.1. *For $n \in \mathbb{N}$, let $Z_n \subseteq \mathcal{D}(A)$ be a linear subspace. Let $E_n : Z_n \to Z$ be the embedding operator and define $P_n : Z \to Z_n$ such that $E_n P_n$ is the orthogonal projection onto the subspace $E_n Z_n$ of Z. Let us equip Z_n with the inner product $\langle x, y \rangle_n = \langle E_n x, E_n y \rangle$ for all $x, y \in Z_n$. Denote by $\{T(t)\}_{t\geq 0}$ the semigroup associated to the delay equation (6.1). Furthermore, define $A_n : Z_n \to Z_n$ as $A_n = P_n A E_n$ and suppose that the two following conditions are satisfied:*

(i) $\lim_{n\to\infty} E_n P_n z = z$ for all $z \in Z$.

(ii) For some $k \geq 1$, we have that

a) $\lim_{n\to\infty} D\psi_n = D\psi$ in $L^2((-\tau, 0), \mathbb{R}^d)$,

b) $\lim_{n\to\infty} L(\psi_n) = L(\psi)$ in $\mathbb{R}^d$

for all $\psi \in \mathcal{C}^k$, where ψ_n is such that $P_n \hat{\psi} = (\psi_n(0), \psi_n)$ and D is the differential operator.

Then each A_n generates a strongly continuous semigroup $\{T_n(t)\}_{t\geq 0}$ on Z_n such that

$$\lim_{n\to\infty} E_n T_n(t) P_n z = T(t)z$$

for all $z \in Z$, uniformly on t-bounded intervals.

In Section 6.2 and Section 6.3, we will choose the spaces $Z_n \subseteq Z$ as subspaces associated to approximations of functions in the function space Z. Then Theorem 6.1.1 tells us i) how to build an approximating semigroup using the function approximation scheme and ii) what sufficient conditions on the function approximation scheme are for the semigroups $\{T_n(t)\}_{t\geq 0}$ to converge to the original semigroup $\{T(t)\}_{t\geq 0}$.

We will prove this theorem on page 42. To prove the theorem, we need the following lemmata:

Lemma 6.1.2. *For $k = 1, 2, \ldots$ the sets*

$$\mathcal{D}^k = \{\hat{\phi} \in \mathcal{C}^k \mid \dot{\phi}(0) = L(\phi)\}$$

and $(\lambda I - A)\mathcal{D}^k$ for $\lambda \in \mathbb{R}$ sufficiently large are dense in Z.

Proof. In order to prove the statement, we first prove that for $k = 1, 2, \ldots$, $\mathcal{C}^{k-1}$ is dense in Z. We note that $\mathcal{D}(A^k) \subseteq \mathcal{C}^{k-1}$: if $(\eta, \phi) \in \mathcal{D}(A^k)$, then $\eta = \phi(0)$ since $(\eta, \phi) \in \mathcal{D}(A)$, and $\phi \in H^k\left([-\tau, 0], \mathbb{R}^d\right)$. By the Sobolev inequality (see [13, Thm. 5.6.6]), this implies that $\phi \in \mathcal{C}^{k-1}$. Thus, we find that $(\eta, \phi) = (\phi(0), \phi) \in \mathcal{C}^{k-1}$. Because $\mathcal{D}(A^k) \subseteq \mathcal{C}^{k-1}$ and $\mathcal{D}(A^k)$ is dense in Z, we conclude that $\mathcal{C}^{k-1}$ is dense in Z.

We note that for $\lambda \in \mathbb{R}$ large enough the operator $(\lambda I - A)^{-1}$ exists and is bounded. We prove that $(\lambda I - A)^{-1} \mathcal{C}^{k-1} \subseteq \mathcal{D}^k$. Indeed, let us take $\hat{\psi} \in \mathcal{C}^{k-1}$ and let $\hat{\phi}$ denote the unique solution to $(\lambda I - A)\hat{\phi} = \hat{\psi}$. Note that $\hat{\phi}$ is of the form $\hat{\phi} = (\phi(0), \phi)$ since $\hat{\phi} \in \mathcal{D}(A)$. The equality $(\lambda I - A)\hat{\phi} = \hat{\psi}$ implies that

$$L(\phi) - \lambda\phi(0) = \psi(0) \tag{6.4}$$

$$\dot{\phi} - \lambda\phi = \psi \tag{6.5}$$

Since $\psi \in C^{k-1}$, the equality (6.5) implies that $\phi \in C^k$. Furthermore, (6.5) also implies that $\dot{\phi}(0) - \lambda\phi(0) = \psi(0)$; together with (6.4) this implies that $L(\phi) = \dot{\phi}(0)$. So we see that $\hat{\phi} \in \mathcal{D}^k$. This implies that $(\lambda I - A)^{-1}\mathcal{C}^{k-1} \subseteq \mathcal{D}^k$.

The statement of the lemma now follows from the identity $(\lambda I - A)^{-1}\mathcal{C}^{k-1} \subseteq \mathcal{D}^k$. Since we have already proven that $\mathcal{C}^{k-1}$ is dense, and since $(\lambda I - A)^{-1}$ is a bounded linear operator, it follows that $(\lambda I - A)^{-1}\mathcal{C}^{k-1}$ is dense; therefore $(\lambda I - A)^{-1}\mathcal{C}^{k-1} \subseteq \mathcal{D}^k$ implies that $\mathcal{D}^k$ is dense. Furthermore, $(\lambda I - A)^{-1}\mathcal{C}^{k-1} \subseteq \mathcal{D}^k$ implies that $\mathcal{C}^{k-1} \subseteq (\lambda I - A)\mathcal{D}^k$; since $\mathcal{C}^{k-1}$ is dense, it follows that $(\lambda I - A)\mathcal{D}^k$ is dense. This proves the claim. $\square$

Since $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on Z, we know by theory of semigroups (see for example [11, Appendix II, Prop. 1.3]) that there exists a $M \geq 1, \omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0. \tag{6.6}$$

We state the following lemma [1, Lemma 2.3]:

Lemma 6.1.3. *Let $\omega \in \mathbb{R}$ be such that* (6.6) *holds for some $M \geq 1$. Then $A - \omega I$ is dissipative on Z, i.e.*

$$\langle Az, z \rangle \leq \omega \|z\|^2 \quad \text{for all } z \in \mathcal{D}(A).$$

Using Lemma 6.1.2 and Lemma 6.1.3, we can now prove Theorem 6.1.1.

Proof. (of Theorem 6.1.1) We prove Theorem 6.1.1 by an application of the Trotter-Kato theorem (Theorem C.0.1). With notation as in Theorem C.0.1, we choose $X = Z, X_n = Z_n \subseteq Z$ for all $n \in \mathbb{N}$, where we equip Z_n with the inner product $\langle x, y \rangle_n = \langle E_n x, E_n y \rangle$ for all $x, y \in Z_n$; we remark that with this inner product, Z_n is a Hilbert space. We note that we have $\|E_n\| \leq 1$ for all $n \in \mathbb{N}$. Since $E_n P_n$ is the orthogonal projection onto $E_n Z_n$, we furthermore have that $\|P_n\| \leq 1$ for all $n \in \mathbb{N}$ and $P_n E_n$ is the identity mapping on Z_n.

Since $P_n : Z \to Z_n$ and $E_n Z_n \subseteq \mathcal{D}(A)$, we have that AE_n is defined on all of Z_n. Because $P_n AE_n$ is the composition of one closed and two bounded operator, we see that it is closed. Thus, by the Closed Graph Theorem, we find that $A_n : Z_n \to Z_n$ is bounded. Therefore, it is the generator a $\mathcal{C}_0$–semigroup $T_n(t) = e^{A_n t}$ on Z_n.

Let $\omega \in \mathbb{R}$ be as in (6.6). To prove that $T_n(t), T(t) \in G(M, \omega)$ for some $M \geq 1$, we first prove that $A_n : Z \to Z_n$ is ω-dissipative. Let $z \in Z_n$; then since $E_n P_n : Z \to Z$ is an orthogonal projection, we find that

$$\langle A_n z, z \rangle_n = \langle E_n P_n AE_n z, E_n z \rangle = \langle AE_n z, E_n P_n E_n z \rangle.$$

Since $P_n E_n$ is the identity operator on Z_n, we see that

$$\langle AE_n z, E_n P_n E_n z \rangle = \langle AE_n z, E_n z \rangle \leq \omega \|E_n z\|^2 = \omega \|z\|_n^2$$

where the inequality follows from Lemma 6.1.3. Since this holds for all $z \in Z_n$, we conclude that A_n is ω-dissipative. This implies that $T_n(t) \in G(M, \omega)$ for all $n \in \mathbb{N}$ and some $M \geq 1$.

To prove consistency of the approximation, we use Theorem C.0.2. Let $k \geq 1$ be as in assumption (ii) of Theorem 6.1.1 and set $D = \mathcal{D}^k$, then by Lemma 6.1.2 we find that D is dense in Z and $(\lambda I - A)D$ is dense in Z for $\lambda \in \mathbb{R}$ large enough. Now, let us fix $\hat{\psi} \in D$ and define $\phi_n = P_n \psi \in \mathcal{D}(A_n)$. Then by Assumption (i), we have that $\lim_{n \to \infty} E_n \phi_n = \psi$. In order to apply Theorem C.0.2, it remains to prove that $\lim_{n \to \infty} E_n A_n \phi_n = A\psi$. Indeed, we have that

$$\|E_n A_n \phi_n - A\psi\| = \|E_n P_n A E_n P_n \psi - A\psi\|$$
$$\leq \|E_n P_n A E_n P_n \psi - E_n P_n A\psi\| + \|E_n P_n A\psi - A\psi\|.$$

We note that $\|E_n P_n A\psi - A\psi\| \to 0$ as $N \to \infty$ by Assumption (i) in Theorem 6.1.1. To estimate the first term, we note that (with notation as in the statement of Theorem 6.1.1)

$$\|E_n P_n A E_n P_n \psi - E_n P_n A\psi\|^2 \leq \|A E_n P_n \psi - A\psi\|^2$$
$$= \|L(\psi_n) - L(\psi)\|^2_{\mathbb{R}^d} + \|D(\psi_n) - D(\psi)\|_{L^2} \to 0 \text{ as } n \to \infty$$

where the last step follows form assumption (ii) of Theorem 6.1.1. We are now in a position to apply Theorem C.0.2 and to conclude that the approximation is consistent. Thus, we can apply Theorem C.0.1 to complete the proof of Theorem 6.1.1. $\qquad\square$

For future reference, we state the following remark:

Remark 6.1.1. If we want to show that Assumption (i) in Theorem 6.1.1 holds, it suffices to proof that $\lim_{n \to \infty} E_n P_n z = z$ for all $z \in \mathcal{C}^k$ for some $k \geq 1$. Indeed, let us assume that $\lim_{n \to \infty} E_n P_n z = z$ for all $z \in \mathcal{C}^k$ for some $k \geq 1$ and let us take an arbitrary $z \in Z$. Because $\mathcal{C}^k$ is dense in Z (see Lemma 6.1.2), we can find a $x \in \mathcal{C}^k$ such that $\|x - z\| \leq \frac{\epsilon}{3}$; furthermore, there exists a $N \in \mathbb{N}$ such that $\|E_n P_n x - x\| \leq \frac{\epsilon}{3}$ for all $n \geq N$. Thus, we find for $n \geq N$ that

$$\|E_n P_n z - z\| \leq \|E_n P_n z - E_n P_n x\| + \|E_n P_n x - x\| + \|x - z\|$$
$$\leq \|z - x\| + \|E_n P_n x - x\| + \|x - z\|$$
$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

which proves that $\lim_{n \to \infty} E_n P_n z = z$.

6.2 Approximation using splines

In this section, we apply Theorem 6.1.1 in the case where the function approximation scheme as introduced in Section 6.1 is the spline approximation scheme (see Section A.1). To avoid our discussion being (too) notationally cumbersome, we study the scalar case, i.e. the case $d = 1$.

Throughout this section, we follow [1].

For $n \in \mathbb{N}, n \geq 1$, let us define

$$t_{j,n} = -\frac{\tau j}{n} \quad \text{for } 0 \leq j \leq n.$$

and let us define the first order spline functions $e_{j,n}$, $0 \leq j \leq n$ as in Section A.1. For $n \in \mathbb{N}$, then we define the following finite dimensional subspace of Z:

$$Z^s_n = (\mathbb{R} \times \text{span}\{e_{0,n}, \ldots, e_{n,n}\}) \cap \mathcal{D}(A)$$
$$= \text{span}\left((e_{0,n}(0), e_{0,n}), \ldots, (e_{n,n}(0), e_{n,n})\right). \tag{6.7}$$

We state the following lemma regarding the convergence of the approximation of the semigroup associated to (6.1).

Lemma 6.2.1. *Let us define Z_n^s as in (6.7), $E_n^s : Z_n^s \to Z$ the embedding operator and $P_n^s : Z \to Z_n^s$ such that $E_n^s P_n^s : Z \to Z$ is the orthogonal projection onto $E_n^s Z_n^s$. Moreover, let us define $A_n^s : Z_n^s \to Z_n^s$ by $A_n^s = P_n^s A E_n^s$. Then A_n^s is the infinitesimal generator of a strongly continuous semigroup $\{T_n^s(t)\}_{t \geq 0}$ on Z_n^s such that*

$$\lim_{n \to \infty} E_n^s T_n^s(t) P_n^s z = T(t) z$$

for all $z \in Z, t \geq 0$ with convergence uniform on t-bounded intervals.

Proof. We show that Conditions (i) and (ii) in the statement of Theorem 6.1.1 are satisfied; the claim then follows by an application of Theorem 6.1.1.

Proof of Condition (i) We recall that by Remark 6.1.1 it suffices to prove that $\lim_{n \to \infty} E_n^s P_n^s z = z$ for all $z \in \mathcal{C}^1$. For $\hat{\psi} = (\psi(0), \psi) \in \mathcal{C}^1$, let us define $Q_n^s \hat{\psi}$ as the first order spline satisfying $(Q_n^s \hat{\psi})(t_{i,n}) = \psi(t_{i,n})$ for $0 \leq i \leq n$. Since $E_n^s P_n^s : Z \to Z$ is an orthogonal projection onto $E_n^s Z_n^s$, we have that

$$\left\| \hat{\psi} - E_n^s P_n^s \hat{\psi} \right\| = \min_{\phi \in E_n^s Z_n^s} \left\| \phi - \hat{\psi} \right\|. \tag{6.8}$$

Since $((Q_n^s \hat{\psi})(0), Q_n^s \hat{\psi}) \in E_n^s Z_n^s$, we find that

$$\left\| \hat{\psi} - E_n^s P_n^s \hat{\psi} \right\| \leq \left\| \hat{\psi} - ((Q_n^s \hat{\psi})(0), Q_n^s \hat{\psi}) \right\|$$

$$= \left(\left\| \psi(0) - (Q_n^s \hat{\psi})(0) \right\|^2 + \left\| \psi - Q_n^s \hat{\psi} \right\|_{L^2}^2 \right)^{1/2}$$

$$= \left\| \psi - Q_n^s \hat{\psi} \right\|_{L^2}$$

where in the last step we used that $(Q_n^s \hat{\psi})(0) = \psi(0)$. Since $\lim_{n \to \infty} \left\| \psi - Q_n^s \hat{\psi} \right\|_{L^2} = 0$ (see Lemma A.1.1), this gives that $\lim_{n \to \infty} E_n^s P_n^s \hat{\psi} = \hat{\psi}$ for $\hat{\psi} \in \mathcal{C}^1$. Thus, using Remark 6.1.1, we conclude that Condition (i) in Theorem 6.1.1 is satisfied.

Proof of Condition (iia) Let $\hat{\psi} = (\psi(0), \psi) \in \mathcal{C}^2$; we want to prove that $\|D\psi_n - D\psi\|_{L^2} \to 0$ as $n \to \infty$ (with notation as in the statement of Theorem 6.1.1). We note that

$$\|D\psi_n - D\psi\|_{L^2} \leq \left\| D\psi_n - DQ_n^s \hat{\psi} \right\|_{L^2} + \left\| D\psi - DQ_n^s \hat{\psi} \right\|_{L^2}. \tag{6.9}$$

By Lemma A.1.1, we find that $\left\| D\psi - DQ_n^s \hat{\psi} \right\|_{L^2} \to 0$ as $n \to \infty$.

If we let $a, b \in \mathbb{R}$, then a quick computation shows that there exists a constant $k > 0$ such that for all $p : [a, b] \to \mathbb{R}$ polynomials of degree 1, we have that

$$\int_a^b \left(p'(x) \right)^2 dx \leq k \frac{1}{(b-a)^2} \int_a^b p^2(x) dx$$

for all $p : [a, b] \to \mathbb{R}$ polynomials of degree 1. Since $\psi_n, Q_n^s \hat{\psi}$ are polynomials of order 1 on the segments $[t_{k,n}, t_{k+1,n}]$, we find that

$$\left\| D\psi_n - DQ_n^s \hat{\psi} \right\|_{L^2}^2 = \int_{-\tau}^0 \left((D\psi_n)(x) - (DQ_n^s \hat{\psi})(x) \right)^2 dx$$

$$= \sum_{k=0}^{n-1} \int_{t_{k+1,n}}^{t_{k,n}} \left((D\psi_n)(x) - (DQ_n^s \hat{\psi})(x) \right)^2 dx$$

$$\leq \sum_{k=0}^{n-1} k_1 \left(\frac{n}{\tau} \right)^2 \int_{t_{k+1,n}}^{t_{k,n}} \left(\psi_n(x) - (Q_n^s \hat{\psi})(x) \right)^2 dx$$

$$= k_1 \left(\frac{n}{\tau} \right)^2 \left\| (\psi_n(x) - (Q_n^s \hat{\psi})(x) \right\|_{L^2}^2.$$

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We have that

$$\left\|(\psi_n(x) - (Q_n^s\hat{\psi})(x)\right\|_{L^2} \le \left\|E_n^s P_n^s\hat{\psi} - ((Q_n^s\hat{\psi})(0), (Q_n^s\hat{\psi}))\right\| \le \left\|E_n^s P_n^s\hat{\psi} - \hat{\psi}\right\| + \left\|\hat{\psi} - ((Q_n^s\hat{\psi})(0), (Q_n^s\hat{\psi}))\right\|$$

$$\le 2\left\|\hat{\psi} - ((Q_n^s\hat{\psi})(0), (Q_n^s\hat{\psi}))\right\|$$

where the last step follows from (6.8). Because $(Q_n^s\hat{\psi})(0) = \psi(0)$, we find that $\left\|\hat{\psi} - ((Q_n^s\hat{\psi})(0), (Q_n^s\hat{\psi}))\right\| = \left\|\psi - Q_n^s\hat{\psi}\right\|_{L^2}$. Since $\left\|\psi - Q_n^s\hat{\psi}\right\|_{L^2} = \mathcal{O}\left(\frac{1}{n^2}\right)$ if $\psi \in C^2$ (see Lemma A.1.1), we conclude that

$$\left\|D\psi_n - DQ_n^s\hat{\psi}\right\|_{L^2} \le \sqrt{k_1}\frac{n}{\tau}\left\|\psi - Q_n^s\hat{\psi}\right\|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

Thus, using (6.9) we conclude that

$$\lim_{n\to\infty} \|D\psi_n - D\psi\| = 0$$

if $\hat{\psi} \in \mathcal{C}^2$.

Proof of Condition (iib) Let us again take $\hat{\psi} \in \mathcal{C}^2$, we now want to prove that $L(\psi_n) \to L(\psi)$ as $n \to \infty$. We note that $L : \mathcal{C}([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ is a bounded linear operator; therefore, it suffices to prove that $\sup_{\theta \in [-\tau, 0]} |\psi_n(\theta) - \psi(\theta)| \to 0$ as $n \to \infty$. To prove this, we note that

$$\psi_n(\theta) = \psi_n(0) + \int_0^\theta D\psi_n(s)ds,$$

$$\psi(\theta) = \psi(0) + \int_0^\theta D\psi(s)ds.$$

Since $\psi_n(0) = \psi(0)$, we find that

$$|\psi_n(\theta) - \psi(\theta)| \le \int_0^\theta |D\psi_n(s) - D\psi(s)|\, ds \le \sqrt{\tau}\, \|D\psi_n - D\psi\|_{L^2}$$

where in the last step we have used Hölders inequality. But since we have already proven that

$$\lim_{n\to\infty} \|D\psi_n - D\psi\|_{L^2} = 0$$

we conclude that $\sup_{\theta \in [-\tau, 0]} |\psi_n(\theta) - \psi(\theta)| \to 0$ as $n \to \infty$ and thus that $L(\psi_n) \to L(\psi)$ as $n \to \infty$. This proves that Assumption (ii) in Theorem 6.1.1 is also satisfied. An application of Theorem 6.1.1 now proves the claim. $\qquad\square$

6.3   Approximation using Legendre polynomials

In this section, we give another example of an application of Theorem 6.1.1, now by using Legendre approximation (see Section A.2) on the state space Z. We study the scalar case, i.e. the case $d = 1$.

Let us denote by $\{p_i\}_{i\in\mathbb{N}}$ the set of Legendre polynomials on $[-\tau, 0]$, normalized such that $\langle p_i, p_j \rangle_{L^2} = \delta_{ij}$ for all $i, j \in \mathbb{N}$. For $n \in \mathbb{N}, n \ge 1$, let us define

$$\begin{aligned}
Z_n^l &= (\mathbb{R} \times \text{span}\{p_0, \ldots, p_n\}) \cap \mathcal{D}(A) \\
&= \text{span}\{(p_0(0), p_0), \ldots, (p_n(0), p_n)\}.
\end{aligned} \tag{6.10}$$

Lemma 6.3.1. *Let us define Z_n^l as in (6.10), $E_n^l : Z_n^l \to Z$ the embedding operator and $P_n^l : Z \to Z_n^l$ such that $E_n^l P_n^l : Z \to Z$ is the orthogonal projection onto $E_n^l Z_n^l$. Moreover, let us define $A_n^l : Z_n^l \to Z_n^l$ by*

$A_n^l = P_n^l A E_n^l$. Then A_n^l is the infinitesimal generator of a strongly continuous semigroup $\{T_n^l(t)\}_{t \geq 0}$ on Z_n such that

$$\lim_{n \to \infty} E_n^l T_n^l(t) P_n^l z = T(t)z$$

for all $z \in Z, t \geq 0$ with convergence uniform on t-bounded intervals.

To prove the lemma, we need the following result:

Lemma 6.3.2. *Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists a constant $C \in \mathbb{R}$ such that*

$$\int_a^b (Dq_n(x))^2 \, dx \leq Cn^4 \frac{1}{(b-a)^2} \int_a^b q_n^2(x) dx$$

for all $q_n : [a, b] \to \mathbb{R}$ polynomials of degree $n \in \mathbb{N}$.

Proof. We first consider the case $a = -1, b = 1$. For $n \in \mathbb{N}$, let us define

$$M_n = \max\{\|Dq_n\|_{L^2}^2 \mid q_n : [-1, 1] \to \mathbb{R} \text{ polynomial of degree } n \text{ with } \|q_n\|_{L^2} = 1\}.$$

If now $q_n : [-1, 1] \to \mathbb{R}$ is a polynomial of degree n, then we have that

$$\int_{-1}^1 (Dq_n(x))^2 dx \leq M_n^2 \int_{-1}^1 q_n(x)^2 dx. \tag{6.11}$$

In [26] it is proven that

$$\lim_{n \to \infty} \frac{M_n^2}{n^4} = \frac{1}{\pi^2}.$$

Thus, there exists a $C > 0$ such that $M_n^2 \leq Cn^4$ for all $n \in \mathbb{N}$; together with (6.11) this implies that

$$\int_{-1}^1 (Dq_n(x))^2 dx \leq Cn^4 \int_{-1}^1 p_n(x)^2 dx \tag{6.12}$$

for all functions $q_n : [-1, 1] \to \mathbb{R}$ that are polynomials of degree n.

If now $a, b \in \mathbb{R}$ with $a < b$ and $q_n : [a, b] \to \mathbb{R}$ is a polynomial of degree n and we define

$$\tilde{q}_n(x) = q_n \left(\frac{2}{b-a} x + \frac{-a-b}{b-a} \right)$$

then $\tilde{q}_n : [-1, 1] \to \mathbb{R}$ is a polynomial of degree n. Then, by (6.12) and the chain rule the statement of the lemma follows. $\square$

We can now prove Lemma 6.3.1.

Proof. (of Lemma 6.3.1) We prove that Condition (i) and (ii) in the statement of Theorem 6.1.1 are satisfied.

Proof of Condition (i) We recall that by Remark 6.1.1 it suffices to prove that $\lim_{n \to \infty} E_n^l P_n^l \hat{\psi} = \hat{\psi}$ for all $\hat{\psi} \in \mathcal{C}^2$. Let $\hat{\psi} = (\psi(0), \psi) \in \mathcal{C}^2$ and let us define

$$Q_n^l(\hat{\psi}) = \sum_{k=0}^n \langle p_k, \psi \rangle_{L^2} p_k$$

then $((Q_n^l(\hat{\psi})(0), Q_n^l(\hat{\psi})) \in Z_n^l$. Since $E_n^l P_n^l : Z \to Z$ is the orthogonal projection onto the subspace $E_n^l Z_n^l$, we have that

$$\left\| \hat{\psi} - E_n^l P_n^l \hat{\psi} \right\| = \min_{z \in Z_n^l} \left\| \hat{\psi} - z \right\| \leq \left\| \hat{\psi} - ((Q_n^l(\hat{\psi})(0), Q_n^l(\hat{\psi})) \right\| \tag{6.13}$$

and

$$\left\|\hat{\psi} - ((Q_n^l(\hat{\psi})(0), Q_n^l(\hat{\psi})))\right\|^2 = \left\|\psi(0) - (Q_n^l(\hat{\psi})(0)\right\|_{\mathbb{R}}^2 + \left\|\psi - Q_n^l(\hat{\psi})\right\|_{L^2}^2.$$

By Lemma A.2.3, we find that $\lim_{n \to \infty} \left\|\psi(0) - (Q_n^l(\hat{\psi})(0)\right\|_{\mathbb{R}}^2 = 0$; since $\{p_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2([-\tau, 0], \mathbb{R})$, we find that $\left\|\psi - Q_n^l(\hat{\psi})\right\|_{L^2} \to 0$ as $n \to \infty$. This proves that assumption (i) in the statement of Theorem 6.1.1 is satisfied.

Proof of Condition (iia) We choose $k = 3$ and let $\hat{\psi} = (\psi(0), \psi) \in \mathcal{C}^k$; we want to prove that $\|D\psi_n - D\psi\|_{L^2} \to 0$ as $n \to \infty$ (with notation as in the statement of Theorem 6.1.1). We note that

$$\|D\psi_n - D\psi\|_{L^2} \leq \left\|D(Q_n^l\hat{\psi}) - D\psi_n\right\|_{L^2} + \left\|D(Q_n^l\hat{\psi}) - D\psi\right\|_{L^2} \tag{6.14}$$

We note that $\left\|D(Q_n^l\hat{\psi}) - D\psi\right\|_{L^2} \leq \left\|Q_n^l\hat{\psi} - \psi\right\|_{H^1}$. For $\psi \in C^3$, we have that $\left\|Q_n^l\hat{\psi} - \psi\right\|_{H^1} \to 0$ as $n \to \infty$ by Lemma A.2.2. We conclude that $\left\|D(Q_n^l\hat{\psi}) - D\psi\right\|_{L^2} \to 0$ as $n \to \infty$.

Since $Q_n^l\hat{\psi}, \psi_n$ are both polynomials of degree n, we find by Lemma 6.3.2 that

$$\left\|D\psi_n - D(Q_n^l\hat{\psi})\right\|_{L^2} \leq Cn^2 \left\|\psi_n - Q^n\hat{\psi}\right\|_{L^2}$$

for some $C > 0$. We remark that

$$\begin{aligned}
\left\|\psi_n - Q^n\hat{\psi}\right\|_{L^2} &\leq \|\psi_n - \psi\|_{L^2} + \left\|Q^n\hat{\psi} - \psi\right\|_{L^2} \\
&\leq \left\|P_n^L\hat{\psi} - \hat{\psi}\right\| + \left\|(Q^n\hat{\psi}(0), Q^n\hat{\psi}) - \hat{\psi}\right\| \\
&\leq 2\left\|(Q^n\hat{\psi}(0), Q^n\hat{\psi}) - \hat{\psi}\right\|
\end{aligned}$$

where the last step follows from (6.13). Thus, we find that

$$\begin{aligned}
\left\|D\psi_n - D(Q_n^l\hat{\psi})\right\|_{L^2} &\leq 2Cn^2 \left\|(Q^n\hat{\psi}(0), Q_n^l\hat{\psi}) - \hat{\psi}\right\| \\
&= 2Cn^2 \left(\left\|(Q_n^l\hat{\psi})(0) - \psi(0)\right\|_{\mathbb{R}}^2 + \left\|Q_n^l\hat{\psi} - \psi\right\|_{L^2}^2\right)^{1/2}.
\end{aligned}$$

Since $\left\|(Q_n^l\hat{\psi})(0) - \psi(0)\right\|_{\mathbb{R}} = \mathcal{O}\left(\frac{1}{n^3}\right)$ if $\psi \in C^2$ (see Lemma A.2.3) and $\left\|Q_n^l\hat{\psi} - \psi\right\|_{L^2} = \mathcal{O}\left(\frac{1}{n^3}\right)$ if $\psi \in \mathcal{C}^3$ (Lemma A.2.1), we find that $\left\|D\psi_n - D(Q_n^l\hat{\psi})\right\|_{L^2} \to 0$ as $n \to \infty$ if $\psi \in C^3$. Since we had already proven that $\left\|D(Q_n^l\hat{\psi}) - D\psi\right\|_{L^2} \to 0$ as $n \to \infty$ for $\psi \in \mathcal{C}^1$, we conclude by (6.14) that

$$\lim_{n \to \infty} \|D\psi_n - \psi\|_{L^2} = 0$$

if $\psi \in C^3$.

Proof of Condition (iib) Again, let $\hat{\psi} \in C^3$; to prove that $L(\psi_n) \to L(\psi)$ as $n \to \infty$, we show that $\sup_{\theta \in [-\tau, 0]} |\psi_n(\theta) - \psi(\theta)| \to 0$ as $n \to \infty$; by continuity of $L : C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ it then follows that $\lim_{n \to \infty} L(\psi_n) = L(\psi)$.

We note that

$$\psi_n(\theta) = \psi_n(0) + \int_0^\theta (D\psi^n)(s)ds,$$

$$\psi(\theta) = \psi(0) + \int_0^\theta (D\psi)(s)ds$$

which gives that

$$|\psi_n(\theta) - \psi(\theta)| \le |\psi_n(0) - \psi(0)| + \int_0^\theta |(D\psi_n)(s) - (D\psi)(s)|\, ds$$

$$\le \left\| E_n^l P_n^l \hat{\psi} - \hat{\psi} \right\| + \int_0^\theta |(D\psi_n)(s) - (D\psi)(s)|\, ds$$

$$\le \left\| E_n^l P_n^l \hat{\psi} - \hat{\psi} \right\| + \sqrt{\tau}\, \|D\psi_n - D\psi\|_{L^2}$$

where in the last step we have used Hölders inequality. We note that $\left\| E_n^l P_n^l \hat{\psi} - \hat{\psi} \right\| \to 0$ and that $\lim_{n\to\infty} \|D\psi_n - D\psi\|_{L^2} = 0$ if $\psi \in C^3$. Therefore, we conclude that

$$\lim_{n\to\infty} \sup_{\theta \in [-\tau, 0]} |\psi_n(\theta) - \psi(\theta)| = 0$$

and thus that $\lim_{n\to\infty} L(\psi_n) = L(\psi)$.

Since we have now proven that Conditions (i) and (ii) hold, the statement of the lemma follows by Theorem 6.1.1. □

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Chapter 7

Trotter-Kato approximation on a subspace

In Section 6.1, we followed [1] and studied Trotter-Kato approximation of the delay equation (6.1) using a systematic definition of the approximating semigroups, where the approximation scheme was based on the approximation of functions in the state space. There is a wide variety of possible approximation schemes to the semigroup associated to (6.1), other than the schemes discussed in Chapter 6. In this chapter, we study an approximation scheme from [22] that is also based upon approximation by Legendre polynomials, but is different form the scheme in Section 6.3. In Section 7.1, we define the approximating semigroups. First, we view the approximating problem as a discretisation of an abstract partial differential equation associated to (6.1), as was done in [22]. Then, we show that the approximating system can also be viewed as a discretisation of the abstract ODE associated to system (6.1), in the spirit of Chapter 6. In Section 7.2, we then prove convergence of the approximating scheme on a dense subspace of the state space.

7.1 Definition of the approximation

We define an approximation scheme, based on Legendre polynomials, as introduced in [22]. First, we introduce the approximation scheme as a discretisation of an abstract PDE associated to (6.1), as was also done in [22]. Then, we show that the approximating scheme can also be viewed as a discretisation of the abstract ODE associated to (6.1), in the spirit of Chapter 6.

The abstract PDE approach

In the following lemma, we show that the delay equation (6.1) can also be reformulated as an abstract PDE on Z.

Lemma 7.1.1. *Let $(\eta, \phi) \in \mathcal{D}(A)$ and denote by $x : [-\tau, \infty) \to \mathbb{R}^d$ the solution of the initial value problem (6.1). Define $z(t, \theta) = x(t + \theta)$ for $t \geq 0, \theta \in [-\tau, 0]$. Then z is a solution of the following partial differential equation:*

$$\frac{\partial z}{\partial t}(t, \theta) = \frac{\partial z}{\partial \theta}(t, \theta) \quad \text{for } t \geq 0, \ \theta \in [-\tau, 0], \tag{7.1}$$

$$\frac{dz}{dt}(t, 0) = L(\theta \mapsto z(t, \theta)). \tag{7.2}$$

Proof. We note that $x(t + \theta) = (T(t)(\eta, \phi))_2 (\theta)$, where $(T(t)(\eta, \phi))_2$ denotes the second component of the couple $T(t)(\eta, \phi) \in \mathbb{R}^d \times L^2 ([-\tau, 0], \mathbb{R}^d)$. Since $(\eta, \phi) \in \mathcal{D}(A)$ and the domain $\mathcal{D}(A)$ is invariant under the

semigroup $T(t)$, we find that x is weakly differentiable on $[-\tau, \infty)$. Moreover, since $z(t, \theta) = x(t + \theta)$ we see that (7.1) holds.

To prove that (7.2) holds, we note that

$$\frac{d}{dt} z(t, 0) = \frac{d}{dt} \left(T(t)(\eta, \phi) \right)_2 (0) = \left(\frac{d}{dt} T(t)(\eta, \phi) \right)_2 (0).$$

But since $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup, we know that $\frac{d}{dt} T(t)(\eta, \phi) = AT(t)(\eta, \phi) \in \mathcal{D}(A)$. Combining this with the definition of $\mathcal{D}(A)$, we find that

$$\frac{d}{dt} z(t, 0) = \left(\frac{d}{dt} T(t)(\eta, \phi) \right)_2 (0) = (AT(t)(\eta, \phi))_2 (0)$$
$$= (AT(t)(\eta, \phi))_1 = L \left(T(t)(\eta, \phi)_2 \right) = L(\theta \mapsto z(t, \theta))$$

which proves the claim. $\qquad\square$

Using the reformulation of delay equation (6.1) as an abstract PDE on Z, we now define an approximation of the semigroup associated to (6.1), as was done in [22]. In particular, we see that the boundary condition of the abstract PDE plays an important role.

Let us fix $n \in \mathbb{N}$ and let us consider the following Legendre polynomial of degree n:

$$z_n(t, \theta) = \sum_{k=0}^{n} a_k(t) p_k(\theta) \tag{7.3}$$

with $a_k(t) \in \mathbb{R}$ for all $0 \leq k \leq n$. We recall that

$$\sum_{k=0}^{n} a_k(t) \dot{p}_k(\theta) = \sum_{k=0}^{n-1} b_k(t) p_k$$

with b_k as defined in (A.3). Thus, if we substitute (7.3) in (7.1), we find that

$$\sum_{k=0}^{n} \dot{a}_k(t) p_k(\theta) = \sum_{k=0}^{n-1} b_k(t) p_k(\theta). \tag{7.4}$$

Using that the set functions $\{p_k\}_{k \in \mathbb{N}}$ are orthogonal, this implies that

$$\dot{a}_k(t) = b_k(t) \quad \forall \, 0 \leq k \leq n - 1.$$

We remark that we (7.4) also implies that $a_n(t) \equiv 0$. However, if we would choose this as our last equation of the n-dimensional system of ODEs, i.e. if we choose as the approximating system

$$\dot{a}_k(t) = b_k(t) \quad \forall \, 0 \leq k \leq n - 1, \tag{7.5}$$
$$\dot{a}_n(t) \equiv 0$$

then the system is independent of the choice of L. Therefore, we do not expect that, given an L, the solutions of the system (7.5) approach the solutions of the intial value problem (6.1).

Instead, we use the boundary condition (7.2) to derive an equation for $\dot{a}_n$. Substituting (7.3) into (7.2) implies that

$$\dot{a}_n(t) = L \left(\sum_{k=0}^{n} a_k(t) p_k \right) - \sum_{k=0}^{n-1} \dot{a}_k(t)$$
$$= \sum_{k=0}^{n} a_k(t) L(p_k) - \sum_{k=0}^{n-1} b_k(t).$$

Thus, we find as the approximating system of ODEs:

$$\dot{a}_k(t) = b_k(t) \quad \forall\, 0 \le k \le n-1,$$

$$\dot{a}_n(t) = \sum_{k=0}^{n} a_k(t) L(p_k) - \sum_{k=0}^{n-1} b_k(t). \tag{7.6}$$

The abstract ODE approach

We show that the approximating problem (7.6) can also be viewed as a discretisation of the abstract ODE (6.3), in the spirit of Chapter 6.

For $n \in \mathbb{N}$, let us define the spaces

$$Z_n = \left(\mathbb{R}^d \times \operatorname{span}(p_0, \ldots, p_n) \right) \cap \mathcal{D}(A)$$

and the operators

$$L_n : Z \to E_n Z_n, \ L_n(\eta, \phi) = \left(\eta, \sum_{k=0}^{n-1} \phi_k p_k + \left(\eta - \sum_{k=0}^{n-1} \phi_k \right) p_n \right)$$

where $\phi_k = \langle \phi, p_k \rangle$.

Using this, we now define the approximating generator $A_n : Z_n \to Z_n$ as $A_n = L_n A E_n$. Using this operator A_n, we can now define the following abstract ODE on Z_n:

$$\frac{d}{dt} v(t) = A_n v(t). \tag{7.7}$$

We now show that this abstract ODE gives rise to the system (7.6) on $\mathbb{R}^{d(n+1)}$. Let $z_n(t)$ be solution of (7.7). We note that, since $v(t) \in Z_n$, we can write

$$z_n(t) = \left(\sum_{k=1}^{n} a_k(t), \sum_{k=1}^{n} a_k(t) p_k \right)$$

with $a_k(t) \in \mathbb{R}$ for $0 \le k \le n$ and $t \ge 0$. We note that $A_n z_n(t)$ is given by

$$A_n z_n(t) = \left(L\left(\sum_{k=1}^{n} a_k(t) p_k \right), \sum_{k=0}^{n-1} b_k(t) p_k + \left(L\left(\sum_{k=1}^{n} a_k(t) p_k \right) - \sum_{k=0}^{n-1} b_k \right) p_n \right)$$

where b_k is defined as in (A.3). Thus, that $z_n(t)$ is a solution of (7.7) implies that

$$\sum_{k=0}^{n} \dot{a}_k(t) p_k = \sum_{k=0}^{n-1} b_k(t) p_k + \left(F\left(\sum_{k=1}^{n} a_k(t) p_k \right) - \sum_{k=0}^{n-1} b_k \right) p_n.$$

Using that the functions $\{p_k\}_{k \in \mathbb{N}}$ are orthogonal, we find the equations (7.6) hold.

7.2 Convergence on a subspace

In this section, we show that the approximating scheme introduced in Section 7.1 converges on $\mathcal{D}(A)$, the domain of the generator. This serves as a proof of principle to show that the Trotter-Kato theorem can also be used to prove convergence on a subspace.

Let us write $X = H^1\left([-\tau, 0], \mathbb{R}^d\right)$ and equip this space with the inner product

$$\langle \phi, \psi \rangle_1 = \langle \phi(0), \psi(0) \rangle_{\mathbb{R}^d} + \int_{-\tau}^{0} \left\langle \dot{\phi}(\theta), \dot{\psi}(\theta) \right\rangle_{\mathbb{R}^d} d\theta$$

and denote by $\|.\|_1$ the induced norm. Moreover, let us define the operator $E : \mathcal{D}(A) \to X$ by $E(\phi(0), \phi) = \phi$. We easily see that E is a bijection, and in fact we have that $(X, \|.\|_1)$ is equivalent to the Banach space $(A, \|.\|_A)$, where $\|.\|_A$ denotes the graph norm (note that $(A, \|.\|_A)$ is a Banach space since A is a closed operator).

We recall from semigroup theory that $\mathcal{D}(A)$ is invariant under the flow $T(t)$ and that the restricted semigroup $(T(t)|_{\mathcal{D}(A)})_{t \geq 0}$ is again a semigroup. To see that this restricted semigroup is actually strongly continuous with respect to the graph norm, we note that for $x \in \mathcal{D}(A)$ we have that

$$\|T(t)x - x\|_A^2 = \|T(t)x - x\|^2 + \|AT(t)x - Ax\|^2$$
$$= \|T(t)x - x\|^2 + \|T(t)Ax - Ax\|^2$$

where the first step follows from the definition of the graph norm and the second step holds true because $x \in \mathcal{D}(A)$. But since $\lim_{t \downarrow 0} \|T(t)x - x\|^2 = \lim_{t \downarrow 0} \|T(t)Ax - Ax\|^2 = 0$, we find that $\lim_{t \downarrow 0} \|T(t)x - x\|_A^2 = 0$ and thus that the semigroup $(T(t)|_{\mathcal{D}(A)})_{t \geq 0}$ is in fact strongly continuous. We denote the generator of the semigroup $(T(t)|_{\mathcal{D}(A)})_{t \geq 0}$ on $(\mathcal{D}(A), \|.\|_A)$ by $\tilde{A}$. With some abuse of notation, we also write $\tilde{A}$ for the unbounded operator $E\tilde{A}E^{-1}$ on X. Furthermore, we write $T(t)|_X$ for the semigroup $E\,T(t)|_{\mathcal{D}(A)}\,E^{-1}$ on X.

To set up a Trotter-Kato framework on the space X (or equivalently on the space $(\mathcal{D}(A), \|.\|_A)$), we write $X_n = EZ_n$ for $n \in \mathbb{N}$. Suppressing the operators E_n for the rest of this section, we introduce the operators

$$\tilde{A}_n : X_n \to X_n, \quad \tilde{A}_n = EA_nE^{-1},$$

$$\mathcal{Q}_n : X \to X_n, \quad \mathcal{Q}_n(\phi) \text{ is such that } (\mathcal{Q}_n\phi)(0) = \phi(0) \text{ and } \frac{d}{d\theta}\mathcal{Q}_n\phi = Q_{n-1}\dot{\phi}$$

where Q_{n-1} is as in Section A.2.

Since $\tilde{A}_n$ is a closed operator (because it is a composition of closed and bounded operators) and defined on all of X_n, we find by the Closed Graph Theorem that $\tilde{A}_n$ is bounded. Thus, it generates a $\mathcal{C}_0$–semigroup $(\tilde{T}_n(t))_{t \geq 0}$ on X_n. With some abuse of notation, we also write $\tilde{T}_n(t)$ for the semigroup $E\tilde{T}_n(t)E^{-1}$ on Z_n.

To prove that the approximating semigroup converges pointwise to the original semigroup on the domain of the generator, we proceed in two steps: first, we prove that we have convergence in the graph norm. This can be done using the Trotter-Kato theorem, since the graph norm turns the domain $\mathcal{D}(A)$ into a Banach space. Then, we proceed by proving that we have pointwise convergence on $\mathcal{D}(A)$ in the original norm on Z.

The pointwise convergence of the approximating scheme in the graph norm is summarised in the following theorem:

Theorem 7.2.1. *For all $\phi \in X$, we have that*

$$\lim_{n \to \infty} \left\| \tilde{T}_n(t)\mathcal{Q}_n\phi - T(t)|_X\,\phi \right\|_1 = 0$$

uniformly for t in bounded intervals.

In [22], the following two lemmata are proven:

Lemma 7.2.2. *There exists a $\omega \in \mathbb{R}$ such that*

$$\left\langle \tilde{A}_n\phi, \phi \right\rangle_1 \leq \omega \|\phi\|_1^2$$

for all $\phi \in X$.

Lemma 7.2.3. *For $k \geq 1, k \in \mathbb{N}$ let us define*

$$\mathcal{D}_k = \{\phi \in X \mid \phi \in H^k\left([-\tau, 0], \mathbb{R}^d\right) \text{ and } \dot{\phi}(0) = L(\phi)\}.$$

Then for $k \geq 2$ and $\lambda \in \mathbb{R}$ sufficiently large, the spaces $\mathcal{D}^k$ and $(\lambda I - \mathcal{A})\mathcal{D}^k$ are dense in X. For $k \geq 5$, it holds that

$$\lim_{n \to \infty} \tilde{A}_nQ_n\phi = \tilde{A}\phi$$

for all $\phi \in \mathcal{D}^k$.

Using the above lemmata, we can prove Theorem 7.2.1.

Proof of Theorem 7.2.1. We first note that for all $\phi \in X$ we have that

$$\|\mathcal{Q}_n\phi - \phi\|_1^2 = \|(\mathcal{Q}_n\phi)(0) - \phi(0)\|_{\mathbb{R}^d}^2 + \left\|\frac{d}{d\theta}\mathcal{Q}_n\phi - \dot{\phi}\right\|_{L^2}$$

$$= \|\phi(0) - \phi(0)\|_{\mathbb{R}^d}^2 + \left\|Q_{n-1}\dot{\phi} - \dot{\phi}\right\|_{L^2}.$$

By the properties of the Legendre polynomials we have that $\left\|Q_{n-1}\dot{\phi} - \dot{\phi}\right\|_{L^2} \to 0$ as $n \to \infty$; therefore, we find that $\lim_{n\to\infty} \|\mathcal{Q}_n\phi - \phi\|_1 = 0$ for all $\phi \in X$. Using the Uniform Boundedness Theorem, we see that there exists a $C > 0$ such that $\|Q_n\| \leq C$ for all $n \in \mathbb{N}$.

Lemma 7.2.2 implies that there exists a $\tilde{\omega} \in \mathbb{R}, M \geq 1$ such that $T(t)|_X, \tilde{T}_n(t) \in G(M, \tilde{\omega})$ for all $n \in \mathbb{N}$. Using Lemma 7.2.3 and the fact that $\lim_{n\to\infty} \|\mathcal{Q}_n\phi - \phi\|_1 = 0$ for all $\phi \in X$, we can apply Theorem C.0.2 to find that the approximation scheme is consistent. An application of the Trotter-Kato theorem now proves the claim. $\qquad\square$

We now proceed by showing that we have also pointwise convergence on $\mathcal{D}(A)$ in the original norm on the state space. Since $E : (\mathcal{D}(A), \|.\|_A) \to (X, \|.\|_1)$ is an isomorphism, Theorem 7.2.1 also implies that

$$\lim_{n\to\infty} \left\|\tilde{T}_n(t)E^{-1}\mathcal{Q}_n E(\phi(0), \phi) - T(t)|_{\mathcal{D}(A)}(\phi(0), \phi)\right\|_A = 0$$

for all $(\phi(0), \phi) \in \mathcal{D}(A)$. Using Hölders inequality, we find that there exists a $C > 0$ such that

$$\|(\psi(0), \psi)\| \leq C \|(\psi(0), \psi)\|_A$$

for all $(\psi(0), \psi) \in \mathcal{D}(A)$. Since also $T(t)|_{\mathcal{D}(A)}(\phi(0), \phi) = T(t)(\phi(0), \phi)$ for $(\phi(0), \phi) \in \mathcal{D}(A)$, we obtain the following corollary:

Corollary 1. *For all $(\phi(0), \phi) \in \mathcal{D}(A)$ it holds that*

$$\lim_{n\to\infty} \left\|\tilde{T}_n(t)E^{-1}\mathcal{Q}_n E(\phi(0), \phi) - T(t)|_{\mathcal{D}(A)}(\phi(0), \phi)\right\| = 0$$

uniformly for t in bounded intervals.

We conclude that for $(\phi(0), \phi) \in \mathcal{D}(A)$ the approximation scheme also converges in the original norm on the state space Z.

Chapter 8

Outlook

In the introduction of this thesis, we mentioned approximation of invariant manifolds and approximation of bifurcation behaviour by way of the pseudospectral method to be interesting topics of study. The approximation of bifurcation behaviour by way of the pseudospectral method is the subject of this outlook. In particular, we look at the approximation of the (direction of) the Hopf bifurcation in the pseudospectral method.

Let us study the parameter-dependent delay equation

$$\dot{x}(t) = F(x_t, \lambda) \tag{8.1}$$

with $F : X \times \mathbb{R} \to \mathbb{C}^d$ satisfying $F(0, \lambda) = 0$ for all $\lambda = 0$ and $X = C\left([-\tau, 0], \mathbb{C}^d\right)$. Let us assume that a Hopf bifurcation (see [11, Theorem X.2.7]) in the system (8.1) occurs at the parameter value $\lambda = 0$, i.e. at $\lambda = 0$ we have a pair of eigenvalues $\pm i\omega_0 \neq 0$ of $DF(0,0)$ crossing the imaginary axis transversely. Using Theorem 1.5.3 and the intermediate value theorem, we see that for $n \in \mathbb{N}$ large enough the nonlinear pseudospectral approximation A_n will also have a Hopf bifurcation for a parameter value near $\lambda = 0$.

A Hopf bifurcation has a direction in the sense that the bifurcation can generically either be subcritical or supercritical (see [11, Chapter X.3]). A natural question to ask is the following: suppose that the Hopf bifurcation in the equation (8.1) is subcritical (supercritical), can we then find a $N \in \mathbb{N}$ such that for all $n \geq N$, the Hopf bifurcation in A_n is also subcritical (supercritical)?

To compute the direction of a Hopf bifurcation, one needs a description of the center manifold up to and including order two, combined with the eigenvectors of the adjoint generator A^* (see [11, Section X.3]). Therefore, to study the approximation of the direction of bifurcation, one should study i) the approximation of the center manifold up to and including order two (as we did in Chapter 5) and ii) the convergence of the eigenvectors of A_n^T.

To study this last problem, one would like to show that the eigenvectors of A_n^T converge, when rightly embedded, to the eigenvectors of the dual problem associated to the linearisation of (8.1). This is, however, not obvious: the convergence of eigenvalues and eigenvectors in Chapters 1 and 4 relied heavily on the interpretation of the characteristic equation $\det(\lambda I - A_n) =$ in terms of collocation solutions. But when we study the eigenvalues- and -vectors of the transposed problem A_n^T, we loose this 'natural interpretation' in terms of collocation polynomials.

Therefore, as an intermediate step, we propose to discretise the operator A^* separately and to study its eigenvectors. This is also an interesting problem in itself, because the operator A^* is also associated to a dynamical system where the state space consists of forcing functions of a renewal equation. The dynamics is defined in the following manner: denote by $NBV := NBV([0, \tau], \mathbb{C}^d)$ the normalised functions of bounded variation on $[0, \tau]$, that take the value 0 at 0 (see [11, Appendix I]). Let $f \in NBV$ and let us extend f to $[h, \infty)$ by setting $f(t) = f(h)$ for all $t \geq h$. Let us study the renewal equation

$$x = f + \zeta * x$$

In this renewal equation, we call f the forcing function. Let us denote by $S(t)f$ the forcing function for the renewal equation that is satisfied by the translate z_t of z, i.e.

$$z_t = S(t)f + \zeta * z_t$$

As it turns out, the semigroup $S(t)$ satisfies $S(t) = T^*(t)$, where for $t \geq 0$ the operator $T^*(t)$ is the dual of the shift operator $T(t)$ associated to the delay equation (8.3). The weak-star generator of the semigroup $\{T^*(t)\}_{t \geq 0}$ is then given by A^* [11].

8.1 Definition of discretised adjoint generator

Let us study the linearised equation of (8.1) at the bifurcation parameter value $\lambda = 0$, i.e. let us study the problem

$$\dot{x}(t) = DF(0,0)x_t, \quad t \geq 0. \tag{8.2}$$

By Riesz Representation Theorem, we can make the identification $X^* \simeq NBV$. Since $DF(0,0) : X \to \mathbb{C}^d$ is a bounded linear map, we can find a $\zeta \in NBV$ such that we can write (8.2) as

$$\dot{x}(t) = \int_0^\tau d\zeta(\theta)x(t-\theta), \quad t \geq 0 \tag{8.3}$$

with $\zeta \in NBV([0,\tau],\mathbb{R}^d)$. Associated to the equation (8.3) is a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators, whose generator is given by

$$\mathcal{D}(A) = \left\{ \phi \in X \mid \phi \in C^1([-\tau,0],\mathbb{C}^d), \ \dot{\phi}(0) = \int_0^\tau d\zeta(\theta)\phi(-\theta) \right\}, \quad A\phi = \dot{\phi}.$$

The dual space X^* can be identified with the space of normalised functions of bounded variation, i.e. $X^* \simeq NBV$. The dual operator is given by

$$\mathcal{D}(A^*) = \left\{ f \in NBV \mid f(t) = \int_0^t g(s)ds \text{ for } t > 0, \ g \in NBV, \ g(h) = 0 \right\}, \quad A^*f = f' + \zeta^T(.)f(0^+).$$

We have the following theorem concerning the eigenvalues of the operator A^* [11, Theorem IV.5.9]:

Theorem 8.1.1. *We have that*

$$\sigma(A) = \sigma(A^*) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\},$$

where the characteristic function $\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}$ *is given by*

$$\Delta(\lambda) = \lambda I - \int_0^\tau d\zeta(\theta)e^{\lambda\theta}.$$

As in [7], we want to construct finite dimensional maps $B_n : \mathbb{C}^{(n+1) \times d} \to \mathbb{C}^{(n+1) \times d}$ that 'approximate' A^* in the sense that the eigenvalues- and vectors of B_n approximate the eigenvalues- and vectors of A^*.

We define the finite dimensional approximations B_n in the following manner:

Definition 8.1.1. For $n \in \mathbb{N}$, let $0 = \theta_{n,0} < \ldots < \theta_{n,n} = \tau$ be a mesh on $[0,\tau]$ and for $x \in \mathbb{C}^{(n+1) \times d}$, denote by $\mathcal{L}_n x$ the interpolating polynomial through x with respect to the chosen mesh. Then the pseudospectral approximation of the adjoint generator A^*, called B_n, is defined as follows:

$$B_n : \mathbb{C}^{(n+1) \times d} \to \mathbb{C}^{(n+1) \times d}$$
$$B_n x = \left((\mathcal{L}_n x)'(\theta_{n,0}) + \zeta^T(0^+)x_0, (\mathcal{L}_n x)'(\theta_{n,1}) + \zeta^T(\theta_{n,1})x_0, \ldots, (\mathcal{L}_n x)'(\theta_{n,n-1}) + \zeta^T(\theta_{n,n-1}), \zeta^T(\theta_{n,n})x_0 \right)$$

This definition can be motivated in the following manner: for $f : [0, \tau] \to \mathbb{C}^d$ a function, let us denote by $X_0 f : [0, \tau] \to \mathbb{C}^d$ the following function:

$$(X_0 f)(\theta) = \begin{cases} 0 & \text{if } \theta = 0, \\ f(\theta) & \text{if } \theta \in (0, \tau]. \end{cases}$$

Now, if we let $x \in \mathbb{C}^{(n+1) \times d}$, then $X_0 \mathcal{L}_n x \in NBV$. The action of the adjoint generator A^* on the element $X_0 \mathcal{L}_n x$ is given by $(\mathcal{L}_n x)' + \zeta^T(.)x_0$. This we can then project back onto $\mathbb{C}^{(n+1) \times d}$ by evaluating at the mesh points, to obtain the vector

$$\left((\mathcal{L}_n x)'(\theta_{n,0}) + \zeta^T(0^+)x_0, (\mathcal{L}_n x)'(\theta_{n,1}) + \zeta^T(\theta_{n,1})x_0, \ldots, (\mathcal{L}_n x)'(\theta_{n,n}) + \zeta^T(\theta_{n,n})x_0 \right).$$

To incorporate the domain condition in the definition of the approximation, we set $(\mathcal{L}_n x)'(\theta_{n,n}) = 0$, thus obtain the map $B_n : \mathbb{C}^{(n+1) \times d} \to \mathbb{C}^{(n+1) \times d}$ as in Definition 8.1.1. See also Figure 8.1.

$$x \in \mathbb{C}^{(n+1) \times d}$$

$\downarrow$ Embed in X^*

$$X_0(\mathcal{L}_n x)$$

$\downarrow$ Action of the generator A^*

$$(\mathcal{L}_n x)' + \zeta^T(.)x_0$$

$\downarrow$ Back to $\mathbb{C}^{(n+1) \times d}$

$$\left((\mathcal{L}_n x)'(\theta_{n,0}) + \zeta^T(0^+)x_0, (\mathcal{L}_n x)'(\theta_{n,1}) + \zeta^T(\theta_{n,1})x_0, \ldots, (\mathcal{L}_n x)'(\theta_{n,n}) + \zeta^T(\theta_{n,n})x_0 \right)$$

$\downarrow$ Domain condition of the generator A^*

$$\left((\mathcal{L}_n x)'(\theta_{n,0}) + \zeta^T(0^+)x_0, (\mathcal{L}_n x)'(\theta_{n,1}) + \zeta^T(\theta_{n,1})x_0, \ldots, (\mathcal{L}_n x)'(\theta_{n,n-1}) + \zeta^T(\theta_{n,n-1}), \zeta^T(\theta_{n,n})x_0 \right)$$

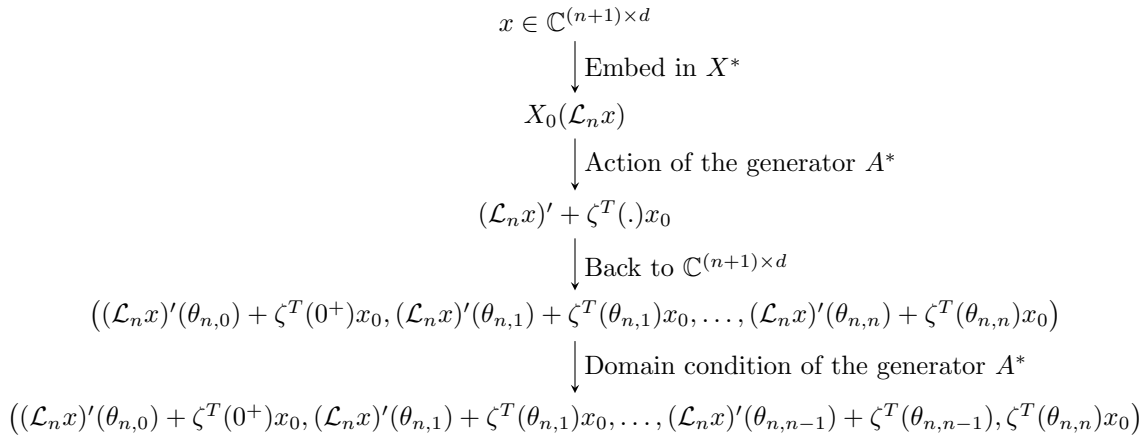Figure 8.1: Schematic representation of definition of B_n.

As a numerical example, let us consider Wright's equation

$$\dot{x}(t) = \alpha x(t-1)\left(x(t) + 1\right), \quad t \geq 0 \tag{8.4}$$

with $x(t) \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ a parameter. The linearisation of system (8.4) around $x = 0$ is given by

$$\dot{x}(t) = \alpha x(t-1), \quad t \geq 0. \tag{8.5}$$

Let us denote by A the generator associated to (8.5). For $\alpha = -\frac{\pi}{2}$, figure 8.2 shows the eigenvalues $\sigma(A) = \sigma(A^*)$ as computed using DDEBiftool together with the eigenvalues of the pseudospectral adjoint approximation B_n to (8.5) for $n = 10$. Moreover, let us denote by λ_n the eigenvalue of the pseudospectral adjoint approximation B_n to (8.5) such that

$$\left| \lambda_n - i\frac{\pi}{2} \right| = \min_{\lambda \in \sigma(B_n)} \left| \lambda - i\frac{\pi}{2} \right|.$$

Then Figure 8.3 shows the error between λ_n and $i\frac{\pi}{2}$ for $1 \leq n \leq 10$.
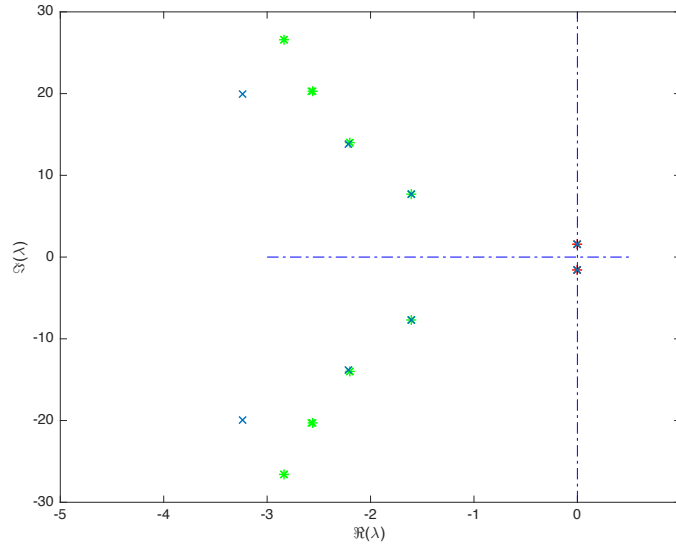
Figure 8.2: Spectrum of (8.5) for $\alpha = -\frac{\pi}{2}$ as computed using DDEBiftool (green and red stars) and the eigenvalues of the pseudospectral adjoint approximation B_n for (8.5) with $\alpha = -\frac{\pi}{2}$ for $n = 10$ (blue crosses).
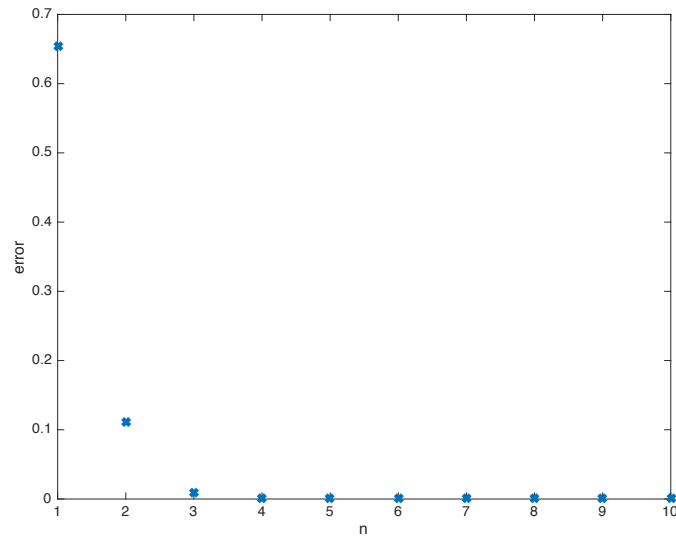

Figure 8.3: Error between eigenvalue $i\frac{\pi}{2}$ of (8.5) for $\alpha = -\pi/2$ and the eigenvalue of the pseudospectral adjoint approximation B_n closest to $i\frac{\pi}{2}$ for $1 \leq n \leq 10$.

Chapter 9

Notes

General remarks

- Since in Part A and Part B we are interested in eigenvalues and normal forms, we will work in Part A and Part B of this thesis with *complex* Banach space. In particular, we will use as the state space the *complex* vector space $X = C\left([-\tau, 0], \mathbb{C}^d\right)$. In the case one starts with a real state space $C\left([-\tau, 0], \mathbb{C}^d\right)$, one can reduce to the situation of complex Banach spaces by complexifying the space $C\left([-\tau, 0], \mathbb{C}^d\right)$; see Section III.7 in [11].

Notes on Chapter 1

- In the main text, we have introduced the pseudospectral matrices $A_n : \mathbb{R}^{(n+1) \times d} \to \mathbb{R}^{(n+1) \times d}$ associated to the delay equation (1.1) as a discretisation of the unbounded (nonlinear) operator $A : \mathcal{D}(A) \subseteq X \to X$.

  We can, however, interpret A_n also as a discretisation of the unbounded operator $A^{\odot *} : \mathcal{D}(A^{\odot *}) \subseteq X^{\odot *} \to X^{\odot *}$ associated to the delay equation (1.1). We recall that

  $$\mathcal{D}(A^{\odot *}) = \{(\alpha, \phi) \in X^{\odot *} \mid \phi \in \mathrm{Lip}(\alpha)\}, \quad A^{\odot *}(\alpha, \phi) = (L\phi, \dot{\phi})$$

  where $\mathrm{Lip}(\alpha) \subseteq L^\infty([-\tau, 0], \mathbb{R}^d)$ exists of all equivalence classes in L^∞ containing a function that is Lipschitz continuous and takes the value α in 0.

  For $n \in \mathbb{N}$, denote by $P_n \subseteq L^\infty([-\tau, 0], \mathbb{R}^d)$ the set of equivalence classes in L^∞ containing a function that is a polynomial of degree n; moreover, let us define the spaces

  $$X_n^{\odot *} = \{(\alpha, \phi) \mid \alpha \in \mathbb{R}^d, \phi \in P_n\},$$
  $$\mathcal{D}_n = X_n^{\odot *} \cap \mathcal{D}(A^{\odot *}).$$

  Moreover, for $n \in \mathbb{N}$, let us define the embedding operator $E_n : \mathbb{R}^{(n+1) \times d} \to \mathcal{D}_n$ and the projection operator $P_n : X_n^{\odot *} \to \mathbb{R}^{(n+1) \times d}$ as

  $$E_n(x) = (x_0, \mathcal{L}_n x)$$
  $$P_n(\alpha, \phi) = (\alpha, \phi(\theta_{n,1}), \dots, \phi(\theta_{n,n}))$$

  Let us now discretise $A^{\odot *}$ by $P_n A^{\odot *} E_n$. Then we note that

  $$P_n A^{\odot *} E_n(x) = P_n A^{\odot *}(x_0, \mathcal{L}_n x) = P_n(L(\mathcal{L}_n x), (\mathcal{L}_n x)') = (L(\mathcal{L}_n x), (\mathcal{L}_n x)'(\theta_{n,1}), \dots, (\mathcal{L}_n x)'(\theta_{n,n}))$$

  i.e. $P_n A^{\odot *} E_n = A_n$.

- As also mentioned in the introduction, the pseudospectral method has the properties that i) it approximates the eigenvalues of the original DDE and ii) the non-linear terms of the approximation correspond to the non-linear terms of the original DDE. This motivates us to think about approximation of i) invariant manifolds and ii) bifurcation behaviour.

- We note that in the proof of Lemma 1.5.1, we have explicitly used that the eigenfunctions are smooth, and that therefore the polynomial interpolation of an eigenfunction converges to that eigenfunction in the supremum-norm. We recall that we can find $\phi \in X$ such that

$$\lim_{n \to \infty} \|\mathcal{L}_n(\phi(\theta_{n,0}), \ldots, \phi(\theta_{n,n})) - \phi\|_\infty \neq 0;$$

  see Section A.3. Therefore, we expect that the semigroup

$$T_n(t) = e^{A_n t}$$

  on $\mathbb{C}^{d \times (n+1)}$ will, when rightly embedded into the state space X, not converge pointwise to the original semigroup $\{T(t)\}_{t \geq 0}$ associated to the delay equation.

- We note that we restrict our convergence analysis to a subset $\sigma_n \subseteq \sigma(A_n)$ of the eigenvalues of the pseudospectral method (as was implicitly also done in [7]). This has the following motivation: in general a delay equation has an infinite number of eigenvalues, but we can write down delay equations that have a finite number of eigenvalues. For example, let us study the ordinary differential equation

$$\dot{x}(t) = Bx(t), \quad t \geq 0, \tag{9.1}$$

  with B a $d \times d$-matrix, and let us view this ODE as a delay equation on the state space $X = C([-\tau, 0], \mathbb{R}^d)$. Denote by A_n the pseudospectral approximation to (9.1). Then $\lambda \in \sigma(A_n)$ if and only if there exists a $x \in \mathbb{C}^{(n+1) \times d}, x \neq 0$ such that

$$\begin{cases} Bx_0 & = \lambda x_0 \\ (\mathcal{L}_n x)'(\theta_{n,i}) & = \lambda x_i, \quad 1 \leq i \leq n \end{cases}$$

  We see that $\sigma_n = \sigma(B)$ for all $n \in \mathbb{N}$. The eigenvalues $\sigma(A_n) \backslash \sigma_n$ are given by eigenvalues of the differentiation matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ a_{10} & a_{11} & \ldots & a_{1(n-1)} & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n0} & a_{n1} & \ldots & a_{n(n-1)} & a_{nn} \end{pmatrix}.$$

  The fact that we only study a subset of the eigenvalues of A_n does not cause too much trouble numerically, because of the following lemma:

Lemma 9.0.1. *Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence with $\lambda_n \in \sigma(A_n) \backslash \sigma_n$ for all $n \in \mathbb{N}$. Then the sequence $(\lambda_n)_{n \in \mathbb{N}}$ cannot be bounded.*

Proof. Suppose that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is bounded; let us write $\rho = \sup_{n \in \mathbb{N}} |\lambda_n| < \infty$. Then by Lemma 1.5.1, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, the collocation problem (1.8) has a unique solution.

Now, let $x_n \in \mathbb{C}^{(n+1) \times d}$ be an eigenvector associated to λ_n, i.e. $A_n x_n = \lambda_n x_n$ and $x_n \neq 0$. Since $\lambda_n \in \sigma(A_n) \backslash \sigma_n$, we have that $(x_n)_0 = 0 \in \mathbb{C}^d$; moreover, it holds that $\mathcal{L}_n x = p_n(\lambda_n, (x_n)_0) = p_n(\lambda_n, 0)$. Now, since for $n \geq N$ the collocation problem (1.8) has a unique solution for $\lambda = \lambda_n$, and $p_n(\lambda_n, 0) \equiv 0$ is a solution, we find that $\mathcal{L}_n x_n \equiv 0$, i.e. $x_n = 0 \in \mathbb{C}^{(n+1) \times d}$. But this contradicts the fact that x is an eigenvector. We conclude that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ cannot be bounded. $\square$

Notes on Chapter 2

- Suppose $\{T_1(t)\}_{t \geq 0}$, $T_1(t) : \mathbb{C}^m \to \mathbb{C}^m$ is the flow associated to the equation (2.1); denote by $\{T_2(t)\}_{t \geq 0}$, $T_2(t) : \mathbb{C}^d \to \mathbb{C}^d$ the flow associated to (2.2). Suppose that (2.3) holds. We claim that the the coordinate transformation $x = P(\theta)$ conjugates the flow associated to (2.2) into the flow associated to (2.1), i.e. $T_1(t)P(\theta) = P(T_2(t)\theta)$. To see this, we define let $\theta \in \mathbb{C}^d$ and define $u(t) = P(T_2(t)\theta)$, then $u(0) = P(\theta)$ and $u(t)$ satisfies

$$u'(t) = DP(T_2(t)\theta)T_2'(t)\theta = DP(T_2(t)\theta)h(T_2(t)\theta) = g(P(T_2(t)\theta)) = g(u(t))$$

  By definition, $\tilde{u}(t) = T_1(t)P(\theta)$ also satisfies $\tilde{u}'(t) = g(\tilde{u}(t))$, $\tilde{u}(0) = P(\theta)$. Thus, by uniqueness of the solution of the initial value problem associated to (2.1), we find that $T_1(t)P(\theta) = P(T_2(t)\theta)$, which proves the claim.

- In the main text, we study the formal conjugation between the equations (2.1) and (2.5), and mention that the existence of an *analytic* conjugation map is in various cases a classical result. However, it is all so possible to study conjugation between the equations (2.1) and (2.5) *up to a certain order*. To do this, much less regularity of the functions g, f is required. This approach we will also pursue in Chapter 3. See also [24].

- In the main text of this chapter, we study the relation between normal form theory and parametrisation method. Using results from normal form theory, we can choose the flow on the tangent space in such a way that the conjugation map (the object of study in the parametrisation method) actually exists, *also in the case of resonances.* However, this knowledge is not directly applicable in the context of rigorous computations. In the case of resonances, we have for the resonant coefficients of our conjugation map a 'line' of possible solutions (see Section 3). In the framework of rigorous computations, one usually uses the contraction mapping principle, and thus one should have a unique solution in a ball with a certain radius – which is not the case for the conjugation map if we have resonances. See also [28].

- In Definition 2.3.1, we define resonant and non-resonant terms for the power series f starting from order 2, since the coefficients of order 1, both of the normal form and the conjugating map, are determined by the relations (2.4).

- We remark that Lemma 2.2.1 is a special case of Lemma 2.3.1: if the eigenvalues of system (2.1) do not have any nontrivial resonances, then Lemma 2.3.1 tells us that system (2.1) is formally conjugate to (2.10) for any choice of power series $f(x) = \sum_{|\alpha|=2}^{\infty} f_\alpha x^\alpha$; for a fixed power series $\sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$ the conjugating map is unique.

- In Section 2.3, we discussed that the non-resonant orders in the normal form are zero. A similar result holds for normal forms on invariant manifolds, resulting in the fact that the normal form on the center manifold for a Hopf bifurcation as used in Section 2.5; see [3, Theorem 3.1].

- To compute the direction of the Hopf bifurcation, one needs a description of the center manifold up to and including order two, combined with the eigenvectors of the adjoint map; see [11, Section X.3]. In Chapter 5, we discuss the approximation of the center manifold up to and including second order in the pseudospectral method. For the approximation of the adjoint problem in the pseudospectral method, see the outlook.

Notes on Chapter 3

- In Section 3.1, we mention that in order to do normal form theory for delay equations, one should extend the nonlinear generator A given by

$$\mathcal{D}(A) = \{\phi \in X \mid \dot{\phi} \in C^1 \text{ and } \dot{\phi}(0) = L\phi + G(\phi)\}, \ A\phi = \dot{\phi},$$

in such a way that the DDE (3.1) appears in the action of the extension, and not in the domain condition. In the main text, we used sun-star calculus for this.

Another possibility is to embed the state space X in the larger space BC, which consists of all functions $f : [-\tau, 0] \to \mathbb{R}^d$ that are uniformly continuous on $[-\tau, 0)$ but have a possible jump at 0. This approach was for example taken in [15] and [14] to study normal form theory for DDEs and in [17] to study the parametrisation method for DDEs. In this approach, the space $BC \equiv X \times \mathbb{R}^d$ is equipped with the norm $\|(\phi, \alpha)\|_{BC} = \|\phi\|_X + \|\alpha\|_{\mathbb{R}^d}$. On BC, we then define the unbounded operator

$$\mathcal{D}(\tilde{A}) = C^1 \subseteq BC, \ \tilde{A}(\phi) = \dot{\phi} + X_0 \left(L\phi + G(\phi) - \dot{\phi}(0) \right).$$

Here, for $x \in \mathbb{R}^d$, the function $X_0 x \in NBV$ is defined via

$$(X_0 x)(\theta) = \begin{cases} 0 \text{ if } \theta \in [-\tau, 0) \\ x \text{ if } \theta = 0. \end{cases}$$

In the case of a linear delay equation, i.e. where we have $G \equiv 0$ in (3.1), we can use the theory from [19] to 'split' the abstract ODE $\dot{x}(t) = \tilde{A}x(t)$ on BC according to the spectrum of $\tilde{A}$, so that we arrive in a situation similar to (2.12). From here, [15] and [14] proceed by giving an algorithm – based on the eigenvalues of $\tilde{A}$ – to transform the splitted abstract ODE on BC into its normal form.

When we work on the state space BC, the delay equation (3.1) does not induce a semigroup on the state space: because the jump discontinuity at 0 'moves along' as we shift along the solution of (3.1), the shifted initial function will in general not be in BC. Since this thesis is written in the language of semigroups, we have chosen to pursue in Chapter 3 an approach that is also formulated in semigroup language, and to work in the sun-star framework rather than in the BC-framework.

Notes on Chapter 5

- On page 5.1, we defined $\lambda_n^\pm$ as the element of σ_n closest to $\pm i\omega_0$. By Theorem 1.5.3, we know that the eigenvalues $\lambda_n^\pm$ are well-defined (i.e. there is a unique closest eigenvalue) for n large enough. For n small, the eigenvalue that is closest to $i\omega_0$ or $-i\omega_0$, respectively, may not be uniquely defined, but this does not matter since we are interested in the limit problem.

- In Chapter 1, we saw that the 'special structure' of the pseudospectral approximation (the first component incorporates the domain condition, the other components represent the action of the generator) allowed us to view the characteristic equation of the pseudospectral approximation as a discretised version of the characteristic equation of the original delay equation (Lemma 1.4.1). In the proof of Lemma 5.2.1, we used the 'special structure' of the pseudospectral approximation to find conditions for $(P_n^\alpha)_0$ (equation (5.11)) that can be viewed as a discretised version of the condition for P_α (equation (3.10)).

Notes on Chapter 6

- Since the Trotter-Kato theorem deals with approximation of semigroups of *linear* operators, we restrict ourselves to *linear* delay equations in Part C (whereas in Part A and B we also looked at nonlinear delay equations).

- We note that in the definition of $\mathcal{D}(A)$ in (6.2), the expression $\phi(0)$ is well-defined, because for $\phi \in H^1([-\tau, 0], \mathbb{R}^d)$, we have by the Sobolev inequality (see [13, Thm. 5.6.6]) that actually $\phi \in C([-\tau, 0], \mathbb{R}^d)$.

- We stress that throughout Chapter 6, we explicitly write the embedding operators $E_n : Z_n \to Z$. This is motivated by the thought that the explicit notation of the embedding operators E_n in the definition of A_n can be useful when one wants to study the adjoint operator A_n^*.

Notes on Chapter 7

- In the construction of the approximating scheme (7.6) as viewed as a discretisation of the abstract PDE (7.1)–(7.2), the choice of the last component of the scheme (7.6) was inspired by the boundary condition of the PDE. This approach is similar to the one in Chapter 1: there, in the definition of A_n viewed as discretisation of the generator $A : \mathcal{D}(A) \to X$, the choice of the first component of A_n was inspired by the domain conditions of the generator A.

    We also note that in both cases, the (maybe somewhat ad hoc) use of the boundary condition can also be circumvented by viewing the approximating problem as a (maybe somewhat more systematic) discretisation of a different problem. The pseudospectral scheme in Chapter 1 can also be viewed as a discretisation of the operator $A^{\odot*} : \mathcal{D}(A^{\odot*}) \to X^{\odot*}$; the scheme (7.6) introduced in Chapter 7 can also be viewed as a discretisation of the abstract ODE (7.7).

- In the construction of the approximating problem (7.6) as a discretisation of the abstract ODE (7.7), we clearly see that the approximating problem is an ODE on the space $Z_n \subseteq \mathcal{D}(A)$. This makes that the approximating scheme is suited to a convergence analysis on $\mathcal{D}(A)$.

- The discussion in Section 7.2 serves as a proof of principle for studying pointwise convergence of approximating semigroups on a subspace of the entire state space. This can be of interest when studying an approximation scheme where convergence of the semigroups on the entire state space is not feasible (as is for example the case in the pseudospectral approximation scheme, see also the remark on page 59).

- In the spirit of the pseudospectral approximation scheme as introduced in Chapter 1, we can also define a pseudospectral approximation scheme to the generator $A : \mathcal{D}(A) \subseteq Z \to Z$. This can be done in the following manner: let us choose nodes $-\theta_{n,n} < \ldots < \theta_{n,0} = 0$. For $x \in \mathbb{R}^{(n+1)\times d}$, let us write $x = (x_0, \ldots, x_n)$ with $x_i \in \mathbb{R}^d$ for $0 \leq i \leq n$. Let us define

$$\iota_n : \mathbb{R}^{d\times(n+1)} \to Z_n, \quad \iota_n(x) = (x_0, \mathcal{L}_n(x),$$

$$p_n : \mathbb{R}^d \times H^1([-\tau, 0], \mathbb{R}^d) \to \mathbb{R}^{d\times(n+1)}, \quad p_n(\eta, \phi) = (\eta, \phi(\theta_{n,1}), \ldots, \phi(\theta_{n,n})).$$

    We now define the operator $\tilde{A}_n : \mathbb{R}^{d\times(n+1)} \to \mathbb{R}^{d\times(n+1)}$ as $\tilde{A}_n = p_n A \iota_n$. This gives that

$$\tilde{A}_n(x) = \left(L(\mathcal{L}_n(x)), \mathcal{L}'_n(x)(\theta_{n,1}), \ldots, \mathcal{L}'_n(x)(\theta_{n,n}) \right).$$

    We note that the resulting approximating generator $\tilde{A}_n : \mathbb{R}^{(n+1)\times d} \to \mathbb{R}^{(n+1)\times d}$ is exactly the same map as the pseudospectral map as introduced in Chapter 1.

    We note that, since Z is a Hilbert space, the spaces Z, $Z^* = Z^{\odot}$ and $Z^{\odot*} = Z^{**}$ are all isomorphic. Therefore, we can define the pseudospectral approximation in terms of embedding and projection operators directly on the Hilbert space Z, whereas when we worked on the state space $X = C\left([-\tau, 0], \mathbb{R}^d\right)$ in Chapter 1, we had to work on the space $X^{\odot*}$ to define the pseudospectral approximation in terms of embedding and projection operators.

Appendices

Appendix A

Approximation Methods

In this Appendix, we give some background approximation of functions via spline approximation, Legendre polynomials and interpolating polynomials. Throughout this chapter, we will write $\|.\|_{L^2}$ as shorthand for $\|.\|_{L^2([-\tau,0],\mathbb{R}^d)}$ and $\|.\|_{H^s}$ as shorthand for $\|.\|_{H^s([-\tau,0],\mathbb{R}^d)}$, where $d \in \mathbb{N}$.

A.1 Splines

Definition

Let us fix $\tau > 0$. For $n \in \mathbb{N}, n \geq 1$, let us define

$$t_{j,n} = -\frac{\tau j}{n} \quad \text{for } 0 \leq j \leq n.$$

Furthermore, set $t_{n+1,n} = -\tau$ and $t_{-1,n} = 0$. We now define the first order spline functions as

$$e_{k,n} = \begin{cases} \frac{\tau}{n}\left(\theta - t_{k+1,n}\right) & \text{if } \theta \in [t_{k+1,n}, t_{k,n}], \\ \frac{\tau}{n}\left(t_{k-1,n} - \theta\right) & \text{if } \theta \in [t_{k,n}, t_{k-1,n}], \\ 0 & \text{else.} \end{cases}$$
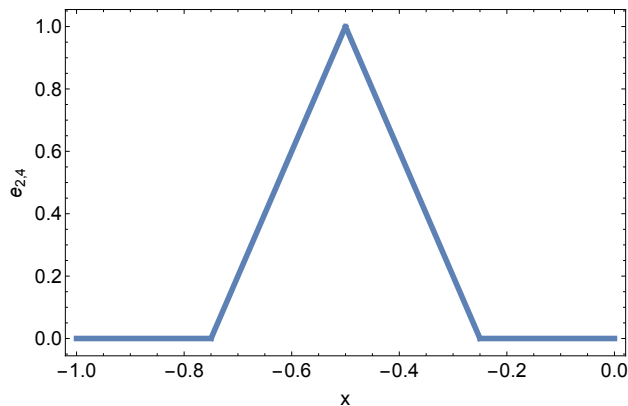
for $0 \leq k \leq n$ [21]. See Figure A.1.

Figure A.1: Plot of $e_{2,4}$ for $\tau = 1$.

Convergence properties

For $n \in \mathbb{N}$ and $\phi \in H^1\left([-\tau, 0], \mathbb{R}^d\right)$, let us define $\Theta_n(\phi) = \sum_{k=0}^{n} e_{k,n}\phi(t_{k,n})$. For $i \in \mathbb{N}$, let us denote by $C^i\left([-\tau, 0], \mathbb{R}^d\right)$ the set of i-times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^d$. Furthermore, let us denote by D the differential operator.

For $t \in \mathbb{N}$ and $1 \le p < \infty$, let us define $PC^{p,t}\left([-\tau, 0], \mathbb{R}^d\right)$ as

$$PC^{p,t}\left([-\tau, 0], \mathbb{R}^d\right) = \{\phi : [-\tau, 0] \to \mathbb{R}^d \mid \phi \in C^{t-1}\left([-\tau, 0], \mathbb{R}^d\right), D^{t-1}\phi \text{ piecewise } C^1 \text{ and } \left\|D^t\phi\right\|_{L^p} < \infty\}.$$

With this notation, we can state the following lemma from [25]; the proof can be found in [25, pp.7-8].

Lemma A.1.1. *Let $\phi \in PC^{2,2}([-\tau, 0])$, then*

$$\|D\phi - D(\Theta_n\phi)\|_{L^2} \le \frac{1}{\pi}\frac{\tau}{n}\left\|D^2\phi\right\|_{L^2}$$

and

$$\|\phi - \Theta_n\phi\|_{L^2} \le \frac{1}{\pi^2}\frac{\tau^2}{n^2}\left\|D^2\phi\right\|_{L^2}.$$

A.2   Legendre polynomials

Definition

Let us fix $\tau > 0$. The set of $\mathbb{R}$-valued polynomials on $[-\tau, 0]$ lies dense in $L^2([-\tau, 0], \mathbb{R})$. Thus, if we define $\phi_k \in L^2([-\tau, 0], \mathbb{R})$ by $\phi_k(x) = x^k$, the set $\{\phi_k\}_{k \in \mathbb{N}}$ forms a basis of the Hilbert space $L^2([-\tau, 0], \mathbb{R})$. This base is not orthogonal, but we can apply a Gram-Schmidt procedure to obtain a set of orthogonal functions $\{\tilde{p}_n\}_{n \in \mathbb{N}}$, where $\tilde{p}_n$ is a polynomial of degree n. In fact, if we normalize the functions $\tilde{p}_n$ such that $\langle \tilde{p}_n, \tilde{p}_m \rangle = \delta_{nm}\frac{2}{2n+1}$, then one can show that $\tilde{p}_n$ is a solution of the initial value problem

$$\frac{\tau^2}{4}\frac{d}{dx}\left(\left(1 - \frac{\tau^2}{4}(x-1)^2\right)\frac{dp}{dx}\right) + n(n+1)p(x) = 0$$

and $\tilde{p}_n(0) = 1$. By setting

$$p_n = \frac{\tilde{p}_n}{\|\tilde{p}_n\|_{L^2([-\tau, 0], \mathbb{R})}} = \sqrt{\frac{2n+1}{2}}p_n$$

we obtain a collection of functions $\{p_n\}_{n \in \mathbb{N}}$ that forms a basis of $L^2([-\tau, 0], \mathbb{R})$ and satisfies $\langle p_i, p_j \rangle_{L^2([-\tau, 0], \mathbb{R})} = \delta_{ij}$. The functions $\{p_n\}_{n \in \mathbb{N}}$ are called the Legendre polynomials. Using Parseval's identity, we find that every $\phi \in L^2([-\tau, 0], \mathbb{R})$ can be written as

$$\phi = \sum_{k=0}^{\infty} \phi_k p_k \tag{A.1}$$

where

$$\phi_k = \langle \phi, p_k \rangle_{L^2} = \int_{-\tau}^{0} \phi(x)p_k(x)dx. \tag{A.2}$$

Using this, we find for $d \in \mathbb{N}$ and $\phi \in L^2([-\tau, 0], \mathbb{R}^d)$ that (A.1) also holds, but where the integral (A.2) is now an $\mathbb{R}^d$-valued integral. [22]

Differentiation

Let f be a finite Legendre polynomial, i.e.

$$f = \sum_{k=0}^{N} a_k p_k$$

with $a_k \in \mathbb{R}$ for $0 \le k \le N$. Since f is smooth, we have for its derivative $\dot{f}$ that $\dot{f} \in L^2\left([-\tau,0],\mathbb{R}^n\right)$ and therefore has again an expansion in terms of Legendre polynomials. In fact, it holds that

$$\dot{f} = \sum_{k=0}^{N-1} b_k p_k$$

where

$$b_k = (2k+1) \sum_{n=k+1, n+k \text{ odd}} a_n. \tag{A.3}$$

See also [22].

Convergence properties

For future reference, we cite (without proof) the following lemmata from [22] concerning the convergence properties of approximation by Legendre polynomials.

For $\phi \in L^2\left([-\tau,0],\mathbb{R}^d\right)$ and $n \in \mathbb{N}$, let us define $Q_n\phi = \sum_{k=0}^{n} \langle \phi, p_k \rangle_{L^2} p_k$, i.e. Q_n is the n-th order Legendre approximation of ϕ.

Lemma A.2.1. *Let $s \in \mathbb{R}, s \ge 1$. Then there exists a $K = K(s) \in \mathbb{R}$ such that*

$$\|\phi - Q_n\phi\|_{L^2} \le Kn^{-s}\|\phi\|_{H^s}$$

for all $\phi \in H^s\left([-\tau,0],\mathbb{R}^d\right)$.

Lemma A.2.2. *Let $s, \sigma \in \mathbb{R}$ be such that $1 \le s \le \sigma$. Then there exists a $K = K(s,\sigma)$ such that*

$$\|\phi - Q_n\phi\|_{H^s} \le Kn^{2s-\sigma-1/2}\|\phi\|_{H^\sigma}$$

for all $\phi \in H^\sigma\left([-\tau,0],\mathbb{R}^d\right)$.

Lemma A.2.3. *Let $m \in \mathbb{N}$, then there exists a $K = K(m)$ such that*

$$|\phi(x) - (Q_n\phi)(x)| \le Kn^{-2m+1}\|\phi\|_{H^{2m}}$$

for all $\phi \in H^{2m}\left([-\tau,0],\mathbb{R}^d\right)$ and all $x \in [-\tau,0]$.

A.3 Polynomial interpolation

Definition

Let us fix $\tau > 0$ and $d \in \mathbb{N}$. For each $n \in \mathbb{N}$, let us choose mesh points $-\tau \le \theta_{n,n} < \ldots < \theta_{n,0} \le 0$ on the interval $[-\tau,0]$.

Lemma A.3.1. *Let $x = (x_0, \ldots, x_n) \in \mathbb{C}^{(n+1) \times d}$. Then there exists a unique polynomial $f_n : [-\tau,0] \to \mathbb{R}^d$ of degree n such that $f_n(\theta_{n,i}) = x_i \in \mathbb{R}^d$ for all $0 \le i \le n$.*

Proof. We first prove existence. For $0 \le i \le n$, let us define the functions

$$\ell_{n,i} : [-\tau, 0] \to \mathbb{R}, \quad \ell_{n,i}(\theta) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{\theta - \theta_{n,j}}{\theta_{n,i} - \theta_{n,j}}.$$

We note that $\ell_{n,i}(\theta_{n,j}) = \delta_{ij}$ for $0 \le i, j \le n$ and that $\ell_{n,i}$ is a polynomial of degree n. Thus, if we define

$$f_n : [-\tau, 0] \to \mathbb{R}^d, \quad f_n(\theta) = \sum_{i=0}^{n} x_i \ell_{n,i}$$

then f_n is a polynomial of degree n satisfying $f_n(\theta_{n,i}) = x_i$. This proves existence f_n.

To prove uniqueness, let us suppose that $f_n, g_n : [-\tau, 0] \to \mathbb{R}^d$ are both polynomials of degree n such that $f_n(\theta_{n,i}) = g_n(\theta_{n,i}) = x_i$ for all $0 \le i \le n$. Then we have that $f_n - g_n$ is a polynomial of degree n satisfying $(f_n - g_n)(\theta_{n,i}) = 0$ for all $0 \le i \le n$. We conclude that $f_n - g_n$ is a polynomial of degree n having $n + 1$ zeros, which implies that $f_n - g_n \equiv 0$. We see that $f_n = g_n$, which proves uniqueness [27]. $\square$

We call the unique polynomial of Lemma A.3.1 the interpolating polynomial through x and denote it by $\mathcal{L}_n x$. We also make the following definition:

Definition A.3.1. Let $f : [-\tau, 0] \to \mathbb{R}^d$ be a function. The *interpolating polynomial of f of order n*, denoted by $\mathcal{L}_n(f)$, is the unique polynomial of order n satisfying $(\mathcal{L}_n(f))(\theta_{n,i}) = f(\theta_{n,i})$ for all $0 \le i \le n$.

(Note that we write $\mathcal{L}_n f$ for an interpolating polynomial through a function and $\mathcal{L}_n x$ for the interpolating polynomial through a vector.)

Convergence properties

We state the following lemma from [27] concerning the pointwise convergence of the sequence of interpolating polynomials.

Lemma A.3.2. *Let $n \in \mathbb{N}$ and assume that $f : [-\tau, 0] \to \mathbb{R}^d$ is $n + 1$ times differentiable. Then for each $\theta \in [-\tau, 0]$ there exists a $\zeta \in (-\tau, 0)$ such that*

$$f(\theta) - (\mathcal{L}_n f)(\theta) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \pi_{n+1}(\theta)$$

where π_{n+1} is defined as

$$\pi_{n+1} : [-\tau, 0] \to \mathbb{R}, \quad \pi_{n+1}(\theta) = (\theta - \theta_{n,0}) \dots (\theta - \theta_{n,n}).$$

A proof of the lemma can be found in [27].

We note that Lemma A.3.2 gives us also estimates on the supremum-norm of $f - \mathcal{L}_n f$. If we denote by $\|.\|_\infty$ the supremum-norm on the interval $[-\tau, 0]$, then we find by Lemma A.3.2 that

$$\|f - \mathcal{L}_n f\|_\infty \le \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} \|\pi_{n+1}\|_\infty . \tag{A.4}$$

We note that the value of $\|\pi_{n+1}\|_\infty$ depends on the choice of mesh points. We can now ask ourselves if we can make a choice of mesh points such that the value of $\|\pi_{n+1}\|_\infty$ is minimized. This is indeed the case: if we define the *Cheybshev nodes* on $[-\tau, 0]$ as

$$\theta_{n,i}^c = \frac{\tau}{2}\left(\cos\left(\frac{i\pi}{n}\right) - 1\right)$$

then it holds that

$$\sup_{\theta \in [-\tau, 0]} \left| (\theta - \theta_{n,0}^c) \dots (\theta - \theta_{n,n}^c) \right| = \min \left\{ \sup_{\theta \in [-\tau, 0]} \left| (\theta - \theta_{n,0}) \dots (\theta - \theta_{n,n}) \right| \mid -\tau \le \theta_{n,n} < \dots < \theta_{n,0} \le 0 \right\}$$

$$= \frac{\tau^n}{2^{2n+1}}.$$

For a proof, see [30].

It is natural to ask whether, for a certain sequence of mesh points $\{\theta_{n,i} \mid 0 \le i \le n\}$, the interpolating polynomial converges in the supremum–norm to the original function. It turns out that for every choice of sequence of mesh points $\{\theta_{n,i} \mid 0 \le i \le n\}$ we can find a continuous function $f : [-\tau, 0] \to \mathbb{R}^d$ such that $\lim_{n \to \infty} \|f - \mathcal{L}_n f\|_\infty \ne 0$ [30].

Appendix B

Numerical Methods for solving Ordinary Differential Equations

In this appendix, we review two methods to numerically approximate the solution to ordinary differential equations. In Sections B.1–B.2, we review the collocation method; in Section B.3, we introduce the Runge-Kutta method. Throughout this chapter, we follow [2].

B.1   Collocation

Fix $d \in \mathbb{N}$ and let us study the initial value problem

$$\begin{cases} \dot{y}(t) & = f(y(t)) \\ y(0) & = y_0 \end{cases} \tag{B.1}$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$ and $y_0 \in \mathbb{R}^d$. For $n \in \mathbb{N}$, let us choose a mesh $0 \leq \theta_{n,1} \leq \ldots \leq \theta_{n,n} \leq 1$ on the interval $[0,1]$. We say that u is a collocation solution of order n of the initial value problem (B.1) on the interval $[0,1]$ if u is a polynomial of degree $n+1$ that satisfies the equation (B.1) at the mesh points, i.e.

$$u'(\theta_{n,i}) = f(u(\theta_{n,i})), \quad 0 \leq i \leq n$$
$$u(0) = y_0.$$

To construct the collocation solution, we write

$$k_i = u'(\theta_{n,i}), \quad 0 \leq i \leq n.$$

Furthermore, we write

$$\ell_{n,i}(\theta) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{\theta - \theta_{n,j}}{\theta_{n,i} - \theta_{n,j}}.$$

Then u' can be written as

$$u' = \sum_{j=0}^{n} k_i \ell_{n,j}. \tag{B.2}$$

For $0 \leq i \leq n$, integrating from 0 to $\theta_{n,i}$ now gives that

$$
\int_0^{\theta_{n,i}} u'(\theta)d\theta = u(\theta_{n,i}) - y_0
$$

$$
= \sum_{j=0}^n k_j \int_0^{\theta_{n,i}} \ell_{n,j}(\theta)d\theta
$$

$$
= \sum_{j=0}^n k_j a_{ij}
$$

where we have set $a_{ij} = \int_0^{\theta_{n,i}} \ell_{n,j}(\theta)d\theta$. This gives that

$$
u(\theta_{n,i}) = y_0 + \sum_{j=0}^n k_j a_{ij}
$$

Since $k_i = u(\theta_{n,i}) = f(u(\theta_{n,i}))$, this implies that

$$
k_i = f\left(y_0 + \sum_{j=0}^n k_j a_{ij}\right), \quad 0 \leq i \leq n \tag{B.3}
$$

Solving the system (B.3) for $k_0, \ldots, k_n$ and substituting into (B.2) gives us the collocation solution u' and then by integration the collocation solution u.

B.2   Iterative collocation

In the previous section, we have described the one-step collocation method, where we have approximated the solution of the initial value problem (B.1) on $[0,1]$ by one single polynomial. Now suppose we have some fixed $t_0 > 0$ and we want to find an approximate solution to (B.1) on the interval $[0, t_0]$. We can of course proceed by rescaling the one-step collocation method to the interval $[0, t_0]$, but we can also split the interval $[0, t_0]$ in different s different subintervals and iteratively apply the one-step collocation method to each of these subintervals, using as an initial condition the result from the collocation method on the previous interval.

Indeed, let us fix $m \in \mathbb{N}$ and $0 = t_1 \leq \ldots \leq t_m = t_0$ and let $0 \leq \theta_{n,1} \leq \ldots \leq \theta_{n,n} \leq 1$ be as in the Section B.1. For simplicity, we assume that the points $t_1, \ldots, t_m$ are equally spaced with $|t_{s+1} - t_s| = h$ for all $1 \leq s \leq m - 1$.

For $s = 1$, we want the function $u_1 : [t_1, t_2] \to \mathbb{R}^d$ to satisfy:

$$
\begin{cases}
u_1'(h\theta_{n,i}) &= f(u_1(h\theta_{n,i})), \quad 0 \leq i \leq n \\
u_1(t_1) &= y_0
\end{cases}
$$

Having solved this collocation problem, we define the update $y_1 = u_1(t_2)$. Repeating this procedure iteratively, we want for $1 \leq s \leq m - 1$ the solution $u_s : [t_s, t_{s+1}] \to \mathbb{R}^d$ to satisfy

$$
\begin{cases}
u_s'(t_s + h\theta_{n,i}) &= f(u_s(t_s + h\theta_{n,i})), \quad 0 \leq i \leq n \\
u_s(t_s) &= y_{s-1}
\end{cases}
$$

and define the update $y_s = y_s(t_{s+1})$. In this way, we find a family of polynomials u_s, each of which approximates the solution to (B.3) on the subinterval $[t_s, t_{s+1}]$.

To compute the polynomial u_s, we proceed as in the one-step collocation method and write

$$u'_s(t_s + h\theta) = \sum_{i=1}^{n} k_{s,i}\ell_{n,i}$$

A similar computation to the one in Section B.1 then gives that $k_{s,i}$ is defined by

$$k_{s,i} = f\left((y_s + h\sum_{j=1}^{n} a_{ij}k_{s,j}\right)$$

with $a_{ij} = \int_0^{\theta_{n,i}} \ell_{n,j}(\theta)d\theta$. The value of $y_{s+1} = y_s(t_{s+1})$ is then given by

$$y_{s+1} = y_s + h\sum_{j=1}^{n} b_j k_{s,j}$$

with $b_j = \int_0^1 \ell_j(\theta)d\theta$.

B.3   Runge-Kutta methods

In the previous section, we studied the multistep collocation method, which could be summarised as

$$k_{s,i} = f\left(y_s + h\sum_{j=1}^{n} a_{ij}k_{s,j}\right) \tag{B.4}$$

$$y_{s+1} = y_s + h\sum_{j=1}^{n} b_j k_{s,j} \tag{B.5}$$

with $a_{ij} = \int_0^{\theta_{n,i}} \ell_{n,j}(\theta)d\theta$ and $b_j = \int_0^1 \ell_j(\theta)d\theta$.

All methods of the form (B.4) – (B.5), for arbritrary coefficients a_{ij}, b_j, are called *Runge–Kutta methods*. If $a_{ij} = 0$ for $i \leq j$, the method is called *explicit*, because in this case the formula (B.4) gives an explicit formula for $k_{s,i}$ in terms of a_{ij} and f. If $a_{ij} \neq 0$ for all $i \leq j$, then the equation is called *implicit*, because in this case we have to solve the equations (B.4) for $k_{s,i}$.

Appendix C

The Trotter-Kato Theorem

We recall that a semigroup $(T(t))_{t \geq 0}$ on a Banach space $(X, \|.\|)$ is in $G(M, \omega)$ for $M \geq 1, \omega \in \mathbb{R}$ if

$$\|T(t)\| \leq M e^{\omega t} \quad \text{for all } t \geq 0.$$

In this case, we write $T(t) \in G(M, \omega)$.

With this terminology, we can now state the following theorem:

Theorem C.0.1 (Trotter–Kato)**.** *Let $(X, \|.\|)$, $(X_n, \|.\|_n)$ be Banach spaces for $n \in \mathbb{N}$. Let $(T(t))_{t \geq 0}$ be a C_0–semigroup of linear operators on X with infinitesimal generator A. For $n \in \mathbb{N}$, let $P_n : X \to X_n$, $E_n : X_n \to X$ be linear maps and let M_1, M_2 be positive constants such that*

$$\|E_n\| \leq M_1, \; \|P_n\| \leq M_2 \quad \text{for all } n \in \mathbb{N}, \tag{A1}$$

and

$$P_n E_n = I_n \quad \text{for all } n \in \mathbb{N} \tag{A2}$$

where I_n is the identity operator on X_n.

Now let $(T_n(t))_{t \geq 0}$ be a C_0–semigroup of linear operators on X_n for $n \in \mathbb{N}$; denote the infinitesimal generator of $(T_n(t))_{t \geq 0}$ by A_n. Assume that there exists a $M \geq 1, \omega \in \mathbb{R}$ such that $T_n(t), T(t) \in G(M, \omega)$ for all $n \in \mathbb{N}$. Then the following statements are equivalent:

(a) There exists a $\lambda_0 \in \rho(A) \bigcap_{n \in \mathbb{N}} \rho(A_n)$ such that, for all $x \in X$,

$$\lim_{n \to \infty} \left\| E_n (\lambda_0 I_n - A_n)^{-1} P_n x - (\lambda_0 I - A)^{-1} x \right\| = 0.$$

(b) For every $x \in X$ and $t \geq 0$,

$$\lim_{n \to \infty} \| E_n T_n(t) P_n x - T(t) x \| = 0$$

uniformly for t in bounded intervals.

Furthermore, if either (a) or (b) holds, then (a) holds for all $\lambda_0 \in \mathbb{C}$ with $\operatorname{Re} \lambda_0 > \omega$.

The condition that there exists a $M \geq 1, \omega \in \mathbb{R}$ such that $T_n, T \in G(M, \omega)$ for all $n \in \mathbb{N}$, is often called the *stability condition*. The condition (a) from Theorem C.0.1 is often called *consistency* of the approximation. To prove this condition, we can use the following theorem:

Theorem C.0.2. *Let (A1), (A2) be satisfied and assume that there exist $M \geq 1, \omega \in \mathbb{R}$ such that $T_n(t) \in G(M, \omega)$ for all $n \in \mathbb{N}$. Then the condition (a) in Theorem C.0.1 holds if and only if*

$$\lim_{n \to \infty} \| E_n P_n x - x \| = 0, \quad \text{for all } x \in X \tag{C.1}$$

and the following two conditions are satisfied:

(i) *There exists a subset $D \subseteq \mathcal{D}(A)$ such that $\overline{D} = X$ and*

$$\overline{(\lambda_0 I - A)D} = X$$

for a $\lambda_0 > \omega$.

(ii) *For all $x \in D$ there exists a sequence $(\overline{x}_n)_{n \in \mathbb{N}}$ with $\overline{x}_n \in \mathcal{D}(A_n)$ such that*

$$\lim_{n \to \infty} E_n \overline{x}_n = x \quad and \quad \lim_{n \to \infty} E_n A_n \overline{x}_n = Ax.$$

A proof of Theorem C.0.1 and Theorem C.0.2 can be found in [20].

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