

Geometric invariance of determining and resonating centers: Odd- and any-number limitations of Pyragas control

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In the spirit of the well-known odd-number limitation, we study failure of Pyragas control of periodic orbits and equilibria. Addressing the periodic orbits first, we derive a fundamental observation on the invariance of the geometric multiplicity of the trivial Floquet multiplier. This observation leads to a clear and unifying understanding of the odd-number limitation, both in the autonomous and the non-autonomous setting. Since the presence of the trivial Floquet multiplier governs the possibility of successful stabilization, we refer to this multiplier as the determining center. The geometric invariance of the determining center also leads to a necessary condition on the gain matrix for the control to be successful. In particular, we exclude scalar gains. Application of Pyragas control on equilibria does not only imply a geometric invariance of the determining center, but surprisingly also on centers which resonate with the time delay. Consequently, we formulate odd- and any-number limitations both for real eigenvalues together with arbitrary time delay as well as for complex conjugated eigenvalue pairs together with a resonating time delay. The very general nature of our results allows for various applications.

Time-delayed feedback control using the Pyragas method is an important tool for the stabilization of equilibria and periodic orbits. However, the time delay generates an infinite dimensional system and general results on the controllability are hard to obtain. With the present paper, we clarify the confusion in the literature on the odd-number limitation. In order to obtain our results, we shift the emphasis from analytic to geometric aspects, and concentrate on properties of the unstable object itself rather than on its dynamical system. In this way, we obtain fundamental geometric invariance principles from which all further theorems are deduced.

I. INTRODUCTION

In a dynamical system given by the ordinary differential equation $\dot{x}(t) = f(x(t))$, $x \in \mathbb{R}^N$, unstable periodic orbits can be stabilized using additive control terms of the form

$$K(x(t) - x(t - T)). \quad (1)$$

Here $T > 0$ is the time delay, and $K \in \mathbb{R}^{N \times N}$ is the weight of the control term, which we call the *gain matrix*. Such control terms were first introduced by Kestutis Pyragas in his work from 1992²². The control term by Pyragas uses the difference between the delayed state $x(t - T)$ and the current state $x(t)$ of the system. Frequently, the time delay T is chosen to be an integer multiple of the period of the periodic orbit $x_*(t)$ of the uncontrolled system. In this case, the control vanishes on the orbit itself, and $x_*(t)$ is also a solution of the controlled

system. We call such a control term *noninvasive* because it does not change the periodic orbit itself, but only affects its stability properties. In the case of equilibria, the time delay T can be chosen arbitrarily to achieve noninvasiveness.

The main advantage of the Pyragas control scheme, also for experimental realizations, is its model-independence; no expensive calculations are needed for its implementation, and the only information needed is the period of the targeted periodic orbit. As a consequence, Pyragas control has many different successful applications, e.g., in atomic force microscopes³², un-manned helicopters²¹, complex robots²⁹, semiconductor lasers^{24,25}, and the enzymatic peroxidase-oxidase reaction¹⁹, among others. The success of Pyragas control has been verified for a large number of specific theoretical models as well, including spiral break-up in cardiac tissues²³, flow alignment in sheared liquid crystals³⁰, near Hopf bifurcation³ and unstable foci^{12,33}, synchrony in networks of coupled Stuart-Landau oscillators^{26,27}, delay equations^{6,7}, in quantum systems^{2,10}, the Duffing oscillator⁵ and Turing patterns¹⁸.

General conditions on the success or failure of Pyragas control are hard to obtain because the time delay adds infinitely many dimensions to the complexity of the dynamical system. In fact, there has been serious confusion in the literature on the so-called “*odd-number limitation*”, which was correctly proven for non-autonomous systems in 1997²⁰; see also Corollary 1 in Section II B. The odd-number limitation states that in non-autonomous periodic ordinary differential equations, hyperbolic periodic orbits with an odd number of real Floquet multipliers larger than one cannot be stabilized using Pyragas control. In a footnote, Nakajima formulated the conjecture that the odd-number limitation also holds in the autonomous case and this was subsequently often wrongly cited as a proven fact. However, in 2007 Fiedler et al. found a counter-example: It is possible to stabilize a periodic orbit near a subcritical Hopf bifurcation with one real positive Floquet multiplier³. A correct version of the odd-number limita-

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tion for autonomous equations was subsequently presented by Hooton and Amann in 2012¹¹.

In the present paper we clarify the confusion on the limitations of Pyragas control. In contrast to previous works, we focus on the *geometric*, rather than the algebraic, multiplicity of Floquet multipliers (for periodic orbits, see Section II A for a precise definition) and eigenvalues (for equilibria). That is, our interest lies in the dimension of the eigenspace and not on the number of solutions of the characteristic equation. Our main results Theorems 1 and 2 show that the geometric multiplicity of the Floquet multiplier 1, or eigenvalue zero, is invariant under control. Whether such a Floquet multiplier 1, or eigenvalue 0, is present in the uncontrolled system or not decides whether the periodic orbit can in principle be stabilized via Pyragas control. Therefore we refer to geometric eigenspace of the Floquet multiplier 1, or the eigenvalue 0, as the *determining center*.

From the main results Theorem 1 and Theorem 2 we obtain several corollaries, among others the odd-number limitation and an any-number limitation for commuting control matrices. Moreover, and rather surprisingly, for steady states, the geometric multiplicities of the *resonating centers* $2\pi ni/T$, with $n \in \mathbb{Z}$ and T the time delay, are also preserved under control. As a corollary, we obtain that not only real eigenvalues are impossible to stabilize using commuting gain matrices, but also those of the form $\lambda \pm 2\pi in/T$. All results are of a qualitative nature, and apply to any ordinary differential equation (ODE) subject to Pyragas control. They do not give any quantitative restrictions on the Floquet multipliers^{4,14} or on the time delay³³.

This paper is organized as follows: In Section II, we investigate Pyragas control of periodic orbits. We formulate the fundamental principle of the invariance of the geometric multiplicity of the trivial, yet determining Floquet multiplier 1. As corollaries, we prove the odd-number limitation as well as any-number limitations for commuting gain matrices, both with real and complex spectrum. In Section III, we study Pyragas control of equilibria. The main invariance principle here concerns the determining as well as resonating centers. From this, we derive an odd-number limitation for equilibria as well as any-number limitations for commuting gain matrices with either real or complex spectrum. We conclude the paper with a discussion on applications and generalizations in Section IV.

II. GEOMETRIC INVARIANCE OF THE DETERMINING CENTER FOR PERIODIC ORBITS

In this section, we focus on feedback stabilization of periodic orbits. The main result concerns invariance of the geometric multiplicity of the Floquet multiplier 1 under Pyragas control (Theorem 1). From this invariance we deduce several limitations on feedback stabilization. Since the presence of the Floquet multiplier 1 determines whether a periodic solution can in principle be stabilized, we refer to it as the *determining center*.

The geometric invariance of the determining center is all

the more striking since we compare a center in a finite-dimensional system, given by eigenvectors, to a center in an infinite-dimensional system, given by eigenfunctions. Still, there is a one-to-one correspondence between these eigenvectors and eigenfunctions, which we explain in the following.

A. Main result concerning periodic orbits

Throughout we consider the ODE

$$\dot{x}(t) = f(x(t), t), \quad t \geq 0, \quad (2)$$

with $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ a C^1 -function. We make the following, very general, assumptions on system (2):

Assumption 1.

1. The function f is periodic with (not necessarily minimal) period $T > 0$ in its time-argument, i.e., $f(x, t + T) = f(x, t)$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$;
2. System (2) has a periodic solution $x_*(t)$ with (again not necessarily minimal) period T .

Note that besides periodic non-autonomous ODE, Assumption 1 also includes autonomous ODE which possess a periodic orbit of period T . In this case, Assumption 1.1 is trivially fulfilled. Indeed, our results in the rest of the section give a unifying approach to both the autonomous and the non-autonomous case.

Linearizing around the periodic orbit $x_*(t)$, we obtain the system

$$\dot{y}(t) = \partial_x f(x_*(t), t)y(t). \quad (3)$$

By $Y_0(t) \in \mathbb{R}^{N \times N}$, $t \geq 0$ we denote the fundamental solution of the linear matrix differential equation:

$$\begin{cases} \frac{d}{dt} Y_0(t) &= \partial_x f(x_*(t), t)Y_0(t), & t > 0; \\ Y_0(0) &= I, \end{cases} \quad (4)$$

where $I: \mathbb{R}^N \rightarrow \mathbb{R}^N$ denotes the identity matrix. If $y_0 \in \mathbb{R}^N$, then $y(t) := Y_0(t)y_0$ solves (3) with initial condition $y(0) = y_0$, and hence the fundamental solution can be viewed as ‘summarizing’ the solution information to (3).

We refer to the matrix $Y_0(T): \mathbb{R}^N \rightarrow \mathbb{R}^N$ as the **monodromy operator** of system (3). We define the **Floquet multipliers** of system (3) as the eigenvalues of the monodromy operator $Y_0(T)$.

Let $\mu \in \mathbb{C}$ be an eigenvalue of $Y_0(T)$, i.e., μ is a Floquet multiplier of (3). Then the linear space

$$\mathcal{N}(\mu I - Y_0(T)) := \{x \in \mathbb{C}^N \mid \mu x - Y_0(T)x = 0\} \quad (5)$$

has dimension at least 1; and moreover, μ satisfies

$$0 = \det(\mu I - Y_0(T)). \quad (6)$$

The **geometric multiplicity** of μ is defined as the dimension of the linear space (5) and the **algebraic multiplicity** is defined as the order of μ as a zero of the function

$z \mapsto \det(zI - Y_0(T))$. The geometric and algebraic multiplicity of a Floquet multiplier can in general be different; however, the algebraic multiplicity is always larger than or equal to the geometric multiplicity. In Theorem 1, we show that the geometric multiplicity of the Floquet multiplier 1 plays a fundamental role in stabilization. Therefore, we will refer to the geometric eigenspace of the Floquet multiplier 1, i.e., to the space

$$\mathcal{N}(1 - Y_0(T)) \quad (7)$$

as the **determining center**. In this definition, we do *not* assume that the space $\mathcal{N}(1 - Y_0(T))$ contains more than the zero vector (i.e. we do *not* assume that 1 is a Floquet multiplier). If indeed the space $\mathcal{N}(1 - Y_0(T))$ is non-trivial, it is a subspace of the center *eigenspace*, but instead we refer to it as ‘center’ for brevity of notation.

We now apply Pyragas control to the system (2) and write the controlled system as

$$\dot{x}(t) = f(x(t), t) + K[x(t) - x(t - T)] \quad (8)$$

with nonzero gain matrix $K \in \mathbb{R}^{N \times N}$. For $t \geq 0$, denote by

$$Y_1(T) : C([-T, 0], \mathbb{R}^N) \rightarrow C([-T, 0], \mathbb{R}^N) \quad (9)$$

the fundamental solution of the linearized equation

$$\dot{y}(t) = \partial_x f(x_*(t), t)y(t) + K[y(t) - y(t - T)]. \quad (10)$$

The map

$$Y_1(T) : C([-T, 0], \mathbb{R}^N) \rightarrow C([-T, 0], \mathbb{R}^N) \quad (11)$$

is a bounded linear operator, which is also compact (see appendix for a proof). Compactness of the operator implies that all non-zero spectral points are eigenvalues of finite algebraic multiplicity. In particular, if $\mu \neq 0$ is an eigenvalue of $Y_1(T)$, then the linear space

$$\mathcal{N}(\mu I - Y_1(T)) := \{\phi \in C([-T, 0], \mathbb{C}^N) \mid \mu\phi - Y_1(T)\phi = 0\} \quad (12)$$

is finite dimensional and the geometric multiplicity of μ equals the dimension of the space (12). The novelty of our approach is that we initially focus on the geometric, rather than the algebraic multiplicity, of the Floquet multipliers. We show that the geometric multiplicity of the Floquet multiplier 1 is preserved under control. This then serves as an determining principle which decides whether the targeted periodic solution can, in principle, be stabilized.

The following main result compares the geometric multiplicity of the eigenvalue 1 of $Y_0(T)$ with the geometric multiplicity of the eigenvalue 1 of $Y_1(T)$. By convention, if 1 is *not* an eigenvalue of $Y_0(T)$ (resp., $Y_1(T)$), we say that the geometric multiplicity of the eigenvalue 1 of $Y_0(T)$ (resp., $Y_1(T)$) is zero.

Theorem 1 (Geometric invariance of the determining center under Pyragas control). *The geometric multiplicity of the Floquet multiplier 1 is preserved under Pyragas control. That is, for any gain matrix $K \in \mathbb{R}^{N \times N}$, the geometric multiplicity of the eigenvalue 1 of $Y_0(T)$ without control is equal to the geometric multiplicity of the eigenvalue 1 of $Y_1(T)$ with control.*

Proof. We show that there is a one-to-one correspondence between eigenvectors to the eigenvalue 1 for $Y_0(T)$ and eigenfunctions to the eigenvalue 1 of $Y_1(T)$. The statement of the claim then follows.

On the one hand, the vector $y_0 \in \mathbb{C}^N \setminus \{0\}$ is an eigenvector of $Y_0(T)$ with eigenvalue $1 \in \mathbb{C}$ if and only if (3) has a solution $y(t)$ that satisfies

$$\begin{cases} y(t+T) = y(t), & t \in \mathbb{R} \\ y(0) = y_0; \end{cases} \quad (13)$$

see also Appendix A 1. On the other hand, $\phi \in C([-T, 0], \mathbb{C}^N) \setminus \{0\}$ is an eigenfunction of $Y_1(T)$ with eigenvalue $1 \in \mathbb{C}$ if and only if (10) has a solution $y(t)$ that satisfies

$$\begin{cases} y(t+T) = y(t), & t \geq 0 \\ y(t) = \phi(t), & t \in [-T, 0]; \end{cases} \quad (14)$$

see also Appendix A 2. But since the control term $K[y(t) - y(t - T)]$ vanishes on T -periodic functions, (10) has a solution of the form (14) *if and only if* (3) has a solution of the form (13). We conclude that there is a one-to-one correspondence between eigenvectors of the eigenvalue 1 of $Y_0(T)$ and eigenfunctions of the eigenvalue 1 of $Y_1(T)$, which proves the claim. \square

B. Corollary: The odd-number limitation

Strictly speaking, our main result does not make any statement about stabilization via Pyragas control, only addressing the seemingly unimportant center. However, we can use it to deduce a number of restrictions, that is, necessary conditions, on Pyragas control. We start with the long-known odd-number limitation, which follows easily and can now be fully understood in this context of geometric multiplicities.

Previous statements of the odd-number limitation are formulated for either autonomous or non-autonomous systems. In contrast, we formulate the odd-number limitation for *non-degenerate periodic orbits*, i.e., periodic orbits that do not have a Floquet multiplier 1. By shifting the focus from (non)-autonomous systems to (non)-degenerate periodic orbits, we clarify the confusion in the literature regarding the odd-number limitation.

For non-degenerate periodic orbits, the absence of a Floquet multiplier 1 in the uncontrolled system forbids stabilization, as no other path for real eigenvalues is possible. In autonomous systems, every periodic orbit (provided it is not an equilibrium) is degenerate, since translation along the periodic orbit leads to a trivial Floquet multiplier. A clever choice of the gain matrix can allow for a change of the algebraic multiplicity while leaving the geometric multiplicity invariant, thus achieving stabilization.

As a technical prerequisite, we first state and prove the odd-number limitation on the linear level. This has the advantage that, in blockdiagonalizing ODE, the linear statement can be applied to individual blocks.

Proposition 1. *Consider the linear system*

$$\dot{y}(t) = A(t)y(t) \quad (15)$$

with $y(t) \in \mathbb{R}^N$ and $A(t) \in \mathbb{R}^{N \times N}$. Assume that there exists a time $T > 0$ such that $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Moreover, assume that system (15) does not have a Floquet multiplier equal to 1 and that it possesses an odd number (counting algebraic multiplicities) of real Floquet multipliers strictly larger than 1.

Then, for all gain matrices $K \in \mathbb{R}^{N \times N}$, the controlled system

$$\dot{y}(t) = A(t)y(t) + K[y(t) - y(t-T)] \quad (16)$$

has at least one real Floquet multiplier larger than 1.

Proof. We first give an intuitive argument: By assumption, the linearized system (15) has an odd number of real Floquet multipliers on the line $(1, \infty)$. But, since we use a real control matrix $K \in \mathbb{R}^{N \times N}$, non-real Floquet multipliers of the controlled system appear in complex conjugated pairs. Therefore, it is impossible to change the parity of eigenvalues on the line $(1, \infty)$ by leaving the real axis and at least one Floquet multiplier stays on the line $(1, \infty)$. This Floquet multiplier can only move into the unit circle by crossing the point $1 \in \mathbb{C}$. However, this is forbidden by the invariance of the determining center (Theorem 1).

To make this argument precise, fix a gain matrix $K \in \mathbb{R}^{N \times N}$ and introduce the homotopy parameter $\alpha \in [0, 1]$:

$$\dot{y}(t) = A(t)y(t) + \alpha K[y(t) - y(t-T)]. \quad (17)$$

The monodromy operator $Y_\alpha(T)$ for (17) is compact for all $\alpha \in [0, 1]$, and the map $\alpha \mapsto Y_\alpha(T)$ is continuous; therefore the Floquet multipliers of (17) (i.e., the eigenvalues of $Y_\alpha(T)$) depend continuously on α (in the sense of¹⁷).

We first show that the number of Floquet multipliers outside the unit circle cannot change by multipliers ‘‘coming from infinity’’. Indeed, let μ be an eigenvalue of $Y_\alpha(T)$ and let ϕ be such that

$$Y_\alpha(T)\phi = \mu\phi, \quad \|\phi\| = 1, \quad (18)$$

i.e., ϕ is an eigenfunction with norm equal to 1. Then

$$|\mu| = \|\mu\phi\| = \|Y_\alpha(T)\phi\| \leq \|Y_\alpha(T)\|. \quad (19)$$

Thus, if μ is an eigenvalue of $Y_\alpha(T)$, then $|\mu| \leq \|Y_\alpha(T)\|$, i.e., the norm of μ can be bounded by the operator norm of $Y_\alpha(T)$. For every $\alpha \in [0, 1]$, the operator $Y_\alpha(T)$ is bounded and the map $\alpha \mapsto Y_\alpha(T)$ is continuous. Therefore, there exists a $0 < C < \infty$ such that

$$\sup_{\alpha \in [0, 1]} \|Y_\alpha(T)\| < C. \quad (20)$$

We conclude that if μ is an eigenvalue of $Y_\alpha(T)$, then $|\mu| < C$. Hence the number of eigenvalues of $Y_\alpha(T)$ that lies outside the unit circle cannot change by an eigenvalue ‘‘coming from infinity’’; the number of eigenvalues of $Y_\alpha(T)$ outside the unit circle can only change by an eigenvalue crossing the unit circle.

Now, for $\alpha \in [0, 1]$, define

$$n_\alpha = \#\{\mu \in \sigma_{pr}(Y_\alpha(T)) \mid \mu \in (1, \infty)\} \quad (21)$$

i.e., n_α is the number of eigenvalues of $Y_\alpha(T)$ lying on the half-line $(1, \infty)$. Note that the parity of n_α can only change by an eigenvalue crossing the point $1 \in \mathbb{C}$: Indeed, if μ is an eigenvalue of $Y_\alpha(T)$, then $\bar{\mu}$ is an eigenvalue as well, so non-real eigenvalues appear in pairs which do not affect the parity of n_α .

The only way left is through the real line, i.e., through $1 \in \mathbb{C}$. However, by assumption, $1 \in \mathbb{C}$ is not a Floquet multiplier of (17) for $\alpha = 0$ and, by Theorem 1, it will not be a Floquet multiplier of (17) for any $\alpha \in (0, 1]$. Hence it is impossible to change the parity of n_α through Pyragas control. Since by assumption, $n_{\alpha=0}$ is odd, we conclude that n_α is odd for all $\alpha \in [0, 1]$, and (16) has at least 1 Floquet multiplier larger than one. \square

Note how the proof combines continuous dependence on parameters of the Floquet multipliers with the geometric invariance of the center to conclude that stabilization is impossible. The assumption that $1 \in \mathbb{C}$ is not a multiplier of the uncontrolled system is crucial here. To facilitate terminology on this essential assumption, we say that x_* is *non-degenerate* as a solution of (2) if the linearization (3) does *not* have a Floquet multiplier 1.

For non-degenerate periodic solutions of non-autonomous ODE, we recover the well-known odd-number limitation²⁰:

Corollary 1 (Nakajima²⁰, '97). *Consider the system (2) satisfying Assumption 1. Assume that x_* is non-degenerate as a solution of (2) and that the linearized equation (3) has an odd number (counting algebraic multiplicities) of real Floquet multipliers larger than 1.*

Then, for every gain matrix $K \in \mathbb{R}^{N \times N}$, the periodic solution x_ is unstable as a solution of the controlled system (8).*

Proof. We apply Proposition 1 with $A(t) = \partial_x f(x_*(t), t)$: By assumption,

$$\dot{y}(t) = \partial_x f(x_*(t), t)y(t) \quad (22)$$

does not have a Floquet multiplier 1 and has an odd number of Floquet multipliers larger than 1. Thus Proposition 1 implies that

$$\dot{y}(t) = \partial_x f(x_*(t), t)y(t) + K[y(t) - y(t-T)] \quad (23)$$

has a Floquet multiplier larger than 1 for any gain K . Therefore x_* is unstable as a solution of (8). \square

For any linear, time-periodic DDE, the eigenvalues of the monodromy operator are also captured by a finite dimensional function called the characteristic matrix function^{15,16,28}. For system (16), where the time delay is equal to the period, the expression for this characteristic matrix function is relatively explicit. In fact, the original proof in²⁰ relies heavily on the explicit form of the characteristic matrix function. This is unnecessary, as the argument here shows: the odd number limitation follows directly from the invariance principle in Theorem 1; it does *not* rely on the fact that in system (16) the delay is equal to the period.

Let us shortly reflect the situation for autonomous systems (2), i.e., if

$$\partial_t f(x, t) = 0 \quad \text{for all } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}. \quad (24)$$

In this case, if the periodic orbit x_* is not an equilibrium, we differentiate the relation

$$\dot{x}_*(t) = f(x_*(t), t) \quad (25)$$

with respect to t to see that $\dot{x}_*(t)$ is a non-zero, T -periodic solution of (3). It follows that (3) has a Floquet multiplier 1 (called the *trivial Floquet multiplier*). Thus, if system (2) is in fact autonomous, the solution x_* is degenerate and the assumptions of Corollary 1 are not satisfied.

Moreover, if (2) is autonomous, Theorem 1 implies that the geometric multiplicity of the trivial Floquet multiplier is preserved. However, its *algebraic multiplicity* is not fixed under control, and changing the algebraic multiplicity is necessary for successful stabilization. Indeed, the results from Hooton and Amann¹¹ on stabilization in autonomous systems have a natural interpretation in terms of the algebraic multiplicity of the trivial Floquet multiplier (as will be discussed in more detail upcoming work by the first author³¹). Also the positive stabilization result³ for an autonomous system shows stabilization through the center generated by the trivial Floquet multiplier 1.

Some of the results up to this point – most notably, the invariance principle and the odd-number limitation – *might* (!) apply to a more general class of noninvasive control terms than ‘only’ Pyragas control. However, before drawing conclusions on different control terms, one should carefully consider the functional analytical framework, in particular, whether the monodromy operator is still a Riesz operator and whether eigenvalues depend continuously on parameters.

C. Corollary: Any-number limitation for commuting gain matrices with real spectrum

In addition to the odd-number limitation, we obtain direct restrictions on the choice of the gain matrix, again directly from the geometric invariance of the determining center. This will be explored in this subsection: In summary, in combination with real Floquet multipliers, stabilization is impossible if the gain matrix commutes with the linearization. In particular, scalar gains are excluded. This statement is independent on the actual number of real Floquet multipliers larger than 1, therefore we call it the *any-number limitation for commuting gain matrices*.

We first formulate our results on a linear level. We use Floquet theory (see also Appendix A 1–A 2) to transform the linear, time-periodic system $y(t) = A(t)y(t)$ into an autonomous one. Assume that the map $t \mapsto A(t)$ is T -periodic; denote by $Y_0(T)$ its monodromy operator. Since $Y_0(T)$ is invertible (i.e., $0 \notin \sigma(Y_0(T, 0))$), there exists a matrix $B \in \mathbb{C}^{N \times N}$ such that

$$Y_0(T) = e^{BT}. \quad (26a)$$

Floquet theory for ODE (see Appendix A 1) gives that the map

$$P(t) := Y_0(t)e^{-Bt} \quad (26b)$$

is T -periodic; moreover, the coordinate transformation $y(t) = P(t)v(t)$ transforms the time-periodic system $\dot{y}(t) = A(t)y(t)$ into the linear, autonomous system

$$\dot{v}(t) = Bv(t). \quad (27)$$

In the next proposition, we consider gain matrices K that are *commutative* in the sense that

$$KP(t) = P(t)K \quad \& \quad KB = BK \quad (28)$$

for all $t > 0$. This seemingly restrictive assumption can easily be fulfilled, as it is trivially true for scalar gains, or, in the case of symmetric systems, for any gain matrix which leaves the periodic orbit invariant pointwise. In fact, in this case, gain matrices fulfilling assumption (28) seem a very natural choice.

In the next proposition, we provide an analogous statement to Proposition 1, but with the above assumptions on the gain matrix. Regarding the Floquet theory of the uncontrolled system, we assume that there *exists* a real Floquet multiplier strictly larger than 1, but we make no assumptions on the number or parity of such multipliers. Moreover, we do not make assumptions on the presence of a Floquet multiplier 1, and thus the result can be applied in both autonomous and non-autonomous settings.

Proposition 2. *Consider the linear, non-autonomous system*

$$\dot{y}(t) = A(t)y(t) \quad (29)$$

and assume that there exists $T > 0$ such that $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Assume that system (29) has at least one real Floquet multiplier larger than 1.

Assume that the gain matrix $K \in \mathbb{R}^{N \times N}$ satisfies $\sigma(K) \subseteq \mathbb{R}$. Moreover, with $P(t)$ and B as in (26a)–(26b), assume that

$$KP(t) = P(t)K \quad \text{for all } t \in \mathbb{R}; \quad (30a)$$

$$KB = BK. \quad (30b)$$

Then the controlled system

$$\dot{y}(t) = A(t)y(t) + K[y(t) - y(t-T)] \quad (31)$$

has at least one real Floquet multiplier larger than 1.

Proof. We divide the proof into three steps:

Step 1: We first transform the DDE (31) into an autonomous DDE. Recall that the coordinate transform $y(t) = P(t)v(t)$ transforms solutions of (29) into solutions of (27). Therefore, the coordinate transformation $y(t) = P(t)v(t)$ transforms solutions of (31) into solutions of

$$\dot{v}(t) = Bv(t) + P(t)^{-1}K[P(t)v(t) - P(t)v(t-T)] \quad (32)$$

$$= Bv(t) + K[v(t) - v(t-T)], \quad (33)$$

where we have used that $P(t+T) = P(t)$ and identity (30a). Thus, the invertible, time-periodic coordinate transformation $y(t) = P(t)v(t)$ transforms solutions of (31) into solutions of

$$\dot{v}(t) = Bv(t) + K[v(t) - v(t-T)]. \quad (34)$$

Step 2: Next we use the commutativity property (30b) to find a common eigenvector for the unstable eigenvalue of the uncontrolled system and the gain matrix. Since by assumption $Y_0(T) = e^{BT}$ has an eigenvalue $\mu_* > 1$, we can choose B such that B has an eigenvalue $\lambda_* > 0$. Now let $y \in \mathbb{C}^N$ be such that $\lambda_* y - By = 0$. Then

$$(\lambda_* I - B)Ky = K(\lambda_* I - B)y = 0 \quad (35)$$

since, by (30b), B and K commute. Therefore the space

$$\mathcal{N}(\lambda_* I - B) := \{y \in \mathbb{C}^N \mid (\lambda_* I - B)y = 0\} \quad (36)$$

is invariant under K . Hence we can find a non-zero $y_* \in \mathcal{N}(\lambda_* I - B)$ and a $k_* \in \sigma(K)$ such that $Ky_* = k_* y_*$. So we conclude we can find a $y_* \in \mathbb{C}^N \setminus \{0\}$ such that

$$By_* = \lambda_* y_*, \quad Ky_* = k_* y_* \quad (37)$$

i.e., y_* is a simultaneous eigenvector for B and K .

Step 3: We reduce to a 1-dimensional, real valued DDE using the common eigenvector from Step 2. Consider the real-valued, scalar DDE

$$\dot{w}(t) = \lambda_* w(t) + k_* [w(t) - w(t-T)]. \quad (38)$$

Since $\lambda_* > 0$, the ODE

$$\dot{w}(t) = \lambda_* w(t) \quad (39)$$

has one Floquet multiplier $e^{\lambda_* T} > 1$ and no trivial Floquet multiplier. Therefore, Proposition 1 implies that (38) has at least one trivial Floquet multiplier μ larger than 1, i.e. (38) has a solution $w_\mu(t)$ with $w_\mu(t+T) = \mu w_\mu(t)$.

If $w(t) \in \mathbb{R}$ is a solution of (38) and y_* is as in (37), then $v(t) = w(t)y_*$ solves

$$\dot{v}(t) = \lambda_* w(t)y_* + k_* [w(t)y_* - w(t-T)y_*] \quad (40)$$

$$= Bv(t) + K[v(t) - v(t-T)]. \quad (41)$$

Thus, if $w(t)$ solves (38), then $v(t) = w(t)y_*$ solves (34). In particular, $v_\mu(t) := w_\mu(t)y_*$ is a solution of (34) with $v_\mu(t+T) = \mu v_\mu(t)$. This implies that $y_\mu(t) := P(t)v_\mu(t)$ is a solution of (31) with $y_\mu(t+T) = \mu y_\mu(t)$, i.e., $\mu > 1$ is a Floquet multiplier of (31). \square

Remarkably, Proposition 2 does not make any assumptions on the multiplicity of the unstable Floquet multiplier in the uncontrolled system. Therefore Proposition 2 has a wide applicability (see also Corollary 2 below). However, if in Proposition 2 we additionally assume that system (29) has a Floquet multiplier larger than 1 *with odd geometric multiplicity*, we can drop the assumption that K has real spectrum.

Indeed, as in the proof of Proposition 2, the space (36) is invariant under K . Hence, if this space is odd-dimensional,

the real matrix K has at least one real eigenvalue in this space. From there, Step 3 of the proof of Proposition 2 implies instability of the controlled system.

If the gain matrix K satisfies the conditions (30b)–(30a), then in particular $KA(t) = A(t)K$ for all t , i.e. the gain matrix commutes with the linear ODE. Under this weaker assumption, a different proof yields the same any-number limitation, see the upcoming work³¹ by the first author.

As an application of Proposition 2, we consider the case $K = kI$ with $k \in \mathbb{R}$. Then the conditions (30a)–(30b) are trivially satisfied. Therefore Proposition 2 leads to the following corollary:

Corollary 2 (Any-number limitation for scalar gain matrices). *Consider the system (2) satisfying Assumption 1. Suppose that the linearized equation (3) has at least one real Floquet multiplier larger than 1.*

Then, for every $k \in \mathbb{R}$, x_ is unstable as a solution of the controlled system*

$$\dot{x}(t) = f(x(t), t) + k[x(t) - x(t-T)]. \quad (42)$$

D. Corollary: Any-number limitation for commuting gain matrices with complex spectrum

In this section, we address more restrictions on the choice of gain matrix. We again consider commutative gain matrices, but in contrast to the results in Section II C, we do not make any assumptions on the spectrum of the matrix K . The main point here is that the limitation on control similar to the formulation in Corollary 2 still holds, but the reasoning behind the limitation is different: the result in this section is no longer a directly corollary of the odd-number limitation, but requires an explicit analysis of the relevant Floquet multipliers.

Proposition 3. *Consider the linear, non-autonomous system*

$$\dot{y}(t) = A(t)y(t) \quad (43)$$

and assume that there exists $T > 0$ such that $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Assume that system (43) has at least one real Floquet multiplier larger than 1.

Let $P(t)$, B be as in (26a)–(26b) and assume that the gain matrix $K \in \mathbb{R}^{N \times N}$ satisfies

$$KP(t) = P(t)K \quad \text{for all } t \in \mathbb{R}; \quad (44a)$$

$$KB = BK. \quad (44b)$$

Then the system

$$\dot{y}(t) = A(t)y(t) + K[y(t) - y(t-T)] \quad (45)$$

has at least one real Floquet multiplier outside the unit circle.

Proof. We divide the proof into five steps. Since the first two steps are identical to the first two steps in the proof of Proposition 2, we will not repeat them here.

Step 1: Let $y(t)$ be a solution of (45) and let $P(t)$, B be as in (26a)–(26b). Then the periodic coordinate transformation $y(t) = P(t)v(t)$ transforms solutions of (45) into solutions of

$$\dot{v}(t) = Bv(t) + K[v(t) - v(t - T)]. \quad (46)$$

Step 2: Since by assumption $Y_0(T) = e^{BT}$ has an eigenvalue $\mu_* > 1$, we can choose B such that B has an eigenvalue $\lambda_* > 0$. Since by assumption (44b) the matrices B and K commute, the space

$$\mathcal{N}(\lambda_* I - B) := \{y \in \mathbb{C}^N \mid (\lambda_* I - B)y = 0\} \quad (47)$$

is invariant under K . Therefore we can find a non-zero $y_* \in \mathcal{N}(\lambda_* I - B)$ and a (possibly complex!) $k_* \in \mathbb{C}$ such that

$$By_* = \lambda_* y_*, \quad Ky_* = k_* y_* \quad (48)$$

i.e., y_* is a simultaneous eigenvector for B and K .

Step 3: We now consider the reduced, scalar-valued (possibly complex!) DDE

$$\dot{w}(t) = \lambda_* w(t) + k_* [w(t) - w(t - T)]. \quad (49)$$

The DDE (49) has a solution of the form $w(t) = e^{\lambda t}$ if and only if $\lambda \in \mathbb{C}$ satisfies

$$\lambda = \lambda_* + k_* (1 - e^{-\lambda T}). \quad (50)$$

If λ satisfies (50), then $y(t) = P(t)e^{\lambda t} y_*$ is a solution of (45) with $y(t + T) = e^{\lambda T} y(t)$. Therefore, to prove that system (45) has a Floquet multiplier outside the unit circle, it suffices to prove that equation (50) has a solution in the right half of the complex plane. To do so, we distinguish between the case where $k_* \in \mathbb{R}$ (Step 4) and the case where $k_* \in \mathbb{C} \setminus \mathbb{R}$ (Step 5).

Step 4: For $k_* = 0$, equation (49) becomes

$$\dot{w}(t) = \lambda_* w(t) \quad (51)$$

which has one Floquet multiplier $e^{\lambda_* T} > 1$. If we assume that $k_* \in \mathbb{R}$, Proposition 2 on real gain matrices implies that (49) has a Floquet multiplier larger than one for all gains k_* . This proves the theorem in the case that $k_* \in \mathbb{R}$.

Step 5: We now consider the case $k_* \in \mathbb{C} \setminus \mathbb{R}$. If $\lambda \in \mathbb{C}$ satisfies (50), then $\bar{\lambda}$ satisfies

$$\bar{\lambda} = \lambda_* + \bar{k}_* (1 - e^{-\bar{\lambda} T}) \quad (52)$$

and both $e^{\lambda T}$ and $e^{\bar{\lambda} T}$ are Floquet multipliers of (45). Hence, for non-real k_* , non-real solutions of (50) in the right half of the complex plane lead to *two* unstable Floquet multipliers of (45), see also Figure 1.

If λ is a solution of (50) on the imaginary axis, then $e^{\lambda T}$ is a Floquet multiplier of (45) on the unit circle. So to search for stability changes of (45), we search for solutions of (50) on the imaginary axis. Equation (50) has a solution of the form $\lambda = i\omega$ if and only if k_* is on the curve

$$k_*(\omega) = \frac{i\omega - \lambda_*}{1 - e^{-i\omega T}}. \quad (53)$$

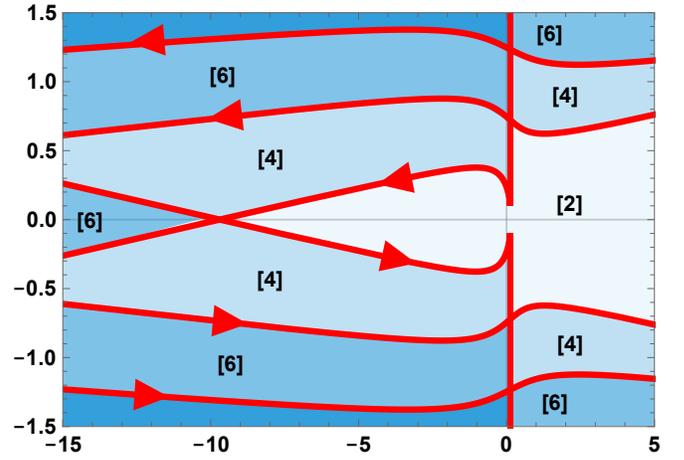


FIG. 1. Hopf curves (red) from eq. (53), horizontal axis: $\text{Re}(k_*)$, vertical axis: $\text{Im}(k_*)$. Increasing ω is indicated by arrows. The unstable dimension is given in brackets and is higher by two to the right of the curves. No control is possible. Parameter values: $\lambda_* = 0.05, T = 2\pi$.

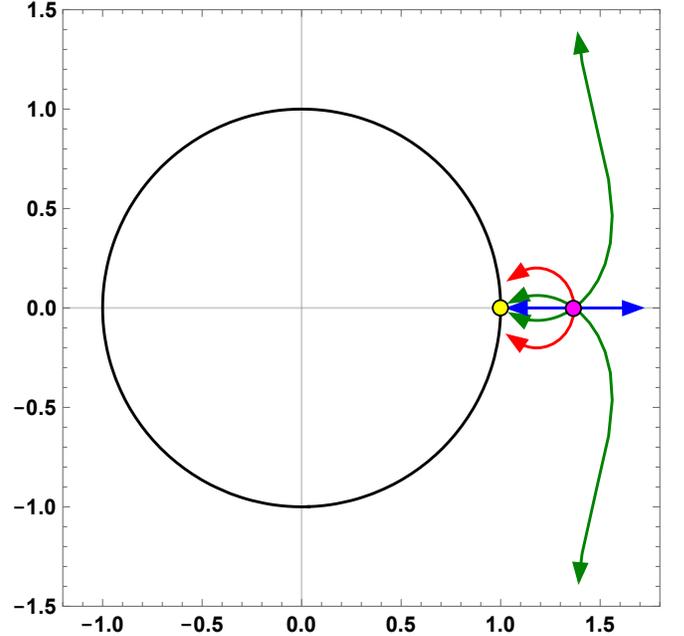


FIG. 2. Unit circle and Floquet multipliers in the complex plane parametrized by different real and complex k_* . Yellow dot: Geometrically invariant determining center. Parameter values: $\lambda_* = 0.05$ (magenta dot), $T = 2\pi$. Blue: real k_* (note the invariance of the real line!), red (dotted): purely imaginary k_* , green: $\arg(k_*) = e^{0.8\pi i}$.

Note how we can write these curves in one single equation but actually obtain a family of curves through the exponential term in the denominator. More precisely, there is one curve for each $\omega \in (2\pi m/T, 2\pi(m+1)/T)$, $m \in \mathbb{Z}$. Each of these curves defines a continuous graph over the real axis: Indeed, the real part is strictly increasing along the curves. Moreover, we remark that (53) is symmetric with respect to the real axis.

It suffices therefore to consider $\omega > 0$, w.l.o.g. Next, notice that the curves, parametrized by ω , are oriented in the direction of decreasing real part of k_* . Moreover, since all the segments are complex differentiable (see also²⁷), complex orientation is preserved and we conclude the number of solutions in the right half plane is higher by 2 to the right of the curve. Therefore, the unstable dimension can only increase if any of the curves (53) are crossed, proving the theorem. \square

Note that in the above proof, in contrast to Proposition 2, the real line does not stay invariant; see Figure 2. In Section III D, we will give an application of Proposition 4 to the stabilization of equilibria.

III. GEOMETRIC INVARIANCE OF RESONATING CENTERS OF EQUILIBRIA

In this section, we focus on Pyragas control of equilibria. The main result concerns invariance of the geometric multiplicity not only of the determining center 0, but also of the resonating centers $\pm 2\pi in/T$ (Theorem 2). From this invariance we deduce several limitations on feedback stabilization. These limitations explicitly depend on the time delay. If the time delay resonates with the eigenvalues, stabilization becomes impossible. We expect these results to be particularly important at (equivariant) Hopf bifurcation, where the time delay T is in resonance with the purely imaginary eigenvalues $\pm 2\pi i/T$ of the equilibrium.

A. Main result concerning equilibria

Throughout this section, consider the autonomous ODE

$$\dot{x}(t) = f(x(t)) \quad (54)$$

with $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ a C^1 -function. We assume that there exists a $x_* \in \mathbb{R}^N$ such that $f(x_*) = 0$, i.e., x_* is an equilibrium of (54).

The linearization

$$\dot{y}(t) = f'(x_*)y(t) \quad (55)$$

has an eigenvalue λ (or, more precisely, the generator of the semigroup associated to (55) has an eigenvalue λ) if and only

$$\det(\lambda I - f'(x_*)) = 0. \quad (56)$$

The geometric multiplicity of the eigenvalue λ is the dimension of the linear space

$$\mathcal{N}(\lambda I - f'(x_*)) = \{x \in \mathbb{C}^N \mid \lambda x - f'(x_*)x = 0\}. \quad (57)$$

and its algebraic multiplicity is given by the order of λ as a zero of the function $z \mapsto \det(zI - f'(x_*))$.

For a fixed time delay $T > 0$, the geometric multiplicity of the eigenvalues $\frac{2\pi ni}{T}$ of (55) plays an important role in the stabilization results in the rest of this section. To facilitate this in notation, and to emphasize the connection to the time

delay, we define the **resonating centers** to be the geometric eigenspace of the eigenvalues $\frac{2\pi ni}{T}$, i.e., the space

$$\mathcal{N}\left(\frac{2\pi ni}{T}I - f'(x_*)\right). \quad (58)$$

For $T > 0$, x_* is again an equilibrium of the controlled system

$$\dot{x}(t) = f(x(t)) + K[x(t) - x(t-T)] \quad (59)$$

with gain matrix $K \in \mathbb{R}^{N \times N}$. Surprisingly, we can find the relevant eigenvalues from a finite-dimensional characteristic equation. Indeed, λ is an eigenvalue of the linearization

$$\dot{y}(t) = f'(x_*)y(t) + K[y(t) - y(t-T)], \quad (60)$$

(or, more precisely, of the generator of the semigroup associated to (60)) if and only if λ satisfies

$$\det(\lambda I - f'(x_*) - K[1 - e^{-\lambda T}]) = 0. \quad (61)$$

The geometric multiplicity of λ equals the dimension of the linear space

$$\mathcal{N}(\lambda I - f'(x_*) - K[1 - e^{-\lambda T}]) \subseteq \mathbb{C}^N \quad (62)$$

and its algebraic multiplicity is the order of λ as a zero of the function

$$z \mapsto \det(zI - f'(x_*) - K[1 - e^{-zT}]); \quad (63)$$

see also¹ (Chapter IV).

In the following theorem, we compare the geometric multiplicity of the resonating eigenvalues of (55) with the geometric multiplicity of the resonating eigenvalues of (60). By convention, if $\lambda \in \mathbb{C}$ is *not* an eigenvalue of (55) (resp. (60)), we say that the geometric multiplicity of the eigenvalue λ of (55) (resp. (60)) is zero.

Theorem 2 (Geometric invariance of resonating centers). *For each $n \in \mathbb{Z}$, the geometric multiplicity of the eigenvalue $\frac{2\pi in}{T}$ is preserved under control of Pyragas type. That is, for $K \in \mathbb{R}^{N \times N}$ and for every $n \in \mathbb{Z}$, the geometric multiplicity of $\frac{2\pi ni}{T}$ as an eigenvalue of the ODE*

$$\dot{y}(t) = f'(x_*)y(t) \quad (64)$$

equals the geometric multiplicity of $\frac{2\pi ni}{T}$ as an eigenvalue of the DDE

$$\dot{y}(t) = f'(x_*)y(t) + K[y(t) - y(t-T)]. \quad (65)$$

Proof. Fix $n \in \mathbb{Z}$. The geometric multiplicity of $\frac{2\pi in}{T}$ as an eigenvalue of (64) is given by

$$\dim \mathcal{N}\left(\frac{2\pi in}{T}I - f'(x_*)\right). \quad (66)$$

Conversely, the geometric multiplicity of $\frac{2\pi in}{T}$ as an eigenvalue of (65) is given by

$$\dim \mathcal{N} \left(\frac{2\pi in}{T} I - f'(x_*) - K \left[1 - e^{-\frac{2\pi in}{T} T} \right] \right) \quad (67)$$

$$= \dim \mathcal{N} \left(\frac{2\pi in}{T} I - f'(x_*) \right). \quad (68)$$

Note that the delay and the resonant eigenvalue cancel in the exponent, causing the contribution of Pyragas control to vanish. We conclude that the geometric multiplicity of $\frac{2\pi in}{T}$ as an eigenvalue of (64) equals the geometric multiplicity of $\frac{2\pi in}{T}$ as an eigenvalue of (65). \square

We briefly reflect on the connection between the two main results on determining and resonating centers. We can interpret the equilibrium x_* of (54) as a periodic solution of arbitrary period $T > 0$. Theorem 2 then implies Theorem 1 for equilibria of autonomous systems, but Theorem 2 on resonating centers is *finer* in the following sense: If we view

$$\dot{y}(t) = f'(x_*)y(t) \quad (69)$$

as a linear, time-periodic equation with trivial time-dependence, its monodromy operator is given by

$$Y_0(T) = e^{f'(x_*)T}. \quad (70)$$

Therefore, $1 \in \sigma(Y_0(T))$, i.e., 1 is a Floquet multiplier, if and only if there is (at least one) $n \in \mathbb{Z}$ such that $\frac{2\pi in}{T}$ as an eigenvalue of (69). In this case the geometric multiplicity of $1 \in \sigma(Y_0(T))$ is given by

$$\sum_{n \in \mathbb{Z}} \dim \mathcal{N} \left(\frac{2\pi in}{T} I - f'(x_*) \right). \quad (71)$$

Note that all the resonating eigenvalues $\frac{2\pi ni}{T}$ for the equilibrium together form the determining Floquet multiplier 1 for the periodic orbit. Theorem 1 only gives information on the geometric multiplicity of the determining Floquet multiplier, thereby losing information on the geometric multiplicity of the individual resonating eigenvalues.

At equivariant bifurcation points, we can expect a higher geometric multiplicity of a resonating center. Note that the invariance principle in Theorem 2 in this case forbids *asymptotic* stabilization.

B. Corollary: Odd-number limitation for equilibria

In this section, we use Theorem 2 to prove limitations for stabilization of equilibria of autonomous systems. Again, the absence of an determining center forbids stabilization. To facilitate this essential point in notation, we say that the equilibrium x_* of the ODE $\dot{x}(t) = f(x(t))$ is *non-degenerate* if 0 is not an eigenvalue of the Jacobian $f'(x_*)$.

The following proposition gives an odd-number limitation for equilibria, analogous to the odd-number limitation for periodic orbits in Corollary 1.

Corollary 3 (Odd-number limitation for equilibria). *Consider the system*

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (72)$$

with $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ a C^1 -function. Suppose that $x_* \in \mathbb{R}^N$ is a non-degenerate equilibrium of (72). Moreover, assume that $f'(x_*)$ has an odd number of eigenvalues (counting algebraic multiplicities) in the strict right half plane.

Then, for all $K \in \mathbb{R}^{N \times N}$ and all $T > 0$, x_* is unstable as a solution of the controlled system

$$\dot{x}(t) = f(x(t)) + K[x(t) - x(t - T)]. \quad (73)$$

Proof. We again start by giving an intuitive argument. Since the matrix $f'(x_*) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is real, its non-real eigenvalues appear in pairs. Therefore, if $f'(x_*)$ has an odd number of eigenvalues in the right half plane, an odd number of these eigenvalues is real. So the assumptions of the statement imply that $f'(x_*)$ has an odd number of eigenvalues on the positive real axis.

Since the gain matrix $K \in \mathbb{R}^{N \times N}$ is real, non-real eigenvalues of the controlled system appear in pairs as well. Therefore, at least one eigenvalue stays on the positive, real axis as control is applied. This eigenvalue can only move into the left half of the complex plane by crossing the point $0 \in \mathbb{C}$. However, this is forbidden, since by Theorem 2 the controlled system never has an eigenvalue 0.

To make the argument more precise, we fix a matrix $K \in \mathbb{R}^{N \times N}$ and a time delay $T > 0$. Moreover, we introduce the homotopy parameter $\alpha \in [0, 1]$:

$$\dot{x}(t) = f(x(t)) + \alpha K[x(t) - x(t - T)]. \quad (74)$$

We first show that the number of eigenvalues with positive real part can only change via a crossing of the imaginary axis: The linearization

$$\dot{y}(t) = f'(x_*)y(t) + \alpha K[y(t) - y(t - T)]. \quad (75)$$

has an eigenvalue $\lambda \in \mathbb{C}$ if and only if λ is a zero of the equation

$$d(\lambda, \alpha) := \det \left(\lambda I - f'(x_*) - \alpha K \left[1 - e^{-\lambda T} \right] \right). \quad (76)$$

Since the function $d(\lambda, \alpha)$ is analytic in λ and continuous in α , the zeros of $\lambda \mapsto d(\lambda, \alpha)$ depend continuously on α (in the sense of¹⁷). Moreover, we can find a $C > 0$ such that

$$\sup \{ \operatorname{Re} \lambda \mid d(\lambda, \alpha) = 0 \} < C \quad (77)$$

for all $\alpha \in [0, 1]$ (see¹ (Section I.4)). Therefore, when varying α , we cannot change the number of zeros of $\lambda \mapsto d(\lambda, \alpha)$ in the strict right half plane by roots “coming from infinity”. Thus the number of roots of $\lambda \mapsto d(\lambda, \alpha)$ in the strict right half plane can only change by a root crossing the imaginary axis. Moreover, if $d(\lambda, \alpha) = 0$, then also $d(\bar{\lambda}, \alpha) = 0$, so non-real roots appear in complex conjugated pairs. In particular, the *parity* of the number of roots with positive real parts can only change if a single eigenvalue crosses zero.

Now, for $\alpha \in [0, 1]$, define

$$n_\alpha = \#\{\lambda \mid d(\lambda, \alpha) = 0 \text{ and } \operatorname{Re} \lambda > 0\} \quad (78)$$

i.e., n_α is the number of roots of $\lambda \mapsto d(\lambda, \alpha)$ in the strict right half of the complex plane. By the previous remarks, the parity of n_α can only change if a root of $\lambda \mapsto d(\lambda, \alpha)$ crosses the point $0 \in \mathbb{C}$. However, by assumption, $0 \notin \sigma(f'(x_*))$; so Theorem 2 implies that

$$d(0, \alpha) \neq 0 \quad (79)$$

for all $\alpha \in [0, 1]$. It follows that the parity of n_α cannot change. Since by assumption, $n_{\alpha=0}$ is odd, we conclude that n_α is odd for all $\alpha \in [0, 1]$. In particular,

$$\dot{y}(t) = f'(x_*)y(t) + K[y(t) - y(t - T)] \quad (80)$$

has an odd number (and thus at least one) of eigenvalues in the right half of the complex plane. This implies that x_* is unstable as a solution of (73). \square

We briefly compare the definition of a non-degenerate periodic orbit with the definition of a non-degenerate equilibrium. We can view the equilibrium x_* of (72) as a periodic solution of arbitrary period $p > 0$. The invariance result of Theorem 1 relies on the fact that the time-delay in the controlled system is the same as the period of the periodic orbit. Therefore, to compare the results in this section with the results on periodic orbits, we view the equilibrium x_* as a periodic orbit of period $p = T$. In this case, the monodromy operator is given by

$$Y_0(T) = e^{f'(x_*)T}. \quad (81)$$

Therefore, x_* is non-degenerate as a periodic orbit of period T if and only if $\frac{2\pi ni}{T}$ is not an eigenvalue of $f'(x_*)$ for every $n \in \mathbb{Z}$. In contrast, for x_* to be non-degenerate as an equilibrium, we only require that 0 is not an eigenvalue of $f'(x_*)$. So the assumption that x_* is non-degenerate as an equilibrium is milder than the assumption that x_* is non-degenerate as a periodic orbit of period $T > 0$.

C. Corollary: Any-number-resonance limitation for commuting gain matrices with real spectrum

The next proposition provides an ‘Any-number limitation’ for equilibria, analogous to the statement in Proposition 2 for time-periodic systems. Here we highlight the case in which the linearization has eigenvalues in the right half plane which are in ‘resonance’ with the delay. In the case of commuting gain matrices with real spectrum, we find that the eigenvalues stay on invariant lines parallel to the real axis.

Corollary 4 (Any-number resonance limitation for equilibria and commuting gain matrices with real spectrum). *Consider the system*

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (82)$$

with $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ a C^1 -function. Let $x_* \in \mathbb{R}^N$ be an equilibrium of (82) and fix a time delay $T > 0$. Suppose that the time

delay is in resonance with one of the unstable eigenvalues, that is, there exist a real $\lambda_* > 0$ and $n \in \mathbb{Z}$ such that

$$\lambda_* + \frac{2\pi ni}{T} \in \sigma(f'(x_*)). \quad (83)$$

Assume that $K \in \mathbb{R}^{N \times N}$ satisfies $\sigma(K) \subseteq \mathbb{R}$ and that the gain matrix commutes with the Jacobian, i.e.,

$$f'(x_*)K = Kf'(x_*). \quad (84)$$

Then x_* is unstable as a solution of the controlled system

$$\dot{x}(t) = f(x(t)) + K[x(t) - x(t - T)]. \quad (85)$$

Proof. We apply Proposition 2 with

$$A(t) := f'(x_*). \quad (86)$$

The fundamental solution of the linear equation

$$\dot{y}(t) = f'(x_*)y(t) \quad (87)$$

is given by

$$Y_0(t) = e^{f'(x_*)t} \quad (88)$$

and hence $B, P(t)$ in (26a)–(26b) are given by

$$B := f'(x_*), \quad P(t) \equiv I. \quad (89)$$

Therefore, $P(t)K = KP(t)$ is trivially satisfied and (84) implies that $KB = BK$. Moreover, (83) implies that the monodromy operator $Y_0(T) = e^{f'(x_*)T}$ has an eigenvalue $e^{\lambda_* T} > 1$. Thus, Proposition 2 implies that

$$\dot{y}(t) = f'(x_*)y(t) + K[y(t) - y(t - T)] \quad (90)$$

has at least one real Floquet multiplier larger than 1 (or, equivalently, at least one eigenvalue in the strict right half plane). Therefore x_* is unstable as a solution of (85). \square

We briefly comment on the connection between Corollary 4 and Proposition 2. In Corollary 4, we assume that the Jacobian $f'(x_*)$ has an eigenvalue $\lambda_* + \frac{2\pi ni}{T}$ with $\lambda_* > 0$. Therefore, if we view x_* as a periodic orbit of period $T > 0$, the monodromy operator

$$Y_0(T) = e^{f'(x_*)T} \quad (91)$$

has a Floquet multiplier $e^{\lambda_* T} > 1$. By Proposition 2, this Floquet multiplier stays on the real line in the controlled system, provided that assumption (84) holds. Therefore, the corresponding eigenvalue should stay on one of the lines

$$\ell_k := \left\{ z \in \mathbb{C} \mid \operatorname{Im} z = \frac{2\pi ki}{T} \right\}. \quad (92)$$

with $k \in \mathbb{Z}$. However, by continuity of the eigenvalues, the eigenvalue cannot ‘jump’ between lines and thus the eigenvalue should stay on the line ℓ_k with $k = n$. So the eigenvalue of the controlled system stays on the same line as the eigenvalue of the uncontrolled system.

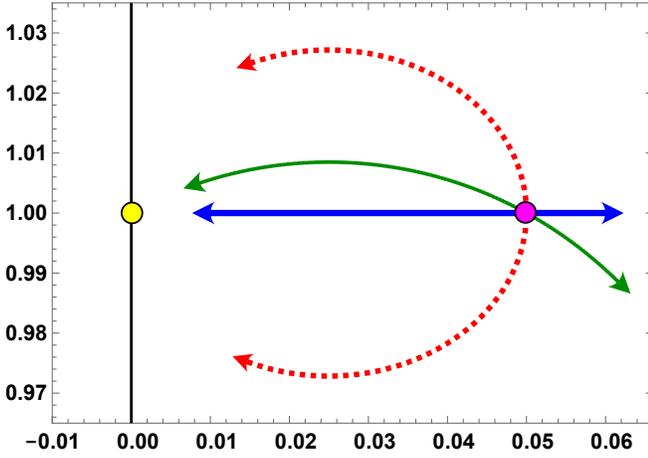


FIG. 3. Imaginary axis and eigenvalues in the complex plane depending of k_* . Yellow dot: Geometrically invariant resonating center at i . Parameter values: $\lambda_* = 0.05$ (magenta dot), $T = 2\pi n = 1$. Blue: real $k_* = \kappa$ (note the invariant line!), red (dotted): purely imaginary k_* , green (thin): $\arg(k_*) = e^{0.8\pi i}$. Note that the picture is seemingly asymmetric, because we have not drawn the complex conjugated resonating center.

D. Corollary: Any-number-resonance limitation on commuting gain matrices with complex spectrum

For the case of commuting gain matrices with *complex* spectrum, we prove an ‘Any-number limitation’ analogous to the statement in Proposition 3 for periodic orbits. The limitation also applies when the uncontrolled system has unstable eigenvalues whose imaginary part is in resonance with the time-delay. See also the result by Hövel and Schöll¹², where the case of an unstable focus was addressed in detail. Note that in contrast to real control gains, but in agreement with Proposition 3 for periodic orbits, the lines (92) are not invariant, see Figure 3.

Corollary 5 (Any-number-resonance limitation for equilibria and commuting gain matrices with complex spectrum). *Consider the system*

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (93)$$

with $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ a C^1 -function. Let $x_* \in \mathbb{R}^N$ be an equilibrium of (93) and fix $T > 0$. Suppose there exists a $\lambda_* > 0$ and $n \in \mathbb{Z}$ such that

$$\lambda_* \pm \frac{2\pi ni}{T} \in \sigma(f'(x_*)). \quad (94)$$

Assume that $K \in \mathbb{R}^{N \times N}$ commutes with the Jacobian, i.e.

$$f'(x_*)K = Kf'(x_*). \quad (95)$$

Then x_* is unstable as a solution of the controlled system

$$\dot{x}(t) = f(x(t)) + K[x(t) - x(t - T)]. \quad (96)$$

Proof. We apply Proposition 3 with

$$A(t) := f'(x_*). \quad (97)$$

The fundamental solution of the linear equation

$$\dot{y}(t) = f'(x_*)y(t) \quad (98)$$

is given by

$$Y_0(t) = e^{f'(x_*)t} \quad (99)$$

and hence $B, P(t)$ in (26a)–(26b) are given by

$$B := f'(x_*), \quad P(t) \equiv I. \quad (100)$$

Therefore, $P(t)K = KP(t)$ is trivially satisfied and (95) implies that $KB = BK$. Moreover, (94) implies that the monodromy operator $X_0(T) = e^{f'(x_*)T}$ has an eigenvalue $e^{\lambda_* T} > 1$. Thus, Proposition 2 implies that

$$\dot{y}(t) = f'(x_*)y(t) + K[y(t) - y(t - T)] \quad (101)$$

has at least one Floquet multiplier outside the unit circle (or, equivalently, at least one eigenvalue in the strict right half plane). Therefore x_* is unstable as a solution of (96). \square

As an application, we consider 2-dimensional systems where both the Jacobian and the control matrix have two complex conjugated eigenvalues. In this case, stabilization is impossible if the time delay is chosen in resonance with the eigenvalues of the Jacobian, see also¹².

Corollary 6. *Consider the system*

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (102)$$

with $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a C^1 -function. Let $x_* \in \mathbb{R}^N$ be an equilibrium of (102). Suppose that the Jacobian $f'(x_*)$ is given by

$$f'(x_*) = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} \quad (103)$$

with $\lambda > 0$ and $\omega > 0$. Moreover, let K be given by

$$K = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad (104)$$

with $\alpha, \beta \in \mathbb{R}$. Then, for $n \in \mathbb{N}$, x_* is unstable as a solution of the controlled system

$$\dot{x}(t) = f(x(t)) + K \left[x(t) - x \left(t - \frac{2\pi n}{\omega} \right) \right]. \quad (105)$$

Proof. The spectrum of $f'(x_*)$ is given by

$$\sigma(f'(x_*)) = \{\lambda \pm i\omega\}. \quad (106)$$

Moreover, with K as in (104), we have that $f'(x_*)K = Kf'(x_*)$. Thus, if we apply Corollary 5 with $T = \frac{2\pi n}{\omega}$ the claim follows. \square

Another application is given by a ring of n coupled Stuart-Landau oscillators, which shows a multitude of ponies-on-a-merry-go-round-solutions. It has been proven²⁷ that only the fully synchronized orbit can be stabilized via Pyragas control.

This is a direct consequence of Corollary 5: The time delay 2π is prescribed by the periodic orbit and in resonance with the imaginary part 1 of all complex eigenvalues near Hopf bifurcation. In the linearization, the system decouples into n complex equations. For all periodic orbits except for the synchronized one, at least one of these equations will have an empty resonating center at i together with an eigenvalue with positive real part²⁷. Therefore, Corollary 5 directly forbids stabilization via Pyragas control, at equivariant Hopf bifurcation. The result in²⁷ shows in addition that this failure of Pyragas control persists close to the Hopf bifurcation point.

IV. CONCLUSION

We proved two fundamental invariance principles of Pyragas control for periodic orbits and equilibria. A number of limitations on Pyragas stabilization of periodic solutions and equilibria follow. Compared to previous literature on this subject^{11,14,20}, our approach provides a new perspective on two crucial points. First of all, in our analysis, we emphasized the geometric rather than the algebraic aspects of the Floquet or eigenvalue problem. Instead of purely calculating algebraic multiplicities, we count the geometric multiplicity, that is, the dimension of the eigenspace. The geometric multiplicity of the determining and resonating centers stay invariant under Pyragas control, while the algebraic number of eigenvalues does not^{3,13}. Moreover, in our formulation of limitations to control, we emphasised properties of the unstable object itself, rather than the properties of the uncontrolled dynamical systems. For controllability, it matters whether there exists an determining or resonating center, but its provenance is irrelevant. These two shifts of perspective provide a clear and unifying understanding of the previously often misinterpreted odd-number limitation, and moreover lead naturally to ‘any-number limitations’ for commutative gain matrices.

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Appendix A: Floquet theory

In Section II C, we have crucially used Floquet theory to transform the linear, time-periodic ODE $\dot{y}(t) = A(t)y(t)$ into a linear, autonomous ODE. For the comfort of the reader, we include a short summary for Floquet theory for ordinary and delay differential equations. A more general treatment for Flo-

quet theory for ODE can, for example, be found in⁸ (Chapter III.7); Floquet theory for DDE is extensively treated in¹ (Chapter XIII) and⁹ (Chapter 8).

1. Floquet theory for ODE

Consider the linear, time-periodic system

$$\dot{y}(t) = A(t)y(t), \quad t \geq 0 \quad (\text{A1})$$

with $y(t) \in \mathbb{R}^N$ and $A(t) \in \mathbb{R}^{N \times N}$. Assume that there exists a $T > 0$ such that $A(t+T) = A(t)$, i.e., the system (A1) is T -periodic. Denote by $Y_0(t) \in \mathbb{R}^{N \times N}$, $t \geq 0$ the fundamental solution of (A1), i.e., $Y_0(t)$ satisfies

$$\begin{cases} \frac{d}{dt}Y_0(t) = A(t)Y_0(t), & t > 0 \\ Y_0(0) = I. \end{cases} \quad (\text{A2})$$

We refer to the operator $Y_0(T) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ as the *monodromy operator* of (A1); we refer to the eigenvalues of $Y_0(T)$ as the *Floquet multipliers* of system (A1).

Since $Y_0(t)$ is invertible for all $t \geq 0$, we in particular have that $0 \notin \sigma(Y_0(T))$. Therefore, there exists a (non-unique!) matrix $B \in \mathbb{C}^{N \times N}$ such that

$$Y_0(T) = e^{BT}. \quad (\text{A3})$$

The following theorem says that there exists a time-periodic transformation that transforms system (A1) into an autonomous, linear system.

Theorem 3 (Floquet). *Consider system (A1) with $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Let $Y_0(t)$ be as in (A2) and let $B \in \mathbb{C}^{N \times N}$ be as in (A3).*

Then the map

$$P(t) := Y_0(t)e^{-Bt} \quad (\text{A4})$$

satisfies $P(t+T) = P(t)$ for all $t \in \mathbb{R}$. Moreover, the coordinate transformation $y(t) = P(t)v(t)$ transforms system (A1) into the system

$$\dot{v}(t) = Bv(t). \quad (\text{A5})$$

Proof. Step 1: We first prove that $P(t)$ as defined in (A4) is T -periodic. Indeed, we have that

$$P(t+T) = Y_0(t+T)e^{-B(t+T)} = Y_0(t+T)e^{-BT}e^{-Bt} \quad (\text{A6})$$

$$= Y_0(t+T)Y_0(T)^{-1}e^{-Bt} \quad (\text{A7})$$

$$= Y_0(t)e^{-Bt} = P(t). \quad (\text{A8})$$

Step 2: We now prove that the transformation $y(t) = P(t)v(t)$ transforms system (A1) into system (A5). Differentiating (A4) gives that

$$\frac{d}{dt}P(t) = A(t)Y_0(t)e^{-Bt} - Y_0(t)e^{-Bt}B \quad (\text{A9})$$

$$= A(t)P(t) - P(t)B. \quad (\text{A10})$$

If $y(t)$ is a solution of (A1), then $y(t) = P(t)v(t)$ satisfies

$$\begin{aligned} A(t)P(t)v(t) &= \left(\frac{d}{dt}P(t) \right) v(t) + P(t) \frac{d}{dt}v(t) & (A11) \\ &= A(t)P(t)v(t) - P(t)Bv(t) + P(t) \frac{d}{dt}v(t). & (A12) \end{aligned}$$

which implies

$$P(t) \frac{d}{dt}v(t) = P(t)Bv(t). \quad (A13)$$

Since $P(t)$ is invertible for all $t \in \mathbb{R}$, this implies that v satisfies (A5). \square

Let y_0 be an eigenvector of $Y_0(T) = e^{BT}$ with eigenvalue μ . Then the decomposition

$$Y_0(t) = P(t)e^{Bt} \quad (A14)$$

implies that $y(t) := P(t)e^{Bt}y_0$ is a solution of (A1) that satisfies $y(t+T) = \mu y(t)$. Vice versa, if $y(t)$ is a solution of (A1), then $y(t) = Y_0(t)y(0)$; so if $y(t+T) = \mu y(t)$, then $y(0)$ is an eigenvector of $Y_0(T) = e^{BT}$ with eigenvalue μ . Hence we conclude:

Lemma 1. *Let $\mu \in \sigma(X(T))$. Then there is a one-to-one correspondence between eigenvectors y_0 of $Y_0(T)$ with eigenvalue μ and solutions of (A1) with $y(t+T) = \mu y(t)$.*

Hence the monodromy operator is given by

$$(Y_1(T)\phi)(\theta) = Y_A(T+\theta)\phi(0) - \int_{-T}^s Y_A(\theta)Y_A(s)^{-1}K\phi(s)ds. \quad (A23)$$

The Arzelà-Ascoli theorem implies that $X(T)$ is a compact operator, and hence its non-zero spectrum consists of isolated eigenvalues of finite multiplicity. \square

2. Floquet theory for DDE

We recall the main results of Floquet theory for DDE, focussing on the DDE

$$\dot{y}(t) = A(t)y(t) + K[y(t) - y(t-T)] \quad (A15)$$

with $T > 0$, $K \in \mathbb{R}^{N \times N}$ and $A(t+T) = A(t)$ for all $t \in \mathbb{R}$.

For a general linear time-periodic, linear DDE, the non-zero spectrum of the monodromy operator consists of eigenvalues of finite multiplicity, see¹ (Chapter VIII). The next lemma proves this for the DDE (A15).

Lemma 2. *Let $Y_1(t), t \geq 0$ be the fundamental solution of system (A15). Then the operator*

$$Y_1(T) : C([-T, 0], \mathbb{R}^n) \rightarrow C([-T, 0], \mathbb{R}^n) \quad (A16)$$

is compact and hence the non-zero spectrum consists of isolated eigenvalues of finite algebraic multiplicity.

Proof. Let $Y_A(t), t \geq 0$ be the fundamental solution of the ODE

$$\dot{y}(t) = A(t)y(t) + Ky(t) \quad (A17)$$

with $Y_A(0) = I$. Fix $\phi \in C([-T, 0], \mathbb{R}^n)$ and consider the initial value problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + K[y(t) - y(t-T)], & t \in [0, T] \\ y(t) = \phi(t), & t \in [-T, 0] \end{cases} \quad (A18)$$

Since $y(t-T) = \phi(t-T)$ for $t \in [0, T]$, Variation of Constants gives that

$$y(t) = Y_A(t)\phi(0) - \int_0^t Y_A(t)Y_A(s)^{-1}K\phi(s-T)ds \quad (A19)$$

or, for $\theta \in [-T, 0]$,

$$y(T+\theta) = Y_A(T+\theta)\phi(0) - \int_0^{T+\theta} Y_A(T+\theta)Y_A(s)^{-1}K\phi(s-T)ds \quad (A20)$$

$$= Y_A(T+\theta)\phi(0) - \int_{-T}^s Y_A(T+\theta)Y_A(T+s)^{-1}K\phi(s)ds \quad (A21)$$

$$= Y_A(T+\theta)\phi(0) - \int_{-T}^s Y_A(\theta)Y_A(s)^{-1}K\phi(s)ds. \quad (A22)$$

For time-periodic ODE, the fundamental solution can be written as the product of a time-periodic and an exponential factor, cf. (A14). For a general DDE, one cannot obtain such a decomposition (see¹ (Chapter VIII, Exercise 4.3). However, a statement similar to Lemma 1 still holds:

Lemma 3. *Let $Y_1(T)$ be the monodromy operator of (A15) and let $\mu \in \sigma_{pt}(X(T))$. Then there is a one-to-one correspondence between eigenfunctions ϕ of $Y_1(T)$ with eigenvalue μ*

and solutions of (A15) with $y(t+T) = \mu y(t)$.

Proof. Let ϕ be an eigenfunction of $Y_1(T)$ associated to the eigenvalue μ . Then $y_t := Y_1(t)\phi$ is a solution of (A15) satisfying $y_T = \mu y_0$ and hence, by uniqueness of solutions $y_{t+T} = \mu y_t$. Vice versa, suppose that $x(t)$ is a solution of (A15), then $x_t = Y_1(t)x_0$. So, if $y(t+T) = y(t)$ for all $t \in \mathbb{R}$, then in particular $y_T = \mu y_0$ and hence $Y_1(T)x_0 = \mu y_0$. \square

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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