Continuous Data Assimilation Using General Interpolant Observables

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Abstract

We present a new continuous data assimilation algorithm based on ideas that have been developed for designing finite-dimensional feedback controls for dissipative dynamical systems, in particular, in the context of the incompressible two-dimensional Navier–Stokes equations. These ideas are motivated by the fact that dissipative dynamical systems possess finite numbers of determining parameters (degrees of freedom) such as modes, nodes and local spatial averages which govern their long-term behavior. Therefore, our algorithm allows the use of any type of measurement data for which a general type of approximation interpolation operator exists. Our main result provides conditions, on the finite-dimensional spatial resolution of the collected data, sufficient to guarantee that the approximating solution, obtained by our algorithm from the measurement data, converges to the unknown reference solution over time. Our algorithm is also applicable in the context of signal synchronization in which one can recover, asymptotically in time, the solution (signal) of the underlying dissipative system that is corresponding to a continuously transmitted partial data.

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1 Introduction

The goal of continuous data assimilation, and signal synchronization, is to use low spatial resolution observational measurements, obtained continuously in time, to accurately find the corresponding reference solution from which future predictions can be made. The motivating application of continuous data assimilation is weather prediction. The classical method of continuous data assimilation, see Daley [10], is to insert observational measurements directly into a model as the latter is being integrated in time. We propose a new approach based on ideas from control theory, see Azouani and Titi [2]. A slightly similar approach in the context of stochastic differential equations, using the low Fourier modes as observables/measurements, appears in a recent work by Blömker, Law, Stuart and Zygalakis [4]. Rather than inserting the measurements directly into the model, i.e. into the nonlinear term, we introduce a feedback control term that forces the model toward the reference solution that is corresponding to the observations. This is motivated by the fact that the measured data is usually obtained as the values of the exact solutions at a discrete set of spatial nodal points, and that it difficult to insert this data directly into the underlying equation, because it is not possible to obtain the exact values of the spatial derivatives. One should observe that in order to guarantee a unique corresponding reference solution one has to supply observational data with enough spatial resolution. This is the object of this paper.

While the classical method of continuous data assimilation is simple in concept, special care has to be taken concerning how the observations are inserted into a model in practice. For example, it is generally necessary to separate the fast and slow parts of a solution before inserting the observations into the model. The method proposed here does not require such a decomposition. Since the observations are not directly inserted into the model, we can rely on the dissipation already present in the dynamics to filter the observed data, i.e. the viscous term will suppress the “spill over” oscillations in the fine scales. The advantage of this approach is that it works for a general class of interpolant observables without modification.

Let $u(t)$ represent the state at time t of the dynamical system in which we are interested, and let $I_h(u(t))$ represent our observations of this system at a coarse spatial resolution of size h. Given observational measurements, $I_h(u(t))$, for $t \in [0,T]$, our goal is to construct an increasingly accurate initial condition from which predictions of $u(t)$, for $t > T$, can be made. We do this by constructing an approximate solution $v(t)$ that converges to $u(t)$ over time.

Suppose the time evolution of u is governed by a given evolution equation of the form

$$\frac{du}{dt} = F(u), \tag{1}$$

where the initial data, u_0, is missing. Our algorithm for constructing $v(t)$ from the observational measurements $I_h(u(t))$ for $t \in [0,T]$ is given by

$$\frac{dv}{dt} = F(v) - \mu I_h(v) + \mu I_h(u), \tag{2}$$

$$v(0) = v_0, \tag{3}$$

where μ is a positive relaxation parameter, which relaxes the coarse spatial scales of v toward those of the observed data, and v_0 is taken to be arbitrary. It is worth stressing that our algorithm is designed to work for general dissipative dynamical systems of the form (1). Such systems are known to have finitely many degrees of freedom in the form of determining parameters of the type $I_h(u)$, see, for example, Cockburn, Jones and Titi [8], Foias, Manley, Rosa and Temam [13], Foias and Prodi [14], Foias and Temam [15], [16], Jones and Titi [21], [22], and references therein. The incompressible two-dimensional Navier–Stokes equations provide a concrete example of a dissipative dynamical system of this type.

We consider here the incompressible two-dimensional Navier–Stokes equations, as a paradigm, because they are amenable to mathematical analysis while at the same time similar to the equations used in realistic weather models. Thus, we shall suppose the evolution of u is is governed by the Navier–Stokes system

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \tag{4}$$

$$\nabla \cdot u = 0, \tag{5}$$

in the physical domain Ω, with either no-slip Dirichlet, or periodic, boundary conditions. Here $u(x,t)$ represents velocity of the fluid at time t at position x, $\nu > 0$ represents the kinematic viscosity, $p(x,t)$ is the pressure and $f(x,t)$ is a time dependent body force applied to the fluid.

In the case of no-slip Dirichlet boundary conditions we take $u = 0$ on $\partial\Omega$. The domain Ω is an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, such that $\partial\Omega$ can be represented locally as the graph of a C^2 function. In the case of periodic boundary conditions we require u and f to be $L-$periodic, in both x and y directions, and take $\Omega = [0,L]^2$ to be the fundamental periodic domain.

Continuous data assimilation, in the context of the incompressible two-dimensional Navier–Stokes equations, was first studied by Browning, Henshaw and Kreiss in [7], later by Henshaw, Kreiss and Yström in [20] and also by Olson and Titi in [24] and [25], motivated by the concept of finite number of determining modes which was introduced for the first time in [14] (see also [13], [24], and references therein). These studies treated the case of periodic boundary conditions, where the observations were given by the low Fourier modes with wave numbers k, such that $|k| \le 1/h$. Since the low modes essentially represent the large spatial scales of the solution, the classical data assimilation algorithm works well for this type of observations. In addition, it is worth mentioning that this method is consistent with some of the signal synchronization algorithms. Most recently, similar idea has also been introduced in [12] to show that the long-time dynamics of the two-dimensional Navier–Stokes equations can be imbedded in an infinite-dimensional dynamical system that is induced by an ordinary differential equations, named *determining form*, which is governed by a globally Lipschitz vector field.

The method of constructing v, given by (2), allows the use of general interpolant observables, given by interpolants $I_h\colon H^1(\Omega) \to L^2(\Omega)$ that are linear and satisfy the following approximation property:

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \le c_0 h^2 \|\varphi\|_{H^1(\Omega)}^2 \tag{6}$$

for every $\varphi \in H^1(\Omega)$. The orthogonal projection onto the low Fourier modes, with wave numbers k such that $|k| \le 1/h$, mentioned above, is an example of such interpolant observable. However, there are many other interpolant observables which satisfy (6).

One physically relevant example of an interpolant which satisfies condition (6) are the volume elements studied in [21] and [22] (see also Foias and

Titi [17]). In this case

$$I_h(\varphi(x)) = \sum_{j=1}^{N} \bar{\varphi}_j \chi_{Q_j}(x) \quad \text{where} \quad \bar{\varphi}_j = \frac{N}{L^2} \int_{Q_j} \varphi(x)\, dx,$$

and the domain $\Omega = [0, L]^2$, for the periodic boundary conditions case, has been divided into N equal squares Q_j, with sides $h = L/\sqrt{N}$. Volume elements generalize to any domain Ω on which the Bramble–Hilbert lemma holds. An elementary discussion of this lemma in the context of finite element methods appears in Brenner and Scott [5].

In addition, we also consider interpolant observables given by linear interpolants $I_h\colon H^2(\Omega) \to L^2(\Omega)$, that satisfy the following approximation property:

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \le \frac{1}{4} c_0^2 h^4 \|\varphi\|_{H^2(\Omega)}^2, \tag{7}$$

for every $\varphi \in H^2(\Omega)$. An example of this type of interpolant is given by measurements at a discrete set of nodal points in Ω. Specifically, let $h > 0$ be given, and let $\Omega = \cup_{j=1}^{N_h} Q_j$, where Q_j are disjoint subsets such that $\operatorname{diam} Q_j \le h$, for $j = 1, 2, \ldots, N_h$, and let $x_j \in Q_j$ be arbitrary points. Then set, for example,

$$I_h(\varphi(x)) = \sum_{k=1}^{N_h} \varphi(x_k) \chi_{Q_j}(x). \tag{8}$$

Following ideas in [22] (see also [16]) one can show that $I_h(\varphi)$ satisfies (7).

Our paper is organized as follows. First, we recall the functional setting of the two-dimensional Navier–Stokes equations necessary for our analysis and then use this setting to formulate our new method of continuous data assimilation. After this we proceed to our main task, that of finding conditions under which the approximate solution obtained by this algorithm of data assimilation converges to the reference solution over time. Section 3 treats the case of smooth, bounded domains with no-slip Dirichlet boundary conditions, while section 4 treats the case of periodic boundary conditions. Our main results may be stated as follows:

Theorem 1. *Let Ω be an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, and let u be a solution to equations (4)–(5) with no-slip Dirichlet boundary conditions. Assume that I_h satisfies (6), with h small enough such that*

$$1/h^2 \ge c_1 \lambda_1 G^2,$$

where c_1 is a constant given in (33). Then there exists $\mu > 0$, given explicitly in Proposition 1, such that $\|v - u\|_{L^2(\Omega)} \to 0$ exponentially, as $t \to \infty$.

Here G denotes the Grashof number

$$G = \frac{1}{\nu^2 \lambda_1} \limsup_{t\to\infty} \|f(t)\|_{L^2} \tag{9}$$

where λ_1 is the smallest eigenvalue of the Stokes operator subject to homogeneous Dirichlet boundary conditions. Let us remark, again, that the constant c_1 is depends only on c_0, given in (6), and the shape, but not the size, of the domain Ω. In particular, c_1 is given by (33) where the constant c is chosen so the bound (16) on the non-linear term holds. Moreover, μ may be chosen equal to $5c^2G^2\nu\lambda_1$ as indicated in Proposition 1, below.

Results similar to Theorem 2 hold when I_h satisfies (7), however, we omit the proof of this result in the case of no-slip Dirichlet boundary conditions and instead proceed directly to the the case of periodic boundary conditions where sharper estimates may be obtained. In particular, we prove

Theorem 2. *Let $\Omega = [0, L]^2$ and let u be a solution to equations (4)–(5) with periodic boundary conditions. Let I_h satisfy either (6) or (7), with h small enough such that*

$$1/h^2 \geq c_2\lambda_1 G(1 + \log(1 + G)),$$

where c_2 is a constant given in (36). Then there exists $\mu > 0$, given explicitly in Proposition 2, such that $\|v - u\|_{H^1(\Omega)} \to 0$ exponentially, as $t \to \infty$.

Let us remark again that c_2 depends only on c_0, and that μ may be chosen as $3c_2\nu\lambda_1 G(1+\log(1+G))/c_0$. In particular, μ is given in Proposition 2 and c_2 is defined in (36) as an increasing function of c_0 and c, where c is chosen large enough so that the bounds in both (22) and (34) hold.

Note that the estimate on the length scale h in Theorem 2 can be compared to previous results reported in [24]. Let $\tilde{v}(t)$ be the approximate solution obtained by the method of continuous data assimilation introduced in [24] for the interpolant observable $I_h(u)$ given by projection onto the Fourier modes with wave numbers $|k| < 1/h$. In [24] it was shown, that for small values of h, such that $1/h^2 \sim \lambda_1 G$, $\|u(t) - \tilde{v}(t)\|_{H^1(\Omega)} \to 0$ exponentially fast, as $t \to \infty$. Up to a logarithmic correction term, Theorem 2 states similar estimates on h for the new algorithm which covers a much wider class of interpolant observables.

The final section of this paper discusses numerical simulations, which are in progress, related works, and closes with a few concluding remarks.

2 Preliminaries

This section reviews the functional setting of the two-dimensional Navier–Stokes equations with no-slip and periodic boundary conditions, recalls some facts that will be used in the remainder of the paper and then gives an explicit formulation of our new method for continuous data assimilation in this context. Following Constantin and Foias [9], Foias, Manley, Rosa and Temam [13], Robinson [26] and Temam [27], we begin by defining a suitable domain Ω and space $\mathcal{V}$ of smooth functions which satisfy each type of boundary conditions.

No-slip Dirichlet Boundary Conditions. Let Ω be an open, bounded and connected domain with C^2 boundary. Define $\mathcal{V}$ to be set of all C^∞ compactly supported vector fields from Ω to $\mathbf{R}^2$ that are divergence free.

Periodic Boundary Conditions. Let $\Omega = [0, L]^2$ for some fixed $L > 0$. Define $\mathcal{V}$ to be the set of all L-periodic trigonometric polynomials from $\mathbf{R}^2$ to $\mathbf{R}^2$ that are divergence free and have zero averages.

Given $\mathcal{V}$ corresponding to either type of boundary conditions let H be the closure of $\mathcal{V}$ in $L^2(\Omega; \mathbf{R}^2)$ and V be the closure of $\mathcal{V}$ in $H^1(\Omega; \mathbf{R}^2)$. The spaces H and V are Hilbert spaces with inner products

$$(u, v) = \int_\Omega u(x) \cdot v(x)\, dx \qquad \text{and} \qquad ((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx,$$

respectively. Denote the norms of H and V by

$$|u| = \sqrt{(u, u)} \qquad \text{and} \qquad \|u\| = \sqrt{((u, u))},$$

and the dual of V by V^* with the pairing $\langle u, v\rangle$ where $u \in V^*$ and $v \in V$.

Define the Leray projector P_σ as the orthogonal projection from $L^2(\Omega; \mathbf{R}^2)$ onto H, and define the Stokes operator $A\colon V \to V^*$, and the bilinear term $B\colon V \times V \to V^*$ to be the continuous extensions of the operators given by

$$Au = -P_\sigma \Delta u \qquad \text{and} \qquad B(u, v) = P_\sigma(u \cdot \nabla v),$$

respectively, for any smooth solenoidal vector fields u and v in $\mathcal{V}$.

Denote the domain of A by $\mathcal{D}(A) = \{\, u \in V : Au \in H \,\}$. The linear operator A is self-adjoint and positive definite with compact inverse $A^{-1}\colon H \to H$. Thus, there exists a complete orthonormal set of eigenfunction w_i in H such that $Aw_i = \lambda_i w_i$ where $0 < \lambda_i \le \lambda_{i+1}$ for $i \in \mathbf{N}$. Writing λ_1 as the smallest eigenvalue of A we have the following Poincaré inequalities:

$$\text{if } u \in V \text{ then } \lambda_1 |u|^2 \le \|u\|^2, \tag{10}$$

$$\text{if } u \in \mathcal{D}(A) \text{ then } \lambda_1 \|u\|^2 \le |Au|^2. \tag{11}$$

Note that for $u \in H$, $|u| = \|u\|_{L^2(\Omega)}$ and for $u \in V$ the Poincaré inequality implies $\|u\|$ is equivalent to $\|u\|_{H^1(\Omega)}$.

The bilinear term B has the algebraic property that

$$\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle \tag{12}$$

for $u, v, w \in V$, and consequently the orthogonality property that

$$\langle B(u,w), w\rangle = 0. \tag{13}$$

Here the pairing $\langle \cdot, \cdot \rangle$ denotes the dual action of V^* on V. Details may be found, e.g., in [9], [13], [26] and [27].

In the case of periodic boundary conditions the bilinear term possesses the additional orthogonality property

$$(B(w,w), Aw) = 0, \qquad \text{for every} \qquad w \in \mathcal{D}(A); \tag{14}$$

and consequently one has

$$(B(u,w), Aw) + (B(w,u), Aw) = -(B(w,w), Au). \tag{15}$$

Note that the bilinear term satisfies a number of inequalities which hold for either no-slip or periodic boundary conditions. These are

$$|\langle B(u,v), w\rangle| \le c|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2}, \tag{16}$$

for every $u, v, w \in V$,

$$|(B(u,v), w)| \le c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \tag{17}$$

for every $u \in V$, $v \in \mathcal{D}(A)$ and $w \in H$, and

$$|(B(u,v), w)| \le c|u|^{1/2}|Au|^{1/2}\|v\||w|, \tag{18}$$

for every $u \in \mathcal{D}(A)$ and $v, w \in V$, where c is a dimensionless constant depending only on the shape, but not the size, of Ω. These inequalities may be obtained from the Hölder's inequality, the Sobolev inequalities and Ladyzhenskaya's inequality, see, e.g., [9], [13], [26] and [27].

With the above notation we write the incompressible two-dimensional Navier–Stokes equations in functional form as

$$\frac{du}{dt} + \nu Au + B(u,u) = f \tag{19}$$

with initial condition $u(0) = u_0$. We have assumed $f \in H$ so that $P_\sigma f = f$. As shown in [9], [13], [26] and [27] these equations are well-posed; and possess a compact finite-dimensional global attractor, when f is time-independent. Specifically, we have

Theorem 3 (Existence and Uniqueness of Strong Solutions)**.** *Suppose $u_0 \in V$ and $f \in L^\infty((0,\infty), H)$. Then the initial value problem (19) has a unique solution that satisfies*

$$u \in C([0,T]; V) \cap L^2((0,T); D(A)) \quad \text{and} \quad \frac{du}{dt} \in L^2((0,T); H),$$

for any $T > 0$.

We now give bounds on solutions u of (19) that will be used in our later analysis. With the exception of inequality (22) due to Dascaliuc, Foias and Jolly [11] these estimates appear in any the references listed above.

Theorem 4. *Fix $T > 0$, and let G be the Grashof number given in (9). Suppose that u is the solution of (19), corresponding to the initial value u_0, then there exists a time t_0, which depends on u_0, such that for all $t \geq t_0$ we have:*

$$|u(t)|^2 \leq 2\nu^2 G^2 \qquad \text{and} \qquad \int_t^{t+T} \|u(\tau)\|^2 d\tau \leq 2(1 + T\nu\lambda_1)\nu G^2. \tag{20}$$

In the case of periodic boundary conditions we also have:

$$\|u(t)\|^2 \leq 2\nu^2 \lambda_1 G^2, \qquad \int_t^{t+T} |Au(\tau)|^2 d\tau \leq 2(1 + T\nu\lambda_1)\nu\lambda_1 G^2; \tag{21}$$

furthermore, if $f \in H$ is time-independent then

$$|Au(t)|^2 \leq c\nu^2 \lambda_1^2 (1 + G)^4. \tag{22}$$

We now write the continuous data assimilation equations (2) for the incompressible two-dimensional Navier–Stokes equations. Let u be a strong solution of (4)–(5), or equivalently (19), as given by Theorem 3, and let I_h be an interpolation operator satisfying (6) or (7). Suppose that u is to be recovered from the observational measurements $I_h(u(t))$, that have been continuously recorded for times t in $[0, T]$. Then, the approximating solution v with initial condition $v_0 \in V$, chosen arbitrarily, shall be given by

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla q = f + \mu (I_h(u) - I_h(v)),$$

$$\nabla \cdot v = 0,$$

on the interval $[0, T]$. Using the above functional setting the above system is equivalent to

$$\frac{dv}{dt} + \nu A v + B(v, v) = f + \mu P_\sigma (I_h(u) - I_h(v)), \tag{23}$$

on the interval $[0, T]$.

If we knew u_0 exactly, then we could take $v_0 = u_0$ and the resulting solution v would be identical to u for all time; this is due to the uniqueness of the solutions of (23) (see Theorem 5, below). However, if we knew u_0 exactly, there would be no need for continuous data assimilation in the first place and one could integrate (19) directly with the initial value u_0. Intuitively speaking it makes sense to take $v_0 = P_\sigma I_h(u(0))$, which is the initial observation of the solution u. However, v_0 chosen in this way may not be an element of V. The main point of the data assimilation method given in (23) is to avoid the difficulties which come from the direct insertion of observational measurements into the approximate solution. A choice for v_0 in agreement with this philosophy is $v_0 = 0$. In fact, our results to hold equally well when v_0 is chosen to be any element of V. In either case we obtain an approximating solution v constructed using only the observations of the solution $I_h(u)$ and the known values of ν and f.

We now show the data assimilation equations (23) are well-posed. When I_h satisfies (6) we show well-posedness for both no-slip Dirichlet and periodic boundary conditions. When I_h satisfies (7) we will deal here, for simplicity, with only the case of periodic boundary conditions.

Theorem 5. *Suppose I_h satisfies (6) and $\mu c_0 h^2 \le \nu$, where c_0 is the constant appearing in (6). Then the continuous data assimilation equations (23) possess unique strong solutions that satisfy*

$$v \in C([0,T];V) \cap L^2((0,T);D(A)) \quad \text{and} \quad \frac{dv}{dt} \in L^2((0,T);H), \tag{24}$$

for any $T > 0$. Furthermore, this solution depends continuously on the initial data v_0 in the V norm.

Proof. Define $g = f + \mu P_\sigma I_h(u)$. Theorem 3 implies $u \in C([0,T];V)$. Consequently

$$|P_\sigma I_h(u)| \le |u - I_h(u)| + |u| \le (c_0^{1/2} h + \lambda_1^{-1/2})\|u\|$$

implies that $P_\sigma I_h(u) \in C([0,T];H)$. Hence $g \in C([0,T];H)$. This means there is a constant M such that $|g|^2 < M$ for every $t \in [0,T]$.

We now show the existence of solutions v to (23) using the Galerkin method. The proof follows the same ideas as the proof of Theorem 3. Let P_n be the n-th Galerkin projector and v^n be the solution to the finite-dimensional Galerkin truncation

$$\begin{cases} \dfrac{dv^n}{dt} + \nu A v^n + P_n B(v^n, v^n) = P_n g - \mu P_n I_h(v^n) \\ v^n(0) = P_n v_0. \end{cases} \tag{25}$$

First, we observe that (25) is a finite system of ODEs, which has short time existence and uniqueness. We focus on the maximal interval of existence, $[0, T_n)$, and show uniform bound for v_n, which are independent of n. This in turn will imply the global existence for (25). Thus, our aim is to find bounds on v^n which are uniform in n. This will then show global existence of solutions to (23). In the estimates that follow, we denote the Galerkin solution v^n by v for notational simplicity.

Begin by taking inner products of (25) with v to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v|^2 + \nu\|v\|^2 &= (g, v) - \mu(I_h(v), v) \\ &= (g, v) + \mu(v - I_h(v), v) - \mu|v|^2 \\ &\le \frac{1}{2\mu}|g|^2 + \frac{\mu}{2}|v|^2 + \frac{\mu}{2}|P_\sigma(v - I_h(v))|^2 + \frac{\mu}{2}|v|^2 - \mu|v|^2 \\ &\le \frac{1}{2\mu}|g|^2 + \frac{\mu c_0 h^2}{2}\|v\|^2. \end{aligned}$$

By hypothesis h is so small that $\mu c_0 h^2 \le \nu$. Therefore,

$$\frac{d}{dt}|v|^2 + \nu\|v\|^2 \le \frac{1}{\mu}|g|^2, \tag{26}$$

and consequently

$$\frac{d}{dt}|v|^2 + \nu\lambda_1|v|^2 \le \frac{1}{\mu}M, \quad \text{for every} \quad t \in [0, T_n). \tag{27}$$

Multiplying (27) by $e^{\nu\lambda_1 t}$ and integrating yields

$$|v(t)|^2 \le |v_0|^2 e^{-\nu\lambda_1 t} + \frac{M}{\mu\nu\lambda_1}\Big(1 - e^{-\nu\lambda_1 t}\Big) \le \rho_H^2, \quad \text{for every} \quad t \in [0, T_n),$$

where

$$\rho_H^2 = |v_0|^2 + \frac{M}{\mu\nu\lambda_1}.$$

As this bound holds uniformly in n for T_n arbitrarily large, we have global existence on the interval $[0, T]$, for all $T \ge 0$. Now, integrating (26) yields

$$|v(t)|^2 - |v_0|^2 + \nu\int_0^t \|v\|^2 \le \frac{t}{\mu}M.$$

It follows that

$$\int_0^t \|v(\tau)\| d\tau \le \sigma_V^2, \quad \text{for every} \quad t \in [0, T],$$

where

$$\sigma_V^2 = \frac{1}{\nu}|v_0|^2 + \frac{T}{\mu\nu}M.$$

Now, take inner products of (25) with Av to obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu|Av|^2 + (B(v,v), Av) = (g, Av) - \mu(I_h(v), Av).$$

Inequality (17) implies

$$\begin{aligned}
|(B(v,v), Av)| &\le c|v|^{1/2}\|v\||Av|^{3/2} \\
&\le \frac{1}{4}\left(\frac{6^{3/4}}{\nu^{3/4}}c|v|^{1/2}\|v\|\right)^4 + \frac{3}{4}\left(\frac{\nu^{3/4}}{6^{3/4}}|Av|^{3/2}\right)^{4/3} \\
&\le \frac{54c^4}{\nu^3}|v|^2\|v\|^4 + \frac{\nu}{8}|Av|^2.
\end{aligned}$$

Furthermore,

$$|(g, Av)| \leq |g||Av| \leq \frac{2}{\nu}|g|^2 + \frac{\nu}{8}|Av|^2$$

and by (6) along with the assumption that $\mu c_0 h^2 \leq \nu$ we obtain

$$\begin{aligned}\mu|(I_h(v), Av)| &\leq \frac{\mu^2}{\nu}|v - I_h(v)|^2 + \frac{\nu}{4}|Av|^2 - \mu\|v\|^2 \\ &\leq \frac{\mu^2 c_0 h^2}{\nu}\|v\|^2 + \frac{\nu}{4}|Av|^2 - \mu\|v\|^2 \leq \frac{\nu}{4}|Av|^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|v\|^2 + \nu|Av|^2 \leq \frac{54c^4}{\nu^3}|v|^2\|v\|^4 + \frac{4}{\nu}|g|^2, \tag{28}$$

and consequently

$$\frac{d}{dt}\|v\|^2 - \frac{54c^4}{\nu^3}|v|^2\|v\|^4 \leq \frac{4}{\nu}|g|^2 \leq \frac{4}{\nu}M, \tag{29}$$

for every $t \in [0, T]$. Define

$$\psi(t) = \exp\Big\{ - \frac{54c^4}{\nu^3}\int_0^t |v|^2\|v\|^2\Big\}. \tag{30}$$

Since

$$\int_0^t |v|^2\|v\|^2 \leq \rho_H^2 \int_0^t \|v\|^2 \leq \rho_H^2\sigma_V^2 < \infty, \quad \text{for every} \quad t \in [0, T],$$

we have that $\psi(t) > 0$ for every $t \in [0, T]$. Multiplying (27) by $\psi(t)$ and integrating yields

$$\|v(t)\|^2 \leq \frac{1}{\psi(t)}\Big\{\|v_0\|^2 + \frac{4}{\nu}M\int_0^t \psi(s)ds\Big\} \leq \rho_V^2, \quad \text{for all} \quad t \in [0, T],$$

where

$$\rho_V^2 = \frac{1}{\psi(T)}\Big\{\|v_0\|^2 + \frac{4T}{\nu}M\Big\}.$$

Now, integrating (28) yields

$$\|v(t)\|^2 - \|v_0\|^2 + \nu\int_0^t |Av|^2 \leq \frac{54c^4}{\nu^3}\int_0^t \Big(|v|^2\|v\|^4 + \frac{4}{\nu}|g|^2\Big) \leq \sigma_{\mathcal{D}(A)}^2,$$

for every $t \in [0, T]$, where

$$\sigma_{\mathcal{D}(A)}^2 = \frac{54c^4 T}{\nu^3} \left\{ \rho_H^2 \rho_V^4 + \frac{4}{\nu} M \right\}.$$

The bounds ρ_V and $\sigma_{\mathcal{D}(A)}$ are uniform in n. Uniform bounds on $|dv/dt|$ then proceed in exactly the same way as for the two-dimensional Navier–Stokes equations. Since the estimates on the Galerkin solutions are uniform in n, Aubin's compactness theorem [1] allows one to extract subsequences in such a way that the limit v satisfies (23) and (24).

Next, we show that such solutions are unique and depend continuously on the initial data. Let v_1 and v_2 be two solutions for (23) both satisfying the conditions in (24). Choose K large enough such that $\|v_1\|^2 \le K$ and $\|v_2\|^2 \le K$ for almost every $t \in [0, T]$. Let $\delta = v_1 - v_2$. Then δ satisfies

$$\frac{d\delta}{dt} + \nu A\delta + B(v_1, \delta) + B(\delta, v_2) = -\mu P_\sigma I_h(\delta).$$

Taking inner product with $A\delta$ yields

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \nu|A\delta|^2 + (B(v_1, \delta), A\delta) + (B(\delta, v_2), A\delta) = -\mu(I_h(\delta), A\delta).$$

Here we used the fact that

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 = \Big(\frac{d\delta}{dt}, A\delta\Big),$$

which can be justified by Lemma 1.2 in Chapter 3 of Temam [28] or Theorem 7.2 in Robinson [26] which is due to Lions–Magenes [23]. Estimate the right-hand side of this equation as

$$\begin{aligned} -\mu(I_h(\delta), A\delta) &= \mu(\delta - I_h(\delta), A\delta) - \mu\|\delta\|^2 \\ &\le \frac{\mu^2}{2\nu}|P_\sigma(\delta - I_h(\delta))|^2 + \frac{\nu}{2}|A\delta|^2 - \mu\|\delta\|^2 \\ &\le \frac{\mu^2 c_0 h^2}{2\nu}\|\delta\|^2 + \frac{\nu}{2}|A\delta|^2 - \mu\|\delta\|^2 \le \frac{\nu}{2}|A\delta|^2, \end{aligned}$$

where we have again used the hypothesis that $\mu c_0 h^2 \le \nu$. It follows that

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \frac{\nu}{2}|A\delta|^2 \le |(B(v_1, \delta), A\delta)| + |(B(\delta, v_2), A\delta)|. \tag{31}$$

The proof of uniqueness and continuity now proceeds as for the two-dimensional Navier–Stokes equations. In particular, estimate the non-linear terms on the right-hand side of (31) using (17) and (18) as

$$|(B(v_1,\delta),A\delta)| \le c|v_1|^{1/2}\|v_1\|^{1/2}\|\delta\|^{1/2}|A\delta|^{3/2} \le \frac{27c^4K^2}{4\nu^3\lambda_1}\|\delta\|^2 + \frac{\nu}{4}|A\delta|^2,$$

and

$$|(B(\delta,v_2),A\delta)| \le c|\delta|^{1/2}\|v_2\||A\delta|^{3/2} \le \frac{27c^4K^2}{4\nu^3\lambda_1}\|\delta\|^2 + \frac{\nu}{4}|A\delta|^2.$$

Therefore,

$$\frac{d}{dt}\|\delta\|^2 \le \frac{27c^4K^2}{2\nu^3\lambda_1}\|\delta\|^2, \quad \text{for all} \quad t\in[0,T].$$

Integrating yields

$$\|\delta(t)\|^2 \le \|\delta_0\|^2 \exp\left\{\frac{27c^4K^2}{2\nu^3\lambda_1}t\right\}.$$

Thus, the solutions v to (23), which satisfy (24), also satisfy $v\in\mathcal{C}([0,T],V)$, and depend continuously on the initial data in the V norm. □

Theorem 6. *In the case of periodic boundary conditions suppose that I_h satisfies (7), and $\mu c_0 h^2 \le \nu$, where c_0 is the constant appearing in (7). Then the continuous data assimilation equations (23) possess unique strong solutions that satisfy (24), for any $T>0$. Furthermore, this solution is in $C([0,T],V)$ and depends continuously on the initial data v_0 in the V norm.*

Proof. The proof is similar to the proof of Theorem 5 but makes use of the identity (14) to obtain estimates on $\|v\|$ and $\int_0^t |Av|^2$ directly. □

The algorithm given by equation (23) for constructing the approximate solution v contains two parameters h and μ. The first parameter h has dimensions of length and corresponds to the resolution of the observational measurements represented by $I_h(u)$. Smaller values of h correspond to spatially more accurate resolved measurements. The relaxation parameter μ controls the rate at which the approximating solution v is forced toward the observable part of the reference solution u. Larger values of μ cause $I_h(v)$ to faster track $I_h(u)$. It is the parameter μ which distinguishes (23) from the previous methods of continuous data assimilation studied in [7], [20], [24] and [25].

The condition that $\mu c_0 h^2 \leq \nu$, given in Theorem 5, places a restriction on the size of μh^2 compared to the viscosity ν, sufficient to ensure the data assimilation equations are well-posed. This restriction is due to the fact that the the interpolant operator μI_h might generate large gradients and spatial oscillations ("spill over" to the fine scales) that need to be controlled (suppressed) by the viscosity term. Notice that in the case when $I_h = P_{m_h}$, where P_{m_h} is the orthogonal projection onto the linear sub-space spanned by the Fourier modes with wave numbers $|k| \leq m_h = \frac{1}{h}$, such oscillations are not generated, since $-(\mu I_h(v), v) = -\mu |P_{m_h} v|^2$ and $-(\mu I_h(v), Av) = -\mu \|P_{m_h} v\|^2$. Consequently, there is no restriction on μh^2 and μ can be taken arbitrary large. In the limit when $\mu \to \infty$ one obtains exactly the same algorithm introduced in [24] (see also [19]). In particular, one has $P_{m_h} v = P_{m_h} u$, and all that one needs to do is to solve for $q = (I - P_{m_h})v$, for which an explicit evolution equation is presented in [24].

Next, our aim is to give further conditions on h and μ which guarantee that the difference between the approximating solution v and the reference solution u converges to zero as $t \to \infty$. To do this we consider the time evolution of $w = u - v$. Since

$$B(u,u) - B(v,v) = B(u,w) + B(w,v) = B(u,w) + B(w,u) - B(w,w)$$

and

$$I_h(u) - I_h(v) = I_h(w)$$

then subtracting equation (23) from equation (19) yields

$$\frac{dw}{dt} + \nu A w + B(u,w) + B(w,u) - B(w,w) = -\mu P_\sigma I_h(w). \qquad (32)$$

This equation serves as the starting point for the proofs of Theorem 1 and Theorem 2 in the proceeding two sections.

3 No-slip Dirichlet Boundary Conditions Case

In this section we prove Theorem 1. We first recall the following generalized Gronwall inequality proved in Jones and Titi [21], see also [17].

Lemma 1 (Uniform Gronwall Inequality)**.** *Let $T > 0$ be fixed. Suppose*

$$\frac{dY}{dt} + \alpha(t) Y \leq 0, \qquad \textit{where} \qquad \limsup_{t\to\infty} \int_t^{t+T} \alpha(s)\,ds \geq \gamma > 0.$$

Then $Y(t) \to 0$ exponentially, as $t \to \infty$.

We now state and prove a lemma leading to our main result.

Proposition 1. *Let Ω be an open, bounded and connected set in $\mathbf{R}^2$ with C^2 boundary, and let u be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on Ω with no-slip Dirichlet boundary conditions. Let v be the approximating solution given by equations (23). Then $|u-v| \to 0$, as $t \to \infty$, provided $\mu c_0 h^2 \le \nu$ and $\mu \ge 5c^2G^2\nu\lambda_1$.*

Proof. Let $w = u - v$. Then w satisfies equation (32) stated above. Taking the inner product with w we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + (B(w,u),w) &= -\mu(I_h(w),w) \\ &= \mu(w - I_h(w), w) - \mu|w|^2 \\ &\le \frac{\mu}{2}|P_\sigma(w - I_h(w))|^2 + \frac{\mu}{2}|w|^2 - \mu|w|^2 \\ &\le \frac{\mu c_0 h^2}{2}\|w\|^2 - \frac{\mu}{2}|w|^2 \le \frac{\nu}{2}\|w\|^2 - \frac{\mu}{2}|w|^2.\end{aligned}$$

Since (16) implies

$$|(B(w,u),w)| \le c\|u\||w|\|w\| \le \frac{c^2}{2\nu}\|u\|^2|w|^2 + \frac{\nu}{2}\|w\|^2,$$

we obtain

$$\frac{d}{dt}|w|^2 + \Big(\mu - \frac{c^2}{\nu}\|u\|^2\Big)|w|^2 \le 0.$$

Denote

$$\alpha(t) = \mu - \frac{c^2}{\nu}\|u\|^2.$$

Taking $T = (\nu\lambda_1)^{-1}$ in Theorem 4 we have for $t \ge t_0$ that

$$\int_t^{t+T} \|v\|^2 \le 2(1 + T\nu\lambda_1)\nu G^2 = 4\nu G^2.$$

Thus

$$\limsup_{t\to\infty} \int_t^{t+T} \alpha(s)ds \ge \frac{\mu}{\nu\lambda_1} - 4c^2G^2 \ge c^2G^2 > 0,$$

and by Lemma 1 it follows that $|w| \to 0$, exponentially, as $t \to \infty$. $\square$

Proof of Theorem 1. The hypothesis of Proposition 1 require that

$$\mu c_0 h^2 \le \nu \qquad \text{and} \qquad \mu \ge 5c^2 G^2 \nu \lambda_1.$$

Therefore

$$\frac{1}{h^2} \ge \frac{\mu c_0}{\nu} \ge c_1 G^2 \lambda_1 \tag{33}$$

where $c_1 = 5c_0c^2$. ◻

4 Periodic Boundary Conditions Case

In this section we prove Theorem 2. We begin with an elementary inequality which will be be referred to in the sequel.

Lemma 2. *Let $\phi(r) = r - \beta(1 + \log r)$ where $\beta > 0$. Then*

$$\min\{\phi(r) : r \ge 1\} \ge -\beta \log \beta.$$

Proof. Note first that

$$\phi(1) = 1 - \beta \qquad \text{and} \qquad \lim_{r\to\infty} \phi(r) = \infty.$$

The derivative $\phi'(r) = 1 - \beta/r$ is zero if and only if $r = \beta$. Therefore

$$\min\{\phi(r) : r \ge 1\} = \begin{cases} 1 - \beta & \text{if } 0 < \beta \le 1 \\ -\beta \log \beta & \text{if } 1 < \beta. \end{cases}$$

Observe that over the interval $0 < \beta \le 1$ we have $1 - \beta \ge -\beta \log \beta$, which concludes our proof. ◻

We now state and prove a lemma leading to the proof of Theorem 2.

Proposition 2. *Let $\Omega = [0, L]^2$, for some fixed $L > 0$. Let u be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on Ω equipped with periodic boundary conditions. Let v be the approximating solution given by equations (23), where I_h satisfies (6). Then $\|u - v\| \to 0$, as $t \to \infty$, provided $\mu c_0 h^2 \le \nu$ and $\mu \ge 3\nu\lambda_1(2c \log 2c^{3/2} + 8c \log(1 + G))G$.*

Proof. The proof makes use of the orthogonality properties (14) and (15) along with the Brézis–Gallouet inequality [6] which may be written as

$$\|u\|_{L^\infty(\Omega)} \le c\|u\|\Big\{1 + \log\frac{|Au|^2}{\lambda_1\|u\|^2}\Big\}, \tag{34}$$

which will allow us to obtain sharper estimates than for the case of no-slip boundary conditions.

Take the inner product of equation (32) with Aw and and use the orthogonality relations (14) and (15) to obtain

$$\frac{1}{2}\frac{d\|w\|^2}{dt} + \nu|Aw|^2 = (B(w,w),Au) - \mu(I_h(w),Aw).$$

Using (34) and the hypothesis $\mu c_0 h^2 \le \nu$ we have

$$|(B(w,w),Au)| \le c\|w\|^2\Big\{1 + \log\frac{|Aw|^2}{\lambda_1\|w\|^2}\Big\}|Au|,$$

and

$$\begin{aligned}-\mu(I_h(w),Aw) &= \mu(w - I_h(w),Aw) - \mu\|w\|^2\\ &\le \mu|P_\sigma(w - I_h(w))||Aw| - \mu\|w\|^2\\ &\le \frac{\mu^2 c_0 h^2}{2\nu}\|w\|^2 + \frac{\nu}{2}|Aw|^2 - \mu\|w\|^2 \le \frac{\nu}{2}|Aw|^2 - \frac{\mu}{2}\|w\|^2.\end{aligned}$$

Therefore,

$$\frac{d\|w\|^2}{dt} + \nu|Aw|^2 \le \Big(2c|Au|\Big\{1 + \log\frac{|Aw|^2}{\lambda_1\|w\|^2}\Big\} - \mu\Big)\|w\|^2,$$

or

$$\frac{d\|w\|^2}{dt} + \Big(\nu\lambda_1\frac{|Aw|^2}{\lambda_1\|w\|^2} - 2c|Au|\Big\{1 + \log\frac{|Aw|^2}{\lambda_1\|w\|^2}\Big\} + \mu\Big)\|w\|^2 \le 0.$$

Now setting

$$\beta = \frac{2c|Au|}{\nu\lambda_1} \qquad \text{and} \qquad r = \frac{|Aw|^2}{\lambda_1\|w\|^2}$$

in Lemma 2, and noting that $r \ge 1$, by Poincaré's inequality (11), we obtain

$$\frac{d\|w\|^2}{dt} + \Big\{\mu - 2c|Au|\log\frac{2c|Au|}{\nu\lambda_1}\Big\}\|w\|^2 \le 0.$$

By (22) we estimate

$$2c \log \frac{2c|Au|}{\nu\lambda_1} \le J,$$

where

$$J = c_3 + c_4 \log(1 + G), \tag{35}$$

$c_3 = 2c \log 2c^{3/2}$ and $c_4 = 8c$. It follows that

$$\frac{d\|w\|^2}{dt} + \Big\{\mu - J|Au|\Big\}\|w\|^2 \le 0.$$

Furthermore, Young's inequality

$$J|Au| \le \frac{J^2}{2\mu}|Au|^2 + \frac{\mu}{2}$$

implies

$$\frac{d\|w\|^2}{dt} + \frac{1}{2}\Big\{\mu - \frac{J^2}{\mu}|Au|^2\Big\}\|w\|^2 \le 0.$$

Denote

$$\alpha(t) = \frac{1}{2}\Big\{\mu - \frac{J^2}{\mu}|Au(t)|^2\Big\}.$$

Taking $T = (\nu\lambda_1)^{-1}$ in Theorem 4 we have for $t \ge t_0$ that

$$\int_t^{t+T} |Au|^2 \le 2(1 + T\nu\lambda_1)\nu\lambda_1 G^2 = 4\nu\lambda_1 G^2.$$

Thus,

$$\limsup_{t\to\infty} \int_t^{t+T} \alpha(s)ds \ge \frac{\mu}{2\nu\lambda_1} - \frac{2\nu\lambda_1}{\mu} J^2 G^2 = \frac{5}{6} JG > 0,$$

and consequently $\|w\| \to 0$ exponentially, as $t \to \infty$. □

Proposition 3. *Let $\Omega = [0, L]^2$, for some fixed $L > 0$. Let u be a solution of the incompressible two-dimensional Navier–Stokes equations (19) on Ω, equipped with periodic boundary conditions. Let v be the approximating solution given by equations (23) where I_h satisfies (7). Then $\|u - v\| \to 0$, as $t \to \infty$, provided $\mu c_0 h^2 \le \nu$ and $\mu \ge 3\nu\lambda_1(2c \log 2c^{3/2} + 8c \log(1 + G))G$.*

Proof. The proof is the same as the proof of Proposition 2 except that the estimate for $-\mu(I_h(w), Aw)$ has to be modified as

$$\begin{aligned}-\mu(I_h(w), Aw) &= \mu(w - I_h(w), Aw) - \mu\|w\|^2 \\ &\le \mu|w - I_h(w)||Aw| - \mu\|w\|^2 \\ &\le \frac{\mu^2 c_0^2 h^4}{4\nu}|Aw|^2 + \frac{\nu}{4}|Aw|^2 - \mu\|w\|^2 \le \frac{\nu}{2}|Aw|^2 - \mu\|w\|^2.\end{aligned}$$

Then, since $-\mu < -\mu/2$ the rest of the proof follows without change. ◻

Proof of Theorem 2. The hypothesis of Proposition 2 or Proposition 3 require that

$$\mu c_0 h^2 \le \nu \qquad \text{and} \qquad \mu \ge 3\nu\lambda_1 JG.$$

Therefore,

$$\frac{1}{h^2} \ge \frac{\mu c_0}{\nu} \ge c_2\lambda_1 G(1 + \log(1 + G)), \tag{36}$$

where $c_2 = 3\max\{c_3, c_4\}$. ◻

5 Conclusions

As shown in this paper, the algorithm given by (2), for constructing $v(t)$ from the observations $I_h(u(t))$, yields an approximation for $u(t)$ such that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \to 0 \qquad \text{exponentially, as} t \to \infty, \tag{37}$$

provided the observations have fine enough spatial resolution. This result has the following consequence. To accurately predict $u(t)$ for time T into the future it is sufficient to have observational data $I_h(u(t))$ accumulated over an interval of time linearly proportional to T in the immediate past.

In particular, suppose it is desired to predict $u(t)$ with accuracy $\epsilon > 0$ on the interval $[t_1, t_1 + T^*]$, where t_1 is the present time and $T^* > 0$ determines how far into the future to make the prediction. Let h be small enough and μ large enough so that Theorem 1 implies (37). Thus, there is $\alpha > 0$ and a constant $C > 0$ such that

$$\|u(t) - v(t)\|_{L^2(\Omega)} \le Ce^{-\alpha t} \qquad \text{for all} \qquad t \ge 0.$$

Now use $v(t_1)$ as the initial condition from which to make a future prediction.

Let w be a solution to (19) with initial condition $w(t_1) = v(t_1)$. Known results on continuous dependence on initial conditions, see, for example, [9], [19], [26] or [27], imply there is $\beta > 0$ such that

$$\|w(t) - u(t)\|_{L^2(\Omega)} \leq \|w(t_1) - u(t_1)\|_{L^2(\Omega)} e^{\beta(t-t_1)} \qquad \text{for} \qquad t \geq t_1.$$

Therefore

$$\|w(t) - u(t)\| \leq Ce^{-\alpha t_1 + \beta T} < \epsilon \qquad \text{for} \qquad t \in [t_1, t_1 + T]$$

provided $\alpha t_1 \geq \beta T + \ln(C/\epsilon)$. Thus $w(t)$ predicts $u(t)$ with accuracy ϵ on the interval $[t_1, t_1 + T]$.

Work is currently underway to numerically test Theorem 2 in the case of determining finite volume elements and nodes. Of particular focus is how to tune the parameter μ. If μ is very large the effects of "spill over" into the fine scales may become significant, whereas if μ is small convergence of the approximate solution may be slow or not happen at all. Numerical simulations performed by Gesho [18] confirm that the continuous data assimilation algorithm given by equation (2) directly works, without additional filtering, for observational measurements at a discrete set of nodal points, where I_h is given by (8). As with previous computational work (cf. [19],[24] and [25]) the approximating solution $v(t)$ converges to the reference solution $u(t)$ under much less stringent conditions than required by our theory.

The main advantage of introducing a control term that forces the approximate solution toward the reference solution is that we can rely on the viscous dissipation, already present in the dynamics, to filter the observational data (that is, to suppress the spatial oscillations, i.e. the "spill over" into the fine scales, that are generated by the coarse-mesh stabilizing term $\mu I_h(v)$). In addition to working for a general class of interpolant observables this technique also allows processing of observational data which contains stochastic noise. In particular, the same algorithm can be used to obtain an approximation $v(t)$ that converges (in some sense) to the reference solution $u(t)$, to within an error of the order of μ times the variance of the noise in the measurements. This work [3] is in progress.

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