Feedback Control of Nonlinear Dissipative Systems by Finite Determining Parameters - A Reaction-diffusion Paradigm

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Abstract

We introduce here a simple finite-dimensional feedback control scheme for stabilizing solutions of infinite-dimensional dissipative evolution equations, such as reaction-diffusion systems, the Navier-Stokes equations and the Kuramoto-Sivashinsky equation. The designed feedback control scheme takes advantage of the fact that such systems possess finite number of determining parameters (degrees of freedom), namely, finite number of determining Fourier modes, determining nodes, and determining interpolants and projections. In particular, the feedback control scheme uses finitely many of such observables and controllers. This observation is of a particular interest since it implies that our approach has far more reaching applications, in particular, in data assimilation. Moreover, we emphasize that our scheme treats all kinds of the determining projections, as well as, the various dissipative equations with one unified approach. However, for the sake of simplicity we demonstrate our approach in this paper to a one-dimensional reaction-diffusion equation paradigm.

Keywords. Reaction-diffusion, Navier-Stokes equations, feedback control, data assimilation, determining modes, determining nodes, determining volume elements.
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1 Introduction

Dissipative dynamical systems, such as the Navier-Stokes equations, the Kuramoto-Sivashinsky equation, the complex Ginzburg-Landau equation and various reaction-diffusion systems are known to have a finite-dimensional asymptotic (in time) behavior (see, e.g., [5], [6], [8], [15], [23], [29], [32], [34], and references therein). This is evident due to the fact that such systems possess finite-dimensional global attractors ([3], [8], [9], [29],[32],[34]), and finite number of determining modes ([17], [16], [15],[27]), determining nodes ([15], [20], [21], [22], [25], [27], [28]), determining volume elements ([22],[26]) and other finite number of determining parameters (degrees of freedom) such as finite elements and other interpolation polynomials ([5],[6],[21].) Moreover, some of these systems, which enjoy the property of separation of spatial scales, are also known to have a finite dimensional inertial manifolds (see, e.g., [8], [9], [18], [19], [34], and references therein). That is, in the presence of separation of spatial scales the long-term dynamics of such a system is equivalent to that of a finite system of ordinary differential equations.

There has been some interesting work on reduction methods, with applications focused on scientific computing and feedback control theory, taking advantage of the finite-dimensional asymptotic behavior of these dissipative dynamical systems (see, e.g., [1],[10],[11], [24], [33] and references therein) . However, there has been very little rigorous analytical work, in particular in the context of feedback control theory, justifying these applications. In the case of separation of spatial scales, and hence the existence of inertial manifolds, the authors of [30] and [31] provide an example of finite-dimensional feedback control (lumped feedback control) that drives the dynamics of one-dimensional reaction-diffusion system to an a priori specified finite-dimensional dynamics. It is worth stressing again that in the case of inertial manifold the dynamics of the underlying evolution equation is equivalent to that of an ordinary different equations to begin with. However, the main challenge is in being able to provide a representation of this ODE system in the relevant parameters dictated by the applications. In [22] and [6] the authors have shown that if a certain dissipative system has separation of scales, and hence an inertial manifold, then such a manifold can be parameterized by any set of adequate parameters, e.g. Fourier modes, nodal values, local volume averages, etc... In the above mentioned work of [30] and [31] the authors employed such an equivalence in the parameterization of the inertial manifolds to show their results.

In this paper we propose a new feedback control for controlling general dissipative evolution equations using any of the determining systems of parameters (modes, nodes, volume elements, etc...) without requiring the presence of separation in spatial scales, i.e. without assuming the existence of an inertial manifold. To fix ideas we demonstrate our idea for a simple reaction diffusion equation, the Chafee-Infante equation, which is the real Ginzburg-Landau equation. It is worth mentioning, however, that this new idea has a far more reaching areas of applications, other than feedback control, such as in data assimilations for weather prediction [2]. In addition, one can use this approach to show that the long time-time dynamics of the underlying dissipative evolution equation, such as the two-dimensional Navier-Stokes equations, can be imbedded in an infinite-dimensional dynamical system that is induced by an ordinary differential equations, named *determining form*, which is governed by a globally Lipschitz vector field, cf. [12], [13] and [14].

In this paper we will use the Chafee-Infante reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha u + u^3 = 0 \tag{1}$$

$$u_x(0) = u_x(L) = 0 \tag{2}$$

for $\alpha > 0$, large enough, as a paradigm to fix ideas and to use the notions of finite number of determining modes, nodes and volume elements to design feedback control to stabilize the $\mathbf{v}(x) \equiv 0$ unstable steady state solution of (1)-(2). Indeed, by linearizing equation (1) about $\mathbf{v} \equiv 0$ one obtains the linear equation

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} - \nu \mathbf{v}_{xx} - \alpha \mathbf{v} &= 0 \\ \mathbf{v}_x(0) = \mathbf{v}_x(L) &= 0 \end{aligned} \tag{3}$$

Solving the equation (1) with initial value $\mathbf{v}_0(x) = A_k \cos(\frac{kx}{L}\pi)$, with $A_k \in \mathrm{I\!R}$, and writting a solution of the form $\mathbf{v}(x,t) = a_k(t) \cos(\frac{k\,x}{L}\pi)$ one obtains

$$\dot{a_k} + \nu a_k\,(\frac{\pi k}{L})^2 - \alpha\, a_k = 0, \tag{4}$$

whose solution is

$$a_k(t) = A_k\, e^{\left(\nu\,(\frac{\pi k}{L})^2 - \alpha\right)t}. \tag{5}$$

Therefore, for $\alpha > 0$, large enough, all the low wave numbers $k^2 < \frac{\alpha\, L^2}{\pi^2 \nu}$ are unstable. Consequently, the dimension of the unstable manifold of $\mathbf{v} \equiv 0$ behaves like $\sqrt{\frac{\alpha\, L^2}{\nu}}$ (see, for instance, [3], [23] and [34] for a similar analysis).

The aim of this paper is to design a feedback control that stabilizes $\mathbf{v} \equiv 0$, for example, either by observing the values of the solutions at certain nodal points, local averages of the solutions in subintervals of $[0, L]$, or by observing finitely many of their Fourier modes. Based on the above discussion, a naive analysis would suggest that one would need about $\sqrt{\frac{L^2 \alpha}{\nu}}$ feedback controllers to stabilize $\mathbf{v} \equiv 0$.

In this paper we will give a rigorous justification to this assertion. First, we demonstrate our result for the case of local averages, which is the most straightforward approach. Later, we present a more general abstract result, that unifies our approach, utilizing all sorts of approximate interpolating polynomials (interpolants), observables and controllers, and show that this abstract approach applies to the Fourier modes, local volume (i. e. local averages) and nodal value as particular examples. It is worth mentioning that the same feedback control scheme can be used to stabilize any other time-dependent solution of (1)-(2). The details of the proof are similar to the ones presented here for stabilizing the zero solution; thus, for the sake of simplicity they will not be provided. Furthermore, similar scheme can be also implemented for feedback control of other nonlinear dissipative dynamical systems, such as the two-dimensional Navier-Stokes equations, the Kuramoto-Sivashinsky equation and reaction-diffusion systems.

2 Finite volume Elements Feedback control

To fix ideas we propose the following feedback control system for (1)-(2) in order to stabilize the steady state solution $\mathbf{v} \equiv 0$,

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha u + u^3 = -\mu \sum_{k=1}^{N} \overline{u}_k\, \chi_{J_k}(x) \tag{6}$$

$$u_x(0) = u_x(L) = 0, \tag{7}$$

where $J_k = \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right]$, for $k = 1, \ldots, N$, $\chi_{J_k}(x)$ is the characteristic function of the interval J_k, and

$$\bar{\varphi}_k = \frac{1}{|J_k|} \int_{J_k} \varphi(x)\ dx = \frac{N}{L} \int_{J_k} \varphi(x)\ dx.$$

Here, the local averages of the solution, $\overline{u}_k$, for $k = 1, ..., N$, are the observables, and they are also used as the feedback controllers in (6). It is easy to observe $\mathbf{v} \equiv 0$ is also a steady state solution for (6)-(7).
For $\varphi \in H^1([0, L])$ we define

$$\|\varphi\|_{H^1}^2 := \frac{1}{L^2} \int_0^L \varphi^2(x)\, dx + \int_0^L \varphi_x^2(x)\, dx. \tag{8}$$

Before showing that (6)-(7) globally stabilizes the steady state $\mathbf{v} \equiv 0$, one has to prove first the global existence and uniqueness of the feedback system (6)-(7). In section 4, we will show in Theorem 4.1 a result concerning global existence and uniqueness for a general family of finite-dimensional feedback control that includes system (6)-(7) as a particular case. Therefore, we will postpone this task of proving the global existence and uniqueness until section 4, and we only show here the global stability of $\mathbf{v} \equiv 0$. This is in order to fix ideas and to demonstrate our general approach.
Next, assuming the global existence and uniqueness of (6)-(7), we will show that every solution u of (6)-(7) tends to zero, as $t \to \infty$, under specific explicit assumptions on N, ν, α, L and μ (see Theorem 2.1 for details). First, we state the following proposition which is needed for our result. It is worth mentioning that similar propositions were introduced and proved in [7], [22], [25], [26] and [27] (see also [30] and [31]). We adapt here similar ideas from [7] for our proof.

Proposition 2.1. *Let $\varphi \in H^1([0, L])$ then*

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k\, \chi_{J_k}(\cdot)\|_{L^2} \;\leq\; h\, \|\varphi_x\|_{L^2} \;\leq\; h\, \|\varphi\|_{H^1}, \tag{9}$$

where $h = \frac{L}{N}$. Moreover,

$$\|\varphi\|_{L^2}^2 \leq \left(\frac{h}{2\,\pi}\right)^2 \left(\gamma^2(\varphi) + \|\varphi_x\|_{L^2}^2\right), \tag{10}$$

where

$$\gamma^2(\varphi) = \sum_{k=1}^{N} \overline{\varphi}_k^2.$$

Proof.

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{J_k}(\cdot)\|_{L^2}^2 = \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{J_k}(x) \right)^2 dx$$
$$= \int_0^L \left(\varphi(x) \sum_{k=1}^{N} \chi_{J_k}(x) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{J_k}(x) \right)^2 dx,$$

where in the last equality we used the fact that $\sum_{k=1}^{N} \chi_{J_k}(x) \equiv 1$, almost everywhere. Therefore,

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{J_k}(\cdot)\|_{L^2}^2 = \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k \,) \, \chi_{J_k}(x)) \right) \left(\sum_{l=1}^{N} (\varphi(x) - \overline{\varphi}_l \,) \, \chi_{J_l}(x) \right) dx$$
$$= \int_0^L \sum_{k,l=1}^{N} (\varphi(x) - \overline{\varphi}_k) \, (\varphi(x) - \overline{\varphi}_l) \, \chi_{J_k}(x) \, \chi_{J_l}(x) dx.$$

Since $\chi_{J_l}(x) \, \chi_{J_k}(x) \equiv \chi_{J_k}(x) \delta_{kl}$, it follows from the above that

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_k(\cdot)\|_{L^2}^2 = \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k) \right)^2 \chi_{J_k}(x) \, dx$$
$$= \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 \, dx. \tag{11}$$

By vertue of Poincaré inequality we have

$$\int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 \, dx \leq \left(\frac{h}{2\pi} \right)^2 \int_{J_k} (\varphi'(x))^2 \, dx. \tag{12}$$

Thus, (11) and (12) imply

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}(\cdot)\|_{L^2}^2 \leq \left(\frac{h}{2\pi} \right)^2 \sum_{k=1}^{N} \int_{J_k} (\varphi'(x))^2 \, dx$$
$$= \left(\frac{h}{2\pi} \right)^2 \int_0^L (\varphi'(x))^2 \, dx, \tag{13}$$

which proves inequality (9) in the Proposition 2.1.
Next, we prove inequaliy (10). From the Poincaré inequality (12) we have

$$\int_{J_k} \varphi^2(x) \, dx - \overline{\varphi}_k^2 \, h \leq \left(\frac{h}{2\pi} \right)^2 \int_{J_k} (\varphi'(x))^2 \, dx. \tag{14}$$

Thus, by summing over $k = 1, \ldots, N$, in the above inequality we conclude inequality (10) of the Proposition 2.1.

Theorem 2.1. *Let N and μ be large enough such that $\mu \geq \nu > \left(\frac{h}{2\pi}\right)^2 max\{\alpha, \mu\}$ where $\alpha > 0$ and $h = \frac{L}{N}$. Then $\|u(t)\|_{L^2}$ tends to zero, as $t \to \infty$, for every solution $u(t)$ of (6)-(7).*

Proof. Taking the L^2 inner product of equation (6) with u, and integrating by part, gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\,\|u_x\|_{L^2}^2 - \alpha\|u\|_{L^2}^2 + \|u\|_{L^4}^4 = -\frac{\mu}{2\pi}\sum_{j=1}^{N}\frac{L}{N}\,\overline{u}_j^2 = -\frac{\mu}{2\pi}\frac{L}{N}\,\gamma^2(u),$$

and a result we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\,\|u_x\|_{L^2}^2 + \mu\left(\frac{h}{2\pi}\right)\gamma^2(u) - \alpha\,\|u\|_{L^2}^2 \leq 0 \tag{15}$$

Using (10), from Proposition 2.1, and the assumption $\mu\,h \geq \nu$ we have

$$\nu\,\|u_x\|_{L^2}^2 + \mu\left(\frac{h}{2\pi}\right)\gamma^2(u) = \nu\,(\|u_x\|_{L^2}^2 + \gamma^2(u)) + (\mu\left(\frac{h}{2\pi}\right) - \nu)\,\gamma^2(u) \geq \frac{4\pi^2\,\nu}{h^2}\,\|u\|_{L^2}^2. \tag{16}$$

Substituting (16) in (15) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (\frac{\nu\,4\,\pi^2}{h^2} - \alpha)\,\|u\|_{L^2}^2 \leq 0.$$

Therefore, by virtue of Gronwall's inequality and the assumption that $\nu > \alpha\,\frac{h^2}{4\,\pi^2}$ one obtains

$$\|u(t)\|_{L^2}^2 \,\leq\, e^{-(\nu(\frac{2\,\pi\,N}{L})^2 - \alpha)\,t}\|u(0)\|_{L^2}^2;$$

and the Theorem follows.

Remark 2.1

It is worth mentioning that if we choose $\mu = O(\alpha)$ then the assumption $N > \sqrt{\frac{L^2\,\alpha}{\nu}}$ in Theorem 2.1 is consistant with the fact that the dimension of the unstable manifold about $\mathbf{v} \equiv 0$ is of order of $\sqrt{\frac{L^2\,\alpha}{\nu}}$. Moreover, in Theorem 5.1 we give a different and more general proof that illustrate this point further.

3 Approximate interpolant feedback controllers

In this section we will consider a general linear map $I_h : H^1([0, L]) \to L^2([0, L])$ which is an approximate interpolant of order h of the inclusion map $i : H^1 \hookrightarrow L^2$, that satisfies the estimate

$$\|\varphi - I_h(\varphi)\|_{L^2} \leq c\,h\,\|\varphi\|_{H^1}, \tag{17}$$

for every $\varphi \in H^1([0, L])$. The last inequality is a version of the well-known Bramble-Hilbert inequality, that usually appears in the context of finite elements

[4]. We propose here to consider the following general feedback system of the form

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha\, u + u^3 = -\mu\, I_h(u) \tag{18}$$

$$u_x(0) = u_x(L) = 0 \tag{19}$$

in order to stablize $\mathbf{v} \equiv 0$. Here one can think of $I_h(u)$ as the observables and controllers that will be used to stabilize our system.
Before we state and prove our general theorems concerning system (18)-(19), we will give some examples of the approximate interpolant $I_h(\varphi)$ which satisfy the approximation property (17). In particular, we are interested in approximate interpolant I_h of finite rank, and whose rank is of the order $O(1/h)$.

3.1 Examples of finite rank approximate interpolants

3.1.1 Finite volume elements

Using the notation of section 2 we consider the approximate interpolant

$$I_h(\varphi) = \sum_{j=1}^{N} \bar{\varphi}_k\, \chi_{J_k}(x), \tag{20}$$

that uses local spatial averages (finite volume elements) for approximating the local values of the underlying function. We observe that the approximate interpolant $I_h(\varphi)$ that is introduced in (20) and implemented in (18) is exactly the same one discussed in detail in section 2. In particular, one can easily see that approximating inequality (17) holds in this case, thanks to Proposition 2.1.

3.1.2 Approximate Interpolants based on nodal values

In this example we consider the approximate interpolant

$$I_h(\varphi) = \sum_{k=1}^{N} \varphi(x_k)\, \chi_{J_k}(x), \tag{21}$$

where J_k and χ_{J_k} are as in section 2, and the points $x_k \in J_k$, for $k = 1, 2, \cdots, N$ are arbitrary. Next, we show that approximate interpolant given in (21) satisfied the approximation property (17). Here again we adopt ideas from [7] to prove the next proposition.

Proposition 3.1. *For every* $\varphi \in H^1([0, L])$

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(\cdot)\|_{L^2} \leq h\, \|\varphi_x\|_{L^2} \leq h\, \|\varphi\|_{H^1}.$$

Proof.

$$\|\varphi(.) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(.)\|_{L^2}^2 = \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(x)\right)^2 dx,$$

and since $\sum_{k=1}^{N} \chi_k(x) \equiv 1$ a.e., it follows that

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \varphi(x_k)\,\chi_{J_k}(\cdot)\|_{L^2}^2 = \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \varphi(x_k))\,\chi_{J_k}(x) \right)^2 dx.$$

As in the proof of the Proposition 2.1, we observe that $\chi_{J_k}(x)\,\chi_{J_l}(x) \equiv \chi_{J_k}(x)\,\delta_{kl}$ and then we obtain

$$\begin{aligned}
\|\varphi(\cdot) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(\cdot)\|_{L^2}^2 &= \int_0^L \sum_{k=1}^{N} (\varphi(x) - \varphi(x_k))^2\,\chi_{J_k}(x)\,dx \\
&= \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \varphi(x_k))^2\,dx \\
&= \sum_{k=1}^{N} \int_{J_k} \left(\int_{x_k}^{x} \varphi'(y)\,dy \right)^2 dx \\
&\leq \sum_{k=1}^{N} \int_{J_k} \left(\int_{J_k} |\varphi'(y)|\,dy \right)^2 dx \\
&\leq h \sum_{k=1}^{N} \left(\int_{J_k} |\varphi'(y)|\,dy \right)^2 dx.
\end{aligned} \tag{22}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|\varphi(.) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(.)\|_{L^2}^2 &\leq \sum_{k=1}^{N} h^2 \int_{J_k} |\varphi'(y)|^2\,dy, \\
&= h^2 \|\varphi_x\|_{L^2}^2;
\end{aligned} \tag{23}$$

which concludes the proof of Proposition 3.1.
In view of (1), (18) and (21) we propose the following feedback controller for stabilizing $\mathbf{v} \equiv 0$

$$\frac{\partial u}{\partial t} - \nu\,u_{xx} - \alpha u + u^3 = -\mu \sum_{k=1}^{N} u(x_k)\,\chi_{J_k}(x) \tag{24}$$

$$u_x(0) = u_x(L) = 0, \tag{25}$$

which is a special case of (18).

3.1.3 Projection onto Fourier modes as an approximate interpolant

Here, we consider the following projection onto the first N Fourier modes as an example of an approximate interpolant;

$$I_h(\varphi) = \sum_{k=1}^{N} \hat{\varphi}_k\,\cos(\frac{k\,\pi x}{L}),\; h = \frac{L}{N}, \tag{26}$$

where the Fourier coefficients are given by

$$\hat{\varphi}_k = \frac{2}{L}\int_0^L \varphi(x)\,\cos(\frac{\pi k x}{L})dx.$$

Next, we observe that inequality (17) holds for the approximate interpolant given in (26).

Proposition 3.2. *Let $\varphi \in H^1([-L,L])$ be an even function, i.e. $\varphi(-x)=\varphi(x)$. Then*

$$\|\varphi(x) - \sum_{k=1}^{N} \widehat{\varphi}_k\,\cos\left(\frac{k\,x\pi}{L}\right)\|_{L^2([0,L])} \le c\,h\,\|\varphi_x\|_{L^2([0,L])}. \qquad (27)$$

Proof. The proof of this proposition is a simple exercise in Fourier series. Thus it will be omitted.

4 Global existence and uniqueness for the closed-loop system

In this section we establish the global existence and uniqueness for the general feedback system introduced in (18)-(19). This will be accomplished under the assumption (17) and the condition

$$\nu \ge \mu\,c^2\,h^2. \qquad (28)$$

To this end one uses the standard Galerkin approximation procedure based on the eigenfunctions of the Laplacian, subject to the Neumann boundary condition, i.e., $\cos(\frac{\pi\,k\,x}{L})$ for $k = 1, 2....$ We will omite the details of this standard procedure and provide only the formal *a-priori* estimates (see, e.g., [34]). These estimates can be obtained rigorously through the Galerkin procedure, by passing to the limit while using the relevant compactness theorems.
Let us now establish the aformentioned formal *a-priori* bounds for the solution which are essential for guaranteeing globlal existence and uniqueness.
System (18)-(19) can be rewritten as

$$\frac{\partial u}{\partial t} - \nu\,u_{xx} + \frac{\nu}{L^2}u - (\alpha + \frac{\nu}{L^2})\,u = -u^3 - \mu I_h(u) \qquad (29)$$

$$u_x(0) = u_x(L) = 0. \qquad (30)$$

Taking the L^2- inner product of (29) with u, integrating by parts and using the Neumann boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx \quad + \quad \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx = -\int_0^L u^4 dx$$
$$+(\alpha + \frac{\nu}{L^2})\int_0^L u^2\,dx - \mu\int_0^L I_h(u)\,u\,dx.$$

Writing

$$I_h(u)\,u = I_h(u)\;(u - I_h(u)) + (I_h(u))^2$$

and applying the Cauchy-Schwarz and Young's inequalities, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\frac{1}{2}\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})^2\frac{L}{2}$$

$$-\mu\int_0^L |I_h(u)|^2\,dx + \mu\left(\int_0^L |I_h(u)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^L |u - I_h(u)|^2\,dx\right)^{\frac{1}{2}}.$$

Using the Young inequality once again we reach

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx &+ \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\frac{1}{2}\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})^2\frac{L}{2}\\ &- \frac{\mu}{2}\int_0^L |I_h(u)|^2\,dx + \frac{\mu}{2}\|u - I_h(u)\|_{L^2}^2\,dx.\end{aligned}$$

Using (17), and the definition of the H^1-norm given in (8) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\frac{1}{2}\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})^2\frac{L}{2}$$

$$-\frac{\mu}{2}\int_0^L |I_h(u)|^2\,dx + \mu\,\frac{c^2\,h^2}{2}\left(\frac{1}{L^2}\int_0^L u^2\,dx + \int_0^L u_x^2\,dx\right).$$

Thanks to the assumption (28) we conclude

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \frac{\nu}{2}\left(\int_0^L u_x^2\,dx + \frac{1}{L^2}\int_0^L u^2\,dx\right) \le (\alpha + \frac{\nu}{L^2})^2\frac{L}{2}$$

$$-\frac{1}{2}\int_0^L u^4 dx - \frac{\mu}{2}\int_0^L |I_h(u)|^2\,dx.$$

Thus, one has

$$\frac{d}{dt}\,\|u\|_{L^2}^2 + \nu\,(\|u_x\|_{L^2}^2 + \frac{1}{L^2}\,\|u\|_{L^2}^2)\ \le (\alpha + \frac{\nu}{L^2})^2\,L. \tag{31}$$

Therefore, by dropping the $\|u_x\|_{L^2}^2$ term from the left-hand side of (31) and applying Gronwall's inequality we have

$$\|u(t)\|_{L^2}^2 \le e^{\frac{-\nu\,t}{L^2}}\,\|u(0)\|_{L^2}^2 + (\alpha + \frac{\nu}{L^2})^2\,\frac{L^3}{\nu}\left(1 - e^{\frac{-\nu\,t}{L^2}}\right) =: K_0(t). \tag{32}$$

Observe that by integrating equation (31) over the interval $[t, t+1]$, for $t \ge 0$, and using (32) one has

$$\begin{aligned}\|u(t+1)\|_{L^2}^2 + \nu\int_t^{t+1}(\|u_x(s)\|_{L^2}^2 + \frac{1}{L^2}\,\|u(s)\|_{L^2}^2)\quad ds\ &\le (\alpha + \frac{\nu}{L^2})^2\,L + \|u(t)\|_{L^2}^2\\ &\le\ \ (\alpha + \frac{\nu}{L^2})^2\,L + K_0(t).\end{aligned} \tag{33}$$

Next, we find estimate for $\|u_x\|^2$. Multiplying equation (18) by $-u_{xx}$ and integrating by parts, using the Neumann boundary conditions (19), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2\,dx+\nu\int_0^L u_{xx}^2\,dx &= \alpha\int_0^L u_x^2\,dx+\int_0^L u^3\,u_{xx}dx+\mu\int_0^L I_h(u)\,u_{xx}\,dx\\ &= \alpha\int_0^L u_x^2dx-3\int_0^L u^2\,u_x^2\,dx+\mu\int_0^L u\,u_{xx}dx\\ &\quad+\mu\int_0^L (I_h(u)-u)\,u_{xx}\,dx.\end{aligned}$$

Integrating by part, using (19), and using the Cauchy-Schwarz and Young inequalities we reach

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2\,dx+\nu\int_0^L u_{xx}^2\,dx &\le \alpha\int_0^L u_x^2dx-3\int_0^L u^2\,u_x^2\,dx-\mu\int_0^L u_x^2dx\\ &+\ \frac{\mu^2}{2\nu}\|u-I_h(u)\|_{L^2}^2+\frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2.\end{aligned}$$

By the assumption (17) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2\,dx+\frac{\nu}{2}\int_0^L u_{xx}^2\,dx &\le \alpha\int_0^L u_x^2dx-3\int_0^L u^2\,u_x^2\,dx-\mu\int_0^L u_x^2dx\\ &+\frac{\mu^2}{2\nu}\,c^2\,h^2\|u\|_{H^1}^2.\end{aligned}$$

Dropping the negative terms in the right-hand side and using the definition of H^1-norm given in (8), will lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2\,dx+\frac{\nu}{2}\int_0^L u_{xx}^2\,dx &\le \alpha\,\|u_x\|_{L^2}^2+\frac{\mu^2\,c^2\,h^2}{2\,\nu}\left(\frac{1}{L^2}\|u\|_{L^2}^2+\|u_x\|_{L^2}^2\right)\\ &\le (\alpha+\frac{\mu^2\,c^2\,h^2}{2\,\nu})\left(\frac{1}{L^2}\|u\|_{L^2}^2+\|u_x\|_{L^2}^2\right). \quad (34)\end{aligned}$$

Therefore, from the above and (34) we have

$$\frac{d}{dt}\|u_x\|_{L^2}^2\le 2\,(\alpha+\frac{\mu^2\,c^2\,h^2}{2\,\nu})\left(\frac{1}{L^2}\,\|u\|_{L^2}^2+\|u_x\|_{L^2}^2\right). \quad (35)$$

Since $K_0(t)$, which is given in (32), is a bounded uniformly, for all $t\ge 0$, then by virtue of Gronwall's inequality one concludes from (35) that $\|u_x(t)\|_{L^2}^2$ remains bounded on every finite interval $[0,T]$ provided $u(0)\in H^1$. However, this estimate will grow exponentially in time. Our immediate goal will be to provide a uniform bound for $\|u_x(t)\|_{L^2}^2$, for all $t\ge 0$.

From the above discussion, following equation (35), we know that one has a uniform bound over the time interval $[0,1]$. Therefore, we will be focusing on the time interval $[1,\infty)$. Now, let $t+1\ge s\ge t\ge 0$. We integrate (35) over the interval $[s,t+1]$, and use (33) to obtain

$$\|u_x(t+1)\|_{L^2}^2\le\|u_x(s)\|_{L^2}^2+\frac{2}{\nu}(\alpha+\frac{\mu^2\,c^2\,h^2}{2\,\nu})\ \left[(\alpha+\frac{\nu}{L^2})\,L+K_0(t)\right].$$

Next, we integrate the above inequality with respect to s over the interval $[t, t{+}1]$, and use (33), to conclude

$$\|u_x(t+1)\|_{L^2}^2 \le \frac{1}{\nu}\left[(\alpha+\frac{\nu}{L^2})\,L+K_0(t)\right]\left[1+2(\alpha+\frac{\mu^2\,c^2\,h^2}{2\,\nu})\right], \tag{36}$$

for all $t\ge 0$. Therefore the H^1-norm of u is bounded uniformly for all $t \ge 0$. Next, we show the continuous dependence of the solutions of (18) on the initial data and the uniqueness, provided the assumptions (17) and (28) hold. Indeed, let u_1, u_2 be two solutions and $w = u_1 - u_2$ of (18). From (18) we find that

$$\frac{\partial w}{\partial t} - \nu w_{xx} - \alpha\, w = u_2^3 - u_1^3 - \mu\, I_h(w).$$

Multiplying by w and integrating with respect to x over $[0, L]$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &= \alpha\int_0^L w^2\,dx - \int_0^L w^2\,\frac{(u_1+u_2)^2+u_1^2+u_2^2}{2}\,dx\\ &\quad -\mu\int_0^L I_h(w)\,w\;dx\\ &\le (\alpha-\mu)\int_0^L w^2\,dx + \mu\int_0^L |I_h(w)-w|\,|w|\,dx.\end{aligned}$$

A straightforward computation, using the Cauchy-Schwarz and Young inequalities and assumption (17), yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &\le (\alpha-\mu)\int_0^L w^2\,dx + \mu\,\|I_h(w)-w\|_{L^2}\,\|w\|_{L^2}\\ &\le (\alpha-\mu)\,\|w\|_{L^2}^2\,dx + \frac{\mu}{2}\,\|I_h(w)-w\|_{L^2}^2 + \frac{\mu}{2}\|w\|_{L^2}^2\\ &\le (\alpha-\frac{\mu}{2})\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\|w\|_{H^1}^2.\end{aligned}$$

Using (8), the definition of the H^1-norm, we reach

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &\le \left(\alpha-\frac{\mu}{2}\right)\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\left(\frac{\|w\|_{L^2}^2}{L^2}+\|w_x\|_{L^2}^2\right)\\ &\le \left(\alpha-\frac{\mu}{2}+\frac{\mu}{2}\,\frac{c^2\,h^2}{L^2}\right)\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\,\|w_x\|_{L^2}^2.\end{aligned}$$

By assumption (28) the above implies

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{\nu}{2}\,\|w_x\|_{L^2}^2 \le \left(\alpha-\frac{\mu}{2}+\frac{\nu\,c^2\,h^2}{2L^2}\right)\,\|w\|_{L^2}^2.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{\nu}{2}\,\|w_x\|_{L^2}^2 \le \left(\alpha-\frac{\mu}{2}+\frac{\nu\,c^2\,h^2}{2L^2}\right)\,\|w\|_{L^2}^2 \;=:\beta\,\|w\|_{L^2}^2,$$

and by Gronwall's inequality we have

$$\|w(t)\|_{L^2}^2 \le e^{\beta\, t}\,\|w(0)\|_{L^2}^2. \tag{37}$$

Thus, if $w(0) = 0$ then $\|w(t)\|_{L^2} \equiv 0$. Moreover, inequaliy (37) implies the continuous dependence of the solutions of (18)-(19) on the initial data. In conclusion, from the above, and in particular thanks to (32) and (36), we have the following theorem:

Theorem 4.1. *Let μ, ν and h be positive parameters satisfying assumption (28); and that I_h satisfies (17). Suppose $T > 0$ and $u_0 \in H^1([0,L])$, then system (18)-(19) has a unique solution $u \in C([0,T], H^1) \cap L^2([0,T], H^2)$ which also depends continuously on the initial data. Moreover,*

$$\limsup_{t\to\infty} \|u(t)\|_{L^2}^2 \le R_0^2 := (\alpha + \frac{\nu}{L^2})^2 \, \frac{L^3}{\nu},$$

and

$$\limsup_{t\to\infty} \|u_x(t)\|_{L^2}^2 \le R_1^2 := \frac{1}{\nu} \left[(\alpha + \frac{\nu}{L^2}^2) \, L + R_0^2 \right] \left[1 + 2(\alpha + \frac{\mu^2 \, c^2 \, h^2}{2\,\nu}) \right].$$

5 Stabilization using $I_h(u)$ as a feedback control

In the previous section we have established the global existence and uniqueness of the general feedback system (18)-(19), under the assumptions (17) and (28). In addition, we gave in section 3 specific examples for the approxiamate interpolant I_h satisfying (17). Now, we are ready to state and prove our main result concerning the general stabilizing feedback system (18)-(19), under the assumptions (17), (28) and (38) below.

Theorem 5.1. *Let $I_h : H^1([0,L]) \to L^2([0,L])$ be a linear map, which is an approximate interpolant of order h of the inclusion map $i : H^1 \hookrightarrow L^2$, that satisfies the approximation inequaliy (17). Moreover, assume that μ is large enough such that*

$$r := \mu - (2\,\alpha + \frac{\nu}{L^2}) > 0 \tag{38}$$

and that h is small enough such that (28) is satisfied.
Then, for every $u_0 \in H^1$, the global unique solution of (18)-(19) decays exponentially to zero as described in (41), below.

Proof. We present here a slightly different proof than the one we gave for Theorem 2.1; taking into account the general form of the approximation property (17). However, we emphasize that the main idea is similar. This proof will also require simpler explicit choices for the parameters $\mu > 0$ and h.
First, let us observe that assumption (28) guarantees the global existence and uniqueness of the solutions to system (18)-(19), because of Theorem 4.1.
We take the L^2 inner product of (18) with u to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx \quad &+ \quad \nu \int_0^L u_x^2\,dx - \alpha \int_0^L u^2\,dx + \int_0^L u^4 dx \\ &= \quad -\mu \int_0^L I_h(u)\,u\,dx. \end{aligned} \tag{39}$$

From property (17), and the definition of the H^1-norm, given in (8), we have

$$-2\int_0^L I_h(u)\,u\,dx \le c^2\,h^2\left(\|u_x\|_{L^2}^2+\frac{1}{L^2}\,\|u\|_{L^2}^2\right)-\|u\|_{L^2}^2-\|I_h(u)\|_{L^2}^2. \quad (40)$$

Substituting (40) into (39) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &+ \nu\,\|u_x\|_{L^2}^2-\alpha\,\|u\|_{L^2}^2+\int_0^L u^4dx+\frac{\mu}{2}\,\|u\|_{L^2}^2\\ &+ \frac{\mu}{2}\|I_h(u)\|_{L^2}^2-\frac{\mu}{2}\,c^2h^2\left(\|u_x\|_{L^2}^2+\frac{1}{L^2}\|u\|_{L^2}^2\right)\le 0.\end{aligned}$$

Thanks to assumption (28) the above implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\frac{\nu}{2}\,\|u_x\|_{L^2}^2 &+ \frac{\mu}{2}\|I_h(u)\|_{L^2}^2+\|u\|_{L^4}^4\\ &+ \left(\frac{\mu}{2}-\alpha-\frac{\mu\,c^2\,h^2}{2\,L^2}\right)\,\|u\|_{L^2}^2\le 0.\end{aligned}$$

By assumption (28) we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\frac{\nu}{2}\,\|u_x\|_{L^2}^2+\frac{\mu}{2}\|I_h(u)\|_{L^2}^2+\|u\|_{L^4}^4+\frac{r}{2}\,\|u\|_{L^2}^2\le 0.$$

Thanks to assumption (38) and Gronwall's inequality we have

$$\|u(t)\|_{L^2}^2\le e^{-r\,t}\,\|u(0)\|_{L^2}^2. \quad (41)$$

Remark 5.1

Let us observe that in order to satisfy assumption (38) one can choose $\mu = O(\alpha)$. As a result, asssumption (28) will hold if we choose $N := \frac{L}{h} = O(\sqrt{\frac{\alpha\,L^2}{\nu}})$, that is the number of feedback controllers is comparable to the dimension of the unstable manifold about $\mathbf{v}\equiv 0$. This is consistent with our earlier observation in the inroduction and in Remark 2.1.

6 Stabilizing in the H^1-norm

In the previous section we have shown that the feedback system (18)-(19) stabilzes the steady state solution $\mathbf{v}\equiv 0$ in the L^2-norm, i.e., $\|u\|_{L^2}\to 0$, as $t\to\infty$, provided assumptions (28) and (38) hold.
Next, we show that we also have $\|u(t)\|_{H^1}\to 0$, as $t\to\infty$. To this end it is enough to show that $\|u_x\|_{L^2}\to 0$, as $t\to\infty$.
Let us rewrite (18)-(19) as

$$u_t+\frac{1}{L^2}\,u-\nu\,u_{xx}-(\alpha+\frac{1}{L^2})\,u+u^3=-\mu\,I_h(u) \quad (42)$$

$$u_x(0)=u_x(L)=0; \quad (43)$$

We take the L^2 inner product of (42) with $-u_{xx}$. Notice that from the proof of global existence and uniqueness of Theorem 4.1, in particular from the uniform boundedness of the H^1-norm and equation (34) we can show that $u \in L^2\left([0,T],H^2\right)$, and consequently $\frac{\partial u}{\partial t} \in L^2\left([0,T],L^2\right)$. Therefore, the above mentioned inner product, between $\frac{\partial u}{\partial t}$ and $-u_{xx}$ makes sense rigorously. Integrating by parts, and using the Neumann boundary conditions (19) we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \nu\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\,\|u_x\|_{L^2}^2 &- (\alpha + \frac{1}{L^2})\|u_x\|^2 \\
&= \int_0^L u^3\,u_{xx}\,dx + \mu\int_0^L I_h(u)\,u_{xx}\,dx \\
&= -3\int_0^L u^2\,u_x^2\,dx + \mu\int_0^L (I_h(u)-u)\,u_{xx}\,dx \\
&+ \mu\int_0^L u\,u_{xx}\,dx \\
&= -3\int_0^L u^2\,u_x^2\,dx + \mu\int_0^L (I_h(u)-u)\,u_{xx}\,dx \\
&- \mu\int_0^L u_x^2\,dx.
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \nu\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha + \frac{1}{L^2})\,\|u_x\|_{L^2}^2 - \mu\,\|u_x\|_{L^2}^2 \\
&+ \mu\,\|I_h(u)-u)\|_{L^2}\,\|u_{xx}\|_{L^2}.
\end{aligned}$$

Applying Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2+\nu\,\|u_{xx}\|_{L^2}^2+\frac{1}{L^2}\|u_x\|_{L^2}^2 \le (\alpha+\frac{1}{L^2}-\mu)\|u_x\|^2+\frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2+\frac{\mu^2}{2\,\nu}\,\|I_h(u)-u)\|_{L^2}^2.$$

Using property (17) and the definition of the H^1-norm in (8), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx + \frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha + \frac{1}{L^2} - \mu)\|u_x\|_{L^2}^2 \\
&+ \frac{h^2\,\mu^2\,c^2}{2\,\nu}\left(\frac{1}{L^2}\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2\right).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx + \frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha + \frac{1}{L^2} + \frac{h^2\,\mu^2\,c^2}{2\,\nu} - \mu)\|u_x\|_{L^2}^2 \\
&+ \frac{h^2\,\mu^2\,c^2}{2\,\nu\,L^2}\,\|u\|_{L^2}^2.
\end{aligned}$$

Notice that $\|u_x\|_{L^2}^2 = -\int_0^L u\,u_{xx}dx \le \|u\|_{L^2}\,\|u_{xx}\|_{L^2}$. Thus from above we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 &\le \left|\alpha + \frac{1}{L^2} + \frac{h^2\,\mu^2\,c^2}{2\,\nu} - \mu\right|\,\|u_{xx}\|_{L^2}\,\|u\|_{L^2} \\
&+ \frac{h^2\,\mu^2\,c^2}{2\,\nu\,L^2}\,\|u\|_{L^2}^2.
\end{aligned}$$

By Young's inequality we have

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx + \frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 \;\leq\; \left((\alpha + \frac{1}{L^2} + \frac{h^2\,\mu^2\,c^2}{2\,\nu} - \mu)^2\frac{1}{\nu} + \frac{h^2\,\mu^2\,c^2}{2\,\nu\,L^2}\right)\|u\|_{L^2}^2 + \frac{\nu}{4}\|u_{xx}\|^2,$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx + \frac{\nu}{4}\,\|u_{xx}\|_{L^2}^2 + \frac{1}{L^2}\|u_x\|_{L^2}^2 \;\leq\; \gamma\,\|u\|_{L^2}^2,$$

where $\gamma = \left((\alpha + \frac{1}{L^2} + \frac{h^2\,\mu^2\,c^2}{2\,\nu} - \mu)^2\frac{1}{\nu} + \frac{h^2\,\mu^2\,c^2}{2\,\nu\,L^2}\right)$. Consequently, we have

$$\frac{d}{dt}\|u_x\|_{L^2}^2\,dx + \frac{2}{L^2}\,\|u_x\|_{L^2}^2 \;\leq\; \gamma\,\|u(t)\|_{L^2}^2.$$

Since

$$\lim_{t\to\infty}\|u(t)\|^2 = 0,$$

then, by Gronwall's inequality it is easy to show that $\|u_x\|_{L^2}^2 \to 0$, as $t\to\infty$ (see, also special Gronwall's type Lemma in [25]). Notice that the sign of γ does not matter in the above argument.

7 Nodal observables and feedback controllers

In this section we propose a different feedback control based on nodal value observables and feedback controllers. Assume that the observables are the values of the solutions $u(\overline{x}_k)$, at the points $\overline{x}_k \in J_k = [(k-1)\,\frac{L}{N}, k\,\frac{L}{N}]$, $k = 1, ..., N$, and that the feedback is at some points $x_k \in J_k$, x_k is not necessarily the same as $\overline{x}_k$. That is the measurements are made at $\overline{x}_k$, while the feedback controllers are at x_k, for $k = 1, 2..., N$. To avoid technical issues that are dealing with boundary conditions, we focus here on the periodic boundary condition case. In this case the feedback system will read

$$\frac{\partial u}{\partial t} - \nu\,u_{xx} - \alpha u + u^3 = -\mu\sum_{k=1}^{N} h\,u(\overline{x}_k)\,\delta(x - x_k), \tag{44}$$

$$u(x,t) = u(x+L,t), \tag{45}$$

where $h = \frac{L}{N}$; and $\delta(x-a) \in H^{-1}_{per}([0,L])$, for $a \in [0,L]$, and is extended periodically such that

$$< \delta(\cdot - a), \varphi >= \varphi(a) \tag{46}$$

for every $\varphi \in H^1_{per}([0,L])$.

The feedback control proposed in (44) -(45) is different than that of (18)-(19), since the right-hand side in (44) is a distribution that belongs to $H^{-1}_{per}([0,L])$, while the right-hand side in (18) belongs to $L^2([0,L])$.

In this section we will show that, under similar assumptions to those in Theorem 5.1, the proposed feedback system (44) stabilizes the steady state $\mathbf{v} \equiv 0$ in the

L^2-norm. One should not expect here a stronger statement, as the one stated in section 6, in which the stabilizing is also valid in the H^1-norm. This is because the solutions of (44)-(45) are weaker than those of (18)-(19), since the right-hand side in (44) is less regular than its counterpart in (18).
Below, we will show the formal steps, which demonstrate simultaneously the global existence and stability. These formal steps and estimates can be justified rigorously by implementing the Galerkin procedure based on the eigenfunction of the Laplacian, subject to periodic boundary conditions, with period L (see, e.g., [34]. First, let us prove the following Lemma, which is basically the embedding of the Hölder space of $C^{\frac{1}{2}}\, CH^1$ (see also [7]).

Lemma 7.1. *Let* $x_k, \overline{x}_k \in J_k = [(k-1)\,h, k\,h], k = 1,..,N,$ *where* $h = \frac{L}{N}$, $N \in \mathbb{Z}^+$. *Then for every* $\varphi \in H^1([0,L])$ *we have*

$$\sum_{k=1}^{N} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 \leq \; h \, \|\varphi_x\|_{L^2}^2, \tag{47}$$

and

$$\|\varphi\|_{L^2}^2 \leq \; 2 \left[h \sum_{k=1}^{N} |\varphi(x_k)|^2 + h^2 \, \|\varphi_x\|_{L^2}^2 \right]. \tag{48}$$

Proof. We prove inequality (47) for $\varphi \in C^1([0,L])$, and by the density of $C^1 \subset H^1$ the result follows for every $\varphi \in H^1$.

$$\begin{aligned} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 \;&\leq\; \left| \int_{\overline{x}_k}^{x_k} \varphi'(s)\, ds \right|^2 \leq \left(\int_{J_k} |\varphi'(s)|\, ds \right)^2 \\ &\leq\; |J_k| \int_{J_k} |\varphi'(s)|^2\, ds = h \int_{J_k} |\varphi'(s)|^2\, ds. \end{aligned}$$

By summing the above inequality over $k = 1,..,N$ we conclude (47).
To prove (48) we observe that for every $x \in J_k$ we have

$$|\varphi(x)| \;\leq\; |\varphi(x_k)| + \int_{J_k} |\varphi'(s)|\, ds.$$

Thus

$$|\varphi(x)|^2 \;\leq\; 2 \left[|\varphi(x_k)|^2 + \left(\int_{J_k} |\varphi'(s)|\, ds \right)^2 \right], \tag{49}$$

and by integrating with respect to x over J_k, and using the Cauchy-Schwarz inequality, we obtain

$$\int_{J_k} |\varphi(x)|^2\, dx \leq \; 2\,h \left[|\varphi(x_k)|^2 + h \int_{J_k} |\varphi'(s)|^2\, ds \right]. \tag{50}$$

Now we conclude (48) by summing over $k = 1, \ldots, N$.

Theorem 7.1. *Let* $\mu > 4\,\alpha$ *and* h *is small enough such that* $\nu \geq 2\,\mu\, h^2$. *Then for every* $T > 0$, *and every* $u_0 \in L^2_{per}[0,L]$ *system (44) has a unique solution*

$$u \in C([0,T]; L^2_{per}[0,L]) \cap L^2\left([0,T]; H^1_{per}\left[\,0,L]\right) \cap L^4\left([0,T]; L^4_{per}[0,L]\right),$$

and

$$\frac{\partial u}{\partial t} \in L^2\left([0,T]; H^{-1}_{per}\right).$$

Moreover,

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{51}$$

Proof. We take the H^{-1} action of (44) on $u \in H^1$, and use Lemma of Lions-Magenes (cf. Chap. III-p.169, [35]), to obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} \quad &+ \quad \nu\,\|u_x\|^2_{L^2} = \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \mu\,h\sum_{k=1}^N u(\overline{x}_k)\,u(x_k) \\
&= \quad \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \mu\,h\sum_{k=1}^N |u(x_k)|^2 + \mu\,h\sum_{k=1}^N \left(u(x_k) - u(\overline{x}_k)\right)\,u(x_k) \\
&\leq \quad \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \frac{\mu}{2}\,h\sum_{k=1}^N |u(x_k)|^2 + \frac{\mu}{2}\,h\sum_{k=1}^N |u(x_k) - u(\overline{x}_k)|^2,
\end{aligned}$$

where in the last step we applied the Young's inequality. Next, we apply (47) and (48) to the right-hand side

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \nu\,\|u_x\|^2_{L^2} \leq \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \frac{\mu}{4}\,h\,\|u\|^2_{L^2} + \mu\,h^2\,\|u_x\|^2_{L^2}.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + (\nu - \mu\,h^2)\,\|u_x\|^2_{L^2} \leq (\alpha - \frac{\mu}{4})\,\|u\|^2_{L^2} - \int_0^L u^4\,dx. \tag{52}$$

Since $\nu \geq 2\mu\,h^2$ and $4\alpha < \mu$ we conclude (52), thanks to Gronwall's inequality. Moreover, by integrating (52) over $[0,T]$, we conclude the regularity of the solution as stated in the theorem. Next, we prove the uniqueness of solutions.
Let u_1 and u_2 be any two solutions. Denote by $w = u_1 - u_2$. Then w satisfies

$$\frac{\partial w}{\partial t} - \nu\,w_{xx} - \alpha\,w + (u_1^2 + u_1\,u_2 + u_2^2)\,w = -\mu\,h\sum_{k=1}^N w(\overline{x}_k)\,\delta(x - x_k) \tag{53}$$

taking the H^{-1} action on $w \in H^1$, and using again Lemma of Lions-Magenes (cf. Chap. III-p.169, [35]), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} + \nu\,\|w_x\|^2_{L^2} - \alpha\,\|w\|^2_{L^2} = \int_0^L (u_1^2 + u_1\,u_2 + u_2^2)\,w^2\,dx - \mu\,h\sum_{k=1}^N w(\overline{x}_k)\,w(x_k).$$

Since $\int_0^L (u_1^2 + u_1\,u_2 + u_2^2)\,w^2\,dx \geq \int_0^L \frac{u_1^2 + u_2^2}{2}\,w^2\,dx \geq 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} + \nu\,\|w_x\|^2_{L^2} - \alpha\,\|w\|^2_{L^2} \leq -\mu\,h\sum_{k=1}^N w(\overline{x}_k)\,w(x_k)$$

From here we follow the same steps as in the proof of the stability to obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + (\nu - \mu\, h^2)\, \|w_x\|_{L^2}^2 \leq (\alpha - \frac{\mu}{4})\, \|w\|_{L^2}^2.$$

Since $\nu \geq 2\mu\, h^2$ and $\mu > 4\,\alpha$ we conclude (52), thanks to Gronwall's inequality.

$$\|w(t)\|_{L^2}^2 \leq e^{(\alpha-\mu/4)\,t}\|w(0)\|_{L^2}^2. \tag{54}$$

Notice that (54) implies the uniqueness of the solutions and their continuous dependence on the initia data.

Remark 7.1

Here again we observe that by choosing $\mu = O(\alpha)$ then the condition of the theorem imply that $N := \frac{L}{h} = O(\sqrt{\frac{\alpha}{\nu}})$ which is comparable to the dimension of the unstable manifold about $\mathbf{v} \equiv 0$.

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