Stabilizing periodic orbits using time–delayed feedback control

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Bachelor Thesis in Mathematics

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Abstract

In this thesis we study Pyragas control, a form of time–delayed feedback control that stabilizes unstable periodic solutions of differential equations. We first give an introduction to the theory of differential delay equations. We then apply the Pyragas control scheme to the normal form of the Hopf bifurcation and show for which values of parameters stability can be achieved.

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Contents

Introduction

Robustness is one of the fundamental principles in engineering sciences, since it is undesirable that disturbances of the system structurally change the state of the system. In the context of the study of motion, this translates into a wish for the observed motion to be stable: if we add a small disturbance to the system, the observed motion should after some time converge to the initial motion of the system.

A way to achieve that motion is stable, is by applying time–delayed feedback control. In this thesis, we study Pyragas control, a form of time–delayed feedback control that is particularly well–suited to stabilize periodic motion.

Let us study the differential equation

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$. We assume that an unstable periodic solution $u(t)$ of period T is known to exist. The idea of Pyragas control is to add a control term, such that the system becomes

$$\dot{x}(t) = f(x(t)) + K\left[x(t) - x(t-T)\right] \tag{2}$$

with K an $n \times n$-matrix. We note that $u(t)$ is still a solution of (2). We can therefore attempt to choose the matrix K in such a way that $u(t)$ becomes in fact a stable solution of (2). [15]

Stability theory

Since robustness is an important concept in engineering, one needs to be able to tell whether a system is robust or not. In the context of Pyragas control, we need to decide upon the (in)stability of motion, and specifically upon the (in)stability of periodic solutions. In this thesis, we will mainly use *linearized stability* to do so.

Since equilibria are the simplest form of periodic solutions, we introduce the concept of linearized stabilty by studying the stability of equilibria. Let A be a $n \times n$-matrix and

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{3}$$

We note that $x = 0$ is an equilibrium of (3). Using the definition of stability for equilibria, one can prove that if all the eigenvalue λ_j satisfy $\operatorname{Re}\lambda_j < 0$, the equilibrium $x = 0$ is stable. If there exists an eigenvalue λ of A such that $\operatorname{Re}\lambda > 0$, then the solution $x = 0$ of (3) is unstable (see Appendix A). Therefore, for we can decide upon the stablity of the equilibrium $x = 0$ of (3) by looking a the eigenvalues of A.

Now, let $\overline{x} \in \mathbb{R}^n$ be a equilibrium of (1), i.e. $f(\overline{x}) = 0$. If f is smooth enough, we can use a Taylor expansion to write

$$\begin{aligned}\dot{x}(t) = f(x(t)) &= f(\overline{x}) + Df(\overline{x})(x(t) - \overline{x}) + \mathcal{O}((x(t) - \overline{x})^2) \\ &= Df(\overline{x})(x(t) - \overline{x}) + \mathcal{O}((x(t) - \overline{x})^2)\end{aligned}$$

Under the assumption that $\operatorname{Re}\lambda_j \neq 0$ for all eigenvalues λ_j of $Df(\overline{x})$, it follows from the Hartman-Grobman Theorem (see Appendix A) that the stability of the fixed point $x = \overline{x}$ of

$$\dot{x}(t) = Df(\overline{x})(x(t) - \overline{x}) \tag{4}$$

is the same as the stability of the fixed point $x = \overline{x}$ of (1). Since (4) is the form of (3), we know how to determine the stability of the fixed point $x = \overline{x}$, that is, by looking at the eigenvalues of $Df(\overline{x})$. Thus, we now have a criterion to determine the stability of the equilibrium $x = \overline{x}$ of (1). Since we arrived at this criterion by studying the linearized problem, this approach is known as linearized stability.

We can extend this approach to periodic solutions. If we assume that $u(t)$ is a periodic solution of (1), we can linearize system (1) around $u(t)$; this yields the so-called linear variational equation. From the linear variational equation we can extract the *characteristic multipliers*, that play the role that eigenvalues have in the study of equilibria. We can determine stability of the periodic orbit $u(t)$ of (1) using these characterstic multipliers. If all the non-trivial characteristic multipliers ρ satisfy $|\rho| < 1$, the periodic solution $u(t)$ of (1) is stable; if one of the characteristic multipliers satisfies $|\rho| > 1$, the periodic solution $u(t)$ of (1) is unstable (see Section A.5 in the Appendix).

Delay equations

Using the stability theory discussed above, we are able to determine the stability of solutions of the ordinary differential equation (1). We note, however, that (2) is not an ordinary differential equation, but a *differential delay equation*. The mathematical treatment of differential delay equations is different from that of ordinary differential equations, the main difference being that the *state space* of a differential delay equation is infinite dimensional.

We define the *state* of a differential equation at time τ as the minimal amount of information needed to give an unique solution for $t \geq \tau$. The *state space* $\mathcal{S}$ is defined as the set consisting of all such possible states. Under the assumption that $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, it is a well-known result from ordinary differential equations that the initial-value problem (1) has an unique solution for t in some open neighbourhood around zero. Thus, given an $x_0 \in \mathbb{R}^n$, (1) has an unique solution for some $t \geq 0$. The state of (1) is therefore given by $x_0 \in \mathbb{R}^n$ and the state space by $\mathcal{S} = \mathbb{R}^n$.

Now let us study the linear differential delay equation

$$\dot{x}(t) = Ax(t) + Bx(t - T), \quad t \geq 0 \tag{5}$$

with A, B $n \times n$-matrices and $T > 0$. Using the variation of constants formula, one finds that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bx(s - T)ds, \quad t \in [0, T] \tag{6}$$

For the right hand side of (5) to be defined for $t \in [0, T]$, one should know at least know the value of $x(t)$ for all $t \in [-T, 0]$. Using (6) we see that this is also sufficient information to give an unique solution of (5) for $t \in [0, T]$. Therefore, the state space of (5) is given by $\mathcal{S} = \mathcal{C}([-T, 0], \mathbb{R}^n)$. We note that this space is infinite dimensional, whereas the dimension of the state space $\mathcal{S} = \mathbb{R}^n$ of (1) is finite.

Using functional analysis, one can extend the stability theory for ordinary differential equations to differential delay equations. This will be explored in Chapter 1.

Pyragas control

Having discussed the necessary theory to discuss stability of ordinary differential equations and differential delay equations, we now turn to feedback–control. Specifically, we study Pyragas control in the context of the concrete model

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)\left|z(t)\right|^2 z(t) \tag{7}$$

For the controlled system we write

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)\left|z(t)\right|^2 z(t) - Ke^{i\beta}\left[z(t) - z(t - T)\right] \tag{8}$$

System (7) is related to travelling wave solutions of the complex Ginzberg-Landau equation, that models a variety of physical phenomena. The model (7) has the nice property that we can find a simple expression for a periodic solution. This periodic solution is unstable and arises from a subcritical Hopf bifurcation. We now want to choose the values of K, β in (8) in such a way that the periodic orbit emmanates from a supercritical Hopf bifurcation. If this is the case, it follows that the periodic solution is stable as a solution of (8). Using this approach, we are able to determine the stability of the periodic solution of (8), although this periodic orbit has an infinite number of characteristic multipliers.

Organization of the thesis

We start by introducing the main concepts of differential delay equations in Chapter 1. In Chapter 2 we study Pyragas control of system (7), where we find conditions for the periodic orbit of (7) to be (un)stable as a solution of (8). In Chapter 3 we study the bifurcation diagram of system (8) in more detail, as to gain more insight in the overall in the dynamics of the system. In Sections 2.2 and 2.4 and in Chapter 3, we mostly follow [10]. In Chapter 4 we present a different approach to the analysis discussed in Chapter 2. Using this, we are able to weaken the conditions found in [10] for the periodic orbit of (7) to be (un)stable as a solution of (8). We conclude by discussing the methods and results of this thesis in Chapter . An overview of stability theory for ordinary differential equations and the Hopf bifurcation theorem for ordinary differential equations can be found in Appendix A.

Chapter 1

Introduction to differential delay equations

As was mentioned in the Introduction, (2) is a differential delay equation. In order to be able to discuss Pyragas control, we therefore introduce the main concepts and the basic theory of differential delay equations.

A difference between differential delay equations and ordinary differential equations is that in the first case the state space is infinite dimensional. Since this is an important concept in the study of differential delay equations, we explore this more carefully in Section 1.1. In Section 1.2, we study the characteristic equation and its relation with stability of equilibria. In Section 1.3 we explore the relation between the flow maps and the characteristic equation; we use this to decompose the state space in subspace where the stability of a fixed point is better known. In these discussions, we mostly follow [8].

1.1 The state space of a differential delay equation

If we study general differential equations, we usually do not only want to know whether a solution exists, but also if this solution is unique given some initial data. We formalize this concept in the following definition:

Definition 1.1.1. We define the *state at time* τ of a differential equation to be the minimal information that is needed to uniquely determine the solution for $t \geq \tau$. The *state space* S is a normed space consisting of all such possible states. [8]

To explore this concept, we start by determining the state space of an ordinary differential equation. If we study the system

$$\dot{x}(t) = f(x(t)) \tag{1.1}$$

with $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, we recall from the theory of ordinary differential equations that when the initial condition $x(\tau) = x_\tau \in \mathbb{R}^n$ is given, we can uniquely extend the solution of (1.1) to some open neighbourhood of τ. [4] In this case, the state space is thus S is given by $\mathbb{R}^n$.

We now want to determine the state space of the differential delay equation

$$\dot{x}(t) = Ax(t) + Bx(t-r) + f(t) \tag{1.2}$$

where A, B are constant $n \times n$-matrices, $r > 0$ and $f : \mathbb{R} \to \mathbb{R}^n$ is continuous. We first explore what the initial value should be in order to give the initial value problem corresponding to (1.2) an unique solution. Inspired by ordinary differential equations, we first try a point in $\mathbb{R}^n$ as initial value:

$$\dot{x}(t) = Ax(t) + Bx(t-r) + f(t), \quad x(0) = x_0 \tag{1.3}$$

We apply the variation of constants method (see for example [6]) to find that $x(t)$ should satisfy:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} \left(Bx(s-r) + f(s)\right) ds, \quad t \in [0, r]$$

From here, we see that for (1.2) to have an unique solution, we should know the value of $x(t)$ for $t \in [-r, 0]$. If we write $x(t) = \phi(t)$ for $t \in [-r, 0]$ and some $\phi \in \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$, we should therefore look at the initial value problem

$$\dot{x}(t) = Ax(t) + Bx(t-r) + f(t), \quad x(t) = \phi(t) \text{ for } t \in [-r, 0] \tag{1.4}$$

We see by the argument above that for $t \in [0, r]$, its (unique) solution is given by

$$x(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)} \left(B\phi(s-r) + f(s)\right) ds, \quad t \in [0, r]$$

Since this defines a continuous function on $[0, r]$, we can repeat this argument to find an unique solution on $[r, 2r]$, etc. This method, also known as the *method of steps*, ensures that the problem (1.4) has an unique solution for $t \in [0, \infty)$.

Denote by $x(t, \phi)$ the solution of (1.2) which agrees with ϕ on $[-r, 0]$. We can then use the above remarks to see that once $x(\tau + \theta, \phi)$ is known for all $\theta \in [-r, 0]$, we can uniquely extend the solution for all $t \geq \tau$. Thus, the state is given by $x_\tau(\phi)$, with

$$x_t(\phi)(\theta) = x(t + \theta, \phi) \quad \text{for } \theta \in [-r, 0]$$

Since $x_\tau(\phi) \in \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$, we conclude that the state space of system (1.2) is given by

$$\mathcal{C} := \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$$

which we equip with the supremum-norm. With this in mind, we will from now on say that $x(t)$ is a solution of (1.2) with initial condition ϕ if $x(t)$ satisfies (1.2) and $x(t) = \phi(t)$ for all $t \in [-r, 0]$.

We note that the state space $\mathcal{S} = \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ of the differential delay equation (1.2) is an infinite–dimensional space; whereas the state space of the ordinary differential equation (1.1) is given by the finite–dimensional space $\mathcal{S} = \mathbb{R}^n$. This difference turns out to be an important concept in the study of differential delay equations. For example, it does not matter with which norm we equip the state space $\mathcal{S} = \mathbb{R}^n$ of an ordinary differential equation, since all norms on $\mathbb{R}^n$ are equivalent. If we study the state space $\mathcal{S} = \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ of (1.2), the choice of norm does matter. We usually choose to equip $\mathcal{S} = \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ with the supremum–norm $\|.\|_\infty$, since this turns $\mathcal{S} = \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ into a Banach–space.

We recall from the theory of ordinary differential equations that one can define flow maps $\phi(t, .)$ from the state space to itself, such that the flow maps translate a state 'along the solution' of the differential equation (see Definition A.1.1). Mimicking this idea for differential delay equations, we define for all $t \geq 0$.

$$T(t) : \mathcal{C} \to \mathcal{C}, \quad T(t)\phi = x_t(\phi)$$

As in the case of ordinary differential equations, we see that

$$T(0) = I$$
$$T(s)T(t) = T(t + s)$$

for $t, s \geq 0$. In the case of ordinary differential equations, we also have the additional property that $\phi(-t, .)\phi(t, .) = I$ (see Appendix A). For differential delay equations, however, backward continuation of a solution is not always possible. If $\phi \in \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ and we want to define the solution $x(t, \phi)$ of Eq. (1.2) for $t \in [-2r, -r]$, it is a necessary condition that $\phi \in \mathcal{C}^1\left([-r, 0], \mathbb{R}^n\right)$. Thus, for general $t \geq 0$, $\phi \in \mathcal{C}\left([-r, 0], \mathbb{R}^n\right)$ the expression $T(t)\phi$ may not be defined. We conclude that $\{T(t) \mid t \geq 0\}$ defines a semi-group, but not a group.[8] We therefore refer to the maps $T(t)$ for $t \geq 0$ as the *semi-flow*. In Section 1.3, we will prove some more properties of the semi-flow for the case where $f = 0$ in Eq. (1.2)

1.2 The characteristic equation and stability of equilibria

In the following, we will look at the differential delay equation

$$\dot{x}(t) = Ax(t) + Bx(t-r) \tag{1.5}$$

with A, B $n \times n$-matrices and $x \in \mathbb{R}^n$. To find a non-trivial solution, we try the function $x(t) = e^{\lambda t}c$, with $\lambda \in \mathbb{C}$ and $c \in \mathbb{R}^n$. For this function to be a solution of (1.5), we should have that

$$\lambda e^{\lambda t}c = Ae^{\lambda t}c + Be^{\lambda t}e^{-\lambda r}c$$

Cancelling the terms $e^{\lambda t}$ on both sides, we find that this equation has a non-trivial solution if and only if the map

$$c \mapsto (\lambda I - A - Be^{-\lambda r})c$$

has a non–trivial kernel, i.e. if and only if $\det\left(\lambda I - A - Be^{-\lambda r}\right) = 0$. This motivates the following definition:

Definition 1.2.1. The *characteristic function* of Eq. (1.5) is defined to be $\Delta(\lambda) = \lambda I - A - Be^{-\lambda r}$. The equation $\det \Delta(\lambda) = 0$ is referred to as the *characteristic equation.*

We can use complex analysis to say something about the distribution of the solutions of $\det \Delta(\lambda) = 0$ in the complex plane, as is summarized in the following lemma:

Lemma 1.2.1. *Denote by $\Delta(\lambda)$ the characteristic function of Eq. (1.5). If there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of zero's of $\det \Delta(\lambda)$ such that $|\lambda_j| \to \infty$ as $j \to \infty$, then $Re(\lambda_j) \to -\infty$ as $j \to \infty$. In this case, there exists an $\alpha \in \mathbb{R}$ such that $\{\lambda \mid \det \Delta(\lambda) = 0\} \subseteq \{z \in \mathbb{C} \mid Re(z) < \alpha\}$. It always holds that only a finite number of zero's of $\det \Delta(\lambda)$ can be contained in any compact set or in a vertical strip of the complex plane.*

Proof. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of zero's $\det \Delta(\lambda)$ such that $|\lambda_j| \to \infty$ as $j \to \infty$. By definition of $\det \Delta(\lambda)$, we find for each $j \in \mathbb{N}$ a $c_j \in \mathbb{R}^n$ such that $\lambda_j c_j - Ac_j - Bc_j e^{-r\lambda_j} = 0$. Without loss of generality, we may assume that $\|c_j\| = 1$. Thus we find that $\|\lambda_j c_j - Ac_j\| = \|Bc_j\| e^{-\mathrm{Re}(\lambda_j)r}$. Since $\|c_j\| = 1$, it follows from the definition of the operator norm that

$$\|B\| e^{-r\mathrm{Re}(\lambda_j)} \geq \|\lambda_j c_j - Ac_j\| = |\lambda_j| \left\| c_j - \frac{1}{\lambda_j}Ac_j \right\|$$

If $|\lambda_j| \to \infty$, then $|\lambda_j| \left\| c_j - \frac{1}{\lambda_j}Ac_j \right\| \to \infty$ (note that $\|c_j\| = 1$ for all $j \in \mathbb{N}$) and thus it follows by the inequality that $\|B\| e^{-r\mathrm{Re}(\lambda_j)} \to \infty$. Since $\|B\| < \infty$, it follows that $\mathrm{Re}(\lambda_j) \to -\infty$ as $j \to \infty$. We can therefore find an $\alpha \in \mathbb{R}$ such that $\{\lambda_j \mid j \in \mathbb{N}\} \subseteq \{z \in \mathbb{C} \mid \mathrm{Re}\, z < \alpha\}$.

From here, we find that $\det \Delta(\lambda)$ can only have finitely many zero's in any compact subset $U \subseteq \mathbb{C}$. If $\det \Delta(\lambda)$ has infintely many zero's in U, then it follows by the Bolzano-Weierstrass Theorem and the fact that U is closed that there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ such that $\det \Delta(\lambda_j) = 0$ and $\lim_{j\to\infty} \lambda_j = j \in U$. Since $\det \Delta(\lambda)$ is continuous, we have that $\Delta(\lambda) = 0$. Since λ is an accumulation point of zero's and $\det \Delta(\lambda)$ is holomorphic on $\mathbb{C}$, it follows that $\det \Delta(\lambda)$ on all of $\mathbb{C}$, which is a contradiction.

Now, let W be a vertical strip of the complex plane, such that W contains an infinite number of roots λ of $\det \Delta(\lambda)$. Then the set $\{\lambda \in W \mid \det \Delta(\lambda) = 0\}$ is clearly either bounded or not bounded. If it is bounded, we can find a compact subset $V \subseteq W$ such that $\lambda \in V$ for all $\lambda \in \{\lambda \in W \mid \det \Delta(\lambda) = 0\}$. This contradicts that any compact set can only contain a finite number of zero's of $\det \Delta(\lambda)$. If the set $\{\lambda \in W \mid \det \Delta(\lambda) = 0\}$ is unbounded, we can find a sequence $(\lambda_j)_{j\in\mathbb{N}} \subseteq \{\lambda \in W \mid \det \Delta(\lambda) = 0\}$ such that $|\lambda_j| \to \infty$ as $j \to \infty$. But then $\mathrm{Re}(\lambda_j) \to -\infty$ as $j \in \infty$, which contradicts the fact that $\lambda_j \in W$ for all $j \in \mathbb{N}$. Thus, W can only contain a finite number of zero's of $\det \Delta(\lambda)$. [8] □

We state the following result without proof:

Theorem 1.2.2. *Let $x(t,\phi)$ be a solution of* (1.5) *with initial condition $\phi \in \mathcal{C}([-r,0])$. If $\alpha_0 = \max\{Re(\lambda) \mid \det \Delta(\lambda) = 0\}$, then for any $\alpha > \alpha_0$, there exists a constant $K(\alpha)$ such that*

$$\|x(t,\phi)\| \leq K e^{\alpha t} \|\phi\|$$

A proof can be found in [8]. We note that by Lemma 1.2.1, the maximum in $\max\{\mathrm{Re}(\lambda) \mid \det \Delta(\lambda) = 0\}$ is well-defined.

If we choose $\phi \in \mathcal{C}$ as $\phi = 0$, then $x(t) = 0$ is a solution of (1.5) with initial value ϕ. The stability of the equilibrium $\phi = 0$ is defined in exactly the same manner as for equilibria of ordinary differential equations (see also Definition A.2.2 in the Appendix). We have defined the state space of system (1.5) to be $\mathcal{C} = \mathcal{C}([-r,0],\mathbb{R}^n)$ equipped with the supremum-norm. Since now the state space is infinite-dimensional and not all norms are equivalent, we stress that the chosen norm is important. If we equip $\mathcal{C}$ with another, non-equivalent norm, stability properties of the equilibrium can be entirely different. As in the case of linear ordinary differential equations, the distribution of roots of the characteristic equation in the complex plane can help us to determine the stability of the equilibrium $x = 0$. This is summarized in the following Corollary, which resembles the one found for linear ordinary differential equations with constant coefficients.

Corollary 1.2.1. *Denote by $\alpha_0 = \max\{Re(\lambda) \mid \det \Delta(\lambda) = 0\}$, where Δ is the characteristic function correpsonding to* (1.5)*. If $\alpha_0 < 0$, the equilibrium $x = 0$ of* (1.5) *is asympotitcally stable. If $\alpha_0 > 0$, then $x = 0$ is an unstable equilibrium of* (1.5)*.*

Proof. Denote by $x(t,\phi)$ the solution of (1.5) with initial condition ϕ. If $\alpha_0 < 0$, we can choose an $\alpha_0 < \alpha < 0$. Using Theorem 1.2.2, we find that there exists a constant K such that

$$\|x(t,\phi)\| \leq K e^{\alpha t} \|\phi\|$$

thus, if $\|\phi - 0\| = \|\phi\| < \frac{\epsilon}{K}$, then $\|x(t,\phi)\| < \epsilon$ for all $t \geq 0$. Furthermore, since $\alpha < 0$, $\|x(t,\phi)\| \to 0$ for all $\phi \in \mathcal{C}$. We conclude that $x = 0$ is asymptotically stable.

If $\alpha_0 > 0$, there exists a λ such that $\det \Delta(\lambda) = 0$ and $\mathrm{Re}(\lambda) > 0$. By definition of Δ, it follows that there exists a $\mu \in \mathbb{R}, c \in \mathbb{R}^n$ with $\|c\| = 1$ such that $x_\mu(t) = \mu e^{\lambda t} c$ is a solution of Eq. (1.5). We note that this solution satisfies the initial condition $\phi_\mu(t) = \mu e^{\lambda t} c$ for all $t \in [-r,0]$. Since $\mathrm{Re}(\lambda) > 0$, we have that $\|\phi_\mu - 0\| = \|\phi_\mu\| = |\mu| \, \|c\| = |\mu|$. For $\epsilon = 1$ and $\delta > 0$ arbitrary, we choose $\mu = \frac{\delta}{2}$. Then $\|\phi_\mu\| = \mu = \frac{\delta}{2} < \delta$. However, since $\mathrm{Re}(\lambda) > 0$, we have that $\|x_\mu(t)\| \to \infty$ as $t \to \infty$, hence there exists a time t_0 such that $\|x_\mu(t_0)\| \geq 1 = \epsilon$. Thus we conclude that $x = 0$ is an unstable equilibrium of Eq. (1.5). □

1.3 The semi–flow and decomposition of the state space

In Section 1.1, the semi-flow maps $T(t)$ were introduced. In this Section, we will first prove some properties of these maps. This will enable us to find a relation between the spectrum of $T(t)$ and the roots of $\det \Delta(\lambda)$. Using this, we will work toward a more geometric approach on stability of a fixed point.

We need the following Theorem:

Theorem 1.3.1. *Let $x(t,\phi)$ be a solution of the initial value problem* (1.4)*. Then there exists constants $a, b \geq 0$ such that*

$$|x(t,\phi)| \leq a e^{bt} \left(\|\phi\| + \int_0^r \|f(s)\| \, ds \right)$$

For the proof, we need the following inequality

Lemma 1.3.2 (Gronwall Inequality)**.** *Let $u, \alpha, \beta \in \mathcal{C}([a,b],\mathbb{R})$, $\beta \geq 0$ such that*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s) u(s) ds, \quad a \leq t \leq b$$

If α is non-decreasing, then it holds that

$$u(t) \leq \alpha(t) \exp\left(\int_a^t \beta(s)ds\right)$$

For a proof, see for example [8].

Proof. (of Theorem 1.3.1) Since $x(t,\phi)$ satisfies 1.4, we find by integration that

$$x(t,\phi) = \phi(0) + \int_0^t Ax(s) + Bx(s-r) + f(s)ds$$

and thus that

$$\|x(t,\phi)\| \leq \|\phi\| + \int_0^t \|A\| \|x(s)\| + \|B\| \|x(s-r)\| + \|f(s)\| \, ds$$

We also see that

$$\begin{aligned}\int_0^t \|B\| \|x(s-r)\| \, ds &= \int_{-r}^0 \|B\| \|x(s)\| \, ds + \int_r^t \|B\| \, x(s-r)ds = \int_{-r}^0 \|B\| \|\phi(s)\| \, ds + \int_0^{t-r} \|B\| \|x(s)\| \, ds \\ &\leq r \|B\| \|\phi\| + \int_0^t \|B\| \|x(s)\| \, ds\end{aligned}$$

where the last inequality holds since $\|B\| \|x(s)\| \geq 0$. Thus we find that

$$\|x(t,\phi)\| \leq (1 + \|B\| r) \|\phi\| + \int_0^t (\|A\| + \|B\|) \|x(s)\| \, ds + \int_0^t \|f(s)\| \, ds$$

If we choose $\alpha(t) = (1+Br) \|\phi\| + \int_0^t \|f(s)\| \, ds$ and $\beta(t) = \|A\| + \|B\|$, then we have that α, β are real-valued continuous functions, α is non–decreasing and $\beta \geq 0$. Applying the Gronwell Inequality, we find that

$$\begin{aligned}\|x(t)\| &\leq \left((1 + \|B\| r) \|\phi\| + \int_0^t \|f(s)\| \, ds\right) \exp\left((\|A\| + \|B\|)\, t\right) \\ &= (1 + \|B\| r) \exp\left((\|A\| + \|B\|)t\right) \|\phi\| + (1 + \|B\| r) \exp\left((\|A\| + \|B\|)t\right) \int_0^t \|f(s)\| \, ds\end{aligned}$$

Setting $a = 1 + \|B\| r$ and $b = \|A\| + \|B\|$ proves the theorem. [8] □

Using this, we can prove the following:

Lemma 1.3.3. *Let $\mathcal{C} = \mathcal{C}([-r,0], \mathbb{R}^n)$ denote the state space of the differential delay equation $\dot{x}(t) = Ax(t) + Bx(t-r)$, where A, B are $n \times n$ matrices and $r > 0$. Denote by $T(t)$ the corresponding semi-flow, i.e. $T(t)\phi = x_t(\phi)$ for all $\phi \in \mathcal{C}$. Then $T(t) : \mathcal{C} \to \mathcal{C}$ is a bounded linear operator for $t \geq 0$ and is compact for $t \geq r$.*

Proof. We note that $x(t,\phi) + x(t,\psi)$ and $x(t, \phi+\psi)$ are solutions of (1.5) with initial condition $\phi+\psi$. Since solutions of (1.5) are unique given an initial condition, it follows that $x(t,\phi)+x(t,\psi) = x(t,\phi+\psi)$. Similarly, we see that $x(t, \lambda\phi) = \lambda x(t,\phi)$. Using this, we easily see that $T(t)$ is a linear operator.

To see that $T(t)$ is bounded for $T > 0$, we use Theorem 1.3.1 to obtain the estimate:

$$\|T(t)\phi\| = \|x_t(\phi)\| = \sup_{\theta\in[-r,0]} \|x(t+\theta, \phi)\| \leq \sup_{\theta\in[-r,0]} ae^{b(t+\theta)} \|\phi\| = ae^{bt} \|\phi\|$$

for some $a, b \geq 0$. We conclude that $\|T(t)\| \leq ae^{bt} < \infty$, i.e. $T(t)$ is a bounded operator.

To show that $T(t)$ is compact for $t \geq r$, we denote by $x(t)$ a solution of (1.5) with initial condition ϕ and remark that

$$\|\dot{x}(t)\| \leq \|A\| \, \|x(t)\| + \|B\| \, \|x(t-r)\| \leq \left(\|A\| \, ae^{bt} + \|B\| \, ae^{b(t-r)}\right) \|\phi\| = k(t) \, \|\phi\|$$

for $t \geq r$ and for some $a, b \geq 0$. It now follows that

$$\|x(t)\| = \left\| x(0) + \int_0^t \dot{x}(s) ds \right\| \leq \|\phi(0)\| + \int_0^t \|\dot{x}(s)\| \, ds \leq \left(1 + \int_0^t k(s) ds\right) \|\phi\| := p(t) \, \|\phi\|$$

where $p(t)$ is a continuous function. Therefore, if $\epsilon > 0$ is given, there exists a $\delta > 0$ such that $|s - t\langle \delta$ implies $\|p(t) - p(s)\| < \epsilon$. If we take $\|\phi\| < 1$, we can use the estimate to find that $\|x(t, \phi) - x(s, \phi)\| < \epsilon$. Using Arzela-Ascoli's Theorem, it follows that $\overline{T(t) \left(B_1^{\mathcal{C}}(0)\right)}$ is a compact subset of $\mathcal{C}$ (where $B_1^{\mathcal{C}}(0)$ denotes the unit ball in $\mathcal{C}$), i.e. the operator $T(t)$ is compact for $t \geq r$. [8] $\square$

It turns out that the spectrum of $T(t)$ is strongly related to the zero's of the characteristic function $\det \Delta(\lambda)$. In order to show this, we introduce the following operator:

Definition 1.3.1. Denote by $T(t) : \mathcal{C} \to \mathcal{C}$ the flow map corresponding to Eq. (1.5). The *infinitesimal generator* A of $T(t)$ is defined as

$$\begin{aligned} \mathcal{D}(A) &= \{\phi \in \mathcal{C} \mid \lim_{t \to 0} \frac{1}{t} \left[T(t)\phi - \phi\right] \text{ exists}\} \\ A\phi &= \lim_{t \to 0} \frac{1}{t} \left[T(t)\phi - \phi\right] \quad \text{if } \phi \in \mathcal{D}(A) \end{aligned}$$

where $\mathcal{D}(A)$ denotes the domain of A.

It turns out that we are able to compute the domain $\mathcal{D}(A)$ and give a more explicit form for the operator A, as is shown in the following Lemma:

Lemma 1.3.4. *Let A be the infinitesimal generator of $T(t)$ as in Definition 1.3.1. Then the following holds:*

$$\begin{aligned} &\mathcal{D}(A) = \{\phi \in \mathcal{C} \mid \phi \in \mathcal{C}^1\left([-r, 0], \mathbb{R}^n\right), \ \dot{\phi}(0) = A\phi(0) + B\phi(-r)\} \\ &A\phi = \dot{\phi} \quad \textit{if } \phi \in \mathcal{D}(A) \end{aligned}$$

Proof. We first prove that if $\phi \in \mathcal{D}(A)$, then $A\phi = \dot{\phi}$. By definition of the semi-flow, we have that

$$\lim_{t \to 0} \frac{1}{t} \left[T(t)\phi - \phi\right] = \lim_{t \to 0} \frac{1}{t} \left[x_t(\phi) - x_0(\phi)\right]$$

We notice that the last expression is a difference quotient; thus, if the limit exists, it should equal

$$\lim_{t \to 0} \frac{1}{t} \left[x_t(\phi) - x_0(\phi)\right] = \frac{d}{dt} \, x_t(\phi)|_{t=0}$$

Since $x(\theta, \phi) = \phi(\theta)$ for $\theta \in [-r, 0]$, we conclude that

$$A\phi = \frac{d}{dt} \, x_t(\phi)|_{t=0} = \dot{\phi}$$

To prove that $\mathcal{D}(A) = \{\phi \in \mathcal{C} \mid \phi \in \mathcal{C}^1\left([-r, 0], \mathbb{R}^n\right), \ \dot{\phi}(0) = A\phi(0) + B\phi(-r)\}$, we first take $\phi \in \mathcal{C}^1\left([-r, 0], \mathbb{R}^n\right), \ \dot{\phi}(0) = A\phi(0) + B\phi(-r)$. We note that by (1.5), $x(t, \phi)$ is continuously differentiable for $t > 0$. Since we have taken, $\phi \in \mathcal{C}^1 := \mathcal{C}^1\left([-r, 0], \mathbb{R}^n\right)$ and $x(t, \phi) = \phi(t)$ for all $t \in [-r, 0]$, we find that $x(t, \phi)$ is continuously differentiable for $t \in (-r, 0)$. Since $x(t, \phi)$ is a solution of (1.5), we find that its right derivative at $t = 0$ equals $Ax(0, \phi) + Bx(-r, \phi) = A\phi(0) + B\phi(-r)$. The condition that $\dot{\phi}(0) = A\phi(-r) + B\phi(-r)$

thus ensures that the left derivative of $x(t,\phi)$ at $t=0$ equals its right derivative. We find that the derivative of $x(t,\phi)$ at $t=0$ exists and, since $x(t,\phi)$ satisfies (1.5), that this derivative is continuous. We conclude that $x(t,\phi)$ is continuously differentiable on $(-r,\infty)$.

Denoting by $\|.\|_\infty$ the supremum-norm, and by $\|.\|$ a norm on $\mathbb{R}^n$, we find that

$$\left\|\frac{1}{t}\left[T(t)\phi-\phi\right]-\dot{\phi}\right\|_\infty = \sup_{\theta\in[-r,0]}\left\|\frac{1}{t}\left[x(t+\theta,\phi)-x(\theta,\phi)\right]-\frac{d}{dt'}\left.x(t',\phi)\right|_{t'=\theta}\right\|$$

We note that

$$\frac{d}{dt'}\left.x(t',\phi)\right|_{t'=\theta}=\lim_{t\to 0}\frac{1}{t}\left[x(t+\theta,\phi)-x(\theta,\phi)\right]$$

for all $\theta\in[-r,0]$. Let $\epsilon>0$ be given. Then we can find for every $\theta\in[-r,0]$ a $\delta>0$ such that if $\|t\|<\delta$, then $\left\|\frac{1}{t}\left[x(t+\theta,\phi)-x(\theta,\phi)\right]-\frac{d}{dt'}\left.x(t',\phi)\right|_{t'=\theta}\right\|<\epsilon$. However, since

$$\theta\mapsto\frac{1}{t}\left[x(t+\theta,\phi)-x(\theta,\phi)\right]-\frac{d}{dt'}\left.x(t',\phi)\right|_{t'=\theta}$$

is continuous by the remarks made above, and since $[-r,0]$ is compact, we can make such an estimate independent of θ, i.e. we can find a $\delta>0$ such that

$$\left\|\frac{1}{t}\left[x(t+\theta,\phi)-x(\theta,\phi)\right]-\frac{d}{dt'}\left.x(t',\phi)\right|_{t'=\theta}\right\|<\epsilon$$

if $\|t\|<\delta$ and $\theta\in[-r,0]$. We conclude that $\lim_{t\to 0}\frac{1}{t}\left[T(t)\phi-\phi\right]=\dot{\phi}$ in the supremum-norm and in particular, the limit exists. Thus, we find that $\phi\in\mathcal{D}(A)$ and $\{\phi\in\mathcal{C}\mid\phi\in\mathcal{C}^1\left([-r,0],\mathbb{R}^n\right),\ \dot{\phi}(0)=A\phi(0)+B\phi(-r)\}\subseteq\mathcal{D}(A)$.

We now prove that $\mathcal{D}(A)\subseteq\{\phi\in\mathcal{C}\mid\phi\in\mathcal{C}^1\left([-r,0],\mathbb{R}^n\right),\ \dot{\phi}(0)=A\phi(0)+B\phi(-r)\}$. The fact that $\lim_{t\to 0}\frac{1}{t}\left[T(t)\phi-\phi\right]$ exists, implies that both the left-sided and right-sided limit exist and equal each other. Arguing as in the first part of the reverse inclusion, we should have that $\phi\in\mathcal{C}^1$ and $\dot{\phi}(0)=A\phi(0)+B\phi(-r)$. Thus, we conclude that $\mathcal{D}(A)\subseteq\{\phi\in\mathcal{C}\mid\phi\in\mathcal{C}^1\left([-r,0],\mathbb{R}^n\right),\ \dot{\phi}(0)=A\phi(0)+B\phi(-r)\}$. Combining this with the reversed inclusion as shown above, we find that $\mathcal{D}(A)=\{\phi\in\mathcal{C}\mid\phi\in\mathcal{C}^1\left([-r,0],\mathbb{R}^n\right),\ \dot{\phi}(0)=A\phi(0)+B\phi(-r)\}$. [8] $\square$

We prove the following lemmata concerning the spectra of $T(t)$ and A:

Lemma 1.3.5. *With A as in Definition 1.3.1, we have that $\sigma(A)=\sigma_{pt}(A)=\{\lambda\in\mathbb{C}\mid\det\Delta(\lambda)=0\}$, where $\sigma_{pt}(A)$ denotes the point spectrum of A.*

Proof. We first prove that $\sigma_{pt}(A)=\{\lambda\in\mathbb{C}\mid\det\Delta(\lambda)=0\}$. We note $\lambda I-A$ is not injective if and only if there exists an $\phi\neq 0$ such that $A\phi=\dot{\phi}=\lambda\phi$. It follows that $\phi(\theta)=ce^{\lambda\theta}$ for some $c\in\mathbb{R}^n, c\neq 0$. But such a $c\in\mathbb{R}^n\backslash\{0\}$ exists if and only if $\det\Delta(\lambda)=\det(\lambda I-A-Be^{-r\lambda})\neq 0$. Thus, we find that $\sigma_{pt}(A)=\{\lambda\in\mathbb{C}\mid\det\Delta(\lambda)=0\}$.

By definition of the point spectrum, we have that $\sigma_{pt}(A)\subseteq\sigma(A)$. Thus, if $\sigma(A)\subseteq\sigma_{pt}(A)=\{\lambda\in\mathbb{C}\mid\det\Delta(\lambda)=0\}$, the statement of the lemma follows. Take a $\lambda\in\mathbb{C}$ such that $\det\Delta(\lambda)\neq 0$. We want to show that $\lambda\in\rho(A)$, where $\rho(A)$ denotes the resolvent set of A. We do so by constructing an inverse: take any $\psi\in\mathcal{C}$. We note that $(\lambda I-A)\phi=\lambda\phi-\dot{\phi}=\psi$ is equivalent to $\dot{\phi}(\theta)=\lambda\phi(\theta)-\psi(\theta)$ for all $\theta\in[-r,0]$. Applying the variation of constants formula, we find that a solution is given by:

$$\phi(\theta)=e^{\lambda\theta}\left(c-\int_0^\theta e^{-\lambda s}\psi(s)ds\right) \tag{1.6}$$

where $c \in \mathbb{R}^n$ is a constant which we want to choose in such a way that $\phi \in \mathcal{D}(A)$. We note that, independent of the choice of c, we have that $\phi \in \mathcal{C}^1([-r,0],\mathbb{R}^n)$. Furthermore, we have that

$$\begin{aligned}
\dot{\phi}(\theta) &= \lambda e^{\lambda\theta}\left(c - \int_0^\theta e^{-\lambda s}\psi(s)ds\right) - e^{\lambda\theta}e^{-\lambda\theta}\psi(\theta) = \lambda e^{\lambda\theta}\left(c - \int_0^\theta e^{-\lambda s}\psi(s)ds\right) - \psi(\theta) \\
\dot{\phi}(0) &= \lambda c - \psi(0) \\
A\phi(0) + B\phi(-r) &= Ac + Be^{-\lambda r}c - e^{-\lambda r}B\int_0^{-r} e^{-\lambda s}\psi(s)ds
\end{aligned}$$

Thus, we find that $\dot{\phi}(0) = A\phi(0) + B\phi(-r)$ if and only if

$$\lambda c - Ac - Be^{-r\lambda}c = \psi(0) - e^{-\lambda r}B\int_0^{-r} e^{-\lambda s}\psi(s)ds \tag{1.7}$$

Since we chose λ such that $\det\Delta(\lambda) \neq 0$, we find that the map $c \mapsto \lambda c - Ac - Be^{-r\lambda}c$ is surjective. In particular, there exists a $c \in \mathbb{R}^n$ that satisfies equality (1.7). For this choice of c, we have by construction that ϕ as defined in Eq. (1.6) satisfies $\phi \in \mathcal{D}(A)$ and $\lambda\phi - A\phi = \psi$. Since the map $c \mapsto \lambda c - Ac - Be^{-r\lambda}c$ is linear, so is its inverse. Thus, for a fixed λ, the choice of c depends continuously on ψ. We can combine this with the expression in (1.6) to see that the map $\psi \mapsto \phi = (\lambda I - A)^{-1}\psi$ is continuous, hence bounded. We conclude that $\lambda \in \rho(A)$ and thus that $\sigma(A) \subseteq \sigma_{pt}(A) = \{\lambda \in \mathbb{C} \mid \det\Delta(\lambda) = 0\}$. Combining this with the inequality $\{\lambda \in \mathbb{C} \mid \det\Delta(\lambda) = 0\} = \sigma_{pt}(A) \subseteq \sigma(A)$, the Lemma follows. [8] $\square$

Lemma 1.3.6. *Let $T(t)$, A be as above. If $\mu(t)$ is an eigenvalue of $T(t)$, then there exists a $\lambda \in \mathbb{C}$ such that $\mu(t) = e^{\lambda t}$. Furthermore, $(e^{\lambda t}, \phi)$ is an eigenpair of $T(t)$ if and only if (λ, ϕ) is an eigenpair of A. For $t > 0$ we find that*

$$\sigma(T(t))\backslash\{0\} = \{e^{\lambda t} \mid \det\Delta(\lambda) = 0\}$$

Proof. If (λ, ϕ) is an eigenpair of A, we have that $A\phi = \lambda\phi$ or $\phi = ce^{\lambda t}$ for some $c \in \mathbb{R}^n\backslash\{0\}$. In the previous Lemma we proved that $\det\Delta(\lambda) = 0$ if and only if λ is an eigenvalue of A; therefore we can use the definition of Δ to see that $x(t,\phi) = ce^{\lambda t} = \phi$ is an eigenfunction of $T(t)$. To compute the eigenvalue, we note that $(T(t)\phi)(\theta) = e^{\lambda t}e^{\lambda\theta}c = e^{\lambda t}\phi(\theta)$, i.e. the eigenvalue is given by $e^{\lambda t}$. Conversely, we see that if $(e^{\lambda t}, \phi)$ is an eigenpair of $T(t)$, then we should have that $\phi(\theta) = e^{\lambda\theta}c$ for some $c \neq 0$. Thus we find that (λ, ϕ) is an eigenpair of A.

Combining this with the previous lemma, we find that

$$\sigma_{pt}(T(t)) = \{e^{\lambda t} \mid \lambda \in \sigma_{pt}(A)\} = \{e^{\lambda t} \mid \det\Delta(\lambda) = 0\}$$

We now recall that for $t \geq r$, the operator $T(t)$ is compact; hence it follows from the general properties for the spectrum of compact operators that $\sigma(T(t))\backslash\{0\}$ consists of point spectrum only. Thus, for $t \geq r$, the claim follows. For any given $0 < t < r$, we note that there exists an $n \in \mathbb{N}$ such that $nt \geq r$. Using the properties of the semi-flow, we have that $\sigma(T(nt)) = \sigma(T(t)^n)$. We can now use the Polynomial Spectral Theorem (see, for example, [13]) to see that

$$\sigma(T(nt)) = \sigma(T(t)^n) = (\sigma(T(t)))^n := \{\lambda^n \mid \lambda \in \sigma(T(t))\}$$

Since $nt \geq r$, we have already proven the claim in this case and we can use this to find that

$$\sigma(T(t))\backslash\{0\} = \{\sqrt[n]{\lambda} \mid \lambda \in \sigma(T(nt)\backslash\{0\}\} = \{\left(e^{\lambda nt}\right)^{1/n} \mid \lambda \in \mathbb{C},\ \det\Delta(\lambda) = 0\} = \{e^{\lambda t} \mid \lambda \in \mathbb{C},\ \det\Delta(\lambda) = 0\}$$

which proves the claim in the case $0 < t < r$. [8] $\square$

We recall from Section 1.2 that if all the zero's of $\det \Delta(\lambda)$ satisfy $\mathrm{Re}\,(\lambda) < 0$, then the fixed point $x = 0$ of (1.5) is asymptotically stable; if there exists a root such that $\mathrm{Re}\,(\lambda) > 0$, then $x = 0$ is unstable. However, if there are roots on the imaginary axis and all the other roots are in the left half plane, we are not yet able to determine the stability of $x = 0$. To tackle this question, we note by the previous lemma's that the spectrum of $T(t)$ is related to the zero's of $\det \Delta(\lambda)$. We can try to decompose the state space $\mathcal{C}$ in two invariant subsets U and V, such that we 'remove' the purely imaginary eigenvalues by restricting $T(t)$ to U. Then the system (1.5) restricted to U is stable. We illustrate this procedure by an example.

Example 1.3.1. We consider the system (1.5) with $A + B = 0$ and $Ar < 1$. One can show that the roots of $\det \Delta(\lambda)$ all satisfy $\mathrm{Re}\,(\lambda) < 0$, except for a simple root at $z = 0$.[8] Because this root at $z = 0$, we cannot apply Corollary 1.2.1 to analyze the stability of the fixed point $x = 0$.

We define $e_0(\theta) = 1 = e^{0\theta}$ for $-r \leq \theta \leq 0$. Furthermore we define:

$$\mathcal{C}_0 = \{\phi \in \mathcal{C} \mid \phi(0) + B \int_{-r}^{0} \phi(\theta) d\theta = 0\}$$

$$\Pi_0(\phi) = \frac{1}{1 + Br} \left[\phi(0) + B \int_{-r}^{0} \phi(\theta) d\theta\right] e_0$$

We first note that Π_0 is a projection, i.e. $\Pi_0^2 = \Pi_0$. If we introduce the notation

$$\alpha = \frac{1}{1 + Br} \left[\phi(0) + B \int_{-r}^{0} \phi(\theta) d\theta\right]$$

we find that

$$\Pi_0^2 \phi = \frac{1}{1 + Br} \left[\alpha + B \int_{-r}^{0} \alpha e_0 d\theta\right] e_0 = \frac{1}{1 + Br} \left[\alpha + Br\alpha\right] e_0 = \alpha e_0 = \Pi_0 \phi$$

If we denote by $\mathcal{R}(\Pi_0)$ the range of Π_0 and by $\mathcal{N}(\Pi_0)$ the kernel, we can write any $\phi \in \mathcal{C}$ as $\phi = \phi_0 + \phi_r$ with $\phi_0 \in \mathcal{N}(\Pi_0)$ and $\phi_r \in \mathcal{R}(\Pi_0)$. Indeed, if we set $\phi_r = \Pi_0 \phi$ and $\phi_0 = \phi - \Pi_0 \phi$, we have that $\phi_r \in \mathcal{R}(\Pi_0)$ and we can use the property of the projection to see that $\phi_0 \in \mathcal{N}(\Pi_0)$. Thus, we find the decomposition $\mathcal{C} = \mathcal{N}(\Pi_0) \oplus \mathcal{R}(\Pi_0)$.

We now show that $\mathcal{C}_0 = \mathcal{N}(\Pi_0)$ is invariant under $T(t)$, i.e.

$$x(t) + B \int_{-r}^{0} x(t + \theta) d\theta = 0, \quad t \geq 0$$

Let $\phi \in \mathcal{C}_0$ and denote by $x(t)$ the solution of (1.5) through ϕ. Since $x(0) = \phi(0)$ we have for $t = 0$ that

$$x(0) + B \int_{-r}^{0} x(\theta) d\theta = \phi(0) + B \int_{-r}^{0} \phi(\theta) d\theta = 0$$

by the assumption that $\phi \in \mathcal{C}_0$. For $t > 0$ we have that

$$\dot{x}(t) + B \int_{-r}^{0} \dot{x}(t + \theta) d\theta = \dot{x}(t) + B(x(t) - x(t - r)) = Ax(t) + Bx(t - r) + B(x(t) - x(t - r)) \quad = (A + B)x(t) = 0$$

by the assumption that $A + B = 0$. This implies that $x(t) + B \int_{-r}^{0} x(t + \theta) d\theta$ is constant, and since $\phi(0) + B \int_{-r}^{0} \phi(\theta) d\theta = 0$ we find that $x(t) + B \int_{-r}^{0} x(t + \theta) d\theta = 0$ for all $t \geq 0$. We conclude that $T(t)\phi \in \mathcal{C}_0$ if $\phi \in \mathcal{C}_0$ for all $t \geq 0$.

Since $\mathcal{C}_0$ is invariant under $T(t)$, we can restrict $T(t)$ to $\mathcal{C}_0$ to obtain the flow map $S(t)$. Since $\Pi_0 e_0 = e_0 \neq 0$, we find that $e_0 \notin \mathcal{C}_0$. Thus by construction, 1 (corresponding to the root $\lambda = 0$ of Δ) is not an eigenvalue of $S(t)$. Since all the other roots of $\det \Delta(\lambda)$ were in the left half plane, the fixed point $x = 0$ is now a stable equilibrium if we restrict the state space of (1.5) to $\mathcal{C}_0$. [8]

Chapter 2

Time delayed feedback control

After discussing the basic concepts in the study of differential delay equations, we now turn to Pyragas control. We study the system

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)\left|z(t)\right|^2 z(t) \tag{2.1}$$

where $\lambda, \gamma \in \mathbb{R}$ are parameters and $z : \mathbb{R} \to \mathbb{C}$. This system can be used to model a various physical phenomena. [3] Furthermore, system (2.2) is the normal form of a subcritical Hopf bifurcation. This implies the existence of an unstable periodic orbit for $\lambda < 0$. This periodic orbit, that we explicitly compute in Section 2.1, will be the target state of the control.

For the controlled system, we write

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)\left|z(t)\right|^2 z(t) - Ke^{i\beta}\left[z(t) - z(t-\tau)\right] \tag{2.2}$$

with $K \in \mathbb{R}$, $\beta \in [0, \pi]$ and τ is an appropriately chosen delay term.

The main idea of the analysis in this chapter is to choose the value of K in such a way that the target periodic orbit still emmanates from a Hopf bifurcation, but now from a supercritical one. The target periodic orbit is then stable for parameter values near the bifurcation point. To do so, we do not fix τ as the period T of the target state from the start, but instead we treat τ as a parameter. For general values of τ, we cannot expect the periodic orbit of (2.1) to be a solution of (2.2), but by studying these values of τ as well, we are able to use the Hopf bifurcation theorem in such a way that it gives us information on the stability of the periodic solution for $\tau = T$.

In Section 2.1, we study the uncontrolled system, (2.1), in more detail. In Section 2.2, we study conditions for the occurence of a Hopf bifurcation of system (2.2); in Section 2.3, we determine whether the Hopf bifurcation at the bifurcation points is sub– or supercritical. Using this, we give conditions for the target periodic orbit to be an (un)stable solution of (2.2) in Section 2.3. In Sections 2.2 and 2.4 we will mostly follow [10].

2.1 Bifurcations of the uncontrolled system

Before turning to the controlled system (2.2), we first study the bifurcation diagram of the uncontrolled system (2.1) in more detail. We first note that $z = 0$ is a fixed point of (2.1). Using stability theory for ordinary differential equations we can determine the stability of this fixed point; the necessary stability theory – if not yet known to the reader – can be found in Appendix A.

Lemma 2.1.1. *The fixed point $z = 0$ of system (2.1) is stable if $\lambda < 0$ and unstable if $\lambda > 0$.*

Proof. Put $z(t) = x(t) + iy(t)$ and rewrite (2.1) as

$$\dot{x}(t) + i\dot{y}(t) = (\lambda + i)(x(t) + iy(t)) + (1 + i\gamma)(x^2(t) + y^2(t))(x(t) + iy(t))$$

which is equivalent to the system $(\dot{x}(t), \dot{y}(t)) = f(x(t), y(t))$, with $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \begin{pmatrix} \lambda x - y + (x^2 + y^2)(x - \gamma y) \\ x + \lambda y + (x^2 + y^2)(y + \gamma x) \end{pmatrix} \tag{2.3}$$

The derivative of (2.3) at the point $(x, y) = (0, 0)$ is given by

$$Df(0, 0) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

whose eigenvalues are

$$\mu_\pm = \lambda \pm i$$

Noting that $\operatorname{Re}(\mu_\pm) = \lambda$ we see that origin is a stable equilibrium if $\lambda < 0$ and an unstable equilibrium if $\lambda > 0$ (see also Corollary A.3.1). ☐

We note that $\mu(0) = \pm i \in \mathbb{C}\backslash\mathbb{R}$ and $\frac{d}{d\lambda}\operatorname{Re}(\mu(\lambda)) = \frac{d\lambda}{d\lambda} = 1 \neq 0$. Therefore, the Hopf bifurcation theorem, Theorem A.6.1, guarantees the existence of a periodic solution of (2.1) for λ near 0. It turns out that we can actually compute this periodic orbit of (2.1).

Lemma 2.1.2. *A periodic solution of* (2.1) *is given by*

$$z(t) = \sqrt{-\lambda}e^{i(1-\gamma\lambda)t}, \qquad \lambda < 0 \tag{2.4}$$

with period

$$T = \frac{2\pi}{1 - \gamma\lambda}$$

Proof. We rewrite (2.1) in polar coordinates:

$$\dot{r}(t)e^{i\theta(t)} + i\dot{\theta}(t)r(t)e^{i\theta(t)} = (\lambda + i)r(t)e^{i\theta(t)} + (1 + i\gamma)r^3(t)e^{i\theta(t)}$$

Cancelling the factor $e^{i\theta(t)}$ on both sites an taking real and imaginary parts (note that both r and θ are real) we find the following two equations:

$$\dot{r}(t) = \lambda r(t) + r^3(t) \tag{2.5}$$

$$\dot{\theta}(t)r(t) = r(t) + \gamma r^3(t) \tag{2.6}$$

The root $r = 0$ of (2.5) corresponds to the equilibrium at the origin. We note that $r(t) = \pm\sqrt{-\lambda}$ are roots of (2.5). Since in polar coordinates we should have that $r(t) \geq 0$,we find that for $\lambda < 0$ there exsits a periodic orbit satisfying $r(t) = \sqrt{-\lambda}$. For $r(t) = \sqrt{-\lambda}$, Eq. (2.6) reduces to $\dot{\theta}(t) = 1 - \gamma\lambda$, i.e. $\theta(t) = (1 - \gamma\lambda)t + t_0$. Thus, a periodic solution of (2.1) is given by

$$z(t) = \sqrt{-\lambda}e^{i(1-\gamma\lambda)t}$$

with minimal period $T = \frac{2\pi}{1-\gamma\lambda}$. ☐

Using the theory of characterstic multipliers (see Appendix A), we are able to determine the stability of the periodic solution (2.4) of system (2.1).

Lemma 2.1.3. *The periodic solution* (2.4) *of system* (2.1) *is unstable if* $\gamma\lambda < 1$ *and stable if* $\gamma\lambda > 1$.

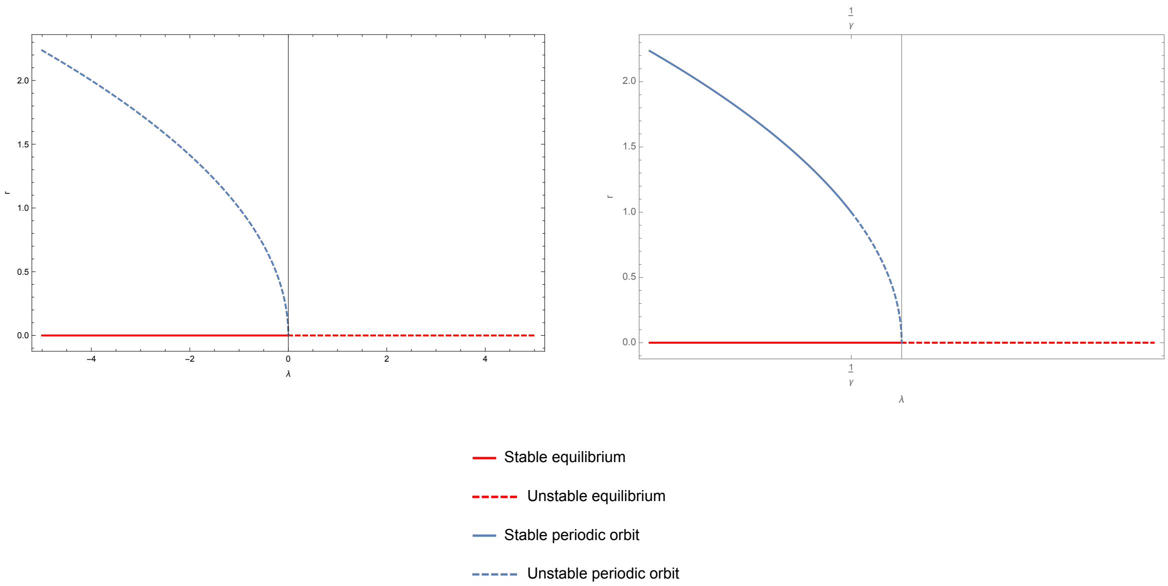


Figure 2.1: Bifurcation diagram of system (2.1) for $\gamma > 0$ (left) and $\gamma < 0$ (right)

Proof. We write $g(r, \theta) = (\lambda r + r^3, 1 + \gamma r^2)$ such that the system (2.5), (2.6) for $r(t) \neq 0$ becomes $(\dot{r}(t), \dot{\theta}(t)) = g(r(t), \theta(t))$. Then in polar coordinates, we find that

$$A(t) = Dg(\sqrt{-\lambda}, (1-\gamma\lambda)t) = \begin{pmatrix} \lambda + 3r(t)^2 & 0 \\ 2\gamma r(t) & 0 \end{pmatrix}\Bigg|_{r=\sqrt{-\lambda}, \theta(t)=(1-\gamma\lambda)t} = \begin{pmatrix} -2\lambda & 0 \\ 2\gamma\sqrt{-\lambda} & 0 \end{pmatrix}$$

Thus, the linear variational equation is given by $\dot{x}(t) = A(t)x(t)$ with $A(t) = Dg(\sqrt{-\lambda}, (1-\gamma\lambda)t)$ as above. Since $A(t) = A$ does not depend on time, we see that the fundamental matrix solution is given by $X(t) = e^{At}$. Thus, we find for the monodromy matrix C (see Definition A.4.2):

$$C = X(0)^{-1}X(T) = e^{AT}$$

The eigenvalues of A are given by $\lambda_1 = 0, \lambda_2 = -2\lambda$. Since $C = e^{AT}$, we find that the eigenvalues $\mu_{1,2}$ of C are given by

$$\begin{aligned} \mu_1 &= e^{\lambda_1 T} = 1 \\ \mu_2 &= e^{\lambda_2 T} = e^{-2\lambda \frac{2\cdot\pi}{1-\gamma\lambda}} \end{aligned}$$

We indeed find the trivial characteristic multiplier, as should be according to the theory of characterstic multipliers (see Appendix A). Since the periodic orbit of interest only exists for $\lambda < 0$, we find that for the non-trivial characteristic multiplier that $\mu_2 > 1$ if $\gamma\lambda < 1$ and $\mu_2 < 1$ if $\gamma\lambda > 1$. We conclude that the periodic orbit (2.4) of (2.1) is unstable if $\gamma\lambda < 1$ and stable for $\gamma\lambda > 1$ (see Theorem A.5.1). Specifically, if we choose $\gamma > 0$, we have that $\gamma\lambda < 0 < 1$ for all values of $\lambda < 0$; thus the periodic orbit is unstable for all values of $\lambda < 0$. For $\gamma < 0$, we find that the periodic orbit is unstable for $\frac{1}{\gamma} < \lambda < 0$ and stable for $\lambda < \frac{1}{\gamma}$. □

The behaviour described in Lemmata 2.1.1 – 2.1.3 is summarized in Figure 2.1.

2.2 Occurence of a Hopf bifurcations in the controlled system

In the previous section, we studied the bifurcation diagram of system (2.1). We found for $\lambda < 0$ and $\gamma\lambda < 1$ that (2.4) is an unstable periodic solution of system (2.1). This periodic orbit is the orbit we want to stabilize via feedback control, see (2.2). We want to choose the parameters K, β in such a way that the periodic orbit (2.4) emmanates from a supercritical Hopf bifurcation, since this implies stability of the periodic orbit (2.4) near the bifurcation point.

In order to do so, we first determine for which parameter values we find a Hopf bifurcation. We introduce the Hopf bifurcation theorem for differential delay equations.

Theorem 2.2.1. *Let us consider the differential delay equation*

$$\begin{cases} \dot{x}(t) = A(\mu)x(t) + B(\mu)x(t-\tau) + g(x_t,\mu) & \text{for } t \geq 0 \\ x(t) = \phi(t) & \text{for } -\tau \leq t \leq 0 \end{cases} \tag{2.7}$$

where $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is such that $x_t(\theta) = x(t+\theta)$. Furthermore, μ is a scalar parameter, $A(\mu), B(\mu)$ $n \times n$-matrices, $\mu \mapsto A(\mu), \mu \mapsto B(\mu)$ smooth maps, $g : \mathcal{C}([-\tau,0],\mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^k$ mapping for $k \geq 2$ and $\phi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$. Assume that $g(0,\mu) = D_1 g(0,\mu) = 0$ for all $\mu \in \mathbb{R}$. Denote by $\Delta(\lambda,\mu)$ the characterstic function of the linearized system of (2.7).

Assume that for $\mu = \mu_0$ there exists an $\omega_0 \in \mathbb{R}\backslash\{0\}$ such that $z = \pm i\omega_0$ are simple roots of $\Delta(\lambda,\mu_0)$ and no other roots of $\Delta(\lambda,\mu_0)$ are of the form $z = i\omega n$ for some $n \in \mathbb{Z}$.

Furthermore, let $p, q \in \mathbb{C}^n$ be such that

$$\begin{aligned} \Delta(i\omega_0,\mu_0)p &= 0 \\ \Delta(i\omega_0,\mu_0)^T q &= 0 \\ qD_1\Delta(i\omega_0,\mu_0)p &= 1 \end{aligned} \tag{2.8}$$

and assume that

$$Re\ (q \cdot D_2\Delta(i\omega_0,\mu_0)p) < 0 \tag{2.9}$$

Then for $\mu = \mu_0$ a Hopf bifurcation of the origin occurs.

The theorem for more general cases of differential delay equations and a proof can be found in [3].

Remark 2.2.1. The second condition in the Hopf bifurcation theorem ensures that the eigenvalue on the imaginary axis moves into the right half plane for $\mu > \mu_0$. If we would have that Re $(q \cdot \Delta_2(i\omega_0,\mu_0)p) > 0$, i.e. the eigenvalue on the imaginary axis moves into the left half plane for $\mu > \mu_0$, we can still ensure that the condition 2.9 is satisfied by studying the system (2.7) with the substitution $\mu \to -\mu$. Instead of the condition (2.9), we could therefore also require that Re $(q \cdot \Delta_2(i\omega_0,\mu_0)p) \neq 0$. The formulation as in Theorem (2.2.1), however, will be more convenient in the study of the direction of the Hopf bifurcation, since it makes the 'direction of the Hopf bifurcation' a well–defined notion.

Remark 2.2.2. By the occurence of a Hopf bifurcation at $\mu = \mu_0$ we mean that there exists a mapping $\epsilon \mapsto \mu^*(\epsilon)$, such that for all ϵ sufficiently small, system (2.7) has a periodic solution for $\mu = \mu^*(\epsilon)$. This periodic solution is a unique periodic orbit for $\mu = \mu^*(\epsilon)$ except for a translation in phase.

We have that the origin $z = 0$ is an equilibrium of system (2.2). Using Theorem 2.2.1, we want to study the values of the parameters λ, τ for which we find a Hopf bifurcation of the origin.

We want to study the Hopf bifurcation with parameters in the (λ,τ)-plane. Theorem 2.2.1 applies to scalar parameters and therefore we should choose a curve through the (λ,τ) plane. There are, of course, different ways to do this. In this Chapter, we will vary the parameter λ and leave τ fixed. A different approach is treated in Chapter 4.

Lemma 2.2.2. *Let us consider the system* (2.2) *where we leave all parameters but* λ *fixed. Let* $(\lambda, \tau) \neq (0,0)$ *be such that*

$$\lambda = K\left[\cos\beta - \cos(\beta - \phi)\right] \tag{2.10}$$

$$\tau = \frac{\phi}{1 - K\left[\sin\beta - \sin(\beta - \phi)\right]} \tag{2.11}$$

for some $\phi \in \mathbb{R}$*. Furthermore, assume that*

$$1 + K\tau e^{i(\beta-\phi)} \neq 0 \tag{2.12}$$

$$1 + K\tau\cos(\beta - \phi) > 0 \tag{2.13}$$

Then a Hopf bifurcation of the origin of system (2.2) *occurs.*

Proof. We note that we can cast system (2.2) in the form of (2.7) by setting

$$A(\lambda) = \lambda + i - Ke^{i\beta}, \quad B(\lambda) = Ke^{i\beta}, \quad g(x_t, \lambda) = (1 + i\gamma)\left|x_t(0)\right|^2 x_t(0)$$

We see that the maps $\lambda \mapsto A(\lambda)$, $\lambda \mapsto B(\lambda)$ are smooth and that g is a smooth map with $g(0,\lambda) = D_1 g(0,\lambda) = 0$.

We note that the linearized system of (2.2) is given by the differential delay equation

$$\dot{z}(t) = (\lambda + i)z(t) - Ke^{i\beta}\left[z(t) - z(t-\tau)\right]$$

whose characteristic function Δ (see Definition 1.2.1) reads

$$\Delta(\mu, \lambda) = \mu - \lambda - i + Ke^{i\beta}\left[1 - e^{-\mu\tau}\right]$$

We want to find the value of (λ, τ) such that we find a non–zero root of the characterstic equation on the imaginary axis. Writing this root as $\mu = i\omega \in \mathbb{R}\backslash\{0\}$, we should have that

$$i\omega = \lambda + i - Ke^{i\beta}\left[1 - e^{-i\omega\tau}\right]$$

Splitting in real and imaginary parts, we find that

$$0 = \lambda - K\left[\cos\beta - \cos(\beta - \omega\tau)\right]$$
$$\omega = 1 - K\left[\sin\beta - \sin(\beta - \omega\tau)\right]$$

Introducing the notation $\phi = \omega\tau$ and rewriting yields (2.10) and (2.11). Thus, for (λ, τ) satisfying (2.10) and (2.11) we find a non–zero root of the characterstic equation on the imaginary axis.

We note that

$$D_1\Delta(i\omega, \lambda) = 1 - Ke^{i\beta}e^{-i\omega\tau}(-\tau) = 1 + K\tau e^{i(\beta-\phi)}$$

Thus, if (2.12) is satisfied, we find that $\mu = i\omega$ is a simple zero of the characteristic equation. By construction, $\mu = i\omega$ is the only zero of the characterstic equation of the form $\mu = i\omega n$, $n \in \mathbb{Z}$.

Set $p = 1, q = \alpha$. If λ, τ satisfies (2.10), (2.11) for $\phi = \omega\tau$, then we have that $\Delta(i\omega, \lambda)p = \Delta(i\omega, \lambda)^T q = 0$. Furthermore, we find that

$$q \cdot D_1\Delta(i\omega, \lambda)p = \alpha\left(1 - Ke^{i\beta}e^{-i\omega\tau}(-\tau)\right) = \alpha\left(1 + K\tau e^{i(\beta-\phi)}\right)$$

Thus, if we choose

$$p = 1, \quad q = \alpha = \frac{1}{1 + K\tau e^{i(\beta-\phi)}}$$

the conditions (2.8) are satisfied. We have that

$$\text{Re } (q \cdot D_2\Delta(i\omega, \lambda)p) = \text{Re } \left(-\frac{1}{1 + K\tau e^{i(\beta-\phi)}} \right) = -\frac{1 + K\tau\cos(\beta - \phi)}{\left|1 + K\tau e^{i(\beta-\phi)}\right|^2}$$

Thus, Re $(q \cdot D_2\Delta(i\omega, \lambda)p) < 0$ if and only if (2.13) is satisfied. Using Theorem 2.2.1, we conclude that if the conditions (2.10) - (2.13) are satisfied a Hopf bifurcation of the origin occurs. □

If we fix the value of K, then a given value of ϕ gives us directly a value of λ via equation (2.10) and a value of τ via (2.11). Thus, we can treat ϕ as a free parameter and we find that the equations (2.10)-(2.11) parametrize a curve in (λ, τ)-space.

Definition 2.2.1. We define the *Hopf bifurcation curve* as the curve in (λ, τ)-parameter space parametrized by equations (2.10) and (2.11).

Using Lemma 2.2.2, we see that the points on the Hopf bifurcation curve are candidate parameter values for the occurence of a Hopf bifurcation. If for a point (λ, τ) on the Hopf bifurcation curve the conditions (2.12)-(2.13) are also satisfied, we indeed find a Hopf bifurcation of the origin of system (2.2).

We can also look at the points in (λ, τ)-parameter space for which (2.4) is a solution of system (2.2). We recall from Lemma 2.1.2 that for $\lambda < 0$, (2.4) is a periodic solution of (2.1) with period $T = \frac{2\pi}{1-\gamma\lambda}$. Hence, for $\tau = T = \frac{2\pi}{1-\gamma\lambda}$, (2.4) is by construction a periodic solution of (2.2). This motivates the following definition:

Definition 2.2.2. We define the *Pyragas curve* in (λ, τ)-parameter space as the graph of

$$\tau(\lambda) = \frac{2\pi}{1 - \gamma\lambda} \tag{2.14}$$

with λ in the domain $(-\infty, 0)$.

For points on the Pyragas curve, we have that the periodic solution (2.4) is a solution of (2.2).

We note that the Pyragas curve ends at the point $(\lambda, \tau) = (0, 2\pi)$. By applying (2.10) and (2.11) with $\phi = 2\pi$, we find that this point also lies on the Hopf bifurcation curve, i.e. the Pyragas curve ends on the Hopf bifurcation curve. The Hopf bifurcation at this point can be sub- or supercritical. If the Hopf bifurcation is subcritical, an unstable periodic orbit occurs to the left hand side of the Hopf bifurcation curve; for a supercritical Hopf bifurcation, a stable periodic orbit is generated to the right hand side of the Hopf bifurcation curve. Thus, we can try to determine the orientation of the Hopf bifurcation curve with respect to the Pyragas curve at $(\lambda, \tau) = (0, 2\pi)$ and combine this with the direction of the Hopf bifurcation (i.e. whether the Hopf bifurcation is sub– or supercritical) at $(\lambda, \tau) = (0, 2\pi)$. This will give us information on the stability of the periodic solution (2.4) of (2.2) for parameter values near $(\lambda, \tau) = (0, 2\pi)$.

To apply this method, we of course need to know the direction of the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$. Determining this direction will be the main topic in the following section.

2.3 Direction of the Hopf bifurcation in the controlled system

In the previous Section, we introduced the Hopf bifurcation curve and the Pyragas curve in the (λ, τ)-plane, see Definitions 2.2.1-2.2.2. Parameter values on the Hopf bifurcation curve are candidates for the occurence of a Hopf bifurcation of system (2.2); for parameter values on the Pyragas curve, (2.4) is a solution of (2.2). We saw that the Pyragas curve ends on the Hopf bifurcation curve. Therefore, we can try to choose the parameters K, β in such a way that the periodic solution (2.4) of (2.2) emmanates from a supercritical Hopf bifurcation; then (2.4) is a stable solution of (2.2) for parameter values near the bifurcation point.

In order to apply this method, we start by determining the direction of the Hopf bifurcation.

Remark 2.3.1. By the 'direction of the Hopf bifurcation' at a point we mean whether the Hopf bifurcation is sub– or supercritical at that point.

In [10], the direction of the Hopf bifurcation for parameter values on the Pyragas curve is determined by performing a normal form reduction on the system (2.2). In literature, we can also find closed–form formulas that determine the direction of the Hopf bifurcation for a general differential delay equation, see for example [3]. Here, we use this closed–form formula to determine the direction of the Hopf bifurcation of (2.2) for parameter values on the Hopf bifurcation curve.

Theorem 2.3.1. *Let us consider the differential delay equation*

$$\begin{cases} \dot{x}(t) = A(\mu)x(t) + B(\mu)x(t-\tau) + g(x_t, \mu) & \text{for } t \geq 0 \\ x(t) = \phi(t) & \text{for } -\tau \leq t \leq 0 \end{cases} \tag{2.15}$$

where μ is a scalar parameter, and $A(\mu)$, $B(\mu)$ and g are as in Theorem 2.2.1. Assume that a Hopf bifurcation of the equilibrium $x = 0$ occurs for $\mu = \mu_0$, i.e. let the conditions in Theorem 2.2.1 be satisfied. Denote by $\Delta(\lambda, \mu)$ the characteristic function of (2.15) and let $\omega \in \mathbb{R}\backslash\{0\}$ be such that $\det \Delta(i\omega_0, \mu_0) = 0$. Let $p, q \in \mathbb{C}^n$ satisfy (2.8). If we introduce

$$\mu_2 = \frac{Re\,(c)}{Re\,(q \cdot D_2\Delta(i\omega_0, \mu_0)p)} \tag{2.16}$$

with

$$\begin{aligned} c = &\frac{1}{2} q \cdot D_1^3 g(0, \mu_0)(\phi, \phi, \overline{\phi}) + q \cdot D_1^2 g(0, \mu_0)(e^{0\cdot}\Delta(0, \mu_0)^{-1} D_1^2 g(0, \mu_0)(\phi, \overline{\phi}), \phi) \\ &+ \frac{1}{2} q \cdot D_1^2 g(0, \mu_0)(e^{2i\omega_0 \cdot}\Delta(2i\omega_0, \mu_0)^{-1} D_1^2 g(0, \mu_0)(\phi, \phi), \overline{\phi}) \end{aligned} \tag{2.17}$$

then for $\mu_2 < 0$, the Hopf bifurcation is subcritical; for $\mu_2 > 0$, the Hopf bifurcation is supercritical.

A proof of the theorem can be found in [3].

We want to apply Theorem 2.3.1 to the system (2.2). Since the differential delay equation (2.15) in Theorem 2.3.1 is real–valued, we give a real–valued representation for (2.2) and its characteristic function.

Lemma 2.3.2. *System (2.2) can also be represented as a system $\mathbb{R}^2$ given by*

$$\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = &\begin{pmatrix} \lambda - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \lambda - K\cos\beta \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t), & x_2(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &+ K \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} \end{aligned} \tag{2.18}$$

whose characteristic function Δ is given by

$$\Delta(\mu, \lambda) = \mu I - \begin{pmatrix} \lambda - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \lambda - K\cos\beta \end{pmatrix} - Ke^{-\mu\tau} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{2.19}$$

Proof. Setting $z = x_1 + ix_2$ in (2.2) and taking real and imaginary parts yields (2.18), whose characteristic function is given by (2.19). □

Define $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^2)$ as $x_t(\theta) = x(t + \theta)$. Then we can describe the non-linear term in (2.18) by the function $g : \mathcal{C}([-\tau, 0], \mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}^2$ given by

$$g(x_t, \lambda) = \langle x_t(0), x_t(0) \rangle \, Cx_t(0) \quad \text{with } C = \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} \tag{2.20}$$

where $\langle . \, , . \rangle$ denotes the standard inner product on $\mathbb{R}^2$. In order to apply formula (2.16) and (2.17), we begin by computing $D_1^2 g(0, \lambda)$ and $D_1^3 g(\phi, \lambda)$.

Lemma 2.3.3. *Define g as in (2.20). For all values of λ, $D_1^2 g(0,\lambda) = 0$ holds. Furthermore, we have that*

$$D_1^3 g(\phi,\lambda)(f_1,f_2,f_3) = \sum_{\sigma\in S_3} \left\langle f_{\sigma(1)}(0), f_{\sigma(2)}(0)\right\rangle C f_{\sigma(3)}(0) \tag{2.21}$$

for all $\phi, f_1, f_2, f_3 \in \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right)$. Here S_3 denotes the permutation group of three objects.

Proof. We introduce

$$T : \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right) \to \mathbb{R}^2, \quad T(\phi) = \phi(0) \tag{2.22}$$

which is a bounded linear operator. We can now write

$$g(\phi,\lambda) = \langle T(\phi), T(\phi)\rangle\, CT(\phi)$$

Denote by

$$h : \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right) \to \mathbb{R}, \quad h(\phi) = \langle T(\phi), T(\phi)\rangle$$

Since T is a linear map, $DT(\phi) = T$ holds for all $\phi \in \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right)$. By combining the derivative of the inner product with the chain rule, we find that

$$Dh(\phi) = \langle T(\phi), T(.)\rangle + \langle T(.), T(\phi)\rangle : \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right) \to \mathbb{R}$$

Using Leibniz' Rule, we find that

$$D_1 g(\phi,\lambda) = \langle T(\phi), T(.)\rangle\, CT(\phi) + \langle T(.), T(\phi)\rangle\, CT(\phi) + \langle T(\phi), T(\phi)\rangle\, CT(.)$$

Denote by $\mathrm{Lin}(X,Y)$ the space of linear maps from X to Y. We have that $D_1 g(\phi,\lambda) \in \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathbb{R}^2)$ and $D_1^2 g(\phi,\lambda) \in \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathbb{R}^2))$. Denoting by .. the argument such that

$$D_1^2 g(\phi,\lambda)(..) : \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right) \to \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathbb{R}^2)$$

we find by the Leibniz rule that

$$\begin{aligned} D_1^2 g(\phi,\lambda) = {} & \langle T(..), T(.)\rangle\, CT(\phi) + \langle T(\phi), T(.)\rangle\, CT(..) + \langle T(.), T(..)\rangle\, CT(\phi) \\ & + \langle T(.), T(\phi)\rangle\, CT(..) + \langle T(..), T(\phi)\rangle\, CT(.) + \langle T(\phi), T(..)\rangle\, CT(.) \end{aligned} \tag{2.23}$$

Since $T(0) = 0$, we find that $D_1^2 g(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$.

Denote by ... the argument such that

$$D_1^3(\phi,\lambda)(...) : \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right) \to \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathrm{Lin}(\mathcal{C}\left([-\tau,0],\mathbb{R}^2\right), \mathbb{R}^2))$$

By (2.23) we have that $D_1^2 g(\phi,\lambda)$ is a linear map in ϕ. Therefore, we find that

$$\begin{aligned} D_1^3 g(\phi,\lambda,\tau) = {} & \langle T(..), T(.)\rangle\, CT(...) + \langle T(...), T(.)\rangle\, CT(..) + \langle T(.), T(..)\rangle\, CT(...) \\ & + \langle T(.), T(...)\rangle\, CT(..) + \langle T(..), T(...)\rangle\, CT(.) + \langle T(...), T(..)\rangle\, CT(.) \end{aligned}$$

By applying this formula, we find that

$$D_1^3 g(\phi,\lambda)(f_1,f_2,f_3) = \sum_{\sigma\in S_3} \left\langle T(f_{\sigma(1)}), T(f_{\sigma(2)})\right\rangle CT(f_{\sigma(3)}) = \sum_{\sigma\in S_3} \left\langle f_{\sigma(1)}(0), f_{\sigma(2)}(0)\right\rangle C f_{\sigma(3)}(0)$$

for all $\phi, f_1, f_2, f_3 \in \mathcal{C}\left([-\tau,0],\mathbb{R}^2\right)$. □

We can apply Lemma 2.3.3 to calculate the value of c as defined in (2.17).

Lemma 2.3.4. *Let (λ, τ) be a point on the Hopf bifurcation curve (see Definition 2.2.1) and let $\phi \in \mathbb{R}$ satisfy* (2.10)–(2.11)*. Then the value of $Re\, c$ with c as defined in* (2.17) *is given by*

$$Re\, c = \frac{4(1 + K\tau\left(\cos(\beta - \phi) + \gamma \sin(\beta - \phi)\right)}{\left|1 + K\tau e^{i(\beta-\phi)}\right|^2} \tag{2.24}$$

Proof. We first calculate p, q as defined in (2.8). Set

$$A = \begin{pmatrix} \lambda - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \lambda - K\cos\beta \end{pmatrix}, \quad B = K\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{2.25}$$

We recall from Lemma 2.2.2 that if (λ, τ) lies on the Hopf bifurcation curve, we find that there exsists an $\omega \in \mathbb{R}$ such that $\Delta(i\omega, \lambda) = 0$. Using Lemma 2.3.2 we can rephrase this condition by saying that $i\omega$ is an eigenvalue of the matrix $A + Be^{-i\omega}$. By definition, $p \in \mathbb{C}^2$ satisfies $\Delta(i\omega, \lambda, \tau)p = 0$ (see (2.8)), i.e. p is an eigenvector of the matrix $A + Be^{-i\omega}$ corresponding to the eigenvalue $i\omega$. Using this, we find for p:

$$p = \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{2.26}$$

Similarly, we find that

$$q = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Here, the renormalization factor $\alpha \in \mathbb{C}$ should be chosen such that

$$q \cdot D_1\Delta(i\omega, \lambda)p = 1 \tag{2.27}$$

We note that

$$D_1\Delta(i\omega, \lambda) = I + K\tau e^{-i\omega\tau}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = I + K\tau e^{-i\phi}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

Here, we write $\phi = \omega\tau$, as we did in Lemma 2.2.2. Thus we find that

$$\begin{aligned} q \cdot D_1\Delta(i\omega, \lambda)p &= \alpha\left((1, i)\begin{pmatrix} 1 \\ -i \end{pmatrix} + K\tau e^{-i\phi}(1, i)\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right) \\ &= 2\alpha\left(1 + K\tau e^{i(\beta-\phi)}\right) \end{aligned}$$

Condition (2.27) thus implies that

$$\alpha = \frac{1}{2(1 + K\tau e^{i(\beta-\phi)})}$$

Applying Lemma 2.3.3 and expression 2.17 we find that

$$\begin{aligned} c &= \frac{1}{2}q \cdot D_1^3 g(0, \lambda)(\phi, \phi, \overline{\phi}) + 0 + 0 \\ &= \frac{1}{2}q \cdot \left(2\left\langle \phi(0), \phi(0)\right\rangle C\overline{\phi(0)} + 2\left\langle \overline{\phi(0)}, \phi(0)\right\rangle C\phi(0) + 2\left\langle \phi(0), \overline{\phi(0)}\right\rangle C\phi(0)\right) \\ &= q \cdot \left(\langle p, p\rangle C\overline{p} + \langle p, \overline{p}\rangle Cp + \langle \overline{p}, p\rangle Cp\right) = \frac{4(1 + i\gamma)}{1 + K\tau e^{i(\beta-\phi)}} \end{aligned}$$

which gives (2.24). □

We can now apply Lemma 2.3.4 to compute the value of μ_2 for values (λ, τ) for which we have a Hopf bifurcation of system (2.2). We note that, in order to apply Theorem 2.3.1 we should only vary one parameter. As in Lemma 2.2.2, we leave K, β, γ and τ fixed and we only vary λ.

Lemma 2.3.5. *Let* (λ, τ) *be a point on the Hopf bifurcation curve and let* $\phi \in \mathbb{R}$ *satisfy* (2.10)-(2.11). *Let us vary* λ *and leave all the other parameters fixed. Then the value of* μ_2 *as defined in 2.16 is given by*

$$\mu_2 = -\frac{4(1 + K\tau\,(\cos(\beta - \phi) + \gamma\sin(\beta - \phi))}{1 + K\tau\cos(\beta - \phi)} \tag{2.28}$$

Proof. Using Lemma 2.3.2, we find that

$$D_2\Delta(i\omega_0, \lambda) = -I$$

Using the values of p, q as determined in the proof of Lemma 2.3.4, we find that

$$q \cdot D_2\Delta(i\omega_0, \lambda)p = -q \cdot p = -2\alpha = -\frac{1}{1 + K\tau e^{i(\beta - \phi)}}$$

which yields

$$\mathrm{Re}\,(q \cdot D_2\Delta(i\omega_0, \lambda)p) = -\frac{1 + K\tau\cos(\beta - \phi)}{\left|1 + K\tau e^{i(\beta-\phi)}\right|^2}$$

Combining this with Lemma 2.3.4 gives (2.28) □

We are now able to determine the direction of the Hopf bifurcation for parameter values (λ, τ) for which a Hopf bifurcation of the origin of system (2.2) occurs.

Corollary 2.3.1. *Let* (λ, τ) *be such that a Hopf bifurcation of the origin of system* (2.2) *occurs, i.e. let the conditions in Lemma 2.2.2 be satisfied.*

$$1 + K\tau\,[\cos(\beta - \phi) + \gamma\sin(\beta - \phi)] > 0 \tag{2.29}$$

then the Hopf bifurcation at (λ, τ) *is subcritical. If*

$$1 + K\tau\,[\cos(\beta - \phi) + \gamma\sin(\beta - \phi)] < 0 \tag{2.30}$$

the Hopf bifurcation at (λ_0, τ_0) *is supercritical.*

Proof. If the conditions in Lemma 2.2.2 are satisfied, we have in particular that (2.13) holds, i.e.

$$1 + K\tau\cos(\beta - \phi) > 0$$

Combining this with Lemma 2.3.5, we find that $\mu_2 < 0$ if (2.29) holds. Applying Theorem 2.3.1, we conclude that the Hopf bifurcation is subcritical. If (2.30) holds, we have that $\mu_2 > 0$ and by Theorem 2.3.1 the Hopf bifurcation is supercritical. □

Corollary 2.3.1 gives us a criterion to decide upon the direction of the Hopf bifurcation. In the next Section, we determine the orientation of the Pyragas curve with respect to the Hopf bifurcation curve. Combining this with Corollary 2.3.1, we can determine for which parameter values the periodic solution (2.4) of (2.2) is (un)stable.

2.4 Stability of the periodic solution

In Section 2.2, we introduced the Hopf bifurcation curve in the (λ, τ) parameter plane (see Definition 2.2.1); points (λ, τ) on the Hopf bifurcation curve are candidate parameter values for the occurence of a Hopf bifurcation. Furthermore, we introduced the Pyragas curve in the (λ, τ)-plane (see Definition 2.2.2); for (λ, τ) on the Pyragas curve, (2.4) is by construction a periodic solution of (2.2). We saw that the Pyragas curve ended on the Hopf bifurcation curve at $(\lambda, \tau) = (0, 2\pi)$. This motivates us to try to choose the parameter values K, β in such a way that the periodic solution (2.4) emmanates from a supercritical Hopf bifurcation. If this is the case, it follows that (2.4) is a stable solution of (2.2) for parameter values near the bifurcation point.

In Section 2.2, we studied the conditions for the occurence of a Hopf bifurcation; in Section 2.3, we determined the direction of the Hopf bifurcation. In this Section, we are interested in the orientation of the Pyragas curve with respect to the Hopf bifurcation curve. In particular, we determine the parameter values for which the Pyragas curve is locally to the right of the Hopf bifurcation curve and for which the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ is supercritical. We will see that for these parameter values, the solution (2.4) of (2.2) is stable for small λ.

We introduce the following terminology:

Definition 2.4.1. We define the *critical value* K_c to be the value of K such that the slope of the Hopf bifurcation curve and the slope of the Pyragas curve coincide. We refer to $(K_c, \tau_c = 2\pi, \lambda_c = 0)$ as the *critical point.*

Lemma 2.4.1. *For $\beta \in (0, \pi)$, the critical value is given by*

$$K_c = -\frac{1}{2\pi\left(\cos\beta + \gamma\sin\beta\right)} \tag{2.31}$$

Proof. Denote by λ_P, τ_P the values of λ, τ on the Pyragas curve as given in Definition 2.2.2 and denote by

$$\frac{d\tau_P}{d\lambda_P} := \left.\frac{d\tau_P}{d\lambda_P}\right|_{(\lambda_P=0, \tau_P=2\pi)}$$

the slope of the Pyragas curve at $(0, 2\pi)$. Similarly, denote by λ_H, τ_H the values of λ, τ on the Hopf bifurcation curve as given by (2.10) and (2.11) and denote by

$$\frac{d\tau_H}{d\lambda_H} := \left.\frac{d\tau_H}{d\lambda_H}\right|_{(\lambda_H=0, \tau_H=2\pi)}$$

the slope of the Hopf bifurcation curve at $(0, 2\pi)$.

We can straightforwardly compute $\frac{d\tau_P}{d\lambda_P}$:

$$\frac{d\tau_P}{d\lambda_P} = \left.\frac{d}{d\lambda}\left(\frac{2\pi}{1-\gamma\lambda}\right)\right|_{\lambda=0} = \left.\frac{2\pi}{(1-\gamma\lambda)^2}\cdot\gamma\right|_{\lambda=0} = 2\pi\gamma \tag{2.32}$$

On the Hopf bifurcation curve, we use (2.10) and (2.11) to find that

$$\begin{aligned}
\left.\frac{d\lambda_H}{d\phi}\right|_{\phi=2\pi} &= -K\sin(\beta-\phi)|_{\phi=2\pi} = -K\sin\beta \\
\left.\frac{d\tau_H}{d\phi}\right|_{\phi=2\pi} &= \left.\left(\frac{1}{1-K\left[\sin\beta - \sin(\beta-\phi)\right]} - \frac{\phi}{\left(1-K\left[\sin\beta-\sin(\beta-\phi)\right]\right)^2}(-K\cos(\beta-\phi))\right)\right|_{\phi=2\pi} \\
&= 1 + 2\pi K\cos\beta
\end{aligned}$$

We note that for $\beta \in (0,\pi)$, (2.10) is invertible for ϕ in a neighbourhood of $\phi = 2\pi$. Combining this with expression (2.11), we find that we can locally write τ as a function over λ around $(\lambda,\tau) = (0,2\pi)$. By the chain rule, we have that

$$\left.\frac{d\tau_H}{d\lambda_H}\right|_{(\lambda_H=0,\tau_H=2\pi)} \left.\frac{d\lambda_H}{d\phi}\right|_{\phi=2\pi} = \left.\frac{d\tau_H}{d\phi}\right|_{\phi=2\pi}$$

If $\left.\frac{d\lambda_H}{d\phi}\right|_{\phi=2\pi} \neq 0$, we can divide both sides by $\left.\frac{d\lambda_H}{d\phi}\right|_{\phi=2\pi}$ to obtain

$$\left.\frac{d\tau_H}{d\lambda_H}\right|_{(\lambda_H=0,\tau_H=2\pi)} = \left.\frac{d\tau_H}{d\phi}\right|_{\phi=2\pi} \Big/ \left.\frac{d\lambda_H}{d\phi}\right|_{\phi=2\pi} = -\frac{1}{K\sin\beta} - \frac{2\pi}{\tan\beta} \tag{2.33}$$

as the slope of the Hopf bifurcation curve at $(0,2\pi)$.

By Definition 2.4.1, we now find that the critical value K_c satisfies

$$2\pi\gamma = -\frac{2\pi}{\tan\beta} - \frac{1}{K_c \sin\beta}$$

(note that $\tan\beta \neq 0, \sin\beta \neq 0$ for $\beta \in (0,\pi)$), which gives the expression

$$K_c = -\frac{1}{2\pi(\cos\beta + \gamma\sin\beta)}$$

□

By looking at the values of $\frac{d\tau_H}{d\lambda_H}$ and $\frac{d\tau_P}{d\lambda_P}$, we can determine whether the Pyragas curve is locally left or right to the Hopf bifurcation curve If $\frac{d\tau_H}{d\lambda_H} < 0$, then the Pyragas curve is locally to the left of the Hopf bifurcation curve if and only if it is locally below it. For $\frac{d\tau_H}{d\lambda_H} > 0$ the Pyragas curve is locally to the left if and only if it locally above the Hopf bifurcation curve (see Figures 2.2 and 2.3). Since the Pyragas curve is only defined for $\lambda < 0$, the inequality $\frac{d\tau_P}{d\lambda_P} < \frac{d\tau_H}{d\lambda_H}$ implies that the Pyragas curve is locally above the Hopf bifurcation curve. Similarly, $\frac{d\tau_P}{d\lambda_P} > \frac{d\tau_H}{d\lambda_H}$ implies that the Pyragas curve is locally below the Hopf bifurcation curve. We can thus distinguish between several cases, as summarized in Table 2.1.

$\frac{d\tau_H}{d\lambda_H} < 0$		$\frac{d\tau_H}{d\lambda_H} > 0$	
$\frac{d\tau_P}{d\lambda_P} < \frac{d\tau_H}{d\lambda_H}$	Pyragas curve right to Hopf curve	$\frac{d\tau_P}{d\lambda_P} < \frac{d\tau_H}{d\lambda_H}$	Pyragas curve left to Hopf curve
$\frac{d\tau_P}{d\lambda_P} > \frac{d\tau_H}{d\lambda_H}$	Pyragas curve left to Hopf curve	$\frac{d\tau_P}{d\lambda_P} > \frac{d\tau_H}{d\lambda_H}$	Pyragas curve right to Hopf curve

Table 2.1: Orientation of the Pyragas curve with respect to the Hopf bifurcation curve

As an example, we apply Lemma 2.4.1 and 2.1 to the case $\gamma = -10,\ \beta = \frac{\pi}{4}$.

Proposition 2.4.2. *Let $\gamma = -10,\ \beta = \frac{\pi}{4}$ and denote by K_0 the value of K such that $\frac{d\tau_H}{d\lambda_H} = 0$. Then for $K > K_c$ and $K < K_0$, the Pyragas curve is locally to the right of the Hopf bifurcation curve. For $K_0 < K < K_c$, the Pyragas curve is locally to the left of the Hopf bifurcation curve.*

Proof. For $\gamma = -10,\ \beta = \frac{\pi}{4}$, we have that $\cos\beta + \gamma\sin\beta < 0$. Using the expressions (2.32), (2.33), we find for $K > K_c$ that $\frac{d\tau_P}{d\lambda_P} < \frac{d\tau_H}{d\lambda_H}$. As seen in Figure 2.4, we have that $\frac{d\tau_H}{d\lambda_H} < 0$ for $K > K_c$. Using Table 2.1, we find that the Pyragas curve is locally to the right of the Hopf bifurcation curve.

A similar analysis shows that the Pyragas curve is to the left of the Hopf bifurcation curve for $K_0 < K < K_c$. For $K < K_0$, we find that the Pyragas curve is locally to the right of the Hopf bifurcation curve at the point $(0,2\pi)$ (see also Figure 2.3). □

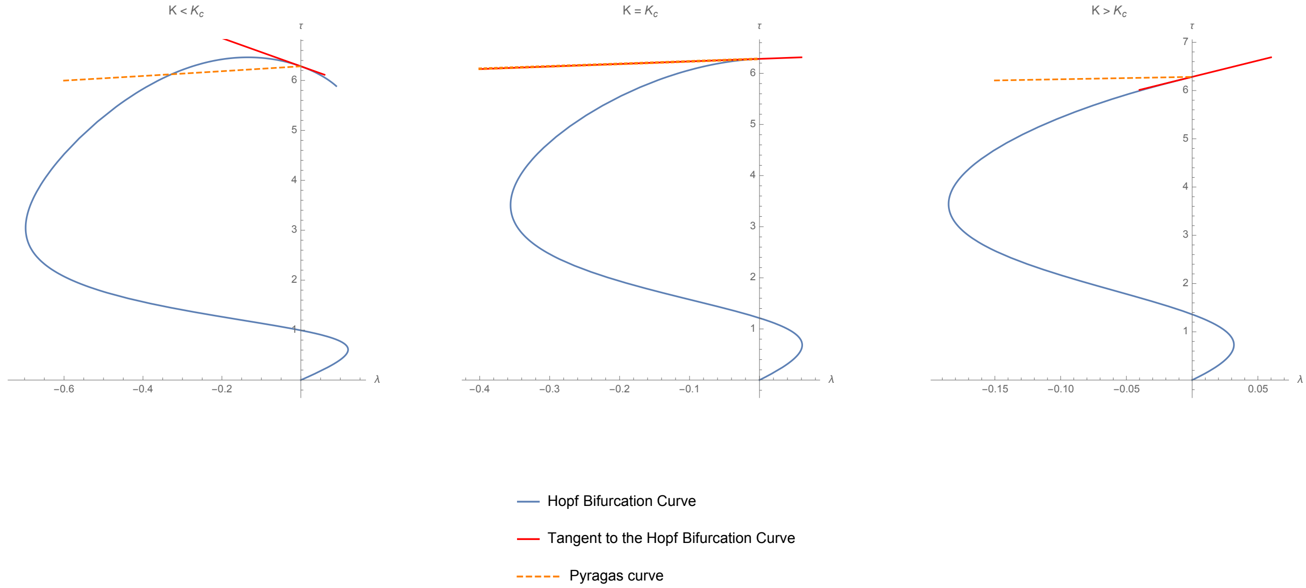


Figure 2.2: Pyragas curve and Hopf Bifurcation curve in (λ, τ)-plane with $K_c < 0$ (in particular, $\gamma = \frac{1}{4\pi}, \beta = \frac{\pi}{4}$) for $K < K_c$ (left), $K = K_c$ (middle) and $K > K_c$ (right)

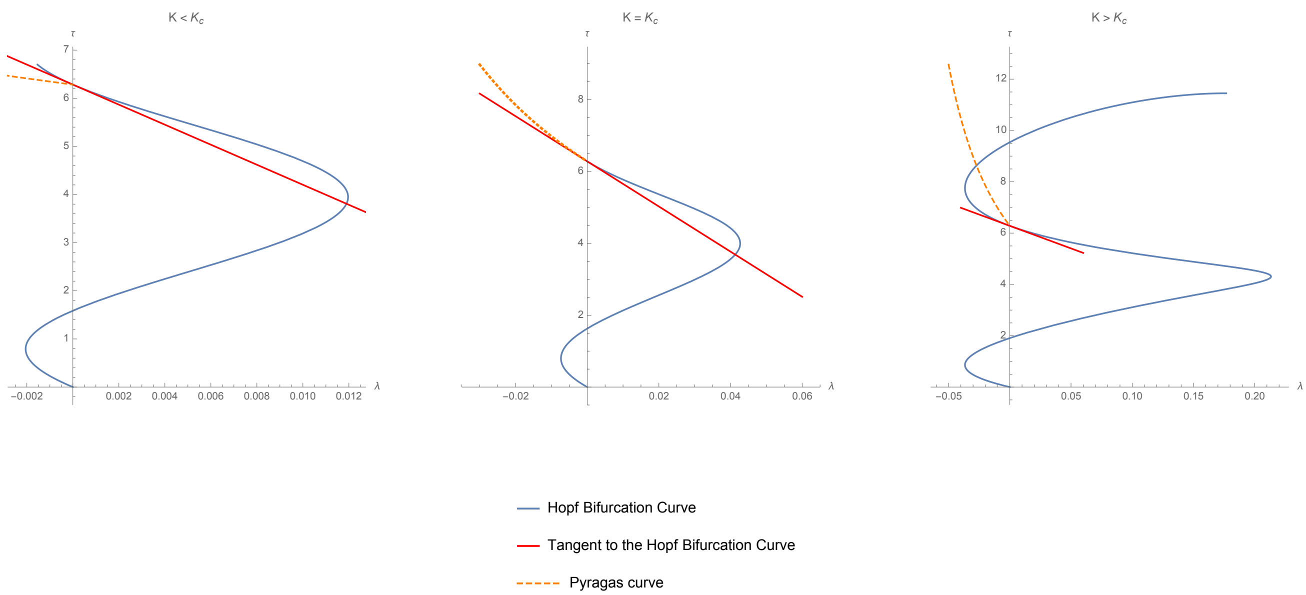


Figure 2.3: Pyragas curve and Hopf Bifurcation curve in (λ, τ)-plane with $K_c > 0$ (in particular, $\gamma = -10, \beta = \frac{\pi}{4}$) for $K < K_c$ (left), $K = K_c$ (middle) and $K > K_c$ (right)

Having determined the orientation of the Pyragas curve with respect to the Hopf bifurcation curve, we can now combine this knowledge with the direction of Hopf bifurcation at $(0, 2\pi)$ to determine parameter values K, β for which the periodic solution (2.4) of (2.2) is stable for small λ.

Corollary 2.4.1. *Let the conditions in Lemma 2.2.2 be satisfied for $\phi = 2\pi$ and let $1+2\pi K [\cos\beta + \gamma \sin\beta] < 0$. Furthermore, assume that the Pyragas curve is locally to the right of the Hopf bifurcation curve. Then the periodic solution (2.4) of (2.2) is stable for small λ.*

Proof. By (2.10) and (2.11), we have that $(\lambda, \tau) = (0, 2\pi)$ is a point on the Hopf bifurcation curve with $\phi = 2\pi$. If we assume that

$$1 + 2\pi K [\cos\beta + \gamma \sin\beta] = 1 + 2\pi K [\cos(\beta - 2\pi) + \gamma \sin(\beta - 2\pi)] < 0$$

we find by continuity of the map $\phi \mapsto 1 + \tau(\phi) K [\cos(\beta - \phi) + \gamma \sin(\beta - \phi)]$ that there exist a neighbourhood U of 2π such that

$$1 + \tau(\phi) K [\cos(\beta - \phi) + \gamma \sin(\beta - \phi)] < 0$$

for all $\phi \in U$. By choosing this neighbourhood small enough, we can ensure that (2.12) and (2.13) are also satisfied for all $\phi \in U$. By applying Corollary 2.3.1, we find a supercritical Hopf bifurcation for all $\phi \in U$. This means that we find a stable periodic solution for λ in a small neighbourhood to the right of the point $\lambda(\phi)$ on the Hopf bifurcation curve for all $\phi \in U$. By Remark 2.2.2, these periodic solutions are unique. Since by assumption the Pyragas curve is locally to the right of the Hopf bifurcation curve, we find for small λ that (2.4) is in fact the periodic orbit coming from the Hopf bifurcation. Therefore, for small λ, the periodic orbit (2.4) of (2.2) is stable. □

Corollary 2.4.2. *Let us assume that the conditions in Lemma 2.2.2 are for $\phi = 2\pi$, satisfied $1+2\pi K [\cos\beta + \gamma \sin\beta] > 0$ and the Pyragas curve is locally to the right of the Hopf bifurcation curve. Then the periodic solution (2.4) of (2.2) is unstable for small λ.*

Proof. A similar argument as in the proof of Corollary 2.4.1 yields the result. □

Remark 2.4.1. By combining Lemma 2.4.1 with Corollary 2.3.1, we find that the critical value $K = K_c$ marks the transition from sub– to supercritical behaviour at the point $(\lambda, \tau) = (0, 2\pi)$.

To conclude this Chapter, we combine the Corollary 2.4.1-2.4.2 with Proposition 2.4.2 to study the case $\gamma = -10,\ \beta = \frac{\pi}{4}$ in more detail. The behaviour for general parameter values is discussed in Section 3.3.

Proposition 2.4.3. *Let us assume that (2.12), (2.13) hold for $\phi = 2\pi$ and denote by K_0 the value of K such that $\frac{d\tau_H}{d\lambda_H} = 0$. For $K_0 < K < K_c$, the solution (2.4) of (2.2) is unstable for small λ. For $K > K_c$, the the solution (2.4) of (2.2) is stable for small λ.*

Proof. By Remark 2.4.1 we have that

$$1 + 2\pi K_c [\cos\beta + \gamma \sin\beta] = 0$$

If we choose $\gamma = -10,\ \beta = \frac{\pi}{4}$, we have that $\cos\beta + \gamma\sin\beta < 0$. This implies that

$$\begin{cases} 1 + 2\pi K_c [\cos\beta + \gamma \sin\beta] < 0 & \text{for} \quad K > K_c \\ 1 + 2\pi K_c [\cos\beta + \gamma \sin\beta] > 0 & \text{for} \quad K < K_c \end{cases}$$

Combining this with Corollary 2.3.1, we find that the Hopf bifurcation at the poin $(0, 2\pi)$ is subcritical for $K < K_c$ and supercritical for $K > K_c$.

As shown in Proposition 2.4.2, we have for $K < K_0$ that the Pyragas curve is locally to the right of the Hopf bifurcation curve at $(\lambda, \tau) = (0, 2\pi)$. The fact that the Hopf bifurcation is subcritical for these values of K, i.e. an unstable periodic orbit appears for parameter values to the left of the Hopf bifurcation curve, gives us thus no information about the stability of the Pyragas curve for $K < K_0$.

For $K_0 < K < K_c$, we recall from Proposition 2.4.2 that the Pyragas curve is locally to the left of the Hopf bifurcation curve at $(\lambda, \tau) = (0, 2\pi)$. Since for $K_0 < K < K_c$ the Hopf bifurcation at the point $(0, 2\pi)$ is subcritical, we can apply Corollary 2.4.2 to see that the solution (2.4) of (2.2) is unstable for small λ.

For $K > K_c$, it was shown in 2.4.2 that the Pyragas curve is locally to the right of the Hopf bifurcation curve at $(\lambda, \tau) = (0, 2\pi)$. Since the Hopf bifurcation at$(\lambda, \tau) = (0, 2\pi)$ is subcritical for $K > K_c$ by Corollary 2.3.1, we find by Corollary 2.4.1 that for $K > K_c$, the solution (2.4) of (2.2) is stable small λ. $\square$

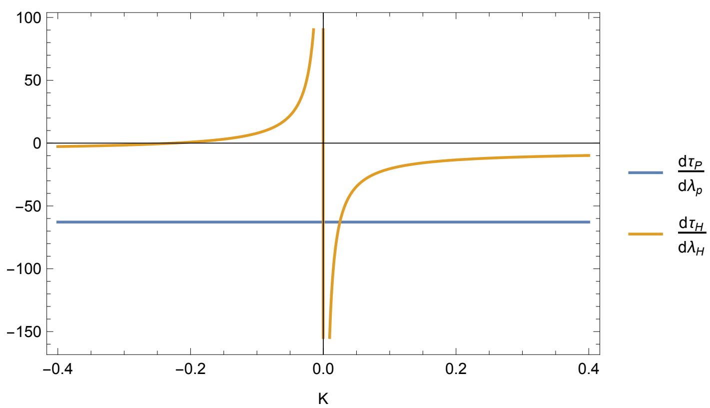


Figure 2.4: Plot of $\frac{d\tau_P}{d\lambda_P} = 2\pi\gamma$ and $\frac{d\tau_H}{d\lambda_H} = -\frac{1}{K \sin\beta} - \frac{2\pi}{\tan\beta}$ for $\gamma = -10,\ \beta = \frac{\pi}{4}$.

Chapter 3

Dynamics of the controlled system

In Chapter 2, we studied the Pyragas control scheme applied to system (2.1). Using the Hopf bifurcation, we were able to find values of K, β such that the Pyragas control scheme is succesfull for small λ, i.e. the periodic solution (2.4) of (2.2) is stable for small λ. As an example, we applied this method to a specific choice of parameters, see Proposition 2.4.3.

In this Chapter, we study the dynamics of the controlled system (2.2) in more detail. This analysis will give us more insight in the mechanisms by which the stabilization of the periodic orbit (2.4) is achieved. In Section 3.3 we combine these results to give a complete overview of the bifurcation diagram and we extend the result from Proposition 2.4.3 to general parameter values. In Sections 3.1 and 3.2 we mostly follow [10].

3.1 Stability on the Pyragas curve

In Chapter 2, we determined the stability of the periodic solution (2.4) of (2.2) by using the Hopf bifurcation theorem. In principle, the stability of periodic solutions can also be determined by looking at the characterstic multipliers (see Section A.5 for a discussion of characteristic multipliers for ordinary differential equations). In this Section, we have a closer look at the characterstic multipliers of the periodic solution (2.4) of (2.2). We find that an infinite number of characterstic multipliers exists. Although we cannot determine all these multipliers analytically, we will study the real multipliers. This gives us an interesting insight in the dynamics of system (2.2).

Lemma 3.1.1. *Suppose there exists a point (λ_t, τ_t) on the Pyragas curve satisfying*

$$\tau_t = -\frac{1}{K(\cos\beta + \gamma \sin\beta)} \tag{3.1}$$

then for the parameter values (K, λ_t, τ_t), the periodic solution (2.4) of (2.2) has a non-trivial characteristic multiplier on the unit circle.

If we furthermore assume that

$$\left|1 + K\tau_t e^{i\beta}\right|^2 + \lambda_t \tau_t > 0$$

and that there exist a neighbourhood U of (λ_t, τ_t) satisfying

$$1 + K\tau(\cos\beta + \gamma \sin\beta) > 0$$

for all $(\lambda, \tau) \in U$, then for points $(\lambda, \tau) \in U$ on the Pyragas curve, the periodic solution (2.4) of system (2.2) is unstable.

Proof. To find the linear variational equation, we look at small deviations around the periodic solution (2.4). We write the small deviations as:

$$z(t) = R_p e^{i\omega_p t}\left[1 + r(t) + i\phi(t)\right]$$

with $r(t), \phi(t) \in \mathbb{R}$ and where R_p, ω_p denote the radius and the angular frequency of (2.4). We expand (2.2) for $z(t)$ to linear order.

Denote by τ_p the period of (2.4), i.e. $\tau_p = \frac{2\pi}{\omega_p}$. For $z(t)$ to be a solution of the system (2.2) with $\tau = \tau_p$, its equation of motion should read

$$\begin{aligned}\dot{z}(t) &= R_p e^{i\omega_p t}\left(i\omega_p\left(1 + r(t) + i\phi(t)\right) + \dot{r}(t) + i\dot{\phi}(t)\right)\\ &= (\lambda + i)R_p e^{i\omega_p t}\left(1 + r(t) + i\phi(t)\right) + (1 + i\gamma)R_p^3 e^{i\omega_p t}\left|1 + r(t) + i\phi(t)\right|^2\left(1 + r(t) + i\phi(t)\right)\\ &\quad - Ke^{i\beta}R_p e^{i\omega_p t}\left[1 + r(t) + i\phi(t) - Re^{-\omega_p\tau_p}\left(1 + r(t - \tau_p) + i\phi(t - \tau_p)\right)\right]\end{aligned} \tag{3.2}$$

which we can simplify by cancelling the terms $R_p e^{i\omega_p t}$ on both sides. Using that we have chosen $r(t), \phi(t) \in \mathbb{R}$, we find that:

$$\begin{aligned}\left|1 + r(t) + i\phi(t)\right|^2\left(1 + r(t) + i\phi(t)\right) &= \left((1 + r(t))^2 + \phi(t)^2\right)\left(1 + r(t) + i\phi(t)\right)\\ &= 1 + 3r(t)^2 + 3r(t) + \phi(t)^2 + r(t)^3 + r(t)\phi(t)^2 + i\phi(t) + i\phi(t)r(t)^2 + 2i\phi(t)r(t) + i\phi(t)^3\end{aligned}$$

Up to order 2, (3.2) reduces to

$$\begin{aligned}i\omega_p(1 + r(t) + i\phi(t)) + \dot{r}(t) + i\dot{\phi}(t) &= \lambda + \lambda r(t) - \phi(t) + i\left(1 + r(t) + \lambda\phi(t)\right) + (1 + i\gamma)R_p^2\left(1 + 3r(t) + i\phi(t)\right)\\ &\quad - Ke^{i\beta}\left[1 + r(t) + i\phi(t) - Re^{-\omega_p\tau_p}\left(1 + r(t - \tau_p) + i\phi(t - \tau_p)\right)\right]\end{aligned}$$

From the fact that $u(t) = R_p e^{i\omega_p t}$ satisfies (2.2) with $\tau = \tau_p$, it follows that

$$i\omega_p = \lambda + i + (1 + i\gamma)R_p^2 - Ke^{i\beta}\left(1 - e^{i\omega_p\tau_p}\right)$$

Using this, in combination with the fact that $R_p^2 = -\lambda$ (see (2.4)) and $e^{i\omega_p\tau_p} = 1$, we find that

$$\begin{aligned}i\omega_p(r(t) + i\phi(t)) + \dot{r}(t) + i\dot{\phi}(t) &= \lambda r(t) - \phi(t) + i\left(r(t) + \phi(t) + \lambda\phi(t)\right) - \lambda(1 + i\gamma)(3r(t) + i\phi(t))\\ &\quad - Ke^{i\beta}\left[r(t) + i\phi(t) - r(t - \tau_p) - i\phi(t - \tau_p)\right]\end{aligned}$$

Taking real and imaginary parts and using that $\omega_p = 1 - \gamma\lambda$ (see (2.4)) results in the linear variational equation:

$$\begin{pmatrix}\dot{r}(t)\\ \dot{\phi}(t)\end{pmatrix} = \begin{pmatrix}-2\lambda & 0\\ -2\lambda\gamma & 0\end{pmatrix}\begin{pmatrix}r(t)\\ \phi(t)\end{pmatrix} - K\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}r(t) - r(t - \tau_p)\\ \phi(t) - \phi(t - \tau_p)\end{pmatrix} \tag{3.3}$$

We note that the linear variational equation is autonomous.

Suppose that for the ordinary differential equation $\dot{x}(t) = f(x(t))$ with periodic solution $u(t)$ the linear variational equation is autonomous, i.e. given by

$$\dot{x}(t) = Ax(t)$$

with A a constant matrix. Then the monodromy matrix C (see also Definition A.4.2 in the Appendix) is given by $C = e^{AT}$ and we can choose the characterstic multipliers as the eigenvalues of A. Extending this idea to differential delay equations, the characteristic exponents of (3.3) are given by the roots of the characteristic equation corresponding to (3.3). This characteristic equation is given by

$$\left(\mu + 2\lambda + K\cos\beta(1 - e^{-\mu\tau_p}\right)\left(\mu + \cos\beta(1 - Ke^{-\mu\tau_p})\right) + \left(2\lambda\gamma + K\sin\beta(1 - e^{-\mu\tau_p})\right)K\sin\beta(1 - e^{-\mu\tau_p}) = 0$$

which we can rewrite as

$$0 = \left(\mu + Ke^{i\beta}(1 - e^{-\mu\tau_p})\right)\left(\mu + Ke^{i\beta}(1 - e^{-\mu\tau_p})\right) + 2\lambda\left(\mu + K(\cos\beta + \gamma\sin\beta)\left[1 - e^{-\mu\tau_p}\right]\right)$$

We note that this is a transcedental equation which, for general parameter values, has an infinite number of solutions. We cannot expect to find all the solutions analytically. To gain more insight in the behaviour of the solutions, we expand the right hand side up to second order terms:

$$\begin{aligned}
0 &= \left(\mu + Ke^{i\beta}(\mu\tau - \frac{1}{2}\mu^2\tau^2 + \ldots)\right)\left(\mu + Ke^{-i\beta}(\mu\tau - \frac{1}{2}\mu^2\tau^2 + \ldots)\right) \\
&+ 2\lambda(\mu + K(\cos\beta + \gamma\sin\beta)(\mu\tau - \frac{1}{2}\mu^2\tau^2 + \ldots) \\
&= \mu^2(1 + Ke^{i\beta}\tau)(1 + Ke^{-i\beta}\tau) + 2\lambda\mu(1 + K(\cos\beta + \gamma\sin\beta))\tau) - \mu^2\tau^2 K\lambda(\cos\beta + \gamma\sin\beta) + \mathcal{O}(\mu^3) \\
&= 2\lambda\mu\left(1 + K\tau(\cos\beta + \gamma\sin\beta)\right) + \mu^2\left(\left|1 + Ke^{i\beta}\right|^2 - \lambda K\tau^2(\cos\beta + \gamma\sin\beta)\right) + \mathcal{O}(\mu^3)
\end{aligned}$$

We note that $\mu = 0$ is always a solution, corresponding to the trivial characteristic multiplier. For (λ_t, τ_t) satisfying (3.1), the coefficient of the first order term vanishes and the characteristic equation reduces further to

$$0 = \mu^2\left(\left|1 + Ke^{i\beta}\right|^2 + \lambda_t\tau_t\right) + \mathcal{O}(\mu^3) \tag{3.4}$$

Thus, for (λ_t, τ_t) on the Pyragas curve satisfying (3.1), we find that $\mu = 0$ is a zero of multiplicity 2, i.e. there is also a non-trivial characteristic multiplier located on the unit circle.

We note that 3.1 implies that

$$\left|1 + Ke^{i\beta}\right|^2 - \lambda_t K\tau_t^2(\cos\beta + \gamma\sin\beta) = \left|1 + K\tau_t e^{i\beta}\right|^2 + \lambda_t\tau_t$$

Thus, if we assume that $\left|1 + K\tau_t e^{i\beta}\right|^2 + \lambda_t\tau_t > 0$, then $\left|1 + Ke^{i\beta}\right|^2 - \lambda K\tau^2(\cos\beta + \gamma\sin\beta) > 0$ in a small enough neighbourhood of the transcritical point. If we now vary the parameters in this neighbourhood in such a way that $1 + K\tau(\cos\beta + \gamma\sin\beta) > 0$, then (3.4) implies that the non-trivial solution becomes positive (note that $\lambda < 0$ on the Pyragas curve). Thus, in a neighbourhood of the transcritical point with $1 + K\tau(\cos\beta + \gamma\sin\beta) > 0$, one of the characteristic multipliers lies outside the unit circle, and for these parameter values the periodic solution (2.4) of system (2.2) is unstable. □

Definition 3.1.1. We define the *transcritical point* to be the point (λ, τ) on the Pyragas curve satisfying (3.1).

Remark 3.1.1. For the critical value $K = K_c$, we find the transcritical point at $(\lambda, \tau) = (0, 2\pi)$.

Remark 3.1.2. We can choose the values of the parameters (K, γ, τ) in such a way that there does not exists a point on the Pyragas curve satisfying (3.1), i.e. the transcritical point does not exist. This is explored in more detail in Section 3.3.

Remark 3.1.3. We note that in Lemma 3.1.1 we have only adressed real characteristic multipliers. The condition for instability given in Lemma 3.1.1 is therefore not a necesary condition; instability can also be caused by non-real multipliers outside the unit circle.

As an example, we apply Lemma 3.1.1 to the case $\gamma = -10,\ \beta = \frac{\pi}{4}$.

Proposition 3.1.2. *Let* $\gamma = -10,\ \beta = \frac{\pi}{4}$*. For* $K < K_c$*, the transcritical point exists. Under the assumptions that* $K < K_c$ *and* $\left|1 + K\tau_t e^{i\beta}\right|^2 + \lambda_t\tau_t > 0$ *the periodic solution* (2.4) *of system* (2.2) *is unstable for parameter values* (λ, τ) *on the Pyragas curve in a neighbourhood to the right of the transcritical point.*

Proof. Since we have chosen $\gamma < 0$ and all points on the Pyragas curve satisfy $\lambda < 0$, we have that

$$\tau = \frac{2\pi}{1-\gamma\lambda} > 2\pi$$

on the Pyragas curve and for all real numbers $r > 2\pi$ we can find a point on the Pyragas curve satisfying $\tau = r$. As stated in Remark 3.1.1, we have for $K = K_c$ that $\tau_t = 2\pi$. For the parameters $\gamma = -10,\ \beta = \frac{\pi}{4}$, $\cos\beta + \gamma \sin\beta < 0$ holds. Applying (3.1), we find that for $K < K_c$ a transcritical point exists on the Pyragas curve, satisfying $\tau_t > 2\pi$.

Since $\gamma < 0$, we have that $\tau < \tau_t$ for points on the Pyragas curve to the right of the transcritical point. Since the transcritical point satisfies (3.1) by definition and $\cos\beta + \gamma\sin\beta < 0$, we find that $1 + K\tau(\cos\beta + \gamma\sin\beta) > 0$ for points on the Pyragas curve to the right of the transcritical point. Thus, if $\left|1 + K\tau_t e^{i\beta}\right|^2 + \lambda_t\tau_t > 0$ is satisfied, we apply Lemma 3.1.1 to find that for paramter values (λ, τ) on the Pyragas curve in a neighbourhood to the right of the transcritical point, the periodic orbit (2.4) has at least one positive multiplier and is thus unstable. $\square$

3.2 Saddle – Node Bifurcations

In Chapter 2, we extensively used the Hopf bifurcations of the fixed point $z = 0$ of system (2.2). For a complete understanding of the bifurcation diagram, we now study the saddle–node bifurcations of system (2.2).

Lemma 3.2.1. *If there exists a $\phi \in \mathbb{R}$ such that the parameters (K, λ, τ) satisfy*

$$K\tau = -\frac{1}{\gamma\sin(\beta-\phi) + \cos(\beta-\phi)} \tag{3.5}$$

$$\tau(1-\gamma\lambda) = \phi - K\tau\left[\gamma\cos\beta - \sin\beta - \gamma\cos(\beta-\phi) + \sin(\beta-\phi)\right] \tag{3.6}$$

$$\lambda \le K\left[\cos\beta - \cos(\beta-\phi)\right] \tag{3.7}$$

then a saddle–node bifurcation of system (2.2) *occurs.*

Proof. Since the periodic solution (2.4) of (2.2) is of the form $z(t) = Re^{i\omega t}$ with $R, \omega \in \mathbb{R}$, we are interested in knowing whether other solutions of this form exist. We insert $z(t) = Re^{i\omega t}$ in (2.2) to obtain:

$$\dot{z}(t) = i\omega Re^{i\omega t} = (\lambda + i)Re^{i\omega t} + (1+i\gamma)R^3e^{i\omega t} - Ke^{i\beta}\left[Re^{i\omega t} - Re^{i\omega t}e^{-i\omega\tau}\right]$$

Cancelling the factors $Re^{i\omega t}$ on both sides and taking real and imaginary parts yields the conditions

$$R^2 = -\lambda + K\left[\cos\beta - \cos(\beta-\phi)\right] \tag{3.8}$$

$$\phi = \tau(1-\lambda\gamma) + K\tau\left[\gamma\cos\beta - \sin\beta - \gamma\cos(\beta-\phi) + \sin(\beta-\phi)\right] \tag{3.9}$$

with $\phi = \omega\tau$. We note that if the diagonal touches the graph generated by the RHS of (3.9) for a certain value of K, changing the value of K will result in either the appearance or disappearance of a solution of (3.9), as is illustrated in Figure 3.1. Thus, if the diagonal touches the graph, a saddle – node bifurcation takes place. Since for a general function f the diagonal touches the graph of f if there exists a ϕ satisfying $\phi = f(\phi)$ and $\frac{d\phi}{d\phi} = 1 = \frac{df}{d\phi}$, we find that a saddle–node bifurcation occurs if there exists a ϕ satisfying (3.5), (3.6). The inequality (3.7) comes from (3.8) in combination with the assumption that $R \in \mathbb{R}$, i.e. $R^2 \ge 0$. $\square$

For a fixed value of K, a given value of ϕ gives us a value of τ via (3.5). The pair (τ, ϕ) gives us a value of λ via (3.6). In combination with the constaint (3.7), we find that the equations (3.5), (3.6), (3.7) parametrize curves in (λ, τ)-parameter space. In analogy to Definition 2.2.1, we now define the following:

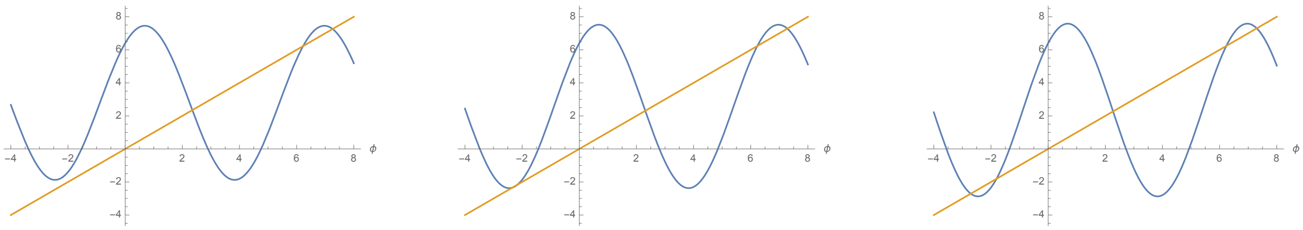
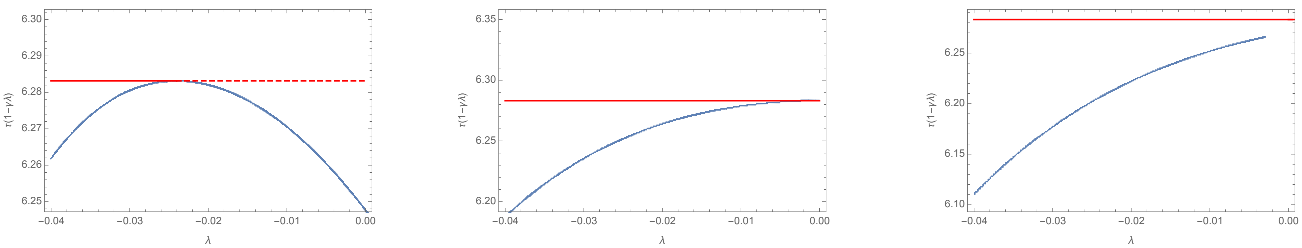


Figure 3.1: Diagonal and RHS of (3.9) for fixed τ, λ, γ and different values of K (see proof of Lemma 3.2.1).



(a) $K < K_c$ (b) $K = K_c$ (c) $K > K_c$

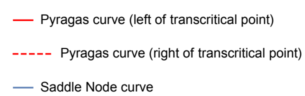


Figure 3.2: Pyragas curve and saddle–node curve around the transcritical point in distorted coordinates for $\gamma = -10,\ \beta = \frac{\pi}{4}$ and different values of K.

Definition 3.2.1. We define the *saddle – node curves* as the curves in (λ, τ)-parameter space parametrized by equations (3.5), (3.6), (3.7).

In Section 2.4, we studied the orientation of the Pyragas curve with respect to the Hopf bifurcation curve. We now turn towards the position of the saddle–node curve with respect to Pyragas curve and the Hopf bifurcation curve.

Lemma 3.2.2. *Let (λ, τ) be a point on the Hopf bifurcation curve where a transition from sub– to supercritical behaviour occurs. Then (λ, τ) also lies on a saddle–node curve.*

Proof. By definition of the Hopf curve, there exists a $\phi \in \mathbb{R}$ such that (2.10) and (2.11) are satisfied. Rewriting the latter, we find for point on the Hopf bifurcation curve that

$$\begin{aligned}
&\phi - K\tau\left[\gamma\cos\beta - \sin\beta - \gamma\cos(\beta-\phi) + \sin(\beta-\phi)\right] \\
&\quad = -K\tau\gamma\left[\cos\beta - \cos(\beta-\phi)\right] + \phi + K\tau\left[\sin\beta - \sin(\beta-\phi)\right] \\
&\quad = -\tau\gamma\lambda + \tau = \tau(1-\gamma\lambda)
\end{aligned}$$

i.e. (3.6) is satisfied. Trivially, (2.10) implies (3.7). Thus, all points on the Hopf bifurcation curve satisfy (3.6) and (3.7). We recall from Corollary 2.3.1 that a transition from sub- to supercritical behaviour on the Hopf bifurcation curve occurs when $1 + K\tau\left[\cos(\beta-\phi) + \gamma\sin(\beta-\phi)\right] = 0$ which is equivalent to (3.5). Using Definition 3.1.1, we conclude that the sub–supercritical Hopf transition points lie on the saddle–node curve. $\square$

Lemma 3.2.3. *A saddle–node curve intersects the Pyragas curve at the transcritical point.*

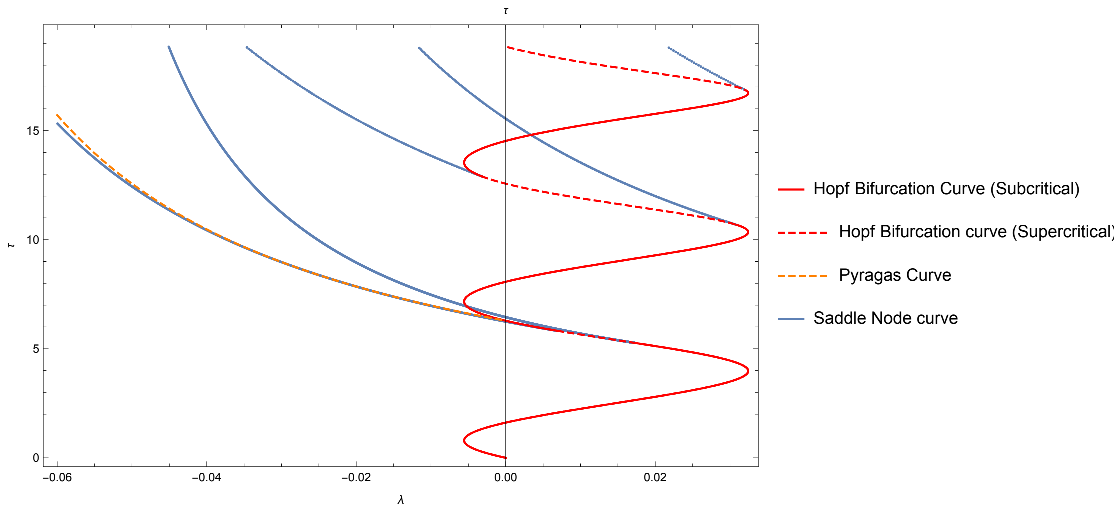


Figure 3.3: Bifurcation diagram in (λ, τ)-plane for $K = 0.019,\ \beta = \frac{\pi}{4},\ \gamma = -10$.

Proof. We recall that for all points on the Pyragas curve satisfy (2.14) with $\lambda < 0$. Thus, they satisfy (3.6) and (3.7) for $\phi = 2\pi$. The transcritical point on the Pyragas curve satisfies (3.1), which is equivalent to (3.5) for $\phi = 2\pi$. We conclude that a saddle–node curve intersects the Pyragas curve at the transcritical point. □

The behaviour described in Lemmata 3.2.2, 3.2.3 is summarized in Figure 3.3: the saddle–node curves emmanate from the Hopf bifurcation curve at points where the direction of the Hopf bifurcation transitions from sub- to supercritical. Furthermore, the transcritical point on the Pyragas curve lies also on the saddle-node curve (see Figure 3.2a).

3.3 Varying parameter values

In the previous Sections, we developed methods to study the bifurcation diagram of system (2.2). In this section, we combine the results from Sections 2.2-3.2 to obtain a complete view of the bifurcation diagram of system (2.2). In Example 3.3.1, we give an overview of the bifurcation diagram with parameter values $\gamma = -10,\ \beta = \frac{\pi}{4}$. Thereupon, we study how the bifurcation diagram changes when we vary the parameters.

Example 3.3.1 (Overview of the bifurcation diagram for $\gamma = -10,\ \beta = \frac{\pi}{4}$)**.** For $0 < K < K_c$, where K_c denotes the critical value, there exists a transcritical point on the Pyragas curve satisfying (3.1). Let us assume that 2.13 holds. For parameter values (λ, τ) on the Pyragas curve in a neighbourhood to the right of the transcritical point, the solution (2.4) of (2.2) is then unstable (see Proposition 3.1.2). For $0 < K < K_c$ the Pyragas curve is locally to the left of the Hopf bifurcation curve (see Proposition 2.4.2). Furthermore, the Hopf bifurcation at the point $(\lambda, \tau) = (0, 2\pi)$ is subcritical, i.e. a unstable periodic orbit is generated to the left of the Hopf bifurcation curve. These remarks imply that the periodic solution (2.4) of (2.2) is also unstable for parameter values (λ, τ) on the Pyragas curve in a neighbourhood to the left of $(\lambda, \tau) = (0, 2\pi)$ (see Proposition 2.4.3).

As $K \to K_c$, the transcritical point moves towards the point $(\lambda, \tau) = (0, 2\pi)$. For $K = K_c$, the transcritical point coincides with the critical point $(0, 2\pi)$ (see Remark 3.1.1) and this point also marks the transition between sub- and supercritical behaviour on the Hopf bifurcation curve (see Remark 2.4.1).

If we increase the value of K further, i.e. $K > K_c$, the saddle–node curve detaches from the Pyragas curve (see Figure 3.2) and no transcritical point exists on the Pyragas curve. At the point $(\lambda, \tau) = (0, 2\pi)$, the Pyragas curve is locally to the right of the Hopf bifurcation curve (Proposition 2.4.2). The point

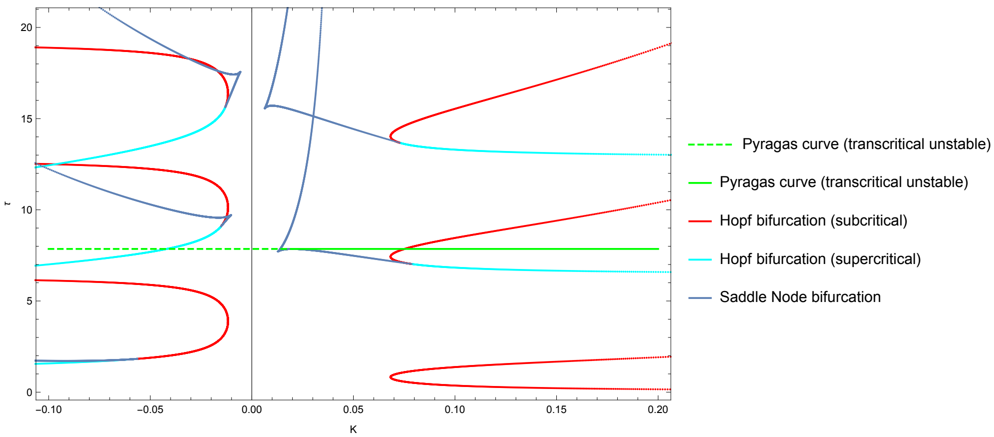


Figure 3.4: Bifurcation diagram of (2.2) in (K, τ) parameter plane for $\gamma = -10,\ \beta = \frac{\pi}{4},\ \lambda = -0.02$.

$(\lambda, \tau) = (0, 2\pi)$ on the Hopf bifurcation curve is now supercritical, i.e. a stable periodic orbit emmanates to the right of the Hopf bifurcation curve. This geometry indicates that the Pyragas curve is now stable near the Hopf bifurcation curve (see Proposition 2.4.3).

So far, we studied the behaviour in the five-dimensional $(\lambda, \beta, K, \gamma, \tau)$-parameter space by looking at the behaviour in (λ, τ)-parameter space for fixed values of $(\gamma,\ \beta,\ K)$; see Figure 3.3. We can also study different two-dimensional sections of the five-dimensional parameter space, for example by looking at (K, τ)-parameter space. This is done in Figure 3.4 and summarizes the behaviour described in Example 3.3.1.

In the following Proposition, we extend the results obtained in Proposition 2.4.3 to general parameter values.

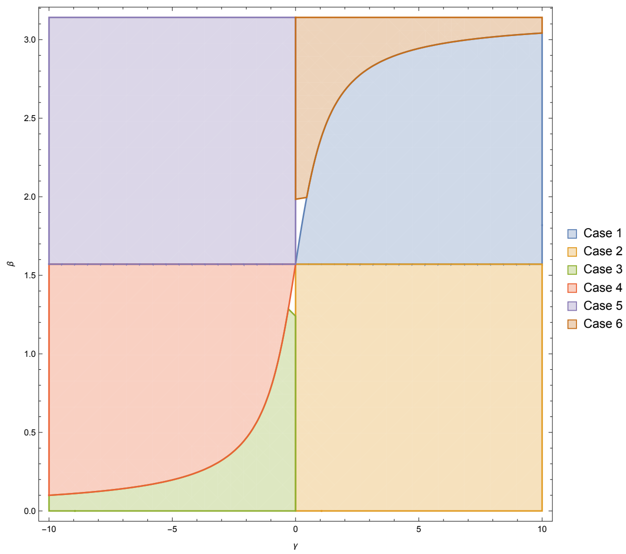
Proposition 3.3.1. *As indicated in Figure 3.5, we can distinguish between six different cases if we want to study the stability of the solution* (2.4) *of* (2.2) *for parameter values* (λ, τ) *near* $(0, 2\pi)$.

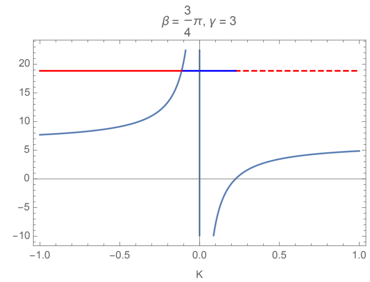
Proof. We recall from Table 2.1 that the orientation of the Pyragas curve with respect to the Hopf bifurcation curve is determine by the sign of $\frac{d\tau_H}{d\lambda_H}$ and the sign of $\frac{d\tau_H}{d\lambda_H} - \frac{d\tau_P}{d\lambda_P}$. We note that the asymptote of $\frac{d\tau_H}{d\lambda_H}(K) = -\frac{2\pi}{\tan\beta} - \frac{1}{K\sin\beta}$ is given by $-\frac{2\pi}{\tan\beta}$. We are therefore able to distinguish between several cases in the (γ, β)-plane. Below the curve $2\pi\gamma - \left(-\frac{2\pi}{\tan\beta}\right) = 0$, we have that $\frac{d\tau_P}{d\lambda_P} = 2\pi\gamma > \frac{d\tau_H}{d\lambda_H} = -\frac{2\pi}{\tan\beta}$, i.e. the tangent to the Pyragas curve is above the asymptote of $\frac{d\tau_H}{d\lambda_H}$. Since both $\frac{d\tau_P}{d\lambda_P}$ and the asymptote of $\frac{d\tau_H}{d\lambda_H}$ can be either negative or positive, we can distinguish between six cases. These different cases are summarized in Figure 3.5, with typical examples of $\frac{d\tau_P}{d\lambda_P}(K),\ \frac{d\tau_H}{d\lambda_H}(K)$. For a given value of K, we can use Table 2.1 to determine in each case whether the Pyragas curve is locally left or right to the Hopf bifurcation curve, as is also indicated in Figure 3.5.

Since by asumption we have $\beta \in (0, \pi)$, we have that $\sin\beta > 0$ for all values of β. We can thus multiply the expression $2\pi\gamma < \frac{2\pi}{\tan\beta}$ by $\sin\beta$ and rearrange terms to find that $2\pi\gamma < \frac{2\pi}{\tan\beta}$ if and only if $\cos\beta + \gamma\sin\beta < 0$. We now apply Corollary 2.3.1 to find that below the curve $2\pi\gamma + \frac{2\pi}{\tan\beta} = 0$ in the (γ, β)-plane, we have $\cos\beta + \gamma\sin\beta > 0$ and thus that the Hopf bifurcation at $(0, 2\pi)$ is subcritical for $K > K_c$ and supercritical for $K < K_c$. Similarly, above the curve $2\pi\gamma + \frac{2\pi}{\tan\beta} = 0$ we have $\cos\beta + \gamma\sin\beta < 0$. Therefore, the Hopf bifurcation is supercritical for $K > K_c$ and subcritical for $K < K_c$ by Corollary 2.3.1.

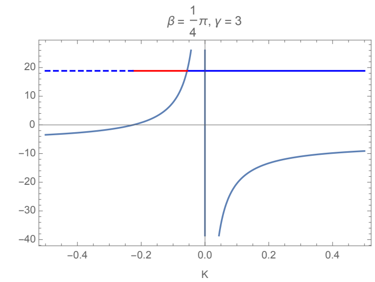
Using the Corollary 2.4.1-2.4.2, we are able to determine upon the stability of 2.4 for parameter values (λ, τ) on te Pyragas curve near $(0, 2\pi)$, see Figure 3.5. □

We note that the results found in Proposition 3.3.1 indeed do depend on the parameters: in some cases, we are not able to determine by the methods used here to determine whether the periodic solution (2.4) of (2.2) is stable for any value of K (see for example Figure 3.5c), whereas in other cases we are. Furthermore,

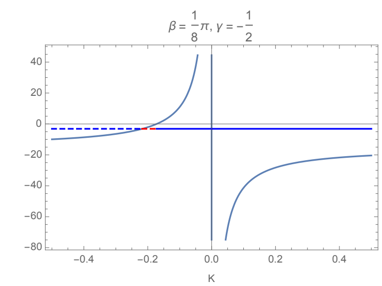




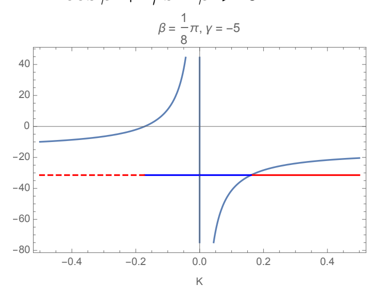
(a) Case 1: $\gamma > 0,\ \beta > \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta > 0$



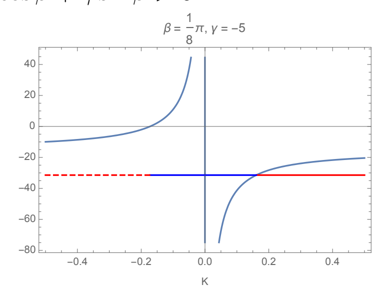
(b) Case 2: $\gamma > 0,\ \beta < \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta > 0$



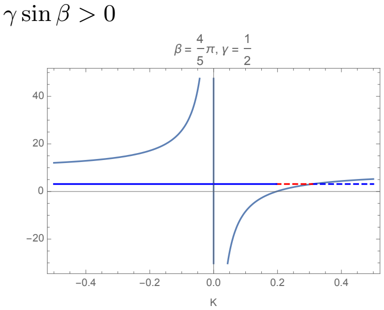
(c) Case 3: $\gamma < 0,\ \beta < \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta > 0$



(d) Case 4: $\gamma < 0,\ \beta < \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta < 0$



(e) Case 5: $\gamma < 0,\ \beta > \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta < 0$



(f) Case 6: $\gamma > 0,\ \beta > \frac{\pi}{2}$, $\cos\beta + \gamma\sin\beta < 0$

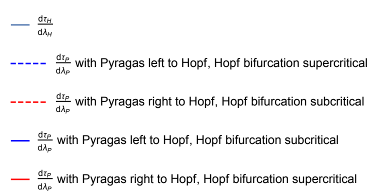


Figure 3.5: Distinction between different cases in (γ, β)-plane with generic examples of these cases. We note that for values of K where the $\frac{d\tau_P}{d\lambda_P}$ is non-dashed, we are able to determine the stability of the Pyragas curve near $(0, 2\pi)$ by studying the direction of the Hopf bifurcation at that point. Red/blue non-dashed lines correspond to stability/instability, respectively. For the dashed regions, we have to resort to other techniques to determine the stability.

we note that for $\cos\beta + \gamma \sin\beta < 0$, i.e. above the line $\cos\beta + \gamma\sin\beta = 0$, stability is achieved for $K > K_c$. However, for $\cos\beta + \gamma\sin\beta > 0$, i.e. below the line $\cos\beta + \gamma\sin\beta = 0$, stablity is achieved for $K < K_c$.

In Proposition 3.1.2 we were able to achieve additional inside in the stabilisation of the Pyragas curve by looking at the characteristic multipliers of (2.2).

As mentioned in Remark 3.1.2, for general values of K the transcritical point need not always to exist. This is studied in more detail in the following Proposition:

Proposition 3.3.2. *Let $\gamma > 0$ and $\cos\beta + \gamma\sin\beta > 0$. Then the transcritical point only exists for $K \leq K_c$.*

Proof. Since $\gamma > 0$ and all points on the Pyragas curve satisfy $\lambda < 0$, we have that

$$0 < \tau = \frac{2\pi}{1 - \gamma\lambda} < 2\pi$$

for all points (λ, τ) on the Pyragas curve. Furthermore, for all real numbers $0 < r < 2\pi$, there exist a point (λ, τ) on the Pyragas curve such that $\tau = r$.

As stated in Remark 3.1.1, we have that $\tau_t = 2\pi$ for $K = K_c$. Since we have by assumption that $\cos\beta + \gamma\sin\beta$, we find that

$$-\frac{1}{K\left[\cos\beta + \gamma\sin\beta\right]} > 2\pi$$

for $K_c < K < 0$. Thus, if a transcritical point would exists for $K_c < K < 0$, it should satisfy $\tau_t > 2\pi$. Since $\tau < 2\pi$ on the Pyragas curve, we conclude that the transcritical point does not exist for $K_c < K < 0$.

Similarly, we see that if a transcritical point exists for $K > 0$, it should satisfy $\tau_t < 0$. Since $\tau > 0$ for all points on the Pyragas curve, we conclude that the transcritical point does not exist for $K > 0$.

For $K \leq K_c$ we find that $-\frac{1}{K[\cos\beta+\gamma\sin\beta]} \leq 2\pi$. Since for all real numbers $0 \leq r \leq 2\pi$ there exists a point (λ, τ) on the Pyragas curve such that $\tau = r$, we find that for all $K < K_c$ there exists a point (λ, τ) on the Pyragas curve satisfying (3.1). For $K = K_c$, we find the transcritical point at $(\lambda, \tau) = (0, 2\pi)$, i.e. at the end of the Pyragas curve. We conlude that for all $K \leq K_c$ the transcritical point exists. ☐

By doing a similar analysis, we can distinguish between the cases indicated in Figure 3.6.

For $\gamma = -10$ $\beta = \frac{\pi}{4}$, we found in Proposition 3.1.2 that a transcritical point existed for $K < K_c$. Under the assumption that (2.12), (2.13) hold, we found that for parameter values (λ, τ) on the Pyragas curve in a neighbourhood to the right of the transcritical point, the solution (2.4) of (2.2) is unstable. This fits nicely with the earlier observation in Proposition 2.4.3 that for $K < K_c$, the solution (2.4) of (2.2) is unstable for parameter values (λ, τ) on the Pyragas curve near $(0, 2\pi)$. By comparing Figures 3.5 and 3.6 we find the existence of parameter values γ, β such that a transcritical point lies on the Pyragas curve, while for (λ, τ) near $(0, 2\pi)$ the solution (2.4) of (2.2) is stable.

Proposition 3.3.3. *Let β, γ be such that $\beta > \frac{\pi}{2}$, $\gamma > 0$ and $\cos\beta + \gamma\sin\beta > 0$ and assume that (2.12), (2.13) hold for $\phi = 2\pi$. Then for $K \leq K_c$, the solution (2.4) of (2.2) is stable for parameter values on the Pyragas curve near $(0, 2\pi)$. For $K \leq K_c$, a transcritical point exists. For K in a small enough neighbourhood of K_c with $K \leq K_c$ and for parameter values (λ, τ) on the Pyragas curve in a neighbourhood to the right of the transcritical point, the solution (2.4) of system (2.2) is unstable.*

Proof. Since the choice of parameter values is an example of Case 1 of Figure 3.5, we find that the solution (2.4) of (2.2) is stable for parameter values on the Pyragas curve near $(0, 2\pi)$.

By Figure 3.6, we find that the transcritical point exists for $K \leq K_c$. As mentioned in Remark 3.1.1, we have for $K = K_c$ that $(\lambda_t, \tau_t) = (0, 2\pi)$. Thus we find that

$$\left|1 + K_c e^{i\beta}\right|^2 - \lambda_t K_c \tau_t^2 (\cos\beta + \gamma\sin\beta) = \left|1 + K_c e^{i\beta}\right|^2 = 0$$

Thus, for K in a small enough neighbourhood of K_c with $K \leq K_c$, we have that

$$\left|1 + K e^{i\beta}\right|^2 - \lambda_t K \tau_t^2 (\cos\beta + \gamma\sin\beta) > 0 \tag{3.10}$$

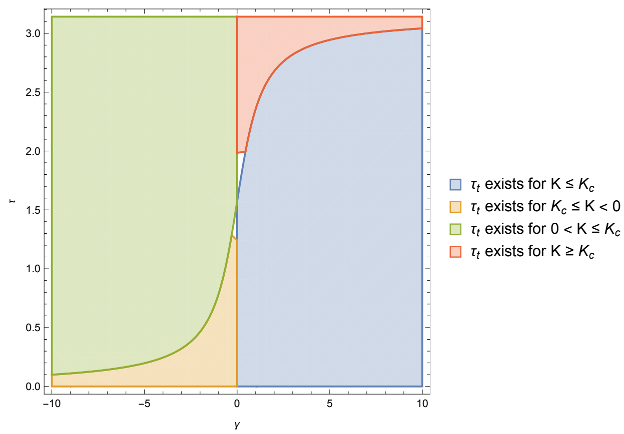


Figure 3.6: The values of the control parameter K for which the transcritical point exists depend on the values of the parameters $\gamma,\ \beta$, as indicated above.

Let us now fix K in this small enough neighbourhood of K with $K \leq K_c$. Since $\gamma > 0$, we have that $\tau > \tau_t$ to the right of the transcritical point. By assumption we have that $\cos\beta + \gamma \sin\beta > 0$ and thus that $1 + K\tau\left[\cos\beta + \gamma\sin\beta\right] > 0$ for τ to the right of τ_t. By choosing τ in a small enough neighbourhood of τ_t, (3.10) implies that

$$\left|1 + Ke^{i\beta}\right|^2 - \lambda K\tau(\cos\beta + \gamma\sin\beta) > 0$$

Thus, for $K \leq K_c$ in a small enough neighbourhood of K_c and by choosing τ in a small enough neighbourhood of τ_t we can apply Lemma 3.1.1. We conclude that for $K \leq K_c$ in a small enough neighbourhood of K_c and (λ, τ) on the Pyragas curve in a small enough neighbourhood to the right of τ_t, the solution (2.4) of (2.2) is unstable. □

Remark 3.3.1. We note that Proposition 3.3.3 implies the existence of a point (λ, τ) on the Pyragas curve between the transcritical point and the point $(0, 2\pi)$, such that the solution (2.4) of (2.2) changes stability. In particular, we find that at this point the multiplier studied in Lemma 3.1.1 moves back into the unit circle, since near $(\lambda, \tau) = (0, 2\pi)$ the solution (2.4) of (2.2) is stable and therefore all the non–trivial multipliers lie inside the unit circle.

Chapter 4

Weaker conditions for stability of the periodic solution

In Section 2.2, we determined parameter values (λ, τ) for which a Hopf bifurcation of the origin of (2.2) occurs. In Section 2.3, we determined the direction of the Hopf bifurcation (i.e. whether the Hopf bifurcation is sub– or supercritical) at the Hopf bifurcation points. Here, we used Theorem 2.3.1. Since this Theorem applies to scalar parameters and we are interested in Hopf bifurcation points in the two–dimensional (λ, τ) plane, we had to choose a curve over which to approach the Hopf bifurcation points. In Section 2.3, we approached the bifurcation points parallel to the λ–axis, i.e. we only varied the parameter λ, as was done in [10].

We note that there are many possible ways to approach the point $(\lambda, \tau) = (0, 2\pi)$ in the (λ, τ)-parameter plane. In general, the value of μ_2 as defined in Theorem 2.3.1 depends on the chosen path of approach. In this Chapter, we approach the bifurcation point (λ, τ) over a different curve than was done in Section 2.3. It turns out that by doing so, we can can considerably weaken the conditions for (2.4) to be a stable solution of (2.2) for small λ.

We recall from Definition 2.2.2 that the Pyragas curve in (λ, τ)-parameter space is given by the graph of $\lambda(\tau) = \frac{2\pi}{1-\gamma\lambda}$ for $\lambda < 0$. We introduce the following terminology:

Definition 4.0.1. We define the *extended Pyragas curve* as the curve in (λ, τ)-space given by the graph of

$$\tau(\lambda) = \frac{2\pi}{1 - \gamma\lambda}$$

with λ in the domain $\mathbb{R}\backslash\{\frac{1}{\gamma}\}$.

We note that the extended Pyragas curve intersects the Hopf bifurcation curve (see Definition 2.2.1) at $(\lambda, \tau) = (0, 2\pi)$. The main idea is now to approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve. In doing so, we treat the delay τ as a parameter. In the statement of Theorems 2.2.1 and 2.3.1, the delay is left fixed. To show that we can apply Theorem 2.3.1 if the delay τ is treated as a parameter, we prove the following Lemma:

Lemma 4.0.1. *For $\tau \neq 0$, system (2.2) is equivalent to the system*

$$\dot{z}(t) = \tau \left[(\lambda + i)z(t) + (1 + i\gamma) |z(t)|^2 z(t) - Ke^{i\beta} [z(t) - z(t-1)] \right] \tag{4.1}$$

Proof. Let $t = \tau\hat{t}$. Using the chain rule, we find

$$\frac{dz(t)}{dt} = \frac{dz(\tau\hat{t})}{d\hat{t}} \frac{d\hat{t}}{dt} = \frac{1}{\tau} \frac{dz(\tau\hat{t})}{d\hat{t}} = (\lambda + i)z(\tau\hat{t}) + (1 + i\gamma) \left|z(\tau\hat{t})\right|^2 z(\tau\hat{t}) - Ke^{i\beta} \left[z(\tau\hat{t}) - z(\tau(\hat{t} - 1))\right]$$

Multiplying both sides by $\tau \neq 0$ and setting $z(\tau\hat{t}) = \hat{z}(\hat{t})$ gives

$$\frac{d\hat{z}(\hat{t})}{d\hat{t}} = \tau\left[(\lambda+i)\hat{z}(\hat{t}) + (1+i\gamma)\left|\hat{z}(\hat{t})\right|^2\hat{z}(\hat{t}) - Ke^{i\beta}\left[\hat{z}(\hat{t}) - \hat{z}(\hat{t}-1)\right]\right]$$

Dropping the hats gives (4.1). Since all transforms are invertible for $\tau \neq 0$, we see that system (4.1) is in fact equivalent to system (2.2) for $\tau \neq 0$. □

We note that in (4.1), τ is no longer the delay but has the role of a 'normal' parameter. The equivalence between (4.1) and (2.2) implies in particular that we can apply Theorems 2.2.1 and 2.3.1 to (2.2) with τ as a parameter.

Lemma 4.0.2. *Let*

$$1 + 2\pi K e^{i\beta} \neq 0 \tag{4.2}$$

If

$$1 + 2\pi K\left[\cos\beta + \gamma\sin\beta\right] > 0 \tag{4.3}$$

then we find a Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyrags curve from the left.

If

$$1 + 2\pi K\left[\cos\beta + \gamma\sin\beta\right] < 0 \tag{4.4}$$

then we find a Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right.

Proof. We only have to check 2.9 in Theorem 2.2.1. That all the other conditions are satisfied if (4.2) holds, follows from the proof of Lemma 2.2.2.

We note that the choice of p, q as in (2.8) is independent of the curve of approach; thus, the values of p, q are as in the proof of Lemma 2.2.2.

If we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left, we can parametrize the path as

$$(\lambda, \tau) = \left(\theta, \frac{2\pi}{1-\gamma\theta}\right), \quad \theta \in \mathbb{R}\backslash\left\{\frac{1}{\gamma}\right\} \tag{4.5}$$

Using (2.19), we find that for parameter values on this curve the characteristic function is given by

$$\Delta(\mu, \theta) = \mu I - \begin{pmatrix} \theta - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \theta - K\cos\beta \end{pmatrix} - Ke^{-\mu\tau(\theta)}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

We are interested in the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$. We note that the path parametrized by (4.5) reaches this point for $\theta = 0$. We recall from (2.10), (2.11) that for $(\lambda, \tau) = (0, 2\pi)$, we find a root of the characteristic equation given by $\mu = i\omega$ with $\omega = 1$. We find that

$$\begin{aligned} D_2\Delta(i, 0) &= -\frac{d}{d\theta}\begin{pmatrix} \theta - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \theta - K\cos\beta \end{pmatrix} - Ke^{-i\tau(0)}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\left(-i\left.\frac{d\tau}{d\theta}\right|_{\theta=0}\right) \\ &= -I + 2\pi i K\gamma\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \end{aligned}$$

We note that

$$\begin{aligned} q\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}p &= \alpha(1, i)\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \alpha(1, i)\begin{pmatrix} \cos\beta + i\sin\beta \\ \sin\beta - i\cos\beta \end{pmatrix} \\ &= 2\alpha(\cos\beta + i\sin\beta) = 2\alpha e^{i\beta} \end{aligned}$$

Using that α is as in the proof of Lemma 2.2.2 with $\tau = 2\pi$, we find that

$$q \cdot D_2\Delta(i,0)p = -2\alpha + 4\pi i K\gamma\alpha e^{i\beta} = \frac{-1 + 2\pi i\gamma K e^{i\beta}}{1 + K\tau e^{i\beta}}$$

which gives

$$\operatorname{Re}\left(q \cdot D_2\Delta(i,0)p\right) = -\frac{1 + 2\pi K(\cos\beta + \gamma\sin\beta)}{|1 + K2\pi e^{i\beta}|^2}$$

If we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right, we can parametrize the path by (4.5) with the replacement $\phi \mapsto -\phi$. Denote by $\tilde{\Delta}$ the characteristic function of system (2.2) for parameter values (λ, τ) on this path. A similar argument then shows that

$$\operatorname{Re}\left(q \cdot D\tilde{\Delta}_2(i,0)p\right) = \frac{1 + 2\pi K(\cos\beta + \gamma\sin\beta)}{|1 + K2\pi e^{i\beta}|^2}$$

If (4.3) holds, we find that $\operatorname{Re}\left(q \cdot D_2\Delta(i,0)p\right) < 0$ and thus that we find a Hopf bifurcation if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from left. If (4.4) holds, we find that $\operatorname{Re}\left(q \cdot D_2\tilde{\Delta}(i,0)p\right) < 0$ and thus that we find a Hopf bifurcation if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from right. $\square$

Having determined conditions for the occurence of a Hopf bifurcation, we can now compute the direction of the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$.

Lemma 4.0.3. *Let us approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left. Then the value of μ_2 as defined in Theorem 2.3.1 is given by*

$$\mu_2 = -4$$

Proof. If we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left, we find as in the proof of Lemma 4.0.2 that

$$\operatorname{Re}\left(q \cdot \Delta_2(i,0)p\right) = -\frac{1 + 2\pi K(\cos\beta + \gamma\sin\beta)}{|1 + K2\pi e^{i\beta}|^2}$$

Since the value of $\operatorname{Re} c$ is independent of the curve of approach and we are interested in the bifurcation at $(\lambda, \tau) = (0, 2\pi)$, we find that $\operatorname{Re} c$ is given as in Lemma 2.3.4 with $\phi = \tau = 2\pi$, i.e.

$$\operatorname{Re} c = \frac{4(1 + 2\pi K\,(\cos\beta + \gamma\sin\beta)}{|1 + 2\pi K e^{i\beta}|^2}$$

The claim now follows. $\square$

Lemma 4.0.4. *Let us approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right. Then the value of μ_2 as defined in Theorem 2.3.1 is given by*

$$\mu_2 = 4$$

Proof. If we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right, we find as in the proof of Lemma 4.0.2 that

$$\operatorname{Re}\left(q \cdot \Delta_2(i,0)p\right) = \frac{1 + 2\pi K(\cos\beta + \gamma\sin\beta)}{|1 + K2\pi e^{i\beta}|^2}$$

Combining this with the result of Lemma 2.3.4, the claim follows. $\square$

Corollary 4.0.1. *Let* $1 + 2\pi K e^{i\beta} \neq 0$. *If*

$$1 + 2\pi K \left[\cos\beta + \gamma \sin\beta\right] > 0 \tag{4.6}$$

then for small λ, (2.4) *is an unstable periodic solution of* (2.2). *Furthermore, if*

$$1 + 2\pi K \left[\cos\beta + \gamma \sin\beta\right] < 0 \tag{4.7}$$

then for small λ, (2.4) *is a stable periodic solution of* (2.2).

Proof. If (4.6) is satisfied, we have by Lemma 4.0.2 that we find a Hopf bifurcation at the point $(\lambda, \tau) = (0, 2\pi)$ if we approach this point over the extended Pyragas curve from the left. Combining Lemma 4.0.3 with Theorem 2.3.1, we find that this Hopf bifurcation is subcritical. Thus, there exists an unstable periodic solution for parameter values (λ, τ) on the (extended) Pyragas curve to the left of the point $(0, 2\pi)$. By the Hopf bifurcation theorem, the periodic solution for these parameter values is unique (see Remark 2.2.2). By definition of the Pyragas curve, (2.4) is a periodic solution of (2.2) for (λ, τ) near $(0, 2\pi)$, i.e. this is the periodic solution generated by the Hopf bifurcation. We conclude that for (λ, τ) on the Pyragas curve near $(0, 2\pi)$, (2.4) is an unstable periodic solution of (2.2), i.e. (2.4) is an unstable periodic solution of (2.2) for small λ.

If (4.7) is satisfied, we have by Lemma 4.0.2 that we find a Hopf bifurcation at the point $(\lambda, \tau) = (0, 2\pi)$ if we approach this point over the extended Pyragas curve from the right. Combining Lemma 4.0.3 with Theorem 2.3.1, we find that this Hopf bifurcation is supercritical. Therefore, we find an unique, stable periodic solution of (2.2) for (λ, τ) on the Pyragas curve near $(0, 2\pi)$. Since (2.4) is a periodic solution of (2.2) for (λ, τ) on the Pyragas curve, we conclude that for (λ, τ) on the Pyragas curve near $(0, 2\pi)$, this solution is in fact stable. $\square$

In Chapter 2, we approached the points on the Hopf bifurcation curve by varying the parameter λ and leaving all the other parameters fixed. In Corollary 2.4.1 we saw that if (2.12), (2.13) are satisfied, $1 + 2\pi K \left[\cos\beta + \gamma \sin\beta\right] < 0$ and the Pyragas curve is locally to the right of the Hopf bifurcation curve, that then the periodic solution (2.4) of (2.2) is stable for small λ. In Corollary 2.4.2 we saw that if (2.12), (2.13) are satisfied, $1 + 2\pi K \left[\cos\beta + \gamma \sin\beta\right] > 0$ and the Pyragas curve is locally to the left of the Hopf bifurcation curve, that then the periodic solution (2.4) of (2.2) is unstable for small λ.

By approaching the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve, we are able to weaken the conditions for (2.4) to be a (un)stable solution of (2.2) for small λ. Comparing Corollary 2.4.1 and Corollary 2.4.2 with Corollary 4.0.1, we see in particular that we can drop the condition (2.13). Furthermore, we no longer have to study the orientation of the Pyragas curve with respect to the Hopf bifurcation curve to determine the stability of the periodic solution (2.4) of (2.2), which simplifies the analysis considerably.

Discussion

To conclude, we discuss some of the aspects of the analysis presented in this thesis. Furthermore, we look at possible extensions and questions related to the topic of Pyragas control.

Local information and the calculation of characteristic multipliers

We note that the Hopf bifurcation theorem gives us local information on the stability of the arising periodic solution, i.e. it only tells whether the periodic solution is (un)stable for parameter values near the bifurcation point. Since we use the Hopf bifurcation theorem to determine the stability of the periodic solution (2.4) of (2.2), we can only determine the stability for parameter values near the bifurcation point, as we saw in Section 2.4.

We could try to circumvent this problem by calculating the characteristic multipliers of (2.2) for general values of λ. In Section 3.1 we were able to give information on a real characteristic multiplier; see Lemma 3.1.1. In order to deterine the stability by looking at the characterstic multipliers, all the characteristic multipliers (including non–real ones) have to be taken into account and we have not determined all those characteristic multipliers. It is virtually impossible to do so by analytical means, since the characteristic multipliers are defined as roots of a transcedental equation (see Section 3.1). Therefore, if we do not want to use numerical methods but stick to analytic means, an analysis in terms of bifurcation theory as discussed here presents a good possibility to give information on the stability of (2.4) for small λ.

Direction of the Hopf bifurcation and weaker conditions for the stability of the periodic orbit

The direction of the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ is a crucial element in the analysis presented in this thesis. In [10], a normal form reduction is used to determine the direction of the Hopf bifurcation that occurs when we only vary λ and leave all the other parameters fixed. In Section 2.3, we reproduce the result from [10], not by performing a normal form reduction, but by using a closed–form formula from [3] that enables us to determine the direction of the Hopf bifurcation. Since we are also able to determine the orientation of the Pyragas curve with respect to the Hopf bifurcation curve, we can combine this with the direction of the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ to determine the stability of the periodic solution (2.4) of (2.2) (see Corollary 2.4.1, 2.4.2).

We stress that varying λ and leaving all the other parameters fixed is one of the many possible ways to approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$. In Chapter 4, we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve (see Definition 4.0.1) and determine the direction of the Hopf bifurcation. The result is that we can weaken the conditions for the periodic solution (2.4) of (2.2) to be (un)stable. In particular, we do not longer have to take into account the orientation of the Pyragas curve with respect to the Hopf bifurcation curve. This considerably simplifies the analysis of the (in)stability of the periodic solution (2.4).

Variations in the control term

Let us return to the general problem of Pyragas control as formulated in the Introduction, i.e. let us study the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{4.8}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$. Let us assume that a periodic solution $u(t)$ of this system exists; denote its period by T. For the Pyragas control scheme, we write

$$\dot{x}(t) = f(x(t)) + K\,[x(t) - x(t-T)] \tag{4.9}$$

There are several variations to this control scheme possible. If $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(0) = 0$, then $u(t)$ is also a periodic solution of the system

$$\dot{x}(t) = f(x(t)) + Kg\,(x(t) - x(t-T)) \tag{4.10}$$

Furthermore, for $h : \mathbb{R}^n \to \mathbb{R}^n$, we have that $u(t)$ is a solution of the system

$$\dot{x}(t) = f(x(t)) + K\,[h(x(t)) - h(x(t-T))] \tag{4.11}$$

We can explore for which values of K the solution $u(t)$ of (4.8) is stable as a solution of (4.10) and (4.11), respectively. In particular, we can ask ourselves whether there exist values of K for which $u(t)$ is unstable as a solution of (4.9) but stable as a solution of (4.10) or (4.11). Furthermore, it is interesting to study the overall dynamics of (4.9), (4.10) and (4.11). For example, if we can find a value of K such that $u(t)$ is stable as a solution of both (4.9) and (4.10), the basin of attraction of $u(t)$ can still be different in the two sytems. By adressing such questions, we gain more insight in the possibilities for succesfull feedback–control mechanisms.

Appendices

Appendix A

Stability theory for ordinary differential equations

Suppose we study a system described by the ordinary differential equation

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{A.1}$$

with $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $x(t) \in \mathbb{R}^n$. In this chapter, we will study the stability of two classes of solutions of this differential equation: equilibria and periodic solutions.

We call x_0 an equilibrium of (A.1) if $f(x_0) = 0$. It turns out that we can often determine the stability of the equilibrium by studying the differential equation $\dot{x}(t) = Df(x_0)x(t)$. Since this is a linear differential equation with constant coefficients, we will first look at this class of differential equations in Section A.2. From there, we will move on to the study of equilibria of general ordinary differential equations in Section A.3.

We call a solution $u(t)$ of (A.1) periodic if there exists a $T > 0$ such that $u(t+T) = u(t)$ for all t. In the same way as we study the stability of the fixed point x_0 by looking at the linearization $\dot{x}(t) = Df(x_0)x(t)$, there exists a relation between the stability of the periodic orbit $u(t)$ of (A.1) and the stability of $\dot{x}(t) = Df(u(t))x(t)$. Since this is a linear differential equation with periodic coefficients, we will study this class of differential equations in Section A.4. In Section A.5 we will then turn to stability of periodic orbits.

We start by making some introductory remarks on ordinary differential equations that will be helpful throughout the rest of the discussion.

A.1 The state space and flow maps

In Section 1.1, we introduced the state space of a differential equation; see Definition 1.1.1. If the state space of a system is given, we can try to define a map that maps a state to another state 'along the solution' of the differential equation. Thus, we can define a family of maps from the state space to itself. In the case of ordinary differential equations, we can use this idea to define the so-called flow maps:

Definition A.1.1. If $x(t)$ is a solution of (A.1) with initial condition $x(0) = x_0$, we define the *flow map* for system (A.1) as

$$\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad \phi(t, x_0) = x(t)$$

We note that for fixed $t \in \mathbb{R}$, the map

$$\phi(t,\ .) : \mathbb{R}^n \to \mathbb{R}^n$$

is a map from the state space to itself. Thus we can construct a family of maps from the state space to itself:

$$\{\phi(t,\ .) \mid t \in \mathbb{R}\}$$

We note that the map $\phi(t,\ .)$ translates along the solution of (A.1). If we assume that for all $x_0 \in \mathbb{R}^n$ the solution of (A.1) is defined for all $t \in \mathbb{R}$, we find that the following properties hold for all $t, s \in \mathbb{R}$:

$$\phi(0,\ .) = I$$
$$\phi(t,\ .)\phi(s,\ .) = \phi(t+s,\ .)$$

where $I : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity map. In particular, we find that

$$\phi(t,\ .)\phi(-t,\ .) = \phi(-t,\ .)\phi(t,\ .) = I$$

Combining these three properties, we find that the set $\{\phi(t,\ .) \mid t \in \mathbb{R}\}$ is in fact a group under composition of functions.

A.2 Equilibria of linear differential equations with constant coefficients

We now turn to the system

$$\dot{x}(t) = Ax(t) \tag{A.2}$$

where A is a $n \times n$-matrix (with respect to the standard basis of $\mathbb{R}^n$). Defining the exponent of a matrix via its power series expansion, the solution of Eq. (A.2) with initial condition $x(0) = x_0$ is given by $x(t) = e^{At}x_0$. [9] To be able to say more about the properties of this solution, we recall the following definition and lemma from linear algebra:

Definition A.2.1. Let A be a $n \times n$-matrix and λ be one of its eigenvalues. Denote by $r(\lambda)$ the least integer k such that that $N((A - \lambda I)^k) = N((A - \lambda I)^{k+1})$, where N denotes the null-space. The *generalized eigenspace* $M_\lambda(A)$ of A for the eigenvalue λ is defined to be the set $N((A - \lambda I)^{r(\lambda)})$.

Lemma A.2.1. *Let A be a $n \times n$-matrix and $\lambda_1, \ldots, \lambda_s$ be its distinct eigenvalues. Then the generalized eigenspaces $M_{\lambda_1}(A), \ldots, M_{\lambda_s}(A)$ are linearly independent, invariant under the matrix A and any $x_0 \in \mathbb{R}^n$ can be represented uniquely as:*

$$x_0 = \sum_{j=1}^{s} x_{0,j} \quad \textit{for} \quad x_{0,j} \in M_{\lambda_j}(A)$$

For a proof of the lemma, see for example [9]. Using this lemma, we can prove the following:

Proposition A.2.2. *With notation as defined above, the solution of Eq.* (A.2) *with initial condition* $x(0) = x_0$ *is given by:*

$$x(t) = \sum_{j=1}^{s} \left(\sum_{k=1}^{r(\lambda_j)-1} (A - \lambda_j I)^k \frac{t^k}{k!} \right) e^{\lambda_j t} x_{0,j}$$

Proof. Using the general properties of exponents of matrices, we note that for $x_{0,j} \in M_{\lambda_j}(A)$ we have that

$$e^{At}x_{0,j} = e^{(A-\lambda_j I)t} e^{\lambda_j t} x_{0,j} = \left(\sum_{k=1}^{\infty} \frac{1}{k!} (A - \lambda_j I)^k t^k \right) e^{\lambda_j t} x_{0,j}$$

Using that the solution of (A.2) can also be written as $x(t) = e^{At}x_0$, that $(A - \lambda_j I)^k = 0$ for $k \geq r(\lambda_j)$ and that $x_0 = \sum_{j=1}^{s} x_{0,j}$, the statement follows. [6] ☐

We can now use this property to study stability of equilibria. Intuitively speaking, an equilibrium solution is stable if all solutions that start nearby x_0, stay nearby x_0. This is made more precise in the following definition (which is stated for autonomous differential equations).

Definition A.2.2. Let $x_0 \in \mathbb{R}^n$ be an equilibrium point of the differential equation $\dot{x}(t) = f(x(t))$ with $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $f(x_0) = 0$. Denote by S its state space of the system and by $\phi(t, x)$ its flow. The equilibrium $x_0 \in \mathbb{R}^n = S$ is said to be *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in B(x_0, \delta) \subseteq \mathbb{R}^n = S$ implies that $\phi(t, x_0) \in B(x_0, \epsilon) \subseteq \mathbb{R}^n = S$ for all $t \geq 0$. The equilibrium x_0 is said to be *asymptotically stable* if it is stable and there exists a $b > 0$ such that $x \in B_b(x_0) \subseteq \mathbb{R}^n = S$ implies that $\lim_{t\to\infty} \phi(t, x) = x_0$. The equilibrium point x_0 is said to be *unstable* if it is not stable.

This definition formalizes the intuitive understanding of the concept of stability, but is often not very useful in concrete situations. For linear differential equations with constant coefficients, the following propositions give a computational tool for determining stability of the equilibrium $x_0 = 0$ of Eq. (A.2).

Proposition A.2.3. *If $Re(\lambda) < 0$ for all eigenvalues λ of A, then the equilibrium $x_0 = 0$ is asymptotically stable. Furthermore, there exist constants $K, \alpha > 0$ such that $\left\|e^{At}x\right\| \leq K^{-\alpha t} \|x\|$.*

Proof. We denote by $\lambda_1, \ldots, \lambda_s$ the distinct eigenvalues of A and write $x = \sum_{j=1}^{s} x_j,\ x_j \in M_{\lambda_j}(A)$. We can define the following norm:

$$\|.\|' : \mathbb{R}^n \to \mathbb{R}, \quad \|x\|' = \sum_{j=1}^{s} \|x_j\|$$

where $\|.\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. Using that the decomposition of x in generalized eigenvectors is unique, and that $\|.\|$ is a norm, we can readily see that the map $x \mapsto \|x\|'$ is well-defined and in fact a norm. Since all norms on $\mathbb{R}^n$ are equivalent, it follows that there exists a constant $C > 0$ such that $\|x\|' \leq C \|x\|$ for all $x \in \mathbb{R}^n$.

Since $\mathrm{Re}(\lambda_j) < 0$ for all j, we have that $\left|\frac{t^k}{k!}e^{\lambda_j t}\right| \to 0$ as $t \to \infty$. By definition, there exists a $\delta_{j,k} > 0$ such that $\left|\frac{t^k}{k!}e^{\lambda_j t}\right| < 1$ for all $t \geq \delta_{j,k}$. We define

$$a_{j,k} = \max\left(\left\{\left|\frac{t^k}{k!}e^{\lambda_j t}\right| \mid t \in [0, \delta_{j,k}]\right\} \cup \{1\}\right)$$

Furthermore, we have that $\left\|(A - \lambda_j I)^k x_j\right\| \leq \left\|(A - \lambda_j I)^k\right\| \|x_j\|$, where $\left\|(A - \lambda_j I)^k\right\|$ denotes the operator norm of $(A - \lambda_j I)^k$. Introducing the notation

$$\begin{aligned} A &= \max_{j=1,\ldots,s} \sum_{k=1}^{r(\lambda_j)-1} \left\|(A - \lambda_j I)^k\right\| \\ a &= \max_{\substack{j=1,\ldots,s \\ k=1,\ldots,r(\lambda_j)-1}} a_{j,k} \end{aligned}$$

Using Proposition A.2.2, we make the following estiame:

$$\begin{aligned} \|\phi(t, x)\| &\leq \sum_{j=1}^{s} \sum_{k=1}^{r(\lambda_j)-1} \left\|(A - \lambda_j I)^k\right\| \left|\frac{t^k}{k!}e^{\lambda_j t}\right| \|x_j\| \leq \sum_{j=1}^{s} \|x_j\| \sum_{k=1}^{r(\lambda_j)-1} \left\|(A - \lambda_j I)^k\right\| a_{j,k} \\ &\leq aA \sum_{j=1}^{s} \|x_j\| = aA \|x\|' \leq aAC \|x\| \end{aligned}$$

Choosing $\delta = \frac{\epsilon}{aAc}$ shows that $x_0 = 0$ is stable. Since $\mathrm{Re}(\lambda_j) < 0$ for all j, it follows that $\frac{1}{k!}(A - \lambda_j I)^k t^k e^{\lambda_j t} x_{0,j} \to 0$ as $t \to \infty$. We conclude that $\phi(t, x) \to 0$ as $t \to \infty$ for any $x \in \mathbb{R}^n$. Thus $x_0 = 0$ is asymptotically stable.

To prove the inequality $\left\|e^{At}x_0\right\| \leq K^{-\alpha t}\left\|x_0\right\|$, we rewrite $e^{\lambda_j t} = e^{\frac{\lambda_j t}{2}} e^{\frac{\lambda_j t}{2}}$. As before, we remark that $\mathrm{Re}(\lambda_j) < 0$ implies that $\lim_{t\to\infty}\left|\frac{t^k}{k!}e^{\frac{\lambda_j t}{2}}\right| = 0$ and thus that there exists a $\delta'_{j,k}$ such that $\left|\frac{t^k}{k!}e^{\frac{\lambda_j t}{2}}\right| < 1$ for all $t \geq \delta'_{j,k}$. We define

$$
\begin{aligned}
a'_j &= \max\left(\left\{\left|\frac{t^k}{k!}e^{\frac{\lambda_j t}{2}}\right| \mid t \in [0, \delta_{j,k}]\right\} \cup \{1\}\right) \\
a' &= \max_{\substack{j=1,\dots,s \\ k=1,\dots,r(\lambda_j)-1}} a'_{j,k} \\
A' &= \max_{j=1,\dots,s} \sum_{k=1}^{r(\lambda_j)-1} \left\|(A-\lambda_j I)^k\right\| \\
-\alpha &= \max_{j=1,\dots,s} \mathrm{Re}\left(\frac{\lambda_j}{2}\right)
\end{aligned}
$$

Note that $\alpha > 0$. We now obtain the estimate:

$$
\begin{aligned}
\|\phi(t,x_0)\| = \left\|e^{At}x_0\right\| &\leq \sum_{j=1}^{s}\sum_{k=1}^{r(\lambda_j)-1} \left\|(A-\lambda_j I)^k\right\| \left|\frac{t^k}{k!}e^{\frac{\lambda_j t}{2}}\right| \left|e^{\frac{\lambda_j t}{2}}\right| \|x_{0,j}\| \\
&\leq \sum_{j=1}^{s} a'A'e^{\frac{\lambda_j t}{2}}\|x_{0,j}\| \leq a'A'e^{-\alpha t}\sum_{j=1}^{s}\|x_{0,j}\| \leq a'A'e^{-\alpha t}C\|x\| = Ke^{-\alpha t}\|x\|
\end{aligned}
$$

where we have set $K = a'A'C > 0$. □

Proposition A.2.4. *If the matrix A has an eigenvalue with $Re(\lambda_j) > 0$, then the equilibrium point $x_0 = 0$ is unstable.*

Proof. We need to prove that there exists an $\epsilon > 0$ such that for all $\delta > 0$, we can find a pair (x, t_0) such that $\|x\| < \delta$ but $\|\phi(t_0, x)\| \geq \epsilon$.

We choose $\epsilon = 1$. Denote by λ_N the eigenvalue with $\mathrm{Re}(\lambda_N) > 0$ and by v_N a corresponding eigenvector. We choose $x = x(\delta) = \frac{v_N}{\|v_N\|}\frac{\delta}{2}$; then $\|x\| < \delta$. Using Proposition (A.2.2), we find that $\phi(t,x) = e^{\lambda_N t}v_N$. But since $\mathrm{Re}(v_N) > 0$, we have that $\lim_{t\to\infty}\left\|e^{\lambda_N t}v_n\right\| \to \infty$. Thus, there exists an t_0 such that $\|\phi(t,x)\| = \left\|e^{\lambda_N t}v_n\right\| \geq 1$ for all $t \geq t_0$, which proves the claim. □

We can use these results to obtain a more geometric view of stability. Intuitively speaking, we would like to decompose the state space in such a way that the origin of the system (A.2) is stable on one part of the decomposition, and unstable on the other part. To formalize this idea, we introduce the following concepts for general ordinary differential equations.

Definition A.2.3. Let x_0 be an equilibrium of $\dot{x}(t) = f(x(t))$. The *stable manifold* of x_0 is defined as:

$$W^s(x_0) = \{x \in \mathbb{R}^n \mid \phi(t,x) \to x_0 \text{ as } t \to \infty\}$$

Similarly, we define the *unstable manifold* as

$$W^u(x_0) = \{x \in \mathbb{R}^n \mid \phi(t,x) \to x_0 \text{ as } t \to -\infty\}$$

[8]

To apply these concepts to the case of a linear system with constant coefficients, denote by $\{\lambda_1, \ldots, \lambda_s\}$ the eigenvalues of A. We choose the indices in such a way that

$$\begin{aligned} \mathrm{Re}\,(\lambda_i) < 0 \quad & \text{for } 1 \leq i < n_1 \\ \mathrm{Re}\,(\lambda_i) > 0 \quad & \text{for } n_1 \leq i < n_2 \\ \mathrm{Re}\,(\lambda_i) = 0 \quad & \text{for } n_2 \leq i < s \end{aligned}$$

Using Lemma A.2.1, we see that we can write the state space $\mathbb{R}^n$ as a direct sum of the generalized eigenspaces, i.e.

$$\mathbb{R}^n = M_{\lambda_1}(A) \oplus \ldots \oplus M_{\lambda_s}(A)$$

In the case that there exists an eigenvalue λ of A on the imaginary axis, we can use Propositions A.2.3 and A.2.4 to see that

$$\begin{aligned} M_{\lambda_1}(A) \oplus \ldots \oplus M_{\lambda_{n_1-1}}(A) &\subseteq W^s(0) \\ M_{\lambda_{n_1}}(A) \oplus \ldots \oplus M_{\lambda_{n_2-1}}(A) &\subseteq W^u(0) \end{aligned}$$

We do not have an equality in this case since the generalized eigenspaces of the eigenvalues on the imaginary axes can belong to the (un)stable manifold. This problem disappears if all eigenvalues λ of A satisfy $\mathrm{Re}\,(\lambda) \neq 0$. In this case $n_2 = s$ and we know completely how to decompose the state space in an unstable and stable manifold:

$$\begin{aligned} W^s(0) &= M_{\lambda_1}(A) \oplus \ldots \oplus M_{\lambda_{n_1-1}}(A) \\ W^u(0) &= M_{\lambda_{n_1}}(A) \oplus \ldots \oplus M_{\lambda_s}(A) \end{aligned}$$

A.3 Equilibria of autonomous differential equations; linearization

In the previous section, we studied the fixed point $x = 0$ of the system $\dot{x}(t) = Ax(t)$, where A was some constant matrix. It turns out that this can help us to determine the stability of an equilibrium point x_0 of the ordinary differential equation $\dot{x}(t) = f(x(t))$. We first need to introduce some terminology (see also [7]).

Definition A.3.1. Let $x_0 \in \mathbb{R}^n$ be an equilibrium of the differential equation $\dot{x}(t) = f(x(t))$ with $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. Then x_0 is called *hyperbolic* if the linear map $Df(x_0)$ has no eigenvalues on the imaginary axis.

Definition A.3.2. Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$. Two differential equations $\dot{x}(t) = f(x(t))$ and $\dot{x}(t) = g(x(t))$ defined on open sets $U, V \subseteq \mathbb{R}^n$, respectively, are called *topologically equivalent* if there is a homeomorphism $h : U \to V$ that maps orbits of f onto orbits of g, preserving the direction of time.

We are now able to state the following theorem:

Theorem A.3.1 (Grobman-Hartman)**.** *Let $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and let x_0 be an hyperbolic equilibrium point of $\dot{x}(t) = f(x(t))$. Then there exists a neighbourhood of x_0 in which f is topologically equivalent to the linear vector fields $\dot{x}(t) = Df(x_0)x(t)$.*

A proof can be found in [2].

This implies that for a hyperbolic equilibrium, we now also have a computational tool to determine the stability of the equilibrium.

Corollary A.3.1. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and let x_0 be an hyperbolic equilibrium of the differential equation $\dot{x}(t) = f(x(t))$. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $Df(x_0)$. If $\mathrm{Re}\,(\lambda_j) < 0$ for all $1 \leq j \leq n$, then x_0 is stable. If there exists a $1 \leq j \leq n$ such that $\mathrm{Re}\,(\lambda_j) > 0$, then x_0 is an unstable equilibrium.*

Proof. Denote by $\phi(t,x)$ the flow of $\dot{x}(t) = f(x(t))$ and by $\psi(t,x)$ the flow of $\dot{x}(t) = Df(x_0)x(t)$. Then we can parafrase the Grobman-Hartman theorem by saying that there exists a neighbourhood U of x_0, a neighbourhood V of 0 and a homeomorphism $h : U \to V$ with $h(x_0) = 0$ such that $\psi(t, h(x)) = h(\phi(t,x))$. Using this, we prove that (in)stability of the equilibrium 0 of $\dot{x}(t) = Df(x_0)x(t)$ implies (in)stability of the equilibrium x_0 of $\dot{x}(t) = f(x(t))$.

Assume that 0 is a stable equilibrium of $\dot{x}(t) = Df(x_0)x(t)$. Let $\epsilon > 0$. Since h^{-1} is continuous and $h^{-1}(0) = x_0$, there exists a $\delta_1 > 0$ such that $\|x\| < \delta_1$ implies that $\left\|h^{-1}(x) - x_0\right\| < \epsilon$. Since 0 is by assumption a stable equilibrium of $\dot{x}(t) = Df(x_0)x(t)$, there exists a $\delta_2 > 0$ such that $\|x\| < \delta_2$ implies that $\|\psi(t,x)\| < \delta_1$ for all $t \geq 0$. Since h is continuous and $h(x_0) = 0$, there exist a $\delta_3 > 0$ such that $\|x - x_0\| < \delta_3$ implies that $\|h(x)\| < \delta_2$. Wrapping things up, we find that $\|x - x_0\| < \delta_3$ implies that $\|\phi(t,x) - x_0\| = \left\|h^{-1}(\psi(t, h(x))) - x_0\right\| < \epsilon$. A similar argument shows that instability of the equilibrium 0 of $\dot{x}(t) = Df(x_0)x(t)$ implies instability of the equilibrium x_0 for $\dot{x}(t) = f(x(t))$.

Thus, if all eigenvalues of $Df(x_0)$ have negative real part, we recall from Proposition A.2.3 that the origin is a stable equilibrium of $\dot{x}(t) = Df(x_0)x(t)$. This implies that x_0 is a stable equilibrium of $\dot{x}(t) = f(x(t))$. Similarly, if an eigenvalue of $Df(x_0)$ has positive real part, we conclude that x_0 is unstable. $\square$

The proof of the Grobman-Hartman theorem is quite involved. However, we can prove a part of the corollary directly (i.e. without using Grobman-Hartman):

Proposition A.3.2. *Let $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ be an equilibrium of the differential equation $\dot{x}(t) = f(x(t))$ with the property that all the eigenvalues of $Df(x_0)$ have strictly negative real part. Then x_0 is asymptotically stable.*

For the proof, we need the following lemma on linear algebra:

Lemma A.3.3. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ a linear map such that there exists a constant $a \in \mathbb{R}$ with the property that Re$(\lambda) < a$ for all eigenvalues λ of A. Then there exists a basis of $\mathbb{R}^n$ such that the induced inner product $\langle . \, , \, . \rangle'$ and norm $\|.\|'$ satisfy $\langle Ax, x\rangle' \leq a \|x\|'^2$ for all $x \in \mathbb{R}^n$.*

For a proof of the lemma, see for example [9].

Proof. (of Proposition A.3.2). [9] Assume without loss of generality that $x_0 = 0$, the general situation can be reduced to this case by appropriate translation of the coordinate system. Denote by $\lambda_1, \ldots, \lambda_s$ the distinct eigenvalues of $Df(x_0)$. By assumption, we can choose $b > 0$ such that $\max_{1,\ldots,s} \mathrm{Re}(\lambda_j) < -b$. It follows by the lemma that there exists a basis of $\mathbb{R}^n$ such that $\langle Df(x_0)x, x\rangle' < -b\|x\|'^2$ for all $x \in \mathbb{R}^n$, where $\langle . \, , \, . \rangle'$ and $\|x\|'$ denote the inner product resp. norm induced by this basis.

We note that if an equilibrium is (un)stable with respect to a norm, it is (un)stable with respect to any equivalent norm. Since all norms on $\mathbb{R}^n$ are equivalent, it is sufficient to prove stability of x_0 with respect to the norm $\|.\|'$.

Since f is differentiable and $x_0 = 0$ an equilibrium point (i.e. $f(x_0) = 0$), we have by definition of the derivative $Df(x_0)$ that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|'}{\|x\|'} = \lim_{x \to 0} \frac{\|f(x) - Df(0)(x)\|'}{\|x\|'} = 0$$

(note that this also holds in the norm $\|.\|'$ since all norms on $\mathbb{R}^n$ are equivalent). Using the Cauchy-Schwarz inequality, we can make the following estimate:

$$\frac{\left|\langle f(x) - Df(0)(x), x\rangle'\right|}{\|x\|'^2} \leq \frac{\|f(x) - Df(0)(x)\|' \, \|x\|'}{\|x\|'^2} = \frac{\|f(x) - Df(0)(x)\|'}{\|x\|'}$$

Thus, we find that $\frac{\langle f(x) - Df(0)(x), x\rangle'}{\|x\|'^2} \to 0$ as $x \to 0$. We now choose $\epsilon > 0$ such that $-b + \epsilon < 0$. Then there exists a $\delta > 0$ such that for $\|x\| < \delta$, we have that $\frac{\langle f(x) - Df(0)(x), x\rangle'}{\|x\|'^2} < \epsilon$. From here we obtain the estimate:

$$\epsilon \|x\|'^2 > \langle f(x) - Df(0)(x), x\rangle' = \langle f(x), x\rangle' - \langle Df(0)(x), x\rangle' \geq \langle f(x), x\rangle' + b\|x\|'^2$$

Thus $\langle f(x), x\rangle' \leq (\epsilon - b)\|x\|'^2$ for $\|x\|' < \delta$. Denoting the flow of $\dot{x}(t) = f(x(t))$ by $\phi(t,x)$ and using the chain rule, we find that

$$\frac{d}{dt}\|\phi(t,x)\|' = \frac{d}{dt}\sqrt{\langle\phi(t,x),\phi(t,x)\rangle'} = \frac{\left\langle \frac{d}{dt}\phi(t,x),\phi(t,x)\right\rangle'}{\|\phi(t,x)\|'} = \frac{\langle f(x),\phi(t,x)\rangle'}{\|\phi(t,x)\|'} \leq (\epsilon - b)\|\phi(t,x)\|'$$

as $\|\phi(t,x)\|' < \delta$. This leads to the inequality $\|\phi(t,x)\|' \leq e^{(\epsilon-b)t}\|x\|'$. We recall that $\epsilon - b < 0$. For any given $\epsilon' > 0$, we choose $\delta' = \min(\epsilon', \delta)$. Then $\|\phi(t,x)\| \leq e^{(\epsilon-b)t}\|x\|' \leq \|x\|' < \epsilon'$ for all $t \geq 0$ if $\|x\|' < \delta'$. Since the estimate $\|\phi(t,x)\|' \leq e^{(\epsilon-b)t}\|x\|'$ holds for $\|x\|' < \delta$, we find that $\phi(t,x) \to 0$ for such x. We conclude that $x_0 = 0$ is asymptotically stable. □

A.4 Linear differential equations with periodic coefficients

After studying equilibria of ordinary differential equations, we now turn to another class of solutions, namely periodic ones. In the previous sections we saw that to analyse the stability of an equilibrium for an ordinary differential equation, it was helpful to study its linearization. Similarly, it turns out that the so-called linear variational equation can help us to determine the stability of a periodic orbit. Since the linear variational equation is a linear differential equation with periodic coefficients, we study this class of ordinary differential equations in more detail. We will mostly follow [6].

We first introduce some terminology.

Definition A.4.1. Let $B(t) : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map for all $t \in \mathbb{R}$ (we make no assumptions on the periodicity of $B(t)$). We call the $n \times n$-matrix a *matrix solution* of $\dot{x}(t) = B(t)x(t)$ if all its columns are solutions of this differential equation. We call $X(t)$ the *fundamental matrix solution* of $\dot{x}(t) = B(t)x(t)$ if it is a matrix solution and satisfies $X(0) = I$.

We summarize some of the properties of the properties of the fundamental matrix solution in the following lemmata.

Lemma A.4.1. *If $X(t)$ is the fundamental matrix solution of $\dot{x}(t) = B(t)x(t)$, then the columns of $X(t)$ are linearly independent for all $t \geq 0$.*

Proof. Denote by $v_1(t), \ldots, v_n(t)$ the columns of $X(t)$ and assume by contradiction that there exists a $t_0 \geq 0$ such that the columns of $X(t_0)$ are not linearly independent. By definition, there exists $\lambda_1, \ldots, \lambda_n$ such that not all λ_i equal zero simultaneously and

$$0 = \lambda_1 v_1(t_0) + \ldots + \lambda_n v_n(t_0)$$

Since $B(t)$ is a linear map, we see that the associated flow map $\phi(t, \,.) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map for all $t \in \mathbb{R}$. By the definition of the fundamental matrix solution, we have that $v_i(t_0) = \phi(t_0, v_i(0)) = \phi(t_0, e_i)$ for all $1 \leq i \leq n$. Thus we find that

$$\begin{aligned}\phi(-t_0, 0) = 0 &= \phi\left(-t_0, \lambda_1 v_1(t_0) + \ldots + \lambda_n v_n(t_0)\right) = \lambda_1\phi(-t_0, v_1(t_0)) + \ldots + \lambda_n\phi(-t_0, v_n(t_0)) \\ &= \lambda_1\phi(-t_0, \phi(t_0, e_1)) + \ldots + \lambda_n\phi(-t_0, \phi(t_0, e_n)) = \lambda_1 e_1 + \ldots + \lambda_n e_n\end{aligned}$$

where we have used that $x = 0$ is an equilibrium of $\dot{x}(t) = B(t)x(t)$. The statement that $0 = \lambda_1 e_1 + \ldots + \lambda_n e_n$ is clearly a contradiction with the assumption that $\lambda_i \neq 0$ for all $1 \leq i \leq n$ simultaneously. We conclude that $v_1(t), \ldots, v_n(t)$ are linearly independent for all $t \geq 0$. □

Lemma A.4.2. *Let $B(t) : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map for all $t \in \mathbb{R}$. If $X(t),\ Y(t)$ are matrix solutions of $\dot{x}(t) = B(t)x(t)$ such that $\det X(0) \neq 0,\ \det Y(0) \neq 0$, then there exists an invertible matrix C such that $Y(t) = X(t)C$.*

Proof. Denote by $\{e_1, \ldots, e_n\}$ a basis of $\mathbb{R}^n$ and fix $i \in \{1, \ldots, n\}$. By definition of the fundamental matrix solution, $Y(t)e_i$ is a solution of the linear differential equation $\dot{x}(t) = B(t)x(t)$ with initial condition $Y(0)e_i$. However, $X(t)X(0)^{-1}Y(0)e_i$ is also a solution of $\dot{x}(t) = B(t)x(t)$ with initial condition $Y(0)e_i$. Since solutions of $\dot{x}(t) = B(t)x(t)$ are uniquely determined once an initial condition is given, we find that $Y(t)e_i = X(t)X(0)^{-1}Y(0)e_i$. Since this argument holds for all basis vectors e_i, and since a linear map is completely determined by its behaviour on the basis vectors, we find that $Y(t) = X(t)X(0)^{-1}Y(0)$. We set $C = X(0)^{-1}Y(0)$. Since $\det X(0), \det Y(0) \neq 0$ by assumption, we have that $C = X(0)^{-1}Y(0)$ is invertible and the claim follows. $\square$

We now turn to the system

$$\dot{x}(t) = A(t)x(t) \tag{A.3}$$

where $A(t) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map for all $t \in \mathbb{R}$ with the property that $A(t+T) = A(t)$ for some $T > 0$. This kind of ordinary differential equation is – not suprisingly – called a linear differential equation with periodic coefficients. We assume that $A(t)$ is integrable, such that the solutions of (A.3) exist for all $t \in \mathbb{R}$ (see [6]).

We recall that in the discussion of linear differential equations with constant coefficents, Proposition A.2.2 gave a general form for the solutions. For the case of linear periodic coefficients, we similarly find the following result:

Theorem A.4.3. *Let $X(t)$ be the fundamental matrix solution of Eq.* (A.3)*. Then there exist matrices $B, P(t)$ such that $P(t+T) = P(t)$, B is constant and*

$$X(t) = P(t)e^{Bt}$$

Proof. Let $X(t)$ be the fundamental matrix solution of Eq. (A.3). Since $A(t) = A(t+T)$, it follows that $X(t+T)$ is also matrix solution of this system. By the remarks made in the previous lemmata, we find that $\det X(t) \neq 0$, $\det X(t) \neq 0$ and there exists an invertible matrix C such that $X(t+T) = X(t)C$ for all t. Since C is invertible, we have that $\det C \neq 0$ and we can use a result from linear algebra to see that there exists a matrix B such that $C = e^{BT}$ (for a proof, see for example [6]), i.e. $X(t+T) = X(t)e^{BT}$. We define $P(t) = X(t)e^{-Bt}$, then we have that $P(t)e^{Bt} = X(t)$. Using the properties of the exponent of a matrix, we furthermore find that

$$P(t+T) = X(t+T)e^{-B(t+T)} = X(t+T)e^{-Bt}e^{-BT} = X(t)e^{BT}e^{-BT}e^{-Bt} = X(t)e^{-Bt} = P(t)$$

Thus, $P(t)$, B have the required properties. [6] $\square$

Using this, we can prove the following corollary:

Corollary A.4.1. *There exists an (invertible) coordinate transform that transforms Eq.* (A.3) *into a linear differential equation with constant coefficients.*

Proof. We set $x(t) = P(t)y(t)$. Recalling that $P(t) = X(t)e^{-Bt}$ and that $X(t)$ is invertible for all $t \geq 0$, we see that $P(t)$ is invertible. Thus this is indeed an (invertible) coordinate transform. To see that it transforms Eq. (A.3) into a linear differential equation with constant coefficients, we first note that

$$\dot{P}(t) = \dot{X}(t)e^{-Bt} + X(t)(-B)e^{-Bt}$$

Since $X(t)$ is the fundamental matrix solution of (A.3) and any matrix commutes with its exponent, we can rewrite this as

$$\dot{P}(t) = A(t)X(t)^{-Bt} - X(t)e^{-Bt}B = A(t)P(t) - P(t)B$$

Using this, we find that

$$\dot{x}(t) = \dot{P}(t)y(t) + P(t)\dot{y}(t) = (A(t)P(t) - P(t)B)\,y(t) + P(t)\dot{y}(t) = A(t)x(t) = A(t)P(t)y(t)$$

We conclude that $P(t)\dot{y}(t) = P(t)By(t)$ and, since $P(t)$ is invertible, that $\dot{y}(t) = By(t)$. Since this a linear differential equation with constant coefficients, the claim follows. [6] $\square$

Using this transformation, we can try to determine the stability of a periodic orbit of Eq. (A.3) using stability theory of equilibria for linear differential equations with constant coefficients. To this end, we introduce the following terminology.

Definition A.4.2. Let $X(t)$ be the fundamental matrix solution of Eq. (A.3). The *monodromy matrix* associated with $X(t)$ is the matrix C such that $X(t+T) = X(t)C$. The eigenvalues ρ of C are referred to as *characteristic multipliers*. A number λ such that $\rho = e^{\lambda T}$ is called a *characteristic exponent* of Eq. (A.3).

We recall that in Lemma A.4.2 we gave an explicit form for C; applying this to C in the definition above, we find that $C = X(T)$. We also note that the characteristic exponents are not uniquely determined. Indeed, let λ, μ be two complex numbers such that $e^{\lambda T} = e^{\mu T} = \rho$, then $e^{(\lambda-\mu)T} = 1$. Thus we find that $(\lambda - \mu)T = 2\pi i k$ for some $k \in \mathbb{Z}$. Hence two characteristic exponents for one characteristic multiplier can only differ by $\frac{2\pi i k}{T}$ for some $k \in \mathbb{Z}$. We recall from complex analysis that the complex logarithm $\log(re^{i\theta}) = \log(r) + i\theta$ satisfies $e^{\log z} = z$. [11] Since C is invertible, we find that $\rho \neq 0$ for any eigenvalue of C, i.e. for any characteristic multiplier. Therefore, we can write $\rho = r_0 e^{i\theta_0}$ with $r_0 \neq 0$. All the corresponding characteristic exponents can now be written as

$$\lambda_k = \frac{1}{T}\left(\log r_0 + i(\theta_0 + 2\pi k)\right), \quad k \in \mathbb{Z}$$

In particular, we see that if $|\rho| < 1$, then all the associated characteristic exponents lie in the left half plane; if $|\rho| > 1$, then all its characteristic exponents lie in the right half plane. We now note that if B has the property that $C = e^{BT}$, then the eigenvalues of B are characteristic exponents. Thus, if all the eigenvalues ρ of C satisfy $|\rho| < 1$, then all the characteristic exponents and therefore all the eigenvalues of B will lie in the left half plane. If there exists an eigenvalue ρ of C such that $|\rho| > 1$, we can always find an eigenvalue of B in the right half plane. Using this, we can prove the following theorem:

Theorem A.4.4. *If all the characteristic multipliers of Eq.* (A.3) *ρ satisfy $|\rho| < 1$, then the equilibrium $x = 0$ is asympotically stable. If one of the characteristic multipliers satisfy $|\rho| > 1$, then the equilibrium $x = 0$ is unstable.*

Proof. The main ingredient for the proof is Corollary A.4.1. Since $X(t)$ is a fundamental matrix solution, the map $t \mapsto X(t)$ is differentiable, hence continuous. Therefore, the transformation $P(t) = X(t)e^{-Bt}$ is continous. Since the map $P(t)$ is also periodic, we have that $\|P(t)\| \leq \sup\{\|P(t)\| \mid t \in [0,T]\} := K < \infty$ for all t. Thus, we find that $\|x(t)\| \leq K\,\|y(t)\|$ and that asymptotically stability of the $y = 0$ for the linear differential equation $\dot{y}(t) = By(t)$ implies asymptotically stability of $x = 0$ for Eq. (A.3). If $|\rho| < 1$ for all eigenvalues ρ of C, the discussion above implies that all the eigenvalues of B are in the left half plane. Using Proposition A.2.3 we find that $y = 0$ is an asymptotically stable solution of $\dot{y}(t) = By(t)$. Thus, $x = 0$ is an asymptotically stable solution of Eq. (A.3).

Since $P(t)$ invertible, we can rewrite the transformation as $y(t) = P(t)^{-1}x(t)$. Since $P(t)$ is continuous and periodic, the same holds for $P(t)^{-1}$ and thus $\left\|P(t)^{-1}\right\| \leq \sup\{\left\|P(t)^{-1}\right\| \mid t \in [0,T]\} := K' < \infty$ for all t. Thus we find that $\|y(t)\| \leq K'\,\|x(t)\|$ for all t. It follows that stability of $x = 0$ implies stability of $y = 0$, and thus by contraposition that instability of $y = 0$ implies instability of $x = 0$. If there exists a characteristic multiplier ρ of Eq. (A.3) with $|\rho| > 1$, it follows by remarks made above that B has an eigenvalue in the right half plane. By Proposition A.2.4 $y = 0$ is an unstable solution of $\dot{y}(t) = By(t)$. We conclude that $x = 0$ is an unstable solution of Eq. (A.3) $\square$

Although this theorem gives us a computational tool to determine the stability of the equilibrium $x = 0$ of Eq. (A.3), performing this computation is in general not simple. To determine the monodromy matrix C, we first need to determine the fundamental matrix, which involves finding n independent solutions of Eq. (A.3). Hence, the case of linear differential equations with periodic coefficients is certainly more involved than the case of linear differential equations with constant coefficients.

A.5 Stability of periodic orbits

Suppose that $u(t)$ is a periodic solution of the ordinary differential equation $\dot{x}(t) = f(x(t))$ with (minimal) period given by $T > 0$. In the following, we will study the stability of this periodic orbit. Although intuitively clear, we will formalize some concepts first.

Definition A.5.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and let $u(t)$ be a periodic solution with period $T > 0$ of $\dot{x}(t) = f(x(t))$, i.e. a solution with the property that $u(t+T) = u(t)$ for all t. We define the corresponding periodic orbit as $\Gamma = \{u(t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^n$.

Definition A.5.2. Let S be the state space of the differential equation (A.1) and $\phi(t,\ .) : S \to S$ the associated flow maps. We call a set $U \subseteq S$ invariant if $\{\phi(t,x) \mid x \in U\} \subseteq U$ for all $t \in \mathbb{R}$.

If Γ denotes a periodic orbit of the ordinary differential equation $\dot{x}(t) = f(x(t))$, then in particular Γ is a invariant set of the state space $\mathbb{R}^n$. To see this, we note that by definition of flow maps $\phi(t, u(0)) = u(t)$ holds for all $t \in \mathbb{R}$. Now, let $p \in \Gamma$, then by definition of Γ there exists a $t_p \in [0, T]$ such that $p = u(t_p) = \phi(t_p, u(0))$. By remarks made in Section A.1, we find that

$$\phi(t,p) = \phi(t, \phi(t_p, u(0))) = \phi(t + t_p, u(0)) = u(t + t_p) \in \Gamma$$

We conclude that $\{\phi(t,x) \mid x \in \Gamma\} \subseteq \Gamma$ for all $t \in \mathbb{R}$, thus Γ is an invariant set.

Furthermore, since $u(t)$ is continuous and $[0,T]$ compact, it follows that Γ is compact and in particularly closed. Thus we can define the following function:

$$\operatorname{dist}(x_0, \Gamma) = \min\{\|x_0 - p\| \mid p \in \Gamma\}$$

Using this, we define stability of a periodic orbit (which is sometimes also called orbital stability).

Definition A.5.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\Gamma \subseteq \mathbb{R}^n$ be a periodic orbit of $\dot{x}(t) = f(x(t))$. We call Γ stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{dist}(x, \Gamma) < \delta$ implies that $\operatorname{dist}(\phi(t,x), \Gamma) < \epsilon$ for all $t \geq 0$. We call Γ asymptotically stable if it is stable and there exist a neighbourhood W of Γ such that $\lim_{t\to\infty} \operatorname{dist}(\phi(t,x), \Gamma) = 0$ for all $x \in W$.

Suppose $u(t)$ is given periodic solution of $\dot{x}(t) = f(x(t))$. We can try to find other solutions to this ODE by adding small pertubation terms to the periodic solutions, i.e. by using the Ansatz $x(t) = u(t) + v(t)$ with $\|v(t)\|$ 'small enough' for all t. For $x(t) = u(t) + v(t)$ to be a solution of the system, it should satisfy

$$\dot{x}(t) = \dot{u}(t) + \dot{v}(t) = f(u(t)) + \dot{v}(t) = f(u(t) + v(t))$$

If f is smooth enough, we can use Taylor's theorem to expand the right hand side as

$$f(u(t) + v(t)) = f(u(t)) + Df(u(t))v(t) + \text{ higher order terms}$$

Cancelling the terms $f(u(t))$, we find that $v(t)$ should satisfy

$$\dot{v}(t) = Df(u(t))v(t) + \text{ higher order terms}$$

Thus, if we can argue that the higher order terms can be disregarded in a small enough neighbourhood of $u(t)$, and $v(t)$ satisfies the linear differential equation $\dot{v}(t) = Df(u(t))v(t)$, we find that $x(t) = u(t) + v(t)$

gives is a solution of $\dot{x}(t) = f(x(t))$ in a neighbourhood of $u(t)$. This prompts us to study the so-called *linear variational equatoin*

$$\dot{x}(t) = Df(u(t))x(t) \tag{A.4}$$

and look if it locally shares properties with the system (A.1).

Since $Df(u(t+T)) = Df(u(t))$ for all t, Eq. (A.4) is a linear differential equation with periodic coefficients. In resemblance with Corollary A.3.1, we will prove the following result in this Section:

Theorem A.5.1. *Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $u(t)$ a periodic solution of $\dot{x}(t) = f(x(t))$ with (minimal) period given by $T > 0$. If $n-1$ characterstic multipliers ρ of Eq.* (A.4) *satisfy $|\rho| < 1$, then the periodic solution $u(t)$ is asymptotically stable.*

We first give an outline of the proof. In a neighbourhood of Γ, we define an orthonormal basis moving along Γ such that one basis vector is tangent to Γ at every point. This enables us to rewrite the differential equation $\dot{x}(t) = f(x(t))$ as a a system of two differential equations, one describing the motion parallel to Γ, the other describing the motion in the plane perpendicular to Γ. We note that to determine the stability of the periodic orbit Γ, it is sufficient to analyze the motion in the plane perpendicular to Γ at each point. It turns out that (in the coordinate frame moving along Γ), we can describe the motion perpendicular to Γ by $\dot{\rho}(\theta) = A(\theta)\rho(\theta) + f(\theta, \rho)$, where $A(\theta + T) = A(\theta)$ for all θ. Using linearization theory, we find that stability of $\dot{\rho}(\theta) = A(\theta)\rho(\theta)$ implies stability of of $\dot{\rho}(\theta) = A(\theta)\rho(\theta) + f(\theta, \rho)$. We will show that 1 is always a multiplier of Eq. (A.4) and that if $1, \mu_2, \ldots, \mu_n$ are the multipliers of Eq. (A.4), then $\mu_2, \ldots, \mu_n$ are multipliers of $\dot{\rho}(\theta) = A(\theta)\rho(\theta)$. Thus, if $n-1$ multipliers $\mu_2, \ldots, \mu_n$ of Eq. (A.4) satisfy $|\mu_i| < 1$, then $\mu_2, \ldots, \mu_n$ are multipliers of $\dot{\rho}(\theta) = A(\theta)\rho(\theta)$ and the $\dot{\rho}(\theta) = A(\theta)\rho(\theta)$ is stable. From there, it follows that the periodic orbit Γ is stable.

We now state the proof formally, using the following lemma:

Lemma A.5.2. *Let $f : \mathbb{R}^n \supset U \to \mathbb{R}^n$, let $u(t)$ be a periodic solution of period T of $\dot{x}(t) = f(x(t))$ and denote by Γ its orbit. If $u \in C^p(\mathbb{R}, \mathbb{R}^n)$, then there exists a moving orthonormal system along Γ which is C^{p-1}, i.e. there exists an orthonormal basis $\{e_1(\theta), \ldots, e_n(\theta)\}$ such that the map $e_i(\theta) \in C^{p-1}(\mathbb{R}, \mathbb{R}^n)$ for each i, $e_1(\theta) = \frac{du(\theta)}{d\theta} \Big/ \left\| \frac{du(\theta)}{d\theta} \right\|$ for all θ and $e_i(\theta + T) = e_i(\theta)$ for all i, θ.*

A proof can be found in [6]. However, for the case $n = 3$ we can also explicitly construct the basis $\{e_1(\theta), \ldots, e_3(\theta)\}$.

Proof. (case $n = 3$) We note that the map $u : [0, T) \to \mathbb{R}^n$, is a C^p parametrization of Γ. Thus, $\frac{du(\theta)}{d\theta}$ is tangent to Γ for all θ. Therefore, we define $e_1(\theta) = \frac{du(\theta)}{d\theta} \Big/ \left\| \frac{du(\theta)}{d\theta} \right\|$. Since $\frac{du(\theta)}{d\theta} \neq 0$ for all θ, it follows that $e_1(\theta) \neq 0$ for all θ. We note that $\langle e_1(\theta), e_1(\theta) \rangle = 1$ for all $\theta \in [0, \omega)$, thus $\frac{d}{d\theta} \langle e_1(\theta), e_1(\theta) \rangle = 0$. We define

$$\begin{aligned} h &: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad h(x, y) = \langle x, y \rangle \\ g &: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad g(x) = (x, x) \end{aligned}$$

Since h, g are (multi)-linear maps, respectively, we recall that their total derivatives are given by

$$\begin{aligned} Dh(x, y)(u, v) &= \langle x, v \rangle + \langle u, y \rangle \\ Dg(x)v &= (v, v) \end{aligned}$$

We note that $\langle e_1(\theta), e_1(\theta) \rangle = h \circ g \circ e_1(\theta)$. Using the chain rule, we thus find that

$$\begin{aligned} 0 = \frac{d}{d\theta} \langle e_1(\theta), e_1(\theta) \rangle &= Dh(g \circ e_1(\theta)) Dg(e_1(\theta)) \frac{de_1(\theta)}{d\theta} = Dh(e_1\theta, e_1\theta)\left(\frac{de_1(\theta)}{\theta}, \frac{de_1(\theta)}{d\theta}\right) \\ &= \left\langle e_1(\theta), \frac{de_1(\theta)}{d\theta} \right\rangle + \left\langle \frac{de_1(\theta)}{d\theta}, e_1(\theta) \right\rangle = 2 \left\langle e_1(\theta), \frac{de_1(\theta)}{d\theta} \right\rangle \end{aligned}$$

We conclude that $\frac{de_1(\theta)}{d\theta} \perp e_1(\theta)$ for all θ. Therefore, we define $e_2(\theta) = \frac{de_1(\theta)}{d\theta} \Big/ \left\| \frac{de_1(\theta)}{d\theta} \right\|$. We note that $e_2(\theta) \in C^{p-1}$. Furthermore, we define $e_3(\theta) = e_1(\theta) \times e_2(\theta)$, then $\|e_3(\theta)\| = 1$. Since taking outer products is a bilinear, hence smooth, operation, we conclude that $e_3 \in C^{p-1}$. Thus, the set $\{e_1(\theta), e_2(\theta), e_3(\theta)\}$ satisfies the requirements. [5] □

We now define $Z(\theta) = [e_1(\theta), \ldots, e_{n-1}(\theta)]$. Using this, we can define a map transform $\mathbb{R} \times \mathbb{R}^{n-1} \ni (\theta, \rho) \mapsto u(\theta) + Z(\theta)\rho$. It is proven in [6] that this is in fact a coordinate transform in a sufficiently small neighbourhood of Γ, and that there exists functions f_1, f_2 such that we can rewrite $\dot{x}(t) = f(x(t))$ as

$$\begin{aligned} \dot{\theta} &= 1 + f_1(\theta, \rho) \\ \dot{\rho} &= A(\theta)\rho + f_2(\theta, \rho) \end{aligned}$$

with $A(\theta)$ given by

$$A(\theta) = Z^T(\theta) \left(-\frac{dZ(\theta)}{d\theta} + Df(u(\theta))Z(\theta) \right)$$

and f_2 such that $f_2(\theta, 0) = 0$ and $D_\rho f_2(\theta, 0) = 0$ for all θ. We note that $A(\theta + T) = A(\theta)$ for all θ, i.e. $\frac{d\rho}{d\theta} = A(\theta)\rho$ is a linear differential equation with periodic coefficients. Using this, we prove the following lemma:

Lemma A.5.3. *At least one of the characteristic multipliers of the linear variational equation* (A.4) *is equal to 1. If $1, \mu_2, \ldots, \mu_n$ are the characteristic multipliers of Eq.* (A.4)*, then the characteristic multipliers of the $(n-1)$-dimensional system*

$$\frac{d\rho}{d\theta} = A(\theta)\rho$$

are given by $\mu_2, \ldots, \mu_n$.

Proof. We first note that $\frac{du(\theta)}{d\theta}$ is a solution of the linear variational equation (A.4). Indeed, since $\frac{du(\theta)}{d\theta} = f(u(\theta))$, we have that:

$$\frac{d^2u(\theta)}{d\theta^2} = \frac{d}{d\theta} f(u(\theta)) = Df(u(\theta)) \frac{du(\theta)}{d\theta}$$

Therefore, a fundamental matrix solution $Y(t)$ of Eq. (A.4) is given by $Y(\theta) = [\frac{du(\theta)}{d\theta}, w_2(\theta), \ldots, w_n(\theta)]$, where $w_2, \ldots, w_n$ are solutions of Eq. (A.4). Denote the monodromy matrix by K. Since $\frac{du(T)}{d\theta} = \frac{du(0)}{d\theta}$, it follows that K can be written as

$$K = \begin{pmatrix} 1 & K_2 \\ 0 & K_1 \end{pmatrix}$$

where K_2 is a $1 \times (n-1)$ block and K_1 a $(n-1) \times (n-1)$ block. It follows that the eigenvalues of K (i.e. the characteristic multipliers) are given by 1 and the eigenvalues of K_1. To complete the proof, we now claim that K_1 is a monodromy matrix for $\frac{d\rho}{d\theta} = A(\theta)\rho(\theta)$. Indeed, since $\frac{du(\theta)}{d\theta} \Big/ \left\| \frac{du(\theta)}{d\theta} \right\|$ and the columns of $Z(\theta)$ constitute an orthonormal system, there exists a unique $\alpha \in \mathbb{R}$, $\rho \in \mathbb{R}^{n-1}$ such that $y = \frac{du(\theta)}{d\theta}\alpha + Z(\theta)\rho$. Thus, if $y(\theta) = \frac{du(\theta)}{d\theta}\alpha(\theta) + Z(\theta)\rho(\theta)$ satisfies (A.4), we find that

$$\begin{aligned} Df(u(\theta))y(\theta) &= Df(u(\theta)) \left(\frac{du(\theta)}{d\theta}\alpha(\theta) + Z(\theta)\rho(\theta) \right) = \frac{dy(\theta)}{d\theta} = \frac{d}{d\theta} \left(\frac{du(\theta)}{d\theta}\alpha(\theta) + Z(\theta)\rho(\theta) \right) \\ &= \frac{d^2u(\theta)}{d\theta^2}\alpha(\theta) + \frac{du(\theta)}{d\theta}\frac{d\alpha(\theta)}{d\theta} + \frac{dZ(\theta)}{d\theta}\rho(\theta) + Z(\theta)\frac{d\rho(\theta)}{d\theta} \end{aligned}$$

Using that $\frac{d^2u(\theta)}{d\theta^2} = Df(u(\theta))\frac{du(\theta)}{d\theta}$ and cancelling terms on both sides, we find that

$$\frac{du(\theta)}{d\theta}\frac{d\alpha(\theta)}{d\theta} + \frac{dZ(\theta)}{d\theta}\rho(\theta) + Z(\theta)\frac{d\rho(\theta)}{d\theta} = Df(u(\theta))Z(\theta)\rho(\theta)$$

However, since $\frac{du(\theta)}{d\theta}$ is perpendicular to the plane spanned by the columns of $Z(\theta)$, we find that the motion in the plane spanned by the columns of $Z(\theta)$ is given by

$$\frac{dZ(\theta)}{d\theta}\rho(\theta) + Z(\theta)\frac{d\rho(\theta)}{d\theta} = Df(u(\theta))Z(\theta)\rho(\theta)$$

Using that $Z(\theta)$ is orthogonal, i.e. $Z(\theta)^{-1} = Z(\theta)^T$, we find that $\rho(\theta)$ satisfies $\frac{d\rho(\theta)}{d\theta} = A(\theta)\rho(\theta)$. We can write

$$Y_1(\theta) = [w_2(\theta), \dots, w_n(\theta)] = \frac{du(\theta)}{d\theta}(\alpha_2(\theta), \dots, \alpha_n(\theta)) + Z(\theta)(\rho_2(\theta), \dots, \rho_n(\theta)) := \frac{du(\theta)}{d\theta}\alpha(\theta) + Z(\theta)\rho(\theta)$$

By the remarks above, $\rho(\theta)$ is a matrix solution of $\frac{d\rho(\theta)}{d\theta} = A(\theta)\rho(\theta)$. Since $Y_1(\theta)$ has linearly independent columns and the coordinate transform is bijective, we find that $\rho(\theta)$ has linearly independent columns. Thus, $\rho(\theta)$ is in fact a fundamental matrix solution. Using the form of the monodromy matrix K, we have that

$$Y_1(T) = \frac{du(T)}{d\theta}\alpha(T) + Z(T)\rho(T) = \frac{du(0)}{d\theta}K_2 + Y_1(0)K_1 = \frac{du(0)}{d\theta}(K_2 + \alpha(0)K_1) + Z(0)\rho(0)K_1$$

Using again that $\frac{du(\theta)}{d\theta}$ and the columns of $Z(\theta)$ form an orthonormal basis we can uncouple the equation to obtain $Z(T)\rho(T) = Z(0)\rho(0)K_1$. Since $Z(T) = Z(0)$, we find that $\rho(T) = \rho(0)K_1$, i.e. K_1 is a monodromy matrix for $\frac{d\rho(\theta)}{d\theta} = A(\theta)\rho(\theta)$. Recalling that the eigenavlues of K are given by 1 and the eigenvalues of K_1, the statement of the theorem follows. [6] □

Before concluding with the proof of Theorem A.5.1, we first state the following Lemma, whose proof can be found in [6].

Lemma A.5.4. *Suppose the equilibrium of $\dot{x}(t) = A(t)x(t)$ is uniformly asymptotically stable for $t \geq \beta$, $\beta \in (-\infty, \infty)$. If $f(t,x)$ is continuous and for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < \delta$ implies that $\|f(t,x)\| \leq \epsilon\|x\|$ for all t, then the solution of $x = 0$ of $\dot{x}(t) = A(t)x + f(t,x)$ is uniformly asymptotically stable for all $t_0 \geq \beta$.*

Proof. (of Theorem A.5.1) Denote by $\mu_1, \dots, \mu_n$ the multipliers of the linear variational equation (A.4). By Lemma A.5.3, exactly one of them, say μ_1, equals one, and $\mu_2, \dots, \mu_n$ are multipliers of $\dot{\rho} = A(\theta)\rho(\theta)$. By the assumptions of the theorem, we have that $|\mu_i| < 1$ for all $2 \leq i \leq n$. We can apply Theorem A.4.4 to see that the equilibrium $x = 0$ of $\frac{d\rho}{d\theta} = A(\theta)\rho(\theta)$ is asymptotically stable.

We recall that in a neighbourhood of Γ, the motion in the plane perpendicular to Γ can be described by $\frac{d\rho}{d\theta} = A(\theta)\rho + f_2(\theta,\rho)$, where $A(\theta+T) = A(\theta)$ for all θ, $f_2(\theta,0) = 0$, $D_\rho f_2(\theta,0) = 0$ and f_2 has continuous derivatives with respect to ρ. Using this, it follows that for every $\epsilon > 0$ we can find a $\delta > 0$ such that $\|D_\rho f(\theta,\rho)\| < \epsilon$ for all ρ with $\|\rho\| < \delta$. Applying the Mean Value Theorem for several variables (see for example [5]), it follows that

$$\|f(\theta,\rho) - f(\theta,0)\| = \|f(\theta,\rho)\| \leq \epsilon\,\|\rho - 0\| = \epsilon\,\|\rho\|$$

Combining this with the stability of the equilibrium $x = 0$ of $\frac{d\rho}{d\theta} = A(\theta)\rho(\theta)$, it follows by Lemma A.5.4 that there exist $K, \alpha, \eta > 0$ such that $\|\phi(\theta,\theta_0,\rho_0)\| \leq Ke^{-\alpha(\theta-\theta_0)}\,\|\rho_0\|$ for all $\theta \geq \theta_0$ and ρ_0 with $\|\rho_0\| < \eta$. Since only the motion in the plane perpendicular to Γ describes the stability, it follows that Γ is asymptotically stable. [6] □

A.6 The Hopf Bifurcation Theorem for ordinary differential equations

In Section A.5, we studied the stability of periodic solutions of a differential equation. Throughout this discussion, we assumed that such a periodic solution was known to exist. Deciding on the existence of a periodic orbit is, however, in general not an easy task. The Hopf bifurcation theorem helps us to decide upon the existence of a periodic solution.

We start by studying the linear case

$$\dot{x}(t) = A(\lambda)x(t), \quad x(0) = x_0 \tag{A.5}$$

where $A(\lambda)$ is a 2×2-matrix for all $\lambda \in \mathbb{R}$. If there exists a $\lambda_0 \in \mathbb{R}$ such that the eigenvalues of $A(\lambda_0)$ are given by $\mu_{\pm} = \pm i\omega$, with $\omega \neq 0$, we find that for all $x_0 \in \mathbb{R}^2$, the solution of (A.5) with $\lambda = \lambda_0$ is periodic: we recall from Section A.2 that the solution of (A.5) with $\lambda = \lambda_0$ is given by

$$x(t) = e^{A(\lambda_0)t}x_0$$

Since $\omega \neq 0$, we have that $A(\lambda_0)$ has two distinct eigenvalues; thus the corresponding eigenvectors $v_{\pm}$ are linearly independent. Therefore, we can write $x_0 = x_1 v_+ + x_2 v_-$ with $x_1, x_2 \in \mathbb{R}$ and the solution of (A.5) is given by

$$x(t) = e^{A(\lambda_0)t}(x_1 v_+ + x_2 v_-) = x_1 e^{i\omega t} v_1 + x_2 e^{-i\omega t} v_2$$

We note that this solution is periodic with period given by $T = \frac{2\pi}{\omega}$.

If we now perturb the system by adding a non-linear term to (A.5) to obtain the system

$$\dot{x}(t) = A(\lambda)x(t) + f(\lambda, x), \quad x(0) = x_0 \tag{A.6}$$

we can ask ourselves the question whether we can still decide upon the existence of a periodic solution by looking at the eigenvalues of $A(\lambda)$. Under certain conditions, this turns out to be the case, as is stated in the following theorem:

Theorem A.6.1 (Hopf Bifurcation Theorem)**.** *Let*

$$\dot{x}(t) = A(\lambda)x + f(\lambda, x)$$

with $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $A(\lambda)$ a $n \times n$-matrix for all $\lambda \in \mathbb{R}$. Let $A(\lambda), f(\lambda, x)$ be such that they have continuous first derivatives and $f(\lambda, 0) = 0$ for all λ. Furthermore, assume that there exists a $\lambda_0 > 0$ such that $D_x f(\lambda, 0) = 0$ for $|\lambda| < \lambda_0$ and that the eigenvalues of $A(\lambda)$ are given by $\mu(\lambda) \pm i\nu(\lambda)$ such that

$$\nu(0) > 0, \quad \mu(0) = 0, \quad D\mu(0) \neq 0$$

Then there are constants $a_0 > 0$, $\lambda_0 > 0$, $\delta_0 > 0$ and functions $\lambda(a) \in \mathbb{R}$, $\omega(a) \in \mathbb{R}$ that are continuously differentiable for $|a| < a_0$ with $\lambda(0) = 0, \omega(0) = 2\pi$. Furthermore, for $|a| < a_0$, there exists a $\omega(a)$-periodic function $x^*(a)$ that is a solution of (A.6) *with $\lambda = \lambda(a)$. For $|a| < \alpha_0, |\omega - 2\pi| < \delta_0$, every $\omega(a)$-periodic solution of* (A.6) *such that $|x(t)| < \delta_0$ is given by $x^*(a)$ except for a possible translation in phase.*

A proof can be found in [2].

Appendix B

Notes

Since the discussion of stability theory for differential delay equations in Chapter 1 is not always sufficient for the discussion in Chapter 2, we do not always completly validate the methods used in Chapter 2. The fact that the state space of a differential delay equation is given by an infinite dimensional Banach space makes the theory for delay equations more involved. However, the results needed in Chapter 2 tend to be roughly the same as those for ordinary differential equations and in Chapter 2 we point out the similarities between the theory of differential delay equations and ordinary differential equations. More details on the theory of differential delay equations can be found in [8] and [3].

To apply the Pyragas control scheme, one needs to know beforehand that a periodic orbit exists and what the period of this orbit is. In Chapter 2, we therefore study the normal form of the subcritical Hopf bifurcation, see (2.2), since we know that an unstable periodic orbit exists for $\lambda < 0$. However, other systems that are known to have an unstable periodic orbit can, and have been, used to study Pyragas control, see for example [14].

We study parameter values satisfying $\gamma\lambda < 1$, since for $\gamma\lambda > 1$, the target periodic orbit (2.4) of (2.1) is already stable. Furthermore, as noted in [1], if we would study (2.2) with $\tau = \frac{2\pi}{1-\gamma\lambda}$ and $\gamma\lambda > 1$, (2.2) would in fact become a 'forward equation' instead of a delay equation.

The Pyragas control scheme can be implemented experimentally. This was, for example, done in [12], where a periodic orbit of a system of the form of (2.1) was stabilized.

The analysis presented in Chapter 2 can be generalized for n-dimensional systems, see for example [1].

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