Analysis, Modeling and Simulation of

Multiscale Problems

Wave Pinning in Strips

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Preprint 142

Schwerpunktprogramm der Deutschen Forschungsgemeinschaft

Wave Pinning in Strips

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In this paper, we give a proof, that pinning of waves in periodic heterogeneous media can only occur in a parameter region, that is exponentially small in the period of media. For a general outline of traveling waves in heterogeneous media, see [Xin00]. Our approach is based on a formulation of related elliptic equations as (ill-posed) spatial dynamical systems, where an unbounded space direction is considered as a time direction. This is based on an idea of Kirchgässner [Kir82]. Another important ingredient is homogenization theory, which gives effective descriptions for heterogeneous elliptic problems. There the heterogenities in the form of rapidly varying coefficients are removed, such that the resulting approximate equation is homogeneous, i.e. the equation is homogenized, for a general reference see e.g. [JKO94]. To show the exponential smallness of certain effects, we use an averaging or homogenization approach, which gives homogeneous approximations up to exponentially small errors of partial differential equations. Averaging up to exponentially small remainder terms was developed by Neishtadt for analytic ordinary differential equations [Nei84]. Extensions to partial differential equations were first developed for the time-averaging of parabolic equations [Mat01]. Here we use and extend a version for the homogenization of certain elliptic problems on strips [Mat00].

We are looking for traveling waves for parabolic equations on strips $(y, x) \in \mathbb{R} \times \Omega$ where $\Omega = [0, 2\pi]^d$. As a main example consider a scalar equation, where the diffusion coefficients and the nonlinearity both depend periodically on the unbounded space direction y

$$\partial_\tau u = d_0(y/\varepsilon)\partial_y^2 u + d(y/\varepsilon)\Delta_x u + f(u, x, y/\varepsilon; \lambda, \varepsilon). \tag{0.1}$$

We impose periodic boundary conditions on $\partial\Omega$. The diffusion coefficients are assumed to be periodic in y/ε, positive, and continously differentiable functions. The nonlinearity is assumed to be an entire function of u, x, and continuous in y/ε, ε. Assume further that f and all its derivatives with respect to u and x are differentiable with respect to the parameter $\lambda \in \Lambda$, where Λ is a bounded, open interval of the real line.

1 Spatial dynamics and main result

We are interested in pinned or standing waves, i.e. in solutions to the static problem

$$d_0(y/\varepsilon)\partial_y^2 u + d(y/\varepsilon)\Delta_x u + f(u, x, y/\varepsilon; \lambda, \varepsilon) = 0 \tag{1.1}$$

where $u(y,.)$ converges to y-independent equilibria u^1 and u^2 for $y \to \pm\infty$. As we will only consider this static problem we can assume without restriction $d_0 \equiv 1$ after dividing by $d_0(y/\varepsilon)$. Associated to the heterogeneous problem (1.1), we consider a homogenized problem

$$\partial_y^2 u + \bar{d}\Delta_x u + \bar{f}(u, x; \lambda, \varepsilon) = 0, \tag{1.2}$$

with $\bar{d}$ a constant computed below in subsection 2.1, and the nonlinearity is averaged over the "fast" variable:

$$\bar{f}(u, x; \lambda, \varepsilon) \quad = \quad \int_0^1 f(u, x, s; \lambda, \varepsilon)\mathrm{d}s. \tag{1.3}$$

We assume the existence of a standing wave $\bar{u}$ in the homogenized problem (1.2) connecting two y-independent equilibria u^1 and u^2. More precisely, we assume:

H1: $\bar{u}, u^1,$ and u^2 solve (1.2) for $\lambda = \lambda_0 \in \Lambda, \varepsilon = 0$. Furthermore, $\partial_y u^{1,2} \equiv 0$.

H2: $\lim_{y\to\infty} \bar{u}(.,y) = u^1(.)$ and $\lim_{y\to-\infty} \bar{u}(.,y) = u^2(.)$.

H3: $-\bar{d}_\Omega \Delta_x u^i - D\bar{f}(u^i, x; \lambda_0, 0), i = 1, 2$ are both strictly positive operators

Remark 1.1 *In the case of a zero-dimensional cross section (i.e., Ω is a point) these hypotheses are satisfied for bistable nonlinearities. For examples of traveling pulses in that case see [Xin00].*

We will use additional transversality assumptions. For example, in the case of of constant diffusion coefficients there exist standing wave solutions u_ε of (1.1) for $\varepsilon > 0$ close to $\bar{u}$ for some λ_ε close to λ_0. This transversality condition is given in terms of the linearisation of (1.2) along $\bar{u}$. To formulate this, we rewrite (1.2) as a 'dynamic' equation in the unbounded spatial direction y, which we will call from now on t. By letting

$$U = \begin{pmatrix} u \\ u_t \end{pmatrix} \tag{1.4}$$

we have

$$U_t = \bar{A}U + \bar{F}(U; \lambda, \varepsilon) \tag{1.5}$$

with

$$\begin{aligned} \bar{A}U &= \begin{pmatrix} 0 & 1 \\ -\bar{d}\Delta_x & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} \\ \bar{F}(U; \lambda, \varepsilon)(x) &= \begin{pmatrix} 0 \\ -\bar{f}(u, x; \lambda, \varepsilon) \end{pmatrix} \end{aligned}$$

In this notation the standing wave equation of the heterogeneous problem (1.1) has the form

$$U_t = A(t/\varepsilon)U + F(U, t/\varepsilon; \lambda, \varepsilon) \tag{1.6}$$

with

$$
\begin{aligned}
A(t/\varepsilon)U &= \begin{pmatrix} 0 & 1 \\ -d(t/\varepsilon)\Delta_x & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} \\
F(U, t/\varepsilon; \lambda, \varepsilon)(x) &= \begin{pmatrix} 0 \\ -f(u, x, t/\varepsilon; \lambda, \varepsilon) \end{pmatrix}
\end{aligned}
$$

Remark 1.2 *The hypotheses [H1]-[H3] on the stationary solution of the averaged problem translate immediately into analogous statements about the solution of (1.5). (Which we will also refer to as [H1]-[H3].)*

We consider (1.5) and (1.6) as infinite-dimensional dynamical systems on a phase space X. The space X is given as a function space on the cross-section Ω of the cylindrical domain, in particular we choose $H^s(\Omega) \times H^{s-1}(\Omega)$ with $s > \max(1, d/2)$ intersected with the boundary conditions on $\partial\Omega$ on the cross-section $\Omega = [0, 2\pi]^d$ of the cylindrical domain. We equip the domain $X^1 = D(A) \subset H^{s+1} \times H^s$ of the unbounded operator A with the norm $|U|_{X^1} = |U|_X + |AU|_X$, then X^1 is a Banach space. A solution of (1.5) resp. (1.6) is defined to be

(i) $U(.) \in BC^0(\mathbb{R}, X^1) \cap BC^1(\mathbb{R}, X)$,

(ii) $U(.)$ satisfies equations (1.5) resp. (1.6) on $\mathbb{R}$ with values in X.

Then the linearisation of (1.5) along the solution $\bar{U}$ is given by

$$
\begin{aligned}
\mathcal{L} &: BC^0(\mathbb{R}, X^1) \cap BC^1(\mathbb{R}, X) \to BC^0(\mathbb{R}, X) \\
\mathcal{L}V(t) &= V_t(t) - AV(t) - D_U \bar{F}(\bar{U}(t); \lambda_0, 0)V(t)
\end{aligned}
\tag{1.7}
$$

We assume the following generic property

$$
\mathrm{Ker}\,(\mathcal{L}) = \mathrm{span}(\dot{\bar{U}}(.)) \tag{1.8}
$$

Physically, this condition just means that the only zero eigenvalue of the linearized problem is the one that comes from the translation invariance of the averaged equation. The transversality assumption is

$$
\mathrm{Rg}\,\mathcal{L} \oplus D_\lambda \bar{F}(\bar{U}(.); \lambda, 0)_{|\lambda=\lambda_0} = BC^0(\mathbb{R}, X), \tag{1.9}
$$

i.e. the rank defect can be removed when varying the parameter.

Solutions of elliptic equations like (1.5) with f an entire function are analytic in x. Here we express such classes of highly regular functions as Gevrey spaces. We define these spaces as follows. Suppose for convenience that we consider periodic boundary conditions on Ω. Then for each $k \in \mathbb{Z}^d \backslash \{0\}$, the operator $\bar{A}$ has eigenfunctions (in X) given by

$$
v_k(x) = \begin{pmatrix} \exp(ik \cdot x) \\ \sqrt{d}|k| \exp(ik \cdot x) \end{pmatrix}
$$

with positive eigenvalue $\sqrt{\bar{d}}|k|$ and

$$w_k(x) = \begin{pmatrix} \exp(ik \cdot x) \\ -\sqrt{\bar{d}}|k| \exp(ik \cdot x) \end{pmatrix}$$

with negative eigenvalue $-\sqrt{\bar{d}}|k|$. (The zero eigenspace is spanned by $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If we expand a function $U \in X$ as $U = \sum_k (a_k v_k + b_k w_k)$ the X-norm is equivalent to $|U|_X^2 = \sum_k \{(1+|k|)^{2s}(|a_k|^2 + |b_k|^2)\}$. We define the Gevrey space $\mathcal{G}_\sigma$ to be those elements of X for which the norm

$$\|U\|_{\mathcal{G}_\sigma}^2 = \sum_k \{(1+|k|)^{2s}(|a_k|^2 + |b_k|^2)e^{2\sigma|k|}\}$$

is finite.

Remark 1.3 *Note that $\mathcal{G}_0 = X$.*

Remark 1.4 *If the nonlinear term, f, is entire (as we assume) and we fix $\sigma_0 > 0$, then every bounded solution of (1.1) lies in all $\mathcal{G}_\sigma$, $0 \le \sigma \le \sigma_0$. (See [Mat00].)*

Theorem 1 *In (1.1) let d_0, d be 1-periodic in y/ε and entire and assume that f is an entire function in x and u and continuous in y/ε and ε. Let f and all its derivatives with respect to u, x be differentiable in $\lambda \in \Lambda$. Assume the existence of a standing wave $\bar{U} = \begin{pmatrix} \bar{u} \\ \bar{u}_t \end{pmatrix}$ of (1.2) satisfying [H1], [H2], [H3], (1.8) and (1.9). Then standing waves U_ε of (1.6) can only exist in an exponentially small parameter interval, i.e. there exists positive constants $c_1, c_2, \varepsilon_0, B$ and b and a function $\lambda_1(\varepsilon)$ such that if $(U_\varepsilon, \tilde{\lambda}_\varepsilon)$ is a standing wave of (1.6) for $0 < \varepsilon \le \varepsilon_0$ that satisfies*

$$\begin{aligned} \|V_\varepsilon - \bar{U}\|_{BC^0(\mathbb{R},X^1)\cap BC^1(\mathbb{R},X)} &\le c_1, \\ |\tilde{\lambda}_\varepsilon - \lambda_1(\varepsilon)| &\le c_2 \end{aligned}$$

then

$$|\tilde{\lambda}_\varepsilon - \lambda_\varepsilon| \le B \exp(-b/\varepsilon^{1/2}). \tag{1.10}$$

Remark 1.5 *The analysis applies directly to systems with the same diffusion coefficients for each component: With $u \in H^s(\Omega, \mathbb{R}^n)$ it is possible to analyze pinned waves of*

$$u_t = d_0(x/\varepsilon)\partial_y^2 u + d(y/\varepsilon)\Delta u + f(u, x, y/\varepsilon; \lambda, \varepsilon).$$

and to systems with diffusion coefficients independent of y/ε, but possibly different for each component

$$u_t = diag(d_0^1, \ldots, d_0^n)\partial_y^2 u + diag(d^1, \ldots, d^n)\Delta u + f(u, x, y/\varepsilon; \lambda, \varepsilon).$$

2 Proof of Theorem 1

We will use an implicit function argument starting with $\bar{u}$ to show the persistence of pinned solutions under bounded perturbations. In particular we will set up a Ljapunov-Schmidt reduction to deal with the rank defect of $\mathcal{L}$. For this we use methods by Peterhof, Sandstede and Scheel [PSS97] on exponential dichotomies for equations of the form

$$U_t = \bar{A}U + B(t)U. \tag{2.1}$$

To show the exponentially small pinning properties, we employ a homogenization method similar to a result in [Mat00]. To apply both of these results, we will first transform (1.6) to remove the fast dependence in the main part $A(t/\varepsilon)$, while creating a rapidly varying, unbounded perturbation, but which is formally small and will be estimated later.

2.1 Time changes and linear transformations

The goal of this subsection is to make a change of variables which eliminates the "fast" time dependence in the linear part of (1.6). By a linear transformation and time-rescaling we will move the fast variable in the main part of the equation

$$U_t = A(t/\varepsilon)U. \tag{2.2}$$

to a higher order in ε.

First we rescale time as

$$t = \tau + \varepsilon\Theta(\tau/\varepsilon)$$

with $\Theta(.) = \Theta(. + 1)$. We then transform the dependent variable according to:

$$U = \Phi(\tau/\varepsilon)V = \begin{pmatrix} 1 & \varepsilon\phi(\tau/\varepsilon) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.3}$$

with $\phi(.) = \phi(. + 1)$. If we reexpress equation (1.6) in the new variables we find

$$\begin{aligned} V_\tau \;&=\; \begin{pmatrix} v_2 + (\Theta'(\frac{\tau}{\varepsilon}) - \phi'(\frac{\tau}{\varepsilon}))v_2 + \varepsilon\Lambda_1 v_1 + \varepsilon^2\Lambda_2 v_2 \\ \\ -(1+\Theta'(\frac{\tau}{\varepsilon}))d(\frac{\tau}{\varepsilon} + \Theta(\frac{\tau}{\varepsilon}))\Delta_x v_1 - \varepsilon\Lambda_1 v_2 \end{pmatrix} + \\ &\quad (1+\Theta'(\frac{\tau}{\varepsilon}))\Phi^{-1}((\tau/\varepsilon)F(\frac{\tau}{\varepsilon} + \Theta(\frac{\tau}{\varepsilon}), \Phi(\frac{\tau}{\varepsilon})V; \lambda, \varepsilon)\ , \end{aligned} \tag{2.4}$$

where

$$\Lambda_1 v = (1+\Theta'(\frac{\tau}{\varepsilon}))d(\frac{\tau}{\varepsilon} + \Theta(\frac{\tau}{\varepsilon}))\phi(\frac{\tau}{\varepsilon})\Delta_x v\ ,$$

and

$$\Lambda_1 v = (1+\Theta'(\frac{\tau}{\varepsilon}))d(\frac{\tau}{\varepsilon} + \Theta(\frac{\tau}{\varepsilon}))(\phi(\frac{\tau}{\varepsilon}))^2\Delta_x v\ ,$$

The leading order term in depending on "fast time" τ/ε is then in the second component:

$$-d(\tau/\varepsilon+\Theta(\tau/\varepsilon))(1+\Theta'(\tau/\varepsilon))\Delta_x v_1$$

In order to eliminate the fast time dependence of this term we choose Θ to satisfy

$$-d(s+\Theta(s))(1+\Theta'(s)) = -\bar{d} \tag{2.5}$$

The properties of Θ are detailed in the following:

Lemma 2.1 *There exists $\bar{d}>0$ and a differentiable, periodic, function Θ such that (2.5) is satisfied. Furthermore, one can choose Θ such that one also has $\int_0^1 \Theta(\theta)\mathrm{d}\theta = 0$.*

Proof. Let Θ be the solution of the ordinary differential equation

$$\begin{aligned}\dot{\Theta}(s) &= \frac{\bar{d}}{d(s+\Theta(s))} - 1 \\ \Theta(s_0) &= 0.\end{aligned} \tag{2.6}$$

The periodicity of Θ, i.e. $\Theta(.) = \Theta(.+1)$ can be ensured by an appropriate choice of $\bar{d}>0$. Indeed the solution $\Theta(s_0+1)$ depends continuously on the parameter $\bar{d}$ and $\Theta(s_0+1)<\Theta(s_0)$ for $\bar{d}=0$ and $\Theta(s_0+1)>\Theta(s_0)$ for $\bar{d}$ large, hence there exists a $\bar{d}>0$, such that $\Theta(s_0+1)=\Theta(s_0)$, which implies $\Theta(.)=\Theta(.+1)$ by periodicity of the differential equation.

We conclude the proof by showing that if we choose s_0 appropriately we can also obtain $\int_0^1 \Theta(\theta)\mathrm{d}\theta = 0$. Begin, by integrating by parts:

$$I(s_0) = \int_{s_0}^{s_0+1} \Theta(s)\mathrm{d}s = -\int_{s_0}^{s_0+1} s\dot{\Theta}(s)\mathrm{d}s = \int_{s_0}^{s_0+1} s\left(\frac{\bar{d}}{d(s+\Theta(s))}-1\right)\mathrm{d}s\ ,$$

where the first equality used the fact that $\Theta(s_0)=0$ to eliminate the boundary terms. We now use the identity $\bar{d}^{-1} = \int_{s_0}^{s_0+1} 1/d(s+\Theta(s))\mathrm{d}s = \int_0^1 1/d(s+\Theta(s))\mathrm{d}s$ to rewrite the last integral as

$$\begin{aligned}&\bar{d}\int_{s_0}^{s_0+1} s\left(\frac{1}{d(s+\Theta(s))} - \int_{s_0}^{s_0+1}\frac{1}{d(\theta+\Theta(\theta))}\mathrm{d}\theta\right)\mathrm{d}s \\ &= \bar{d}\int_{s_0}^{s_0+1} s\left(\frac{1}{d(s+\Theta(s))} - \int_0^1\frac{1}{d(\theta+\Theta(\theta))}\mathrm{d}\theta\right)\mathrm{d}s \\ &= \bar{d}\int_0^1 (s+s_0)\left(\frac{1}{d(s+s_0+\Theta(s+s_0))} - \int_0^1\frac{1}{d(\theta+\Theta(\theta))}\mathrm{d}\theta\right)\mathrm{d}s \\ &= \bar{d}\int_0^1 s\left(\frac{1}{d(s+s_0+\Theta(s+s_0))} - \int_0^1\frac{1}{d(\theta+\Theta(\theta))}\mathrm{d}\theta\right)\mathrm{d}s\end{aligned}$$

Now integrate $I(s_0)$ with respect to s_0:

$$\int_0^1 I(s_0)\mathrm{d}s_0 = \bar{d}\int_0^1 s\int_0^1 \left(\frac{1}{d(s+s_0+\Theta(s+s_0))} - \int_0^1\frac{1}{d(\theta+\Theta(\theta))}\mathrm{d}\theta)\right)\mathrm{d}s_0\mathrm{d}s = 0,$$

the last equality following from the periodicity of d and Θ. Hence there is a choice of s_0 such that $I(s_0)=0$ and consequently the corresponding solution of (2.6) has $\int_0^1\Theta(s)\mathrm{d}s = 0$. ∎

Having fixed Θ in this way, we next eliminate the term $(\Theta'(\frac{\tau}{\varepsilon}) - \phi'(\frac{\tau}{\varepsilon}))v_2$ in the first component by choosing $\phi(s) = \Theta(s)$.

If we now rename V as U again, we see that in the new variables (1.6) takes the form

$$U_\tau = \bar{A}U + B(\tau/\varepsilon)U + \bar{F}(U;\lambda,\varepsilon) + \tilde{F}(U,\tau/\varepsilon;\lambda,\varepsilon) \tag{2.7}$$

with $\bar{A}$ as in (1.5) and using (2.6)

$$B(\tau/\varepsilon;\varepsilon)U \quad = \quad \varepsilon \begin{pmatrix} \Theta(\tau/\varepsilon)\bar{d}\Delta_x u_1 + \varepsilon\Theta^2(\tau/\varepsilon)\bar{d}\Delta_x u_2 \\ -\Theta(\tau/\varepsilon)\bar{d}\Delta_x u_2 \end{pmatrix}$$

and

$$\begin{aligned} \tilde{F}(U,\tau/\varepsilon;\lambda,\varepsilon)(x) \quad &= \quad (1+\Theta'(\tau/\varepsilon))F(\Phi(\tau/\varepsilon)U, \tau/\varepsilon + \Theta(\tau/\varepsilon);\lambda,\varepsilon) \\ &\quad - \begin{pmatrix} \varepsilon\Theta(\tau/\varepsilon)f(u_1+\varepsilon\phi(\tau/\varepsilon)u_2, \tau/\varepsilon+\Theta(\tau+\varepsilon), x;\lambda,\varepsilon) \\ 0 \end{pmatrix} - \bar{F}(U;\lambda,\varepsilon) \end{aligned}$$

The averages over a period are denoted as $\langle g(.)\rangle = \int_0^1 g(\theta)\mathrm{d}\theta$ and can be computed for B using our choice of Θ

$$\langle B(.)U\rangle = \varepsilon^2 \begin{pmatrix} \int_0^1 \Theta^2(\theta)\mathrm{d}\theta\bar{d}\Delta_x u_2 \\ 0 \end{pmatrix}. \tag{2.8}$$

2.2 Persistence

The transformations of the previous section resulted in a homogenized equation (1.5), perturbed by rapidly varying terms $B(\tau/\varepsilon)$ and $\tilde{F}(U,\tau/\varepsilon;\lambda,\varepsilon)$. In section 3 we will apply an averaging method to show that the effects of $\tilde{F}$ can be made exponentially small. With this in mind, the present section is devoted to studying the persistence of standing waves in the averaged equation when it is subjected to small perturbations. Using a Lyapunov-Schmidt argument based on the theory of exponential dichotomies (see [PSS97]) we show that under the non-degeneracy and transversality hypotheses we made in the introduction, standing waves in the averaged model persist under small, bounded, homogeneous and heterogeneous perturbations. Furthermore we derive some necessary conditions for the persistence of standing waves under unbounded, but formally small, perturbations like the term $B(\tau/\varepsilon)U$ in (eq:lintransf).

Consider an equation of the form

$$U_\tau = \bar{A}U + \bar{F}(U;\lambda,\varepsilon) + \mathcal{P}(U,\tau/\varepsilon;\lambda,\varepsilon)\ . \tag{2.9}$$

(We supress the dependence of $\bar{F}$ and $\mathcal{P}$ on x since it plays little role in this section.)

Our principal interest is in the case considered in the previous subsection where $\mathcal{P} = BU + \tilde{F}$, but is will be of use in the following sections to consider this slightly more general case.

The persistence proof is based the functional analytic framework of exponential dicotomies as in [PSS97], which we follow closely where possible. To prove the existence of a pinned wave U_ε, we introduce again a new variable V by letting $U(\tau) = \bar{U}(\tau) + V(\tau)$. Using the fact that $\bar{U}$ satisfies (1.5) with $\lambda = \lambda_0$ and $\varepsilon = \varepsilon_0$ we find

$$V_\tau = \bar{A}V + D\bar{F}(\bar{U}(\tau); \lambda_0, 0)V + G(V, \tau/\varepsilon; \lambda, \varepsilon) \tag{2.10}$$

with

$$G(V, \tau, \tau/\varepsilon; \lambda, \varepsilon) \quad = \quad \bar{F}(\bar{U}(\tau) + V; \lambda, \varepsilon) - \bar{F}(\bar{U}; \lambda_0, 0) - D\bar{F}(\bar{U}; \lambda_0, 0)V + \mathcal{P}(\bar{U} + V, \tau/\varepsilon; \lambda, \varepsilon)\ .$$

To again make the connection with the material of the previous section note that for an equation of the form (2.7), G will have the more explicit form:

$$\begin{aligned} G(V, \tau, \tau/\varepsilon; \lambda, \varepsilon) \quad &= \quad \bar{F}(\bar{U}(\tau) + V; \lambda, \varepsilon) + \tilde{F}(\bar{U}(\tau + V, \tau/\varepsilon; \lambda, \varepsilon) \\ &\quad + \varepsilon B(\tau/\varepsilon)(\bar{U}(\tau) + V) - \bar{F}(\bar{U}(\tau); \lambda_0, 0) - D\bar{F}(\bar{U}(\tau); \lambda_0, 0)V \end{aligned}$$

Next we will show, that the linear part of (2.10) has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$. First we consider the linearized equations around equilibria $u^i = \begin{pmatrix} u^i \\ 0 \end{pmatrix}$, $i = 1, 2$ at $\lambda = \lambda_0$ and $\varepsilon = 0$:

$$U_t = \bar{A}U + D\bar{F}(u^i; \lambda_0, 0)U$$

The respective projections on stable and unstable subspaces are given by

$$P_{i,\pm} = \frac{1}{2}\begin{pmatrix} I & \pm(L_i)^{-1/2} \\ \pm(L_i)^{1/2} & I \end{pmatrix} \tag{2.11}$$

where $L_i = -\bar{d}\Delta_x - D\bar{f}(u^i, x; \lambda_0, 0)$ is positive and self-adjoint. For the existence of these projections, the assumption H3 that $-\bar{d}\Delta_x - D\bar{f}(u^i, x; \lambda_0, 0)$ is a positive operator in (1.2) is crucial, as it implies that the equilibria U^1 and U^2 are hyperbolic equilibria of the spatial dynamics equation. Using [PSS97, Theorem1,Corollary1], we see that the linearisation along the standing wave

$$U_\tau(\tau) = \bar{A}U(\tau) + D\bar{F}(\bar{U}(\tau); \lambda_0, 0)U(\tau) \tag{2.12}$$

has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ denoted by $P(\tau)$ and $Q(\tau)$ respectively. The intersection $\mathrm{Rg}\,(P(0)) \cap \mathrm{Rg}\,(Q(0))$ of solutions bounded in forward and backward time is then one-dimensional by assumption (1.8)

The solution operators of the exponential dichotomies are denoted as

$$\begin{aligned} &\phi^s_{1,2}(\tau, \theta)z, \tau \geq \theta \text{ for the stable part,} \\ &\phi^u_{1,2}(\tau, \theta)z, \tau \leq \theta \text{ for the unstable part.} \end{aligned}$$

As in [PSS97, Theorem4] we decompose

$$\begin{aligned} \mathrm{Rg}\,(\phi^s_1(0,0)) \quad &= \quad Y_1 \oplus\ \mathrm{span}(\dot{U}(0)) \\ \mathrm{Rg}\,(\phi^u_2(0,0)) \quad &= \quad Y_2 \oplus\ \mathrm{span}(\dot{U}(0)). \end{aligned}$$

We are looking for solutions of the full perturbed equation (2.10), which are bounded on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively. This is equivalent to the existence of $\xi_1 \in Y_1$ and $\xi_2 \in Y_2$, such that

$$\begin{aligned} V_1(\tau) &= \phi_1^s(\tau,0)\xi_1 + \int_0^\tau \phi_1^s(\tau,\theta)G(V_1(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \\ &\quad - \int_\tau^\infty \phi_1^u(\tau,\theta)G(V_1(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \text{ for } \tau > 0 \\ V_2(\tau) &= \phi_2^u(\tau,0)\xi_1 + \int_0^\tau \phi_2^u(\tau,\theta)G(V_2(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \\ &\quad + \int_{-\infty}^\tau \phi_2^s(\tau,\theta)G(V_2(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \text{ for } \tau < 0. \end{aligned}$$

Hence for all $\xi_1 \in Y_1$ and $\xi_2 \in Y_2$ small, we get bounded solutions $V_1(\tau;\xi,\lambda,\varepsilon)$ and $V_2(\tau;\xi,\lambda,\varepsilon)$ by the implicit function theorem. Now for small $\varepsilon > 0$ we try to find $\xi = \xi_1 + \xi_2 \in Y_1 \oplus Y_2$ and λ such that $V_1(0;\xi,\lambda,\varepsilon) = V_2(0;\xi,\lambda,\varepsilon)$. For bounded perturbations G, this holds if and only if there is a solution to

$$\begin{aligned} (\phi_1^s(0,0) - \phi_2^u(0,0))\xi &= \int_{-\infty}^0 \phi_2^s(0,\theta)G(V_2(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \\ &\quad + \int_0^\infty \phi_1^u(0,\theta)G(V_1(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta. \end{aligned} \tag{2.13}$$

On the other hand, if G is unbounded and we have a bounded solution $V(\tau) \in X$, such that $G(V(\tau),\tau,\tau/\varepsilon;\lambda,\varepsilon)$ is bounded in X, then we can also write

$$\begin{aligned} V_1(\tau) &= \phi_1^s(\tau,0)\xi_1 + \int_0^\tau \phi_1^s(\tau,\theta)G(V_1(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \\ &\quad - \int_\tau^\infty \phi_1^u(\tau,\theta)G(V_1(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \text{ for } \tau > 0 \\ V_2(\tau) &= \phi_2^u(\tau,0)\xi_1 + \int_0^\tau \phi_2^u(\tau,\theta)G(V_2(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \\ &\quad + \int_{-\infty}^\tau \phi_2^s(\tau,\theta)G(V_2(\theta),\theta,\theta/\varepsilon;\lambda,\varepsilon)\mathrm{d}\theta \text{ for } \tau < 0. \end{aligned}$$

with $V_1(0) = V_2(0)$. Thus we can proceed as for bounded G to obtain (2.13), noting that the results are necessary conditions only.

Next we show that the left-hand side $L = \phi_1^s(0,0) - \phi_2^u(0,0)$ of (2.13) is a Fredholm operator of index 0. By [PSS97, Theorem1]

$$\begin{aligned} \phi_1^s(0,0) &= P_{1,+} + P_{1,-}(S_1 + K_1) \\ \phi_2^u(0,0) &= P_{2,-} + P_{2,+}(S_2 + K_2), \end{aligned}$$

with S_1 and S_2 arbitrary small and K_1 and K_2 compact in $L(X)$. Thus

$$\begin{aligned} P(0) - Q(0) &= \phi_1^s(0,0) - \phi_2^u(0,0) \\ &= P_{1,+} - P_{1,-} + (P_{1,-} - P_{2,-}) + P_{1,-}(S_1 + K_1) - P_{2,+}(S_2 + K_2) \end{aligned}$$

is a Fredholm operator of index 0, since $P_{1,+} - P_{1,-}$ is invertible in $L(X)$ and thus Fredholm of index 0. This persists under the small perturbations $P_{1,-}S_1 - P_{2,+}S_2$ and the compact perturbations $P_{1,-}K_1 - P_{2,+}K_2$ as well as under the compact perturbation $P_{1,-} - P_{2,-}$, its compactness follows from a direct calculation using (2.11) and the Sobolev embedding theorem. The null space is $\mathrm{Ker}\,(L) = \mathrm{span}(\dot{\bar{U}}(0))$ by (1.8), furthermore $\mathrm{Rg}\,(L) = \{\eta \in X, \langle\psi(0), \eta\rangle = 0\}$, where ψ is the unique bounded solution of the adjoint of (2.12), up to a multiple. Thus, the Lyapunov-Schmidt reduction method implies that if G is bounded, and (2.13) has a solution $(\xi(\varepsilon), \lambda(\varepsilon))$ for $\varepsilon > 0$ then

$$\Gamma(V, \lambda, \varepsilon) = \big(\psi(0), \{\int_{-\infty}^{0} \phi_2^s(\tau, \theta) G(V_2(\theta; \xi(\varepsilon)), \theta, \theta/\varepsilon; \lambda, \varepsilon)\mathrm{d}\theta + \int_{0}^{\infty} \phi_1^u(\tau, \theta) G(V_1(\theta; \xi(\varepsilon)), \theta, \theta/\varepsilon; \lambda, \varepsilon)\mathrm{d}\theta\}\big) \qquad (2.14)$$

Here, the notation $V_{1,2}(\theta; \xi(\varepsilon))$ is simply to remind the reader of the dependence of $V_{1,2}$ on the initial condition ξ, and we define $V : \mathbb{R} \to X$ to be the function which is equal to $V_2(t)$ for t negative and $V_1(t)$ for t positive.

We assume that Γ is a continuous function of ε for $\varepsilon > 0$. Suppose now that

$$\lim_{\varepsilon \to 0} \Gamma(0, \lambda_0, \varepsilon) \equiv \Gamma(0, \lambda_0, 0) = 0\ . \qquad (2.15)$$

Remark 2.2 *Note that for the particular perturbation $\mathcal{P}$ that arises from (2.7), the Riemann-Lebesgue Lemma implies that equation (2.15) is satisfied.*

Suppose further that

$$D_\lambda \Gamma(0, \lambda, 0)|_{\lambda=\lambda_0} = \lim_{\varepsilon \to 0^+} D_\lambda \Gamma(0, \lambda, \varepsilon)|_{\lambda=\lambda_0} \neq 0\ . \qquad (2.16)$$

Then the Lyapunov-Schmidt method guarantees that for ε sufficiently small and positive there exists $(V_\varepsilon(\tau), \lambda_\varepsilon)$ such that (2.13) holds. Thus we have:

Proposition 1 *Assume that $\bar{U}$ is a standing wave of (1.5) satisfying [H1], [H2], [H3], (1.8) and (1.9). Let Γ be defined by (2.14). If (2.15) and (2.16) are satisfied there there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists $\lambda_\varepsilon \in \Lambda$ and $V_\varepsilon \in X$ such that $\bar{U} + V_\varepsilon$ is a standing wave solution of (2.9) with $\lambda = \lambda_\varepsilon$. Furthermore, we can make $|V_\varepsilon|_X$ and $|\lambda_0 - \lambda_\varepsilon|$ arbitrarily small by choosing ε_0 small enough.*

Proof. The proof of existence was discussed just prior to the statement of the proposition. The fact that $|V_\varepsilon|_X$ and $|\lambda_0 - \lambda_\varepsilon|$ can be made arbitrarily small follows from the fact that Γ is smooth with respect to V and λ and continuous with respect to ε. ■

Remark 2.3 *The solution $\bar{U} + V_\varepsilon$ constructed in Proposition 1 will actually lie not just in X but in $\mathcal{G}_\sigma$ by the results of [Mat00].*

Remark 2.4 *The same analysis holds for $U_{\varepsilon,\beta}$ near the shifted $\bar{U}(. + \beta)$. Then typically*

$$\int_{-\infty}^{\infty} (\psi(\tau), D_\beta G(U_{\varepsilon,\beta}(\theta), \theta, \theta/\varepsilon; \lambda_\varepsilon, \varepsilon))\mathrm{d}\theta \neq 0$$

and the standing wave persists under changing λ. However, as we demonstrate in the next section, it can persist over only a very small range of parameters λ.

2.3 Exponentially small pinning

To estimate the parameter interval, in which pinning can occur, we consider equation (2.7) again. We can apply [Mat00, Theorem2] to equation (2.7), if $B + \tilde{F} + \bar{F}$ are bounded on bounded sets and analytic both on X and Gevrey spaces $\mathcal{G}_\sigma$. Then there exists a time-periodic transformation of the phase space X, such that the transformed equation has the form

$$V_\tau = AV + \bar{F}_{exp}(V; \lambda, \varepsilon) + R(V, \tau/\varepsilon; \lambda, \varepsilon) \tag{2.17}$$

with an estimate on the remainder

$$|R(V(\tau), \tau/\varepsilon; \lambda, \varepsilon)|_X \leq C \exp(-c/\varepsilon^{1/2})$$

uniformly in τ for all bounded solutions of (2.17) with $\|U(.)\|_{BC^0(\mathbb{R},X)} \leq R$. Here we need such results with the unbounded G in (2.10) coming from (2.7), if the solutions are regular enough.

Proposition 2 *Under the assumptions of theorem 1, there exists $\varepsilon_0, \sigma_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there is a change of coordinates $U = \Phi(V) = V + \varepsilon W(V, \tau/\varepsilon; \lambda, \varepsilon)$ analytic and invertible on bounded sets in both in X and the Gevrey space $\mathcal{G}_\sigma$, $0 \leq \sigma < \sigma_0$. If U evolves according to (1.6), then $V = \Phi^{-1}(U)$*

$$V_\tau = AV + \bar{F}_{exp}(V; \lambda, \varepsilon) + R(V, \tau/\varepsilon; \lambda, \varepsilon) \ . \tag{2.18}$$

Furthermore, there exists $C_r, c > 0$ such that for every V with $\|V\|_{\mathcal{G}_\sigma} < r$ and $\lambda \in \Lambda$, such that for $U \in \mathcal{G}_\sigma$ there exists a positive

$$|R(U, \tau/\varepsilon; \lambda, \varepsilon)|_X \leq C_r \exp(-c/\varepsilon^{1/2}).$$

The homogenized term, F_{exp} is close to $\bar{F}$ in the sense that for every $M > 0$ there exists $C_M > 0$ such that

$$|F(U; \lambda, \varepsilon) - \bar{F}_{exp}(U; \lambda, \varepsilon)|_X \leq C_1 \varepsilon^{1/2}.$$

Finally, the transformation, Φ, is near to the identity in the sense that W is uniformly bounded on bounded sets in either X or $\mathcal{G}_\sigma$.

The proposition is proved in the next section. We complete the proof of the main theorem first.

Proof. (of Theorem 1) Begin by noting that if we set $\mathcal{P} = F_{exp} - \bar{F}$ in (2.9) then Proposition 1 implies that there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$there exists a standing wave $(\bar{U}_\varepsilon, \lambda_1(\varepsilon))$ for

$$U_t = AU + \bar{F}_{exp}(U; \lambda, \varepsilon) \tag{2.19}$$

close to $(\bar{U}, \lambda_0)$. Let $\tilde{U}_\varepsilon \in \mathcal{G}_\sigma, \tilde{\lambda}_\varepsilon$ be another standing wave solution of (1.6) close to $\bar{U}$, i.e. $|\tilde{U}_\varepsilon - \bar{U}|_X < c_1$, where c_1 will be specified below. Then by (2.14),

$$\int_{\mathbb{R}} (\Psi(\tau), G(V_\varepsilon(\tau), \tau/\varepsilon; \tilde{\lambda}_\varepsilon, \varepsilon)) \mathrm{d}\tau = 0$$

Applying the transformation Φ to $\tilde{U}_\varepsilon$, (and still denoting the transformed variable by by $\tilde{U}_\varepsilon$), we see that

$$\tilde{U}_{\varepsilon,\tau} = \bar{A}\tilde{U}_\varepsilon + \bar{F}(\tilde{U}_\varepsilon; \tilde{\lambda}_\varepsilon, \varepsilon) + \{F_{\exp}(\tilde{U}_\varepsilon; \tilde{\lambda}_\varepsilon, \varepsilon) - \bar{F}(\tilde{U}_\varepsilon; \tilde{\lambda}_\varepsilon, \varepsilon) + R(\tilde{U}_\varepsilon; \tilde{\lambda}_\varepsilon, \varepsilon)\} \ .$$

Then (2.15) implies that

$$\begin{aligned}
I_1(V_\varepsilon(.), \tilde{\lambda}_\varepsilon, \varepsilon) + I_2(V_\varepsilon(.), \tilde{\lambda}_\varepsilon, \varepsilon) &= 0 \quad \text{with} \\
I_1(V(.), \lambda, \varepsilon) &= \int_{\mathbb{R}} (\Psi(\theta), \bar{F}_{exp}(V_\varepsilon(\theta), \theta/\varepsilon; \lambda, \varepsilon) - \bar{F}(\bar{U}(\theta); \lambda_0, 0) \\
&\qquad - D\bar{F}(\bar{U}(\theta); \lambda_0, 0)(V_\varepsilon(\theta) - \bar{U}))\mathrm{d}\theta \\
I_2(V(.), \lambda, \varepsilon) &= \int_{\mathbb{R}} (\Psi(\theta), R(V(\theta), \theta/\varepsilon; \lambda, \varepsilon))\mathrm{d}\theta
\end{aligned}$$

The second integral is exponentially small for all λ near λ_0:

$$I_2(V(.), \lambda, \varepsilon) \leq C \exp(-c/\varepsilon^{1/2})$$

for all $\lambda \in \Lambda$, $0 < \varepsilon < \varepsilon_0$ and all bounded solutions $V(.)$ of equation (1.6)

The first integral is zero for $(\bar{U}_\varepsilon, \lambda_1(\varepsilon))$. But

$$|\int_{\mathbb{R}} (\Psi(\theta), D_\lambda \bar{F}_{exp}(U(\theta), \theta/\varepsilon; \lambda, 0)_{|\lambda=\lambda_0})\mathrm{d}\theta| > C$$

for all U close to $\bar{U}$ – we choose the constant c_1 that bounds the size of $|\bar{U} - \tilde{U}_\varepsilon|_X$ so that $\tilde{U}_\varepsilon$ lies in the neighborhood where this is true. Thus there exists C such that for $|\lambda - \lambda_1(\varepsilon)| > C \exp(-c/\varepsilon^{1/2})$ we obtain $|I_1(V, \lambda, \varepsilon)| > |I_2(V, \lambda, \varepsilon)|$ and therefore $|\tilde{\lambda}_\varepsilon - \lambda_1(\varepsilon)| < C \exp(-c/\varepsilon^{1/2})$, which completes the proof of Theorem 1. ∎

3 Exponential Homogenization

The proof of proposition 2 is based on a combination of Galerkin approximation, an iterative averaging procedure and the use of Gevrey class regularity. A similar problem was treated in [Mat00, Mat01], but the presence of the unbounded term $\varepsilon B(\tau/\varepsilon)V$ in our problem causes additional difficulties in the construction of the averaging transformation.

More precisely, we proceed as follows:

Step 1: We begin by considering (1.5) restricted to a finite dimensional subspace of X. (The dimension of this subspace will be chosen in an ε-dependent fashion.) For this finite dimensional system we show that we can make a change of variables (an "averaging" transformation) such that in terms of the new variables the dependence on the "fast" time τ/ε occurs only in terms that are exponentially small in ε.

Step 2: We then relate the solutions of the finite dimensional model to the evolution of the full system (2.7). We prove that if the standing wave solution of the averaged system is very smooth then the effects of the "fast" time dependence are also exponentially small in the full system.

Remark 3.1 *Throughout the proof we suppress dependencies on the parameter λ, because the estimates on the averaging transformations are all uniform in λ. Also, for simplicity we will assume that we have periodic boundary conditions on the boundary of Ω. The modifications needed to treat Dirichlet or Neumann boundary conditions are straightforward.*

Step 1: Finite dimensional transformation As in [Mat00] we use an ε dependent Galerkin approximation with approximation space

$$H_N = \mathrm{span}\{\begin{pmatrix} \exp(\mathrm{i}kx) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \exp(\mathrm{i}kx) \end{pmatrix} \text{ with } k \in \mathbb{Z}^d, |k|^2 = \sum_{j=1}^{d} k_i^2 \leq N^2\}$$

with orthogonal projection P_N in X, note that $\|P^N A\| = N$ on X and the Gevrey spaces $\mathcal{G}_\sigma$. Then noting $AP^N = P^N A$ and $BP^N = P^N B$ we define the evolution of $U_N = P^N U$ by

$$\dot{U}_N \quad = \quad \bar{A}U_N + \varepsilon B(\tau/\varepsilon)U_N + P^N \bar{F}(U_N;\varepsilon) + P^N \tilde{F}(U_N, \tau/\varepsilon;\varepsilon) \ , \tag{3.1}$$

where we choose N depending on ε. We will couple N and ε by

$$N(\varepsilon) = \left[\varepsilon^{-1/2}\right]. \tag{3.2}$$

We now transform (3.1) by adapting the proof of Neishtadt's theorem [Nei84] about averaging of exponential order for finite dimensional ODE. We make successive coordinate changes, such that the nonautonomous terms are formally of higher order in ε in the transformed equation

$$\dot{V}_N = \bar{A}V_N + P^N \bar{F}(V_N;\varepsilon) + \bar{F}_j(V_N;\varepsilon) + R_j(V_N, \tau/\varepsilon;\varepsilon). \tag{3.3}$$

Then we will give estimates on the transformed equation that are uniform in ε and $N(\varepsilon)$. In the transformed equation the nonautonomous terms are exponentially small in ε.

We describe the formal coordinate changes needed to remove nonautonomous terms. For a moment we suppress the dependence of V on N. We work on the complex extension of the projected ball in $\mathcal{G}_\sigma$. More precisely, if $B_{\mathcal{G}_\sigma}$ is the ball of radius $\mathcal{R}$ in $\mathcal{G}_\sigma$, define $D_{N,\sigma} = P^N(B_{\mathcal{G}_\sigma})$, and

$$\mathcal{N}_\delta^\sigma(D_{N,\sigma}) := \{U \in H_N \mid \inf_{V \in D_N} |U - V|_{\mathcal{G}_\sigma} < \delta\} \tag{3.4}$$

Remark 3.2 *We could choose the radius, $\mathcal{R}$, of the ball on which we define our changes of variables to be as large as we want, though obviously the constants that appear in the proof would depend on $\mathcal{R}$. For convenience, we fix $\mathcal{R}$ once and for all to be twice the max of $\bar{U}(t)$ for $\bar{U}$ the standing wave solution of (1.5).*

We define the supremum norm on these neighborhoods by

$$\|f\|_{\mathcal{N}_\delta^\sigma(D)} = \sup_{U \in \mathcal{N}_\delta^\sigma(D)} |f(U)|_{\mathcal{G}_\sigma}$$

Since σ will be fixed throughout this section we will typically denote this norm in the much simpler fashion:

$$\|f\|_{\mathcal{N}_\delta^\sigma(D)} = \|f\|_\delta$$

if there is no possibility of confusion.

We construct the averaging transformation inductively and at each step of the induction we shrink the size of the complex neighborhood slightly. There will be r steps in our induction (r will be fixed below) and we define a sequence of constants $\delta = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_r > \frac{1}{2}\delta$.

After j inductive steps, the equations will have the form

$$\dot{U} = AU + P^N \bar{F}(U;\varepsilon) + \bar{F}_j(U;\varepsilon) + R_j(U, \frac{t}{\varepsilon}; \varepsilon) \ . \tag{3.5}$$

We will estimate the size of the various terms on the complex domain $\mathcal{N}^{\sigma}_{\delta_j}(D_{N,\sigma})$. The "remainder" term R_j will be chosen to have zero average $\langle R_j\rangle(V,\varepsilon) = \int_0^1 R_j(U,\theta,\varepsilon)d\theta = 0$ and to start the induction we will take $\bar{F}_0 = \langle P^N\tilde{F} + P^N BU\rangle$ and $R_0 = (P^N\tilde{F} + P^N BU) - \langle P^N\tilde{F} + P^N BU\rangle$.

Starting with (3.5) the consider a coordinate change of the form

$$U = V + \varepsilon W_j(V, t/\varepsilon; \varepsilon) \tag{3.6}$$

with $W(\cdot, t/\varepsilon;\varepsilon) : P^N X \to P^N X$, periodic in t/ε with period 1. Substitution into (3.5) yields to

$$\begin{aligned}\dot{V} + \varepsilon\frac{\partial}{\partial V}W_j(V,\tau,\varepsilon)\dot{V} + \frac{\partial}{\partial\tau}W_j(V,\tau,\varepsilon) &= \bar{A}(V+\varepsilon W_j(V,\tau,\varepsilon)) + P^N\bar{F}(V+\varepsilon W_j(V,\tau,\varepsilon)) \\ &+\bar{F}_j(V+\varepsilon W_j(V,\tau,\varepsilon),\varepsilon) + R_j(V+\varepsilon W_j(V,\tau,\varepsilon),\tau,\varepsilon)\end{aligned}$$

We will choose W_j to "kill" (formally) largest term which depends on the "fast" time. In the present case, that is R_j and to remove it we pick:

$$W_j(V,\tau,\varepsilon) = \int_0^{\tau} R_j(V,\theta,\varepsilon)d\theta \tag{3.7}$$

and obtain

$$\dot{V} = \bar{A}V + \bar{F}(V;\varepsilon) + \bar{F}_j(V;\varepsilon) + a(V,\tau/\varepsilon,\varepsilon) \tag{3.8}$$

where

$$\begin{aligned} a(V,\theta,\varepsilon) \ = \ & (1+\varepsilon\frac{\partial}{\partial V}W_j)^{-1}\{\varepsilon P^N\bar{A} + P^N(\bar{F}(V+\varepsilon W_j) - \bar{F}(V)) + P^N(\bar{F}_j(V+\varepsilon W_j) - \bar{F}_j(V)) \\ & +P^N(R_j(V+\varepsilon W_j,\theta,\varepsilon) - R_j(V,\theta,\varepsilon))\} \ . \end{aligned} \tag{3.9}$$

(We have supressed the dependence of many of the functions in this expression on variables like ε and λ to prevent the formula from becoming too unwieldy.) Then we choose

$$\bar{F}_{j+1}(V,\varepsilon) = \bar{F}_j(V) + \langle a(V,.,\varepsilon)\rangle, \quad R_{j+1}(V,\tau,h) = a(V,\tau,\varepsilon) - \langle a(V,.,\varepsilon)\rangle. \tag{3.10}$$

and we see that (3.8) has the form needed for our inductive argument.

We now begin the task of rigorously estimating the various terms that result from the above procedure. We begin by bounding the terms $\bar{F}_0$ and $\bar{R}_0$ that appear in our original equation.

By construction we have $\|\bar{F}_0(\cdot;\varepsilon)\|_\delta \leq \ |\langle P^N \tilde{F}(\cdot,\varepsilon)\rangle\|_\delta + \ |\langle P^N BV\rangle\|_\delta$ With the aid of (2.8) the second term on the right hand side can be bounded by $C\varepsilon^2 N^2$. To bound the first term we note that $\langle P^N \tilde{F}(\cdot,\varepsilon)\rangle = P^N \langle \tilde{F}(\cdot,\varepsilon)\rangle$ and write

$$\begin{aligned} \langle \tilde{F}\rangle &= \langle (1+\Theta'(\theta)) \begin{pmatrix} 0 \\ -f(v_1, \cdot + \Theta(\cdot);\varepsilon) \end{pmatrix} \rangle \\ &- \langle \begin{pmatrix} 0 \\ -f(v_1,\cdot;\varepsilon) \end{pmatrix} \rangle - \langle \begin{pmatrix} \varepsilon\Theta(.) f(v_1 + \varepsilon\phi(.)v_2, . + \Theta(.), x; \lambda, \varepsilon) \\ 0 \end{pmatrix} \rangle \ . \end{aligned}$$

The smoothness of f allows us to bound the $\|\cdot\|_\delta$ norm of the last term by $C\varepsilon$. To bound the difference of the first two terms note that by the change of variables formula

$$\langle f(v_1,\cdot,\varepsilon)\rangle = \langle (1+\Theta'(\cdot)) f(v_1, \cdot + \Theta(\cdot),\varepsilon)\rangle \ .$$

With this identity and the smoothness of f we can bound the difference of the first two terms in the expression for $\langle \tilde{F}\rangle$ by $C\varepsilon N$.

Applying analogous estimates to R_0 we obtain

Lemma 3.3 *There exists $\varepsilon_0 > 0$ and $C > 0$ such that if $0 < \varepsilon < \varepsilon_0$,*

$$\begin{aligned} \|\bar{F}_0(\cdot,\varepsilon)\|_\delta \leq C(\varepsilon N + \varepsilon^2 N^2) &\leq C\varepsilon^{1/2} \\ \sup_{\psi\in[0,1]} \|R_0(\cdot,\psi,\varepsilon)\|_\delta &\leq C \ . \end{aligned}$$

Remark 3.4 *The fact that the norm of R_0 is only bounded by a constant rather than $C\varepsilon^{1/2}$ is due to the presence of the term*

$$\varepsilon\Theta(\psi)\bar{d}\Delta_x u_1 \ ,$$

in BU. This term has average zero and hence does not contribute to F_0 but it is present in R_0 and can only be bounded by $C\varepsilon N^2 \leq C$.

We now construct inductively a sequence of changes of variables which successively transforms (3.1) into (3.5). Begin by defining the change of variables

$$U = \Phi_0(V) \equiv V + \varepsilon W_0(V, t/\varepsilon, \varepsilon) \ , \tag{3.11}$$

with $W_0(V,\xi,\varepsilon) = \int_0^\xi R_0(V,s,\varepsilon)\mathrm{d}s$. From the bound on R_0 in Lemma 3.3 we have

$$\sup_{\xi\in[0,1]} |W_0(\cdot,\xi,\varepsilon)\|_\delta \leq M_0$$

for some $M_0 > 0$. Furthermore, recalling that f is an entire function we see that R_0, and hence W_0 is analytic on $\mathcal{N}^\sigma_\delta(D_{N,\sigma})$ and thus we can use Cauchy estimates to bound its derivatives (see [Mat01, lemma7]):

$$\|\varepsilon\frac{\partial W_0}{\partial u}\|_{\delta_j - K_0} \leq \frac{\varepsilon\|W_0\|_\delta}{K_0} \leq \frac{M_0\varepsilon}{K_0}. \tag{3.12}$$

This estimates, plus the inverse function theorem implies that if we choose $K_0 = \delta/4$ and set $\delta_1 = \frac{3}{4}\delta$ we have:

Lemma 3.5 *There exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, (3.11) defines an analytic and invertible change of variables from $\mathcal{N}^{\sigma}_{\delta_1}(D_{N,\sigma})$ into $\mathcal{N}^{\sigma}_{\delta_0}(D_{N,\sigma})$.*

Writing out the differential equation satisfied by V we have

$$\dot{V} = \bar{A}V + \bar{F}(V; x, \varepsilon) + \bar{F}_1(V, x, \varepsilon) + R_1(V, t/\varepsilon, x, \varepsilon) ,$$

where $\bar{F}_1$ and R_1 are given in terms of $\bar{F}_0$ and R_0 by (3.10). Applying the Cauchy estimates as above we see see that we can bound terms like:

$$\|R_0(\cdot + \varepsilon W_0(\cdot), \xi, \varepsilon - R_0(\cdot, \xi, \varepsilon)\|_{\delta_1} \leq \varepsilon \frac{\|R_0(\cdot, \xi, \varepsilon)\|_{\delta_0}}{(\delta_0 - \delta_1)} \|W_0\|_{\delta_1} \leq C\varepsilon/\delta \ .$$

All other term in $\bar{F}_1$ and R_1 can be bounded in a similar fashion, the most difficult being the term $\|\varepsilon P^N \bar{A} W_0\|_{\delta_1}$ which is bounded by $C\varepsilon N \leq C\varepsilon^{1/2}$. The details are left as an exercise, and we find:

Lemma 3.6 *There exist $\varepsilon_0 > 0$ and positive constants B_1, B_2, such that*

$$\begin{aligned} \|\bar{F}_1(\cdot, \varepsilon)\|_{\delta_1} &\leq B_1 \varepsilon^{1/2} \\ \sup_{\psi \in [0,1]} \|R_1(\cdot, \psi, \varepsilon)\|_{\delta_1} &\leq B_2 \ . \end{aligned}$$

We now proceed inductively. Define $K(\varepsilon) = \tilde{K}\varepsilon^{1/2}$ for $\tilde{K}$ fixed and set $\delta_{j+1} = \delta_j - K(\varepsilon)$ for $j = 1, 2, \ldots, r$. We want $\delta_r \geq \delta/2$ and so we choose

$$r = \left[\frac{\delta}{2K(\varepsilon)} \right] = \left[\frac{\delta}{2\tilde{K}\varepsilon^{1/2}} \right]$$

We then have:

Proposition 3 *There exist $\varepsilon_0 > 0$ and positive constants B_1, B_2, such that for $j = 1, 2, \ldots, r$ there exists an analytic and invertible change of variables*

$$U = \Phi_j(V) = V + \varepsilon W_j(V, \tau/\varepsilon, \varepsilon)$$

from $\mathcal{N}^{\sigma}_{\delta_{j+1}}(D_{N,\sigma})$ into $\mathcal{N}^{\sigma}_{\delta_j}(D_{N,\sigma})$ and such that V satisfies the differential equation

$$\dot{V}_N = \bar{A}V_N + P^N \bar{F}(V_N; \varepsilon) + \bar{F}_{j+1}(V_N; \varepsilon) + R_{j+1}(V_N, \tau/\varepsilon; \varepsilon). \tag{3.13}$$

with

$$\|\bar{F}_{j+1}(\cdot, \varepsilon)\|_{\delta_{j+1}} \leq B_1 \varepsilon^{1/2} \tag{3.14}$$

$$\sup_{\psi \in [0,1]} \|R_{j+1}(\cdot, \psi, \varepsilon)\|_{\delta_{j+1}} \leq M_{j+1} \equiv 2^{-(j+1)} B_2 \varepsilon^{1/2} \ . \tag{3.15}$$

Proof.

Assume that:

$$\begin{aligned} \|\bar{F}_j\|_{D_N^j} &\leq B_1 && (3.16)\\ \|R_j\|_{D_N^j} &\leq M_j \text{ with } M_j = 2^{-j} B_2 \varepsilon^{1/2} && (3.17) \end{aligned}$$

for constants B_1, B_2 chosen below. Lemma 3.6 implies that these estimates hold for $j = 1$ and we will prove that (3.14)-(3.15) follow from them, completing the induction.

Remark 3.7 *To simplify notation we suppress the arguments of W and the dependence on time and parameters N,ε in the functions.*

Define the $j+1$'st change of variables by

$$U = \Phi_j(V) = V + \varepsilon W_j(V, t/\varepsilon, \varepsilon)$$

where W_j is defined by (3.7). The inductive estimates on R_j, plus the Cauchy estimates imply that

$$\|\varepsilon W_j\|_{\delta_j} \leq M_j \varepsilon, \tag{3.18}$$

and

$$\|\varepsilon \frac{\partial W_j}{\partial u}\|_{\delta_j - K(\varepsilon)} \leq \frac{M_j \varepsilon}{K(\varepsilon)}. \tag{3.19}$$

from which we see (via the analytic inverse function theorem) that Φ_j defines an analytic and invertible change of variables from $\mathcal{N}^{\sigma}_{\delta_{j+1}}(D_{N,\sigma})$ into $\mathcal{N}^{\sigma}_{\delta_j}(D_{N,\sigma})$.

We can estimate the correction term a in (3.6).

$$\begin{aligned} \|a\|_{\delta_j - K(\varepsilon)} \leq\ & \|(D_V \Phi_j)^{-1} \left\{ \varepsilon \bar{A} W_j + P^N \bar{F}(V + \varepsilon W_j) - P^N F(V) \right. \\ & + \bar{F}_j(V + \varepsilon W_j) - \bar{F}_j(V) + R_j(V + \varepsilon W_j) - R_j(V) \\ & \left. - \varepsilon \frac{\partial}{\partial V} W_j \left[AV + P^N F(V) + \bar{F}_j(V) \right] \right\} \|_{\delta_j - K(\varepsilon)} \end{aligned}$$

For $V \in \mathcal{N}^{\sigma}_{\delta_{j+1}}(D_{N,\sigma})$, $\|AV\|_{\delta_{j+1}}$ is bounded by by $N \|V\|_{\delta_{j+1}}$. Using the inverse function theorem to estimate $(D_V \Phi_j)^{-1}$ and the mean value theorem we have

$$\begin{aligned} \|a\|_{\delta_j - K(\varepsilon)} \leq\ & \sum_{k=0}^{\infty} \|\varepsilon \frac{\partial}{\partial V} W_j\|^k_{\delta_j - K(\varepsilon)} \left\{ \|\varepsilon W_j\|_{\delta_j} \left[N + \|\frac{\partial}{\partial V} P^N F\|_{\delta_j - K(\varepsilon)} \right. \right. \\ & \left. + \|\frac{\partial}{\partial V} \bar{F}_j\|_{\delta_j - K(\varepsilon)} + \|\frac{\partial}{\partial V} R_j\|_{\delta_j - K(\varepsilon)} \right] \\ & \left. + \|\varepsilon \frac{\partial}{\partial V} W_j\|_{\delta_j - K(\varepsilon)} \left[N\|V\|_{\delta_j} + \|P^N F\|_{\delta_j} + \|\bar{F}_j\|_{\delta_j} \right] \right\} \end{aligned}$$

Applying the Cauchy estimate again gives

$$\|a\|_{\delta_j - K(\varepsilon)} \leq 2 \left\{ \varepsilon M_j \left[N + \frac{B_1}{K(\varepsilon)} + \frac{2B_1}{K(\varepsilon)} \right] + \frac{M_j \varepsilon}{K(\varepsilon)} \left[NM + B_1 + 2B_1 \right] \right\},$$

where M is the radius of D_N, see (3.4). Recalling the definition of $K(\varepsilon)$ we obtain

$$\|a\|_{\delta_j - K(\varepsilon)} \leq M_j \left[2\varepsilon^{1/2} + 6\frac{B_1}{\tilde{K}}\varepsilon^{1/2} + 2\frac{M}{\tilde{K}} + 6\frac{B_1}{\tilde{K}}\varepsilon^{1/2} \right] \leq \frac{1}{4}M_j$$

for $\tilde{K}$ large enough uniformly in $0 < \varepsilon < \varepsilon_0$ and $N(\varepsilon)$. Therefore

$$\|R_{j+1}\|_{\delta_{j+1}} < \frac{M_j}{2} = M_{j+1} \text{ and } \|\bar{F}_{j+1} - \bar{F}_j\|_{\delta_{j+1}} < \frac{M_j}{4}.$$

Hence

$$\begin{aligned} \|\bar{F}_{j+1}\|_{\delta_{j+1}} &\leq \|\bar{F}_0\|_{\delta_{j+1}} + \sum_{k=0}^{j} \|\bar{F}_{k+1} - \bar{F}_k\|_{\delta_{j+1}} \\ &\leq B_2\varepsilon^{1/2} + \frac{1}{4}\sum_{k=0}^{j} M_k \leq B_2\varepsilon^{1/2} + \frac{1}{2}B_2\varepsilon^{1/2} \leq B_1\varepsilon^{1/2}. \end{aligned} \tag{3.20}$$

Thus the inductive statements (3.16) and (3.17) are satisfied for $j+1$ for $0 < \varepsilon < \varepsilon_0$.

■

We now define $\Phi_* = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_r$. Then the inductive estimates, plus the definitions of δ_j and r imply that Φ_* is an analytic and invertible change of variables from $\mathcal{N}^{\sigma}_{\frac{1}{2}\delta}(D_{N,\sigma})$ into $\mathcal{N}^{\sigma}_{\delta}(D_{N,\sigma})$. Furthermore, this change of variables transforms (3.1) into

$$\dot{V} = \bar{A}V + \bar{F}(V;\varepsilon) + \bar{F}_*(V;\varepsilon) + R_*(V, \tau/\varepsilon; \varepsilon) \ , \tag{3.21}$$

from the inductive estimates we obtain estimates on the norms of F_* and R_*, uniform in $\varepsilon \to 0, N(\varepsilon) \to \infty$, on $\mathcal{N}^{\sigma}_{\delta/2}(D_{N,\sigma}) \subset \mathcal{N}^{\sigma}_{\delta_r}(D_{N,\sigma})$. More precisely, we obtain:

Corollary 1 *There exists $\varepsilon_0 > 0$ and $C, c_1, c_2 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, there exists an analytic and invertible change of variable*

$$U = \Phi_*(V) = V + \varepsilon W_*(V)$$

from $\mathcal{N}^{\sigma}_{\frac{1}{2}\delta}(D_{N,\sigma})$ into $\mathcal{N}^{\sigma}_{\delta}(D_{N,\sigma})$ which transforms (3.1) into (3.21). The terms F_*, R_* and W_* can be estimated by:

$$\sup_{\xi\in[0,1]} \|R_*(\cdot,\xi;\varepsilon)\|_{\delta/2} < 2^{-r}B_2\varepsilon^{1/2} < c_2\exp(-c_1\varepsilon^{-1/2}), \tag{3.22}$$

$$\|\bar{F}_*(\cdot;\varepsilon)\|_{\delta/2} < C\varepsilon^{1/2}. \tag{3.23}$$

$$\|\bar{W}_*(\cdot;\varepsilon)\|_{\delta/2} < C. \tag{3.24}$$

Step 2: Transformation of the full system

Next we extend our bounds on the finite dimensional subsystem to the full, infinite dimensional system. We extend the change of variables Φ_* to a (complex neighborhood) of the ball in $\mathcal{G}_\sigma$ by defining

$$U = V + \varepsilon W_*(P^N V, \tau/\varepsilon; \varepsilon),$$

W is bounded on bounded sets of X and $\mathcal{G}_\sigma$ as the estimates above were uniform in $\sigma \geq 0$ and $X = \mathcal{G}_0$ We write the equation for V as in (2.18):

$$\dot{V} = \bar{A}V + \bar{F}_{exp}(V;\varepsilon) + R(V,\tau/\varepsilon;\varepsilon) \tag{3.25}$$

with

$$\bar{F}_{exp}(V;\varepsilon) + R(V,\tau/\varepsilon;\varepsilon) = \bar{F}(V;\varepsilon) + \bar{F}_*(V;\varepsilon) + R_*(V,\tau/\varepsilon;\varepsilon) + b(V,\tau/\varepsilon;\varepsilon)$$

The additional correction term $b = b_1+b_2+b_3$ arises from the infinitely many degrees of freedom that were ignored in (3.1), as well as their interactions with the modes in $P^N X$ and the terms that appear there fall naturally into one of three classes.

The infinite-dimensional corrections b_1 and b_2 come from the modes that were neglected in the transformation Φ_*. From the bounded part of the nonlinearity we have

$$\begin{aligned} & b_1(V,\tau/\varepsilon;\varepsilon) \\ = \; & (I-P^N)\left\{\bar{F}(V+\varepsilon W(P^N V)) - \bar{F}(V) + \tilde{F}(V+\varepsilon W(P^N V),\tau/\varepsilon;\varepsilon)\right\} \;, \end{aligned} \tag{3.26}$$

while the unbounded part contributes:

$$b_2(V,\tau/\varepsilon) = (I-P^N)B(\tau/\varepsilon)(V+\varepsilon W(P^N V)) = (I-P^N)B(\tau/\varepsilon)V$$

We get also a correction b_3 in the finite dimensional Galerkin space, as there is an error in the lower modes due to the neglected influence of the higher Galerkin modes to the lower ones. It is given by

$$\begin{aligned} & b_3(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) \\ = \; & (D_V^N\Phi_*)^{-1}P^N\left\{\bar{F}(V+\varepsilon W(P^N V)) - \bar{F}(P^N V+\varepsilon W(P^N V))\right. \\ & \left.+\tilde{F}(V+\varepsilon W(P^N V),\tau/\varepsilon;\varepsilon) - \tilde{F}(P^N V+\varepsilon W(P^N V),\tau/\varepsilon;\varepsilon)\right\}. \end{aligned} \tag{3.27}$$

Note that $V^N = P^N V$.

We define

$$\bar{F}_{exp}(V;\varepsilon) = \bar{F}(V;\varepsilon) + \bar{F}_*(V;\varepsilon) + \langle b_1(V,.;\varepsilon)\rangle \tag{3.28}$$

$$R(V,\tau/\varepsilon;\varepsilon) = R_*(V,\tau/\varepsilon;\varepsilon) + b(V,\tau/\varepsilon;\varepsilon) - \langle b_1(V,.;\varepsilon)\rangle, \tag{3.29}$$

We start by estimating the size of $\bar{F}_{exp}(.;\varepsilon) = \bar{F}(.;\varepsilon) + \bar{F}_*(.;\varepsilon) + \langle b_1(.,.;\varepsilon)\rangle$. For the finite dimensional part we have $|\bar{F}_*(V,\varepsilon,N)|_X \leq c_3\varepsilon^{1/2}$ by (3.23). Estimating $\langle b_1\rangle$, noting $\langle \tilde{F}(V,.;\varepsilon)\rangle = 0$ and $\langle B(.)V\rangle = 0$ we obtain for $V \in X = \mathcal{G}_0$:

$$\begin{aligned} & |\langle b_1(V,.,\varepsilon,N)\rangle|_X \\ \leq \; & |(I-P^N)\left\{\bar{F}(V+\varepsilon W(P^N V)) - F(V) + \tilde{F}(V+\varepsilon P^N W(V),\tau/\varepsilon;\varepsilon)\right\}|_X \\ \leq \; & \sup_{U=V+rW(P^N V),r\in[0,\varepsilon]} |\bar{F}'(U;\varepsilon)|_{L(X,X)}|\varepsilon W(P^N V)|_X && (3.30) \\ & + \sup_{U=V+rW(P^N V),r\in[0,\varepsilon]} |\tilde{F}'(U;\varepsilon)|_{L(X,X)}|\varepsilon W(P^N V)|_X && (3.31) \\ \leq \; & C\varepsilon && (3.32) \end{aligned}$$

because $|\varepsilon W(P^N V)|_X \leq C\varepsilon$. Thus we obtain the desired estimate on $\bar{F}_{exp}$:

$$|F(U;\lambda,\varepsilon) - \bar{F}_{exp}(U;\lambda,\varepsilon)|_X \leq C_1\varepsilon^{1/2}.$$

Then the additional terms can be estimated very easily using the following observation: If $V(t) \in \mathcal{G}_\sigma$, then the difference between V and its projection into H_N is bounded in the X norm by:

$$|V(t) - P^N V(t)|_X \leq |V(t)|_{\mathcal{G}_\sigma} \exp\left(-\sigma N\right). \tag{3.33}$$

Since we consider Gevrey regular solutions of (3.25), this allows us to estimate the remaining terms.

$$|R|_X = |R_* + b_1 + b_2 + b_3 - \langle b_1\rangle|_X.$$

For the first part we have by (3.22):

$$|R_*(V,\tau/\varepsilon;\varepsilon)|_X \leq c_2 \exp(-c_1\varepsilon^{-1/2})$$

For $b_1 - \langle b_1\rangle$ we use the same analysis as for $\langle b_1\rangle$ above, except we also note that V is bounded in the Gevrey norm $\mathcal{G}_\sigma$. Hence $b_1(V) \in \mathcal{G}_\sigma$ and thus

$$\begin{aligned}
& |b_1(V,\tau/\varepsilon;\varepsilon) - \langle b_1(V,.;\varepsilon)\rangle|_X \\
\leq\ & |(I-P^N)\left\{\bar{F}(V+\varepsilon W(P^N V)) - F(V) + \tilde{F}(V+\varepsilon P^N W(V),\tau/\varepsilon;\varepsilon)\right\}|_X \\
\leq\ & C(|V|_{\mathcal{G}_\sigma})\exp(-\sigma N).
\end{aligned} \tag{3.34}$$

Next we estimate $|b_2(V,\tau/\varepsilon)|_X = |(I-P^N)B(\tau/\varepsilon)V|_X$. Although B is unbounded as an operator on X, it is bounded as a linear operator $\mathcal{G}_\sigma \to \mathcal{G}_{\sigma/2}$. Thus

$$|b_2(V,\tau/\varepsilon)|_X \leq C\exp(-(\sigma/2)N) \tag{3.35}$$

Finally we deal with the term b_3 created by the Galerkin approximation in the finite dimensional part. For $V \in \mathcal{G}_\sigma$ define the line segment $K_V = \{rV + (1-r)P^N V + \varepsilon W(P^N V) | r \in [0,1]\}$, we then obtain

$$\begin{aligned}
& |b_3(V,.,\varepsilon,N)|_X \\
\leq\ & |(D_{V^N}\Phi_*)^{-1}P^N\left\{\tilde{F}(V+\varepsilon W(V),\tau/\varepsilon;\varepsilon) - \tilde{F}(P^N V+\varepsilon W(P^N V),\tau/\varepsilon;\varepsilon)\right\}|_X \\
& \quad + \sup_{U\in K_V} |D\tilde{F}(V)|_{L(X,X)}|V-P^N V|_X) \\
\leq\ & C|V-P^N V|_X \leq C\exp\left(-\sigma N\right),
\end{aligned} \tag{3.36}$$

where we used (3.33).

We now combine the exponential estimates of Step 1 and Step 2, using $N(\varepsilon) = [\varepsilon^{-1/2}]$, the estimates (3.34, 3.36) can be expressed in ε. Then we obtain

$$\begin{array}{ll}
\exp(-c_1\varepsilon^{-1/2}) & \text{from (3.22)} \\
\exp(-\sigma\varepsilon^{-1/2}) & \text{from (3.34,3.35, 3.36)}
\end{array}$$

Then the estimates are of order $\exp(-c\varepsilon^{-\frac{1}{2}})$ and the proof is complete. □

Acknowledgement

Financial support of the Deutsche Forschungsgemeinschaft (DFG) for KM in the Schwerpunktprogramm 'Modellierung, Analysis und effektive Simulation von Mehrskalenproblemen' is gratefully acknowledged.

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