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Topological Stable Manifold Theorems

Diplomarbeit

October 27, 2010

Freie Universität Berlin

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Chapter 0
Introduction

In this work we introduce a generalization of the stable manifold theorem by topological means. This result is motivated by the study of differential equations with heteroclinic attractors, that is an attractor consisting of heteroclinic orbits $\gamma_i(t)$ with $\lim_{t\to-\infty}\gamma_i(t) = x_i$ and $\lim_{t\to+\infty}\gamma_i(t) = x_{i+1}$, as they arise in Bianchi cosmologies (c.f. e.g. [HU09]). One goal of these studies is to extend the understanding of the dynamics of the heteroclinic attractor to its basin of attraction.

The main tool of analysis near a heteroclinic chain is using a sequence of Poincaré-sections, i.e. codimension 1 submanifolds, which are transverse to the heteroclinic orbits. Often, it is useful to use Poincaré sections $\Sigma_{i,in}$ and $\Sigma_{i,out}$ near the equilibrium x_i, which are transverse to γ_{i-1} respective γ_i. This leads then to a sequence of local return-maps $\Phi_{i,loc} : \Sigma_{i,in} \to \Sigma_{i,out}$, which describe the passage near x_i, and global return-maps $\Phi_{i,glob} : \Sigma_{i,out} \to \Sigma_{i+1,in}$. With this construction one can now study the time-discrete, non-autonomous dynamical system defined by $\left(\Phi_{i,\mathrm{loc}}, \Phi_{i,\mathrm{glob}}\right)_i$. One of the central questions is, whether there exist initial conditions, which converge under the flow of the differential equation to the heteroclinic chain "in phase", i.e. whose iterates under the return-maps never leave the domains of the return-maps.

The classical stable manifold theorem classifies the set of initial conditions, which converge to a hyperbolic fixed point of a diffeomorphism. However, this theorem and its non-autonomous versions cannot be applied directly to the return-maps, since the local maps $\Phi_{i,loc}$ are not defined in a complete neighborhood of the heteroclinic points $x_{i,in} \in \gamma_i \cap \Sigma_{i,in}$ and are not differentiable near $x_{i,in}$. Nevertheless, it is sometimes possible to apply the techniques, which lead to the stable manifold theorem, namely the graph transform. This approach requires subtle higher order estimates on $\Phi_{i,loc}$ and has been done in e.g. [LHWG10].

A different approach to the problem of the basin of attraction of a heteroclinic chain is of a more geometric or topological nature: in many cases, the geometric relations between $\operatorname{Ran}\Phi_{i,loc}$ and $\Phi^{-1}_{i+1,glob}[\operatorname{dom}\Phi_{i+1,loc}]$ alone will suffice to prove existence of a nontrivial basin of attraction by topological methods. This approach is the main focus of this work.

The primary advantage of this topological approach is that it requires only C^0-estimates on $\Phi_{i,loc}$, which are far easier to obtain. This allows one to focus intead on another difficulty of heteroclinic chains $(x_i)_i$: any analysis will require bounds on the spectral gaps at x_i, the domains of definition dom $\Phi_{i,loc}$ and on the diffeomorphisms $\Phi_{i,glob}$. In many cases, however, there are no uniform bounds available for $i \to \infty$, but only growth estimates.

In the first chapter we will review the classical stable manifold theorem in order to provide a context for generalizations. Since we aim for generalizations and modifications, we will mainly focus on the methods used to prove the theorem, rather than on an elegant formulation.

In the second chapter, we will introduce a topological generalization of the stable manifold theorem. This topological generalization, Theorem 2.19, allows us to prove existence of stable sets, which are almost manifolds. The theory developed in the second chapter is closely related to Conley Index theory (c.f. e.g. [RW10], [FR00]). Even while Conley Index theory has the advantage of being more elegant and more powerful in many cases, the theory developed in Chapter 2 is "elementary", i.e. does not require a sophisticated topological framework to use, especially since we will focus on relatively closed but noncompact sets. If we are only interested in proving existence of a nontrivial basin of attraction, instead of building a grand theory of topological invariants of dynamical systems, our approach is sufficient.

In the third chapter we will apply the topological techniques to the example of a 3-dimensional system with a homoclinic orbit, where the stable manifold theorem is not applicable because of the non-differentiability of the Poincaré map. The application of the topological techniques then shows that the basin of attraction of the homoclinic orbit is "topologically separating" (Definition 2.9). This application serves to demonstrate that the topological techniques are especially useful in conjuction with direct calculations, since they require only C^0-estimates, which are generally easier to prove directly.

We will close by discussing possible application of the approach to heteroclinic chains in n-dimensional systems as well as future applications in mathematical cosmology.

Chapter 1
Invariant Manifolds

The goal of this chapter is to review the classical stable manifold theorem in order provide a context for the topological theorems which will be introduced as a lower regularity substitute for the classical theorem in Chapter 2 and which will be applied to a toy model in Chapter 3. The setup of this chapter was guided by two principles: Firstly, since we eventually aim for generalizations of the stable manifold theorem, we focus on the methods of proof rather than concise and elegant formulations. Secondly, to give a different perspective on regularity results and for reasons concerning generalizations discussed in detail in Section 1.1.3, we will refrain as much as possible from using the implicit function theorem to prove regularity results. Instead we will use a combination of a contraction mapping principle and apriori bounds on higher derivatives, as outlined in 1.17.

We will start by reviewing some basic facts and definitions about dynamical systems in the first section, among which is the Picard-Lindeloef Theorem 1.14. We will then use the Perron-method and the variations-of-constants formula to prove a first version of the stable manifold theorem in the second section. In the third section, we will prove several invariant manifold theorems with the graph transform approach.

1.1 Dynamical Systems and Invariant Sets

Dynamical systems broadly fall into four categories: One can study time-discrete or time-continuous systems, which can be either autonomous (time-independent) or non-autonomous. Even though we are mainly interested in time-continuous autonomous systems, i.e. flows or solutions to differential equations $\dot{x} = f(x)$, we will need tools used for the study of the other types of systems. Therefore we will start by giving some definitions on all four of these classes. We will then prove the Picard-Lindeloef Theorem 1.14, which establishes the solutions of many differential equations as flows. This section will close by reviewing Poincaré-Sections, which allow to apply methods from time-discrete systems to time-continuous ones.

1.1.1 Time-discrete Systems

The simplest case of a dynamical system is a continuous map $F: X \to X$ on a topological space X. Note that we do not require the map F to be injective or even bijective. Let $I = \mathbb{N}$ or $I = \mathbb{Z}$.

Definition 1.1. Let x_k be a sequence in X, i.e. $(x_k)_{k\in I} \in X^I$. We call it an *orbit* of F, if

$$x_{k+1} = F(x_k) \quad \forall k \in I$$

Since we mainly study orbits, we will sometimes write x_{k+1} as a shorthand for $F(x_k)$, implicitly assuming that we are dealing with an orbit. One of the main points of interest are invariant sets.

Definition 1.2. Let $M \subset X$. M is called *forward invariant* with respect to F if $F[M] \subset M$. For $U \subset X$, define the *maximal foward invariant subset* of U as

$$\operatorname{inv}_F^+(U) = \bigcap_{n\in\mathbb{N}} F^{-n}[U],$$

where $F^{-1}[U]$ denotes the complete preimage of U under F.

Proposition 1.3. *The set* $\operatorname{inv}_F^+(U)$ *is forward invariant and is maximal, i.e. has the property that every forward invariant* $M \subset U$ *is a subset of* $\operatorname{inv}_F^+(U)$.

Proof. $\operatorname{inv}_F^+(U)$ is forward invariant, since

$$\begin{aligned} F\left(U \cap \bigcap_{n\geq 1} F^{-n}[U]\right) &\subset F(U) \cap \bigcap_{n\geq 1} F(F^{-n}(U)) \\ &= F(U) \cap \operatorname{inv}_F^+(U). \end{aligned}$$

Furthermore, $F(M) \subset M$ implies for all $n \in \mathbb{N}$ that $F^n(M) \subset M \subset U$ and therefore $M \subset F^{-n}(U)$. Taking the intersection over all $n \in \mathbb{N}$ proves the assertion. □

Forward invariant sets are the subjects of the stable manifold theorem and will be studied in detail later on.

Remark 1.4. There is also a comparable definition of backward invariance. Since this definition is more subtle for non-injective maps and is not required for the remainder of this work, we will refrain from a lengthy discussion. However, for the sake of completeness we will give the definition:

Definition 1.5. M is called *backward invariant* with respect to F if $M \subset F[M]$.

One often encounters non-autonomous systems, where the map $F_\bullet = (F_n)_{n\in I}$ and the spaces $X_\bullet = (X_n)_{n\in I}$ depend on n.

Definition 1.6. Let $X_\bullet = (X_n)_{n\in I}$ be a sequence of topological spaces and$F_\bullet$ be a sequence of continuous maps $F_n : X_n \to X_{n+1}$. We call a sequence $x_\bullet \in \prod_{k\in I} X_k$ an *orbit*, if

$$x_{k+1} = F_k(x_k) \quad \forall k \in I.$$

Again, we are interested in forward invariant sets. In general the definition of invariance for non-autonomous systems is more subtle. One prominent approach for example is the co-chain approach (cf. e.g. [Sel67]). By considering the sequence $\mathbb{Z} \to F_k$ as a map into a function space this approach provides structures via a pull back, e.g. pulling back the topology allows for compactedness arguments. Since we do not need this kind of structures in the context of this work it suffices to employ the following very simple generalization of the definitions of invariance in the autonomous case.

Definition 1.7. Let $M_\bullet \subset X_\bullet . M_\bullet$ is called *forward invariant* if $F_k[M_k] \subset M_{k+1}$ for all $k \in I$. Define the *maximal forward invariant subset* of $M_\bullet$ as

$$\mathrm{inv}^+_{F_\bullet}(M_\bullet)_k = M_k \cap \bigcap_{n\in\mathbb{N}} F_k^{-1} \circ \ldots \circ F_{k+n}^{-1}[M_{k+n+1}].$$

The forward invariance and maximality of the thus defined sequence of sets can be proved analogously to the autonomous case.

Remark 1.8. One way of viewing this definition is by considering the extended phase-space

$$X = \coprod_{k\in I} X_k = \{(k, X_k) : k \in I\},$$

and the induced map $F : X \to X$

$$F(k,x) = (k+1, F_k(x)) \quad \text{with } x \in X_k.$$

Then the non-autonomous notions of forward invariance and orbits coincide with the autonomous notions on the extended phase-space, i.e. a seqeuence of sets $M_\bullet \subset X_\bullet$ is forward (or backward) invariant with respect to the sequence $F_\bullet$ if and only if the union $M = \bigcup_{k\in I} M_k \subset X$ is forward invariant with respect to F.

The term “forward invariant” may be misleading: A forward invariant sequence of sets is not really forward invariant, since we have no natural way of comparing X_{k+1} with X_k– “covariant” might be a better word than “invariant”. One example is $M_k = \{x_k\}$ where (x_k) is an orbit of (F_k).
One of the main applications of these non-autonomous concepts will be to orbits of autonomous systems.

Example 1.9. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism. Let $x_\bullet$ be an orbit of F. Consider now the non-autonomous change of coordinates $y_k = x - x_k$ and the transformed system

$$F_k(y) = F(y + x_k) - x_{k+1} \quad \forall k \in \mathbb{Z}$$

The orbit $x_\bullet$ is transformed into a fixed point at $y = 0$, i.e. $F_k(0) = 0$. Let $B_r^X(x)$ denote the ball with radius r around x in the space X. If we set $M_k = B_1^n(x_k)$, we can see that

$$\mathrm{inv}_F^+(M_\bullet) = \mathrm{inv}_{F_\bullet}^+(B_1^n(0))$$

Therefore the study of non-autonomous fixed points includes the study of orbits of autonomous system from a local point of view.

1.1.2 Time-continuous Dynamical Systems

Time-continuous dynamical systems are described by flows and semiflows. We will again start with autonomous systems. Let X be a topological space.

Definition 1.10. Let $\phi : \mathbb{R}_{\geq 0} \times X \to X$ be continuous. We write $\phi^t = \phi(t, \cdot)$ as an abbrevation. We say that ϕ is a *semiflow*, if the following holds:

$$\begin{aligned} \phi^0 &= id \\ \phi^{s+t} &= \phi^s \circ \phi^t \qquad && \forall s,t > 0 \end{aligned}$$

If additionally ϕ^t is a homeomorphism for each $t > 0$, call ϕ a flow and extend its domain to $\mathbb{R} \times X$ in the canonical way via $\phi^{-t} = (\phi^t)^{-1}$.

We again are interested in forward invariant sets.

Definition 1.11. Let $M \subset X$. M is called *forward invariant* if $\phi^t[M] \subset M$ for all $t > 0$. For $U \subset X$, define the *maximal foward invariant subset* of U as

$$\mathrm{inv}_\phi^+(U) = \bigcap_{t>0} \left(\phi^t\right)^{-1}[U].$$

The generalization of these definitions for autonomous time-continuous systems to non-autonomous ones will be similar to the time-discrete case. We will however only consider the case of a single space X as the domain, since this case suffices for most applications. A similar distinction between the spaces X_k as in the time-discrete case would require the language of vector-bundles and does not lie in the focus of this work.

Definition 1.12. Let X be a topological space and let $T = \{(t,s) : t \geq s\} \subset \mathbb{R}^2_{\geq 0}$. Let $\phi : T \times X \to X$ be a continuous map. We write $\phi^{t,s} = \phi(t,s,\cdot)$ as an abbrevation. We say ϕ is a *semi-evolution* if the following holds:

$$\begin{aligned} \phi^{t,t} &= id \qquad && \forall t \geq 0 \\ \phi^{t_2,t_0} &= \phi^{t_2,t_1} \circ \phi^{t_1,t_0} \qquad && \forall t_2 \geq t_1 \geq t_0 \geq 0 \end{aligned}$$

If $\phi^{t,s}$ is a homeomorphism for all $t \geq s \geq 0$, we call ϕ an *evolution* and extend the domain of ϕ to $\mathbb{R}^2_{\geq 0} \times X$ in the canonical way via $\phi^{s,t} = (\phi^{t,s})^{-1}$.

Again, we define invariant sets:

Definition 1.13. Let $M \subset \mathbb{R} \times X$ and let $M_s = \{x \in X : (s,x) \in M\}$ denote the time sections.

M is called *forward invariant* if $\phi^{t,s}[M_s] \subset M_t$ for all $t \geq s \geq 0$. Define the maximal foward invariant subset of M as

$$\operatorname{inv}^+_\phi (M)_s = \bigcap_{t>s} \left(\phi^{t,s}\right)^{-1} [M_t].$$

1.1.3 Differential Equations

Differential equations are one of the most-studied classes of dynamical systems. Using the Picard-Lindeloef Theorem we can solve many differential equations and the solution is an evolution or a flow. Since the proof contains ideas which will be extended later, we will review this fundamental theorem.

Theorem 1.14 (Picard Lindeloef Theorem). *Let $X = \mathbb{R}^n$ and $f : \mathbb{R} \times X \to X$ be a Lipschitz continuous vector field with Lipschitz bound $|f|_{C^{0,1}} \leq L_f$ in both variables and global bound $|f|_\infty < \infty$. Then there is a Lipschitz continuous evolution $\phi^{t,s}$ which is differentiable in t and solves*

$$\mathrm{D}_t \phi^{t,s}(x) = f(t, \phi^{t,s}(x)) \quad \forall t, s \in \mathbb{R}.$$

The solution ϕ is almost as smooth as the vector field, i.e. for $f \in C^{N+1}$ we get $\phi \in C^{N,1}$. The solution is unique, in the sense that any differentiable curve $x : [t_1, t_2] \to X$, which solves the differential equation $\dot{x}(t) = f(t, x(t))$ for all $t \in (t_1, t_2)$, coincides with $x(t) = \phi^{t,t_1}(x(t_1))$.

Remark 1.15. We have will prove that the solution of a differential equation is $C^{N,1}$ in x if the right hand side f is C^{N+1}. This regularity is not optimal: the solution is even C^{N+1}. The usual approach of using the implicit function theorem for the Picard iteration (1.1.3.2) does provide optimal regularity, basically by explicitly calculating the higher derivatives (cf. e.g. [Zei95, Chapter 4.9] for a proof using the implicit function theorem). We have, however, chosen to use a different approach, mainly because it is easier to extend to more complicated settings, like the stable manifold theorem for non-autonomous systems 1.29.

The proof will use the contraction mapping principle, which we will state here for convenience:

Theorem 1.16 (Contraction Mapping Principle). *Let Z be a complete metric space and $T : Z \to Z$ be a map. Suppose that there exists a constant $\alpha < 1$ such that*

$$d(Tx,Ty) \le \alpha d(x,y).$$

Then there exists a unique fixed point $x^* \in Z$ with $T(x^*) = x^*$. This fixed point can be achieved as the limit for $n \to \infty$ of $T^n x$ for any $x \in Z$. Furthermore, let $M \subset X$ be a nonempty forward invariant subset of X. Then the fixed point lies in $\overline{M}$.

Proof. We can estimate

$$\begin{aligned} d(T^n x, T^{n+m} y) &\le \alpha^n d(x, T^m y) \\ &\le \alpha^n \left(d(x,y) + d(y,Ty) + \dots d(T^{m-1}y, T^m y) \right) \\ &< \alpha^n \left(d(x,y) + \frac{d(y,Ty}{1-\alpha} \right). \end{aligned}$$

Therefore, $T^n x$ is a Cauchy-sequence and converges to the same fixed point of T for all $x \in X$. If $x \in M$, then all iterates $T^n x$ lie in M. Therefore their limit lies in $\overline{M}$. □

Before we will state the general proof idea, we need to fix some notation. We use the notation $|\cdot|$ for the norm in $\mathbb{R}^n$ and the operator norm of matrices in $\mathbb{R}^n$. In the following table, we will summarize the notation for the most common norms in function spaces. Let $\phi : X \to Y$.

Name	Definition	Notation
Supremum norm	$\sup_{x \in X} \lvert\phi(x)\rvert$	$\lVert\phi\rVert_\infty, \lvert\phi\rvert_\infty$, $\lvert\phi\rvert_{C^0}$ or $\lvert\phi\rvert_{C^0}$
C^k-seminorm	$\sup_{x \in X} \lvert D^k \phi(x)\rvert$	$\lvert\phi\rvert_{C^k}$
C^k-norm	$\max(\lvert\phi\rvert_{C^0}, \dots \lvert\phi\rvert_{C^k})$	$\lVert\phi\rVert_{C^k}$
Hoelder-bracket	$\sup_{x \neq y} \lvert x-y\rvert^{-\alpha} \lvert\phi(x) - \phi(y)\rvert$	$\lfloor \phi \rfloor_\alpha$
$C^{k,\alpha}$-Hoelder-seminorm	$\lfloor D^k \phi \rfloor_\alpha$	$\lvert\phi\rvert_{k,\alpha}$

Proof Outline 1.17 (Existence and Regularity with the Contraction Mapping Principle). In order to get regularity and existence of maps $\phi : X \to Y$ which have certain properties, we will follow the following steps:

Fixed-point Formulation. We construct a (partially defined) iteration $T : Y^X \to Y^X$, such that fixed points of T have the desired property.

Well definedness. We find a nonempty closed set $Z \subseteq Y^X$ such that $T : Z \to Z$ is well defined. In order to get good uniqueness results, we will try to chose the set Z as large as possible.

Contraction estimates. In order to get existence and uniqueness of a solution ϕ^* via the contraction mapping principle 1.16, we prove an estimate of the form

$$\lVert T(\psi) - T(\psi')\rVert \le \alpha \lVert \psi - \psi' \rVert \quad \forall \psi, \psi' \in M_0$$

with some $\alpha < 1$.

Regularity estimates. In order to archieve higher regularity up to C^{N+1} for some $N \ge 0$, we firstly show that there is some forward invariant set $M_1 \subset C^{N+1}$. Then we need to prove apriori bounds for $\phi \in M_1$ of the form

$$\lVert T^n \phi \rVert_{C^{N+1}} \le const(\phi) \quad \forall n \in \mathbb{N}.$$

Such estimates follow by induction over k from simpler estimates of the following form, when $\alpha_k < 1$:

$$|T\phi|_{C^{k+1}} \leq \alpha_{k+1}|T\phi|_{C^{k+1}} + C(k, \|\phi\|_{C^k}) \quad \forall 0 \leq k \leq N.$$

By the contraction mapping principle, the fixed point ϕ^* of T therefore lies in $\mathrm{cl}_{C^0}\left(M_1 \cap B^{C^{k+1}}_{C(\phi}(0)\right)$. The embedding $\iota_{C^{N,1} \to C^0}$ is closed and we have for $R > 0$:

$$\mathrm{cl}_{C^0} B_R^{C^{k+1}}(0) = B_R^{C^{k,1}}(0).$$

Therefore, $\phi^* \in B_R^{C^{k,1}}(0)$ for some $R > 0$.

Higher order convergence. The iterates $T^n\phi$ actually converge even in the C^N-norm. This can be proven by combining the regularity estimates and the following well known interpolation estimate (cf. e.g. [GT98, Chapter 4]) for $R > 0$ and $\phi \in C^{k+1}(B_R^n(0))$:

$$|\phi|_{C^k} \leq 4\varepsilon^{-1}|\phi|_{C^{k-1}} + \varepsilon|f|_{C^{k+1}} \quad \forall 0 < \varepsilon < \min(1, R). \tag{1.1.3.1}$$

Now we will prove the Picard-Lindeloef Theorem.

Proof. We will apply the previously outlined proof idea 1.17. At first we will choose the space Z "as large as possible" in order to get good uniqueness results. Then we will chose closed invariant subspaces in order to get regularity results. Let $L > L_f$. Consider the complete metric space

$$Z = \{\phi : \mathbb{R}^2 \times X \to X, \text{ such that}\, d(id, \phi) < \infty,\, \phi \text{ measurable}\}$$

with the metric

$$d(\phi_1, \phi_2) = \sup_{t,s,x} e^{-L|t-s|}|\phi_1^{t,s}x - \phi_2^{t,s}x|.$$

We will use the iteration of the map $T : Z \to Z$ given by

$$T\phi^{t,s}(x) := x + \int_s^t f(\tau, \phi^{\tau,s}x)\mathrm{d}\tau. \tag{1.1.3.2}$$

This map is also called Picard-Iteration. It is from $Z \to Z$ because

$$e^{-L|t-s|}|T\phi^{t,s}(x) - x| \leq e^{-L|t-s|}|f|_\infty|t-s| < L^{-1}|f|_\infty.$$

The iteration is a contraction, since

$$
\begin{aligned}
e^{-L|t-s|}|T\phi^{t,s}(x)-T\tilde{\phi}^{t,s}(x)| &= e^{-L|t-s|}\left|\int_s^t f(\tau,\phi^{\tau,s}(x))-f(\tau,\tilde{\phi}^{\tau,s}(x))\mathrm{d}\tau\right| \\
&\le e^{-L|t-s|}\left|\int_s^t L_f e^{L|\tau-s|} d(\phi,\tilde{\phi})\mathrm{d}\tau\right| \\
&\le \frac{L_f}{L} d(\phi,\tilde{\phi}).
\end{aligned}
$$

Therefore there exists a unique fixed point ϕ_* of T. We will now prove the remaining assertions by finding appropriate forward invariant subsets of Z.

Uniqueness. Let $x:[t_0,t_2]\to X$ be a solution of the differential equation. Let

$$
M=\{\phi\in Z:\phi^{\tau,t_0}x(t_0)=x(t)\,\forall\tau\in[t_0,t_2]\}.
$$

The set M is a closed nonempty subset of Z and differentiation by τ shows forward invariance. Therefore $\phi_*^{\tau,t_0}x(t_0)$ coincides with $x(\tau)$.

Lipschitz continuity in x. Define

$$
\|\phi\|_{L,Lip}=\sup_{x,y,s,t} e^{-L|s-t|}|x-y|^{-1}|\phi^{t,s}(x)-\phi^{t,s}(y)|.
$$

Suppose $\|\phi\|_{L,Lip}$ is finite. Then we can estimate

$$
\begin{aligned}
&e^{-L|t-s|}|x-y|^{-1}|T\phi^{t,s}(x)-T\phi^{t,s}(y)| \\
&\qquad\le e^{-L|t-s|}\left(1+\int_s^t L_f|x-y|^{-1}|\phi^{\tau,s}(x)-\phi^{\tau,s}(y)|\mathrm{d}\tau\right) \\
&\qquad\le 1+\frac{L_f}{L}\|\phi\|_{L,Lip}.
\end{aligned}
$$

Therefore, the set $M_{\mathrm{Lip}}=\{\phi\in Z:\|\phi\|_{L,\mathrm{Lip}}\le(1-L_f/L)^{-1}\}$ is forward invariant. By an $\varepsilon/3$-argument it is also closed in Z.

Differentiability in t. Suppose that ϕ is continuous in t. Then $T\phi$ is differentiable in t with

$$
\mathrm{D}_t T\phi^{t,s}x=f(t,\phi^{t,s}x)
$$

Therefore, the set $M=\{\phi:\phi\text{ continuous in }t\}$ is forward invariant. It is closed by an $\varepsilon/3$ argument and therefore ϕ_* is continuous in t. Then ϕ_* is continuously differentiable in t and solves the differential equation.

Evolution property. Let ϕ_* be the unique fixed point of T and $x\in X$. Then the orbits $x(t)=\phi_*^{t,r}\phi_*^{r,s}x$ and $\tilde{x}(t)=\phi_*^{t,s}(x)$ both solve the differential equation and coincide at $t=s$. By uniqueness, $\phi_*^{t,r}\phi_*^{r,s}x=\phi_*^{t,s}x$.

Higher regularity in x. We will prove via induction over $0\le k\le N$ that $\phi\in C^{N,1}$. Suppose that f is C^{k+1} and $\|f\|_{C^{k+1}}<\infty$. Suppose there are some constants $C_1,\dots,C_k>0$ such that for $j=1,\dots k$ the following sets M_j are forward invariant:

$$M_j = M_{j-1} \cap \{\phi : \sup_{x,t,s} e^{-L|t-s|} |\mathrm{D}_x^j \phi^{t,s} x| \le C_j\}$$

Then we can calculate for $\mathrm{D}^{k+1}\phi$:

$$\begin{aligned} e^{-L|t-s|}|\mathrm{D}_x^{k+1} T\phi^{t,s}x| &= e^{-L|t-s|} \left| \int_s^t \mathrm{D}_x^{k+1} f(\tau, \phi^{\tau,s}x) \mathrm{d}\tau \right| \\ &\le e^{-L|t-s|} \int_s^t |\partial_x f(\tau, \phi^{\tau,s}x) \mathrm{D}_x^{k+1} \phi^{\tau,s} x| \mathrm{d}\tau \\ &\quad + e^{-L|t-s|} \int_s^t e^{L|\tau-s|} \mathrm{d}\tau \cdot C(C_1, \ldots, C_k, \|f\|_{C^{k+1}}) \\ &\le CL^{-1} + \frac{L_f}{L} \sup_{x,\tau,s} e^{-L|\tau-s|} |\mathrm{D}_x^{k+1} \phi^{\tau,s}(x)| \end{aligned}$$

with a constant $C = C(C_1, \ldots C_k, \|f\|_{C^k})$. Therefore, there is a constant C_{k+1} which makes M_{k+1} forward invariant. By induction over k, we can see that $\phi_* \in \overline{M_{N+1}}$, i.e. $\phi_* \in C^{N,1}$.

□

Remark 1.18. Besides from the suboptimal regularity, there is another caveat in this version of the Picard-Lindeloef Theorem: The global assumptions on F, especially boundedness, are in many cases not fulfilled. The main reason for stating such a global theorem is to avoid having to deal with finite times of existence. The system can be localized:

When we have a vectorfield $F : U \to \mathbb{R}^n$ defined on an open subset $U \subset \mathbb{R} \times \mathbb{R}^n$ we consider a regular continuation $\tilde{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $\tilde{F}_{|U} = F$. Then we can apply the global Picard-Lindeloef theorem. By uniqueness of solutions, the evolutions for two different continuations of F coincide as long as $(t,x) \in U$.

1.1.4 Poincaré Sections

The Poincaré section is a useful construction, which allows to compress large times into a single map and connecting the time-continuous and time-discrete theory. One of the most widely used applications is to periodic orbits, i.e. orbits $\gamma(t)$ with $\gamma(t+T) = \gamma(t)$ for some $T > 0$. We will use a combination of multiple Poincaré sections and topological methods in Chapter 3 in order to prove existence of solutions converging to a homoclinic orbit.

Definition 1.19. Let $\phi^t : \mathbb{R}_{\ge 0} \times X \to X$ be a C^k-semiflow. Let Σ be an embedded submanifold of X, i.e. let $\Sigma = U \cap \{x : \psi(x) = 0\}$ with $U \subset X$ open and $\psi \in C^k$ and 0 be a regular value of ψ. Suppose that Σ is transversal to the orbits of the semiflow, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \psi(\phi^t x) \neq 0 \quad \text{for } x \in \Sigma$$

Define the *return time* by

$$\tau_\Sigma(x) = \inf\{t > 0 : \phi^t(x) \in \Sigma\}$$

and the *return map* by

$$\Phi_\Sigma(x) = \phi^{\tau_\Sigma(x)}(x).$$

Note that the return map is only defined on a subset of the phase space.

There is a simple result for periodic orbits, which makes similar use of Poincaré sections:

Proposition 1.20. *Let ϕ^t be a C^k semiflow on $X = \mathbb{R}^n$. Let γ be a periodic orbit of ϕ^t and Σ be a transversal section with $\gamma(\mathbb{R}) \cap \Sigma = \{x_0\}$. Then for some $\varepsilon > 0$:*

$$\{x : \sup_{t>0} \mathrm{d}(\phi^t(x), \gamma) < \varepsilon\} \cap \Sigma \subset \mathrm{inv}^+_{\Phi_\Sigma}(\Sigma)\,.$$

This means that we can use the stable manifold theorem 1.29 for the equilibrium x_0 of the iteration given by Φ_Σ in order to study points which converge to the periodic orbit in the continuous system.

1.2 The Perron Method for Stable Manifolds

In the previous section, we have reviewed some basic definitions on dynamical systems and especially the Picard-Lindeloef Theorem 1.14, which gives the solution of certain differential equations as flows. In this section, we will give a proof of the classical stable manifold theorem via the Perron method. Other proofs can be found in e.g. [KH95] and in Section 1.3 via the graph transform. We study a dynamical system of the form

$$(\dot{x}, \dot{y}) = (Ax + f(x,y),\ By + g(x,y))\,, \tag{1.2.0.1}$$

where $f(0,0) = g(0,0) = 0$ and $f, g \in C^k$ for $k \geq 1$ and A and B are linear. Let ϕ^t denote the solution flow. The stable manifold theorem classifies the stable set

$$W_s = \{(x,y) : \lim_{t\to\infty} \phi^t(x,y) = 0\}. \tag{1.2.0.2}$$

One variant of this important theorem can be stated as following:

Theorem 1.21 (Stable Manifold Theorem). *Consider the system* (1.2.0.1). *Assume that*

$$\mathrm{Re}\,\sigma(A) < \alpha < 0 < \beta < \mathrm{Re}\,\sigma(B)$$

and that $|e^{At}| \leq e^{\alpha t}$ *and* $|e^{-Bt}| \leq e^{-\beta t}$ *for all* $t \geq 0$ *(when the spectral bounds hold, then the last inequalities can be achieved with a choice of norm on the vector spaces). Assume that the Lipschitz bound L fulfills* $\max(|f|_{0,1}, |g|_{0,1}) \leq L < \min(-\alpha, \beta)$. *Then there is a* $C^{k-1,1}$*-map* $\psi_s^0 : X \to Y$*, such that*

$$W_s = \text{graph } \psi_s^0.$$

The Perron method, which we will use to prove this result, is basically a variation of the method used to prove the Picard-Lindeloef Theorem. The main idea is to split off the linear part of the system, which can be solved explicitely, and then formulate the stable manifold as the solution to a boundary-value problem. The proof rests on the variations-of-constants formula:

Proposition 1.22 (Variations-of-Constants Formula). *Let* $T > 0$ *and consider the differential equation*

$$\dot{x} = Ax + f(x), \tag{1.2.0.3}$$

where A is linear, f is continuous and $x \in \mathbb{R}^n$. *Consider also the integral equation*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(x(s)) \mathrm{d}s \quad \forall t \in [0,T]. \tag{1.2.0.4}$$

Then a trajectory $x : [0,T] \to \mathbb{R}^n$ *solves the differential equation* (1.2.0.3) *if and only if it solves the integral equation* (1.2.0.4).

Proof. Let $x : [0,T] \to \mathbb{R}^n$ be a solution to the integral equation (1.2.0.4). Then differentiation shows that x is a solution to the differential equation (1.2.0.3).

In order to show the other direction, let $x : [0,T] \to \mathbb{R}^n$ be a solution to the differential equation (1.2.0.3). Consider

$$\tilde{y}(t) = e^{At}x(0) + \int_0^t s^{A(t-s) f(x(s)) \mathrm{d}s}.$$

Differentiation shows that $\dot{x}(t) - \dot{y}(t) = A(x-y)(t)$. By the Picard-Lindeloef Theorem and by $(x-y)(0) = 0$, we can conclude that $(x-y)(t) = 0$. Therefore, x solves the integral equation (1.2.0.4). □

Proof (Stable Manifold Theorem 1.21). We will closely follow the Method 1.17 and the proof of the Picard-Lindeloef Theorem 1.14. Consider the norm $\|\psi\| = \sup_{x,t}(1+|x|)^{-1}|\psi^t x|$ and the space

$$Z = \{\psi : [0,\infty) \times X \to X \times Y, \|\psi\| < \infty, \psi \text{ measurable}\}.$$

The iteration we use will use the variation-of-constants formula. Heuristically, we solve the equation forward in time in x and backward in time in y via a variant of the Lindeloef-Iteration. Define the map $T : Z \to Z$ as

$$T\psi^t x = \left(e^{At}x + \int_0^t e^{A(t-\tau)} f(\psi^\tau x)\mathrm{d}\tau \, , \, -\int_t^\infty e^{B(t-\tau)} g(\psi^\tau x)\mathrm{d}\tau \right).$$

The map is from $Z \to Z$ because

$$(1+|x|)^{-1}|T\psi^t x| \leq 1 + (1+|x|)^{-1} \max\left(\int_0^t e^{\alpha(t-\tau)}|f|_{C^0}\mathrm{d}\tau \, , \, \int_t^\infty e^{\beta(t-\tau)}|g|_{C^0}\mathrm{d}\tau \right)$$
$$< \infty$$

The map T is a contraction, since

$$(1+|x|)^{-1}|T\psi^t x - T\tilde{\psi}^t x| \leq \max\left(\int_0^t e^{\alpha(t-\tau)}|f|_{C^1}\mathrm{d}\tau \, , \, \int_t^\infty e^{\beta(t-\tau)}|g|_{C^1} \right) \|\psi - \tilde{\psi}\|$$
$$\leq \max\left(\frac{|f|_{C^1}}{|\alpha|}, \frac{|g|_{C^1}}{\beta} \right) \|\psi - \tilde{\psi}\|.$$

Thus there exists a unique fixed point ψ_s of T. We will now show various properties of the fixed point :

Differentiability in t. Suppose that ψ is continuous in t. Then straightforward calculation shows thet $T\psi$ is differentiable in t and

$$\mathrm{D}_t T\psi^t(x) = \left(A\pi_X T\psi^t(x) + f(\psi^t(x)) \, , \, B\pi_Y T\psi^t(x) + g(\psi^t x) \right)$$

Therefore the set of all ψ, which are continuous in t is forward invariant and the fixed point ψ_s is continuous in t. Hence, ψ_s is differentiable in t and $\psi_s^t(x)$ solves the differential equation.

Uniqueness. Let $(x,y) : [0,\infty) \to X \times Y$ be a bounded solution of the differential equation. We will show that the set

$$M = \{\psi \in Z : \psi^t(x(0)) = (x(t), y(t)) \quad \forall t \geq 0\}$$

is closed and forward invariant under T. In order to show this, we expand $(x,y)(t)$ with the variations-of-constant formula 1.22. This yields for any $T_0 > t$:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(x(s), y(s))$$
$$y(t) = e^{-B(T_0-t)}y(T_0) - \int_t^{T_0} e^{B(s-t)g(x(s),y(s))\mathrm{d}s}. \tag{1.2.0.5}$$

Now let $\psi \in M$. Taking the limit $T_0 \to \infty$, which exists since we assumed that (x,y) is bounded, we can see that $T\psi \in M$. Therefore, M is forward invariant. Since it is obviously closed, the contraction mapping principle yields $\psi_s^t(x(0)) = (x(s), y(s))$ for all $s \geq 0$.

Semiflow property. Let $x \in X$ and $t_0 \geq 0$. Then $t \to \psi_s^{t+t_0}x$ is a solution to the differential equation which stays bounded. By uniqueness, we therefore get $\psi_s^t \pi_X \psi_s^{t_0} x = \psi_s^{t+t_0} x$.

Growth Conditions. Obviously, $W_s \subset W_B$. By the uniqueness proof we also have $W_B \subset \psi_s^0(X)$. In order to prove the last inclusion $\psi_s^0(X) \subset W_s$, consider the set

$$M = \{\psi \in Z : \lim_{t\to\infty} \psi^t(x) = 0 \,\forall x \in X\}.$$

It is easy to see that M is forward invariant under T, since $f(0,0) = 0$ and $g(0,0) = 0$. Since M is closed in the $\|\cdot\|$-norm, the assertion is proved.

Regularity in x. We can mimic the regularity proof of the Picard-Lindeloef theorem.

□

Remark 1.23. The same Remarks with respect to locality (Remark 1.18) and regularity (Remark 1.15) apply as for the Picard-Lindeloef Theorem.

1.3 The Graph Transform Approach

There is another widely used approach to the stable manifold theorem, which is called the Hadamard graph transform. This mehtod is especially well suited for time-discrete and non-autonomous systems. It can also be found in e.g. [KH95, Capter 6.2]. The approach is motivated by the following observation: Consider a map $F : X \times Y \to X' \times Y'$, which is contracting in X-direction and expanding in Y-direction. When we have the graph of a map $\psi' : X' \to Y'$, its preimage will be contracted in the Y direction, while it will be expanded in the X-direction. The preimage will under certain assumptions again be the graph of a map, which is then called the preimage graph transform of ψ', and it will be "smoothened out". By iteratedly taking the preimage and using a contraction mapping principle we can then get a limit graph, which is invariant under the preimage graph transform. This limit graph is the stable manifold.

1.3.1 The Preimage Graph Transform

In this section, we will define the preimage graph transform and proceed to show well-definedness and estimates under some linear conditions.

Definition 1.24. Let X, Y, X' and Y' be toplological spaces and let $F : X \times Y \to X' \times Y'$ be a continuous map. Let $\psi' : X' \to Y'$ be a continuous map, which is defined everywhere. If the preimage set $F^{-1}[\operatorname{graph}\psi']$ is the graph of some map $\psi : X \to Y$, then we call $\psi = F^*\psi'$ the preimage graph transform of ψ' under F, i.e.

$$F[\operatorname{graph} F^*\psi'] = \operatorname{graph}\psi'.$$

Proposition 1.25. *Suppose that X, Y, X' and Y' are Banach spaces and $F : X \times Y \to X' \times Y'$ with*

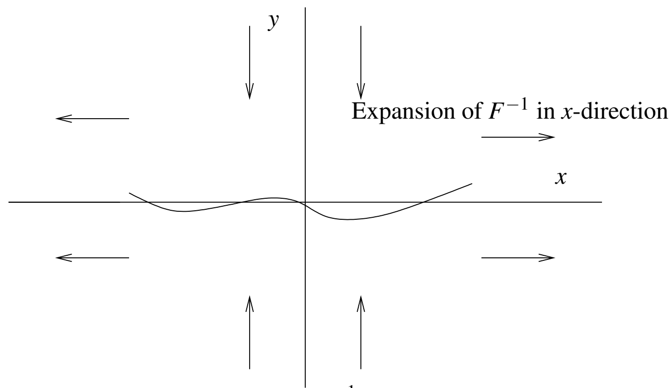

Fig. 1.1 The preimage contraction of the Graph transform

$$F(x,y) = (Ax + f(x,y), By + g(x,y)),$$

where A and B are bounded and linear, B has bounded inverse and f and g are C^1-maps with $f(0,0) = 0$ and $g(0,0) = 0$. Let $\pi_{X'}$ and $\pi_{Y'}$ denote the canonical projections. Then the graph transform is equivalently defined as

$$\pi_{Y'} F(x, \psi(x)) = \psi'(\pi_{X'} F(x, \psi(x)))$$

or as

$$\psi(x) = B^{-1} \left[\psi'(Ax + f(x, \psi(x))) - g(x, \psi(x)) \right] \tag{1.3.1.1}$$

The proposition follows from direct application of the definition and direct calculation. We can use the right-hand-side of this equation to get an iteration, which yields the preimage as a graph under suitable conditions on F and ψ'. We will consider maps which are only locally defined, i.e. f and g are defined on $B^X_{R_x}(0) \times B^Y_{R_y}(0)$.

Theorem 1.26. *Let X, Y, X' and Y' be Banach spaces and $F(x,y) = (Ax + f(x,y), By + g(x,y))$, where A and B are bounded and linear, B has bounded inverse and f and g are C^1-maps with $f(0,0) = 0$ and $g(0,0) = 0$.*
Suppose $\psi' : B^{X'}_{R'_x}(0) \to Y'$, $\psi' \in C^1$ and the following bounds hold:

$$\begin{aligned} &R_x(\|A\| + \|f\|_{C^1}) \leq R'_x \\ &\|B^{-1}\|(\|\psi'\|_\infty| + \|g\|_{C^1}) \leq R_y \\ &|B^{-1}| \left(\|\psi'\|_{C^1} \|f\|_{C^1} + \|g\|_{C^1} \right) < 1. \end{aligned} \tag{1.3.1.2}$$

Then the preimage $F^{-1}[\text{graph}\, \psi']$ is the graph of a C^1-function $\psi = F^* \psi' : B^X_{R_x}(0) \to Y$ and

$$\mathrm{D}_x\psi(x) = -B^{-1}\left[1+\partial_y g B^{-1} - \mathrm{D}_{x'}\psi'\partial_y f B^{-1}\right]^{-1}\left(\partial_x g - \mathrm{D}_{x'}\psi'(A+\partial_x f)\right). \tag{1.3.1.3}$$

The expression for $\mathrm{D}_x\psi(x)$ *is well defined because of* (1.3.1.2) *and the convergence of the geometric operator series* $(1-V)^{-1} = \sum_{n\geq 0} V^n$ *with* $V = -\partial_y g B^{-1} + \mathrm{D}_{x'}\psi'\partial_y f B^{-1}$.

Proof. Consider the iteration

$$T[\psi](x) = B^{-1}\left[\psi'(Ax+f(x,\psi(x))) - g(x,\psi(x))\right]$$

By the first two inequalities, the iteration is well defined. Furthermore

$$(T[\psi_1]-T[\psi_2])(x) \leq |B^{-1}|\left(|\psi'|_{C^1}|f|_{C^1}+|g|_{C^1}\right)|\psi_1-\psi_2|_\infty.$$

Therefore the contraction mapping principle 1.16 yields a unique L^∞-function ψ, whose graph is the complete preimage of the graph of ψ'. The remaining assertions follow from the implicit function theorem. □

Now that we have established the well-definedness of the graph transform, we can consider the contraction estimates needed for contraction mapping principles. Application of the formula (1.3.1.3) for $\mathrm{D}_x\psi$ yields

$$\begin{aligned} |F^*\psi'|_{C^1} \leq & \frac{|B^{-1}|(|A|+|f|_{C^1})}{1-|B^{-1}|(|g|_{C^1}+|f|_{C^1}|\psi'|_{C^1})}|\psi'|_{C^1} \\ & + \frac{|g|_{C^1}}{1-|B^{-1}|(|g|_{C^1}+|f|_{C^1}|\psi'|_{C^1})}. \end{aligned} \tag{1.3.1.4}$$

We can now consider contraction properties of the graph transform. We get

$$\begin{aligned} |\psi_1-\psi_2|(x) \leq & |B^{-1}||[\psi_1'(Ax+f(x,\psi_1(x))) - \psi_1'(Ax+f(x,\psi_2(x))) \\ & + \psi_1'(Ax+f(x,\psi_2(x))) - \psi_2'(Ax+f(x,\psi_2(x))) \\ & - g(x,\psi_1(x)) + g(x,\psi_2(x))] \\ \leq & |B^{-1}|\left[|\psi_1'|_{C^1}|f|_{C^1}|\psi_1-\psi_2|_\infty + |\psi_1'-\psi_2'|_\infty + |g|_{C^1}|\psi_1-\psi_2|_\infty\right] \end{aligned}$$

Solving for $|\psi_1-\psi_2|_\infty$ yields

$$|\psi_1-\psi_2|_\infty \leq \frac{|B^{-1}|}{1-|B^{-1}|(|\psi_1'|_{C^1}+|g|_{C^1})}|\psi_1'-\psi_2'|_\infty. \tag{1.3.1.5}$$

1.3.2 General construction of invariant manifolds

In order to construct stable manifolds, we will again roughly follow the Idea 1.17. We will however need a nonautonomous version of the Contraction Mapping Principle 1.16 in order to construct the stable manifolds.

Theorem 1.27 (Nonautonomous Contracting Mapping Principle). *Consider a dynamical system $F_\bullet$ with $I = \{\dots,-2,-1,0\}$and*

$$F_{k-1} : Z_{k-1} \to Z_k \quad \forall k \in I,$$

where the Z_k are metric spaces. Suppose that

$$d_{k+1}(F_k(x_k), F_k(\tilde{x}_k)) \leq \lambda_k d_k(x_k, \tilde{x}_k) \quad \forall k < 0, \tag{1.3.2.1}$$

where all $\lambda_k > 0$. Let $\overline{Z_k}$ denote the metric completion of the spaces Z_k and let $\overline{F_k} : \overline{Z_k} \to \overline{Z_{k+1}}$ denote the unique continuous extension of F_k. The continuous extension exists and is unique by virtue of (1.3.2.1) *(c.f. e.g. [RS80, Theorem 1.7]).*

Suppose further that the metric spaces Z_k all have a uniformly bounded diameter, i.e. $d_k(x_k, y_k) \leq C$ for all $k \leq 0$ and $x_k, y_k \in Z_k$, and that

$$\lim_{n\to\infty} \prod_{k=-n}^{-1} \lambda_k = 0. \tag{1.3.2.2}$$

Then there exists a unique orbit $x_\bullet = (x_k)_{-k\in I} \in \overline{Z_\bullet}$ of $\overline{F_\bullet}$. Furthermore, for any forward invariant sequence of sets $M_\bullet \subset \overline{Z_\bullet}$ in the sense of Definition 1.7, we have $x_\bullet \in \overline{M_\bullet}$.

Proof. Let $\tilde{Z} = \prod_{k\leq 0} \overline{Z}_k$ be the trajectory space. Define $T : \tilde{Z} \to \tilde{Z}$ as

$$(Tx_\bullet)_k = \overline{F_{k-1}}(x_{k-1}).$$

Let $x_\bullet, \tilde{x}_\bullet \in \tilde{Z}$. Then we have

$$\begin{aligned} d_k\left((T^n x_\bullet)_k, (T^n \tilde{x}_\bullet)_k\right) &= d_k\left(\overline{F_{k-1}} \circ \dots \circ \overline{F_{x-n}} x_{k-n}, \overline{F_{k-1}} \circ \dots \circ \overline{F_{x-n}} \tilde{x}_{k-n}\right) \\ &\leq \lambda_{k-1}\lambda_{k-2}\dots\lambda_{k-n} C. \end{aligned}$$

Since $\prod_{k<0} \lambda_k = 0$, $(T^n x)_k$ is a Cauchy-sequence in n and converges pointwise to a unique fixed point x_k^* of T. This fixed point is an orbit and it is unique. By considering an initial $x_\bullet \in M_\bullet$ it follows directly that $x_k^* \in \overline{M}_k$. □

When we modify the proof idea 1.17 with the preimage graph transform and the Nonautonomous Contraction Mapping Principle 1.27, we get the following proof idea for stable manifold theorems:

Proof Outline 1.28 (Stable Manifolds via Preimage Graph Transform). Consider a dynamical system

$$F_k = (A_k + f_k, B_k + g_k) : X_k \times Y_k \to X_{k+1} \times Y_{k+1} \quad \forall k \in \mathbb{N}$$

where the f_k and g_k are defined on boxes $B^{X_k}_{R_{k,x}}(x) \times B^{Y_k}_{R_{k,y}}(0)$.

Well definedness and C^1-bounds. At first we need to construct a sequence of spaces $Z_k \subseteq B^{C^1(X_k,Y_k)}_{R_k}(0)$ such that the preimage graph-transform fulfills $F_k^* : Z_{k+1} \to Z_k$ and is well defined on Z_k for all $k \in \mathbb{N}$. This construction will make use of Theorem 1.26 and estimate (1.3.1.4) and will require some fine-tuning of constants.

Contraction esitmates. Using the estimate (1.3.1.5), we will get contraction estimates on $F_k^* : Z_{k+1} \to Z_k$ in the $|\cdot|_{C^0}$-norm. Furthermore, in order to apply the Nonautonomous Contraction Mapping Principle, we will need to check the condition (1.3.2.2).

Application of the Nonautonomous Contraction Mapping Principle. We will then apply the Nonautonomous Contraction Mapping Prinicple 1.27. The contraction estimates will allow us to extend the $\overline{F_k}$ to the closure $\mathrm{cl}_{C^0} Z_k \subset B^{C^{0,1}(X_k,Y_k)}_{R_k}(0)$, which will consist of Lipschitz-continuous functions. The Theorem 1.27 will yield a unique Lipschitz-Graph, which is invariant under the preimage graph transform and therefore forward invariant as a set.

Higher regularity bounds. Differentiation of the Formula (1.3.1.3) leads to apriori estimates by the same idea, which was employed in 1.17.

Other Properties. In order to prove other properties of the stable manifold, we can construct forward invariant sequences. We will use non-constant forward invariant sequences even in autonomous systems, where $F_k = F_0$ for all $k \in \mathbb{N}$.

1.3.3 Exploiting the approach: The stable, strong stable and pseudo stable manifold theorems

We will now apply the method 1.28, which has been outlined in the previous section, in order to archieve three stable manifold theorems. The same results can be found in [KH95]. The spectral setting for the following three applications of this method is illustrated in 1.2. A pseudo-stable manifold is also called center-stable manifold when the closure of the spectral gap contains the unit circle. The intersection of a pseudo-stable and a pseudo-unstable manifold is often called a slow manifold and the intersection of a center-stable and a center-unstable manifold is often called center manifold.

1.3.3.1 The Stable Manifold Theorem and a weak λ-Lemma

We can use the method 1.28 to immediately prove a theorem about stable manifolds. The basic setting is similar to the continuous-time case: We have a contracting and

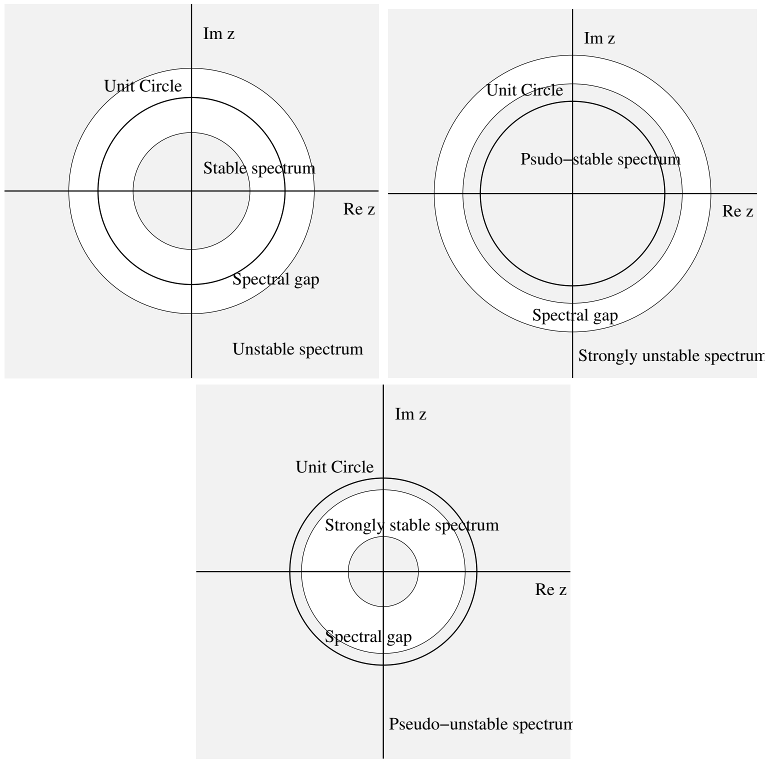

Fig. 1.2 The spectral setting for the stable, the strong stable and the pseudostable manifold theorem

an expanding direction of a dynamical system and look for solutions, which converge to the equilibrium.

Theorem 1.29. *Let $F : X \times Y \to X \times Y$ be continuous with*

$$F(x,y) = (Ax + f(f,y), By + g(x,y))$$

where A and B are linear and $|A| < 1$ as well as $|B^{-1}| < 1$. Suppose further $f,g \in C^k$ with $f(0,0) = 0$ and $g(0,0) = 0$ and that $|f|_{C^1}, |g|_{C^1} < \delta$ with

$$\delta \leq \frac{1}{4}(1 - \max(|A|, |B^{-1}|))$$

Then there exists a Lipschitz function $\psi_s : B_1^X(0) \to B_1^Y(0)$ which is at least $C^{k-1,1}$ and invariant under the graph transform. We call graph ψ_s *the stable manifold. Set*

$$Z = \{\psi : B_1^X(0) \to B_1^Y(0) : |\psi|_{C^1} < 1, \psi \in C^k\}$$

Then every function in Z converges to ψ_s in the C^{k-1}-Norm.

Proof. We proof the theorem in the previously outlined manner. At first we show well-definedness of the preimage graph transform. By the estimate (1.3.1.4) on the graph transform, we have for $\psi \in Z$:

$$\|F^*\psi\|_{C^1} \leq \frac{|B^{-1}|(|A|+\delta)+\delta}{1-2|B^{-1}|\delta} \leq \frac{1-3\delta}{1-2\delta}$$

Higher order apriori bounds can be proved similar as before by induction over n and using (1.3.1.3):

$$\begin{aligned}\|F^*\psi\|_{C^{n+1}} &\leq \frac{|B^{-1}|(|A|+\delta)}{1-2\delta}\left((|A|+\delta)^n + \delta\frac{(|A|+\delta)^{n-1}}{1-2\delta}\right) \\ &\quad + C(\|f\|_{C^{n+1}}, \|g\|_{C^{n+1}}, \|\psi\|_{C^n}) \\ &\leq \frac{|B^{-1}|(|A|+\delta)}{(1-2\delta)}\frac{(|A|+2\delta)}{1-2\delta} + C \\ &\leq (1-4\delta)\|\psi\|_{C^{n+1}} + C(\|f\|_{C^{n+1}}, \|g\|_{C^{n+1}}, \|\psi\|_{C^n})\end{aligned}$$

Therefore, the space Z is forward invariant under the preimage graph transform and uniform bounds on the C^k-Norm hold.

The contraction estimate (1.3.1.5) for the preimage graph transform yields:

$$|F^*\psi_1 - F^*\psi_2|_\infty \leq \frac{|B^{-1}|}{1-2\delta|B^{-1}|}|\psi_1-\psi_2|_\infty \leq \frac{1-4\delta}{1-2\delta}|\psi_1-\psi_2|_\infty$$

The results on ψ_s follows since

$$\lim_{n\to\infty} \|\psi_s - (F^*)^n\psi\|_\infty = 0$$

and $\|(F^*)^n\psi\|_{C^k} < C(\|f\|_{C^k}, \|g\|_{C^k}, \|\psi\|_{C^k})$ for all iterates n and therefore

$$\lim_{n\to\infty} \|\psi_s - (F^*)^n\psi\|_{C^{k-1}} = 0.$$

□

Remark 1.30. With different methods, it can be shown that the stable manifold ψ_s is actually C^k and not just $C^{k-1,1}$. Furthermore the same estimates work if k is not an integer and the norms are interpreted as Hölder norms.

There is another characterization of the stable manifold, which justifies its name: It contains all points which converge to the equilibrium.

Proposition 1.31. *Set*

$$\begin{aligned} W &= \mathrm{inv}_F^+\left(B_1^X(0)\times B_1^Y(0)\right) \\ W_C &= \mathrm{inv}_F^+\left(B_1^X(0)\times B_1^Y(0)\cap\{(x,y) : |y|\leq|x|\}\right) \\ W_s &= \left\{(x,y) : \lim_{n\to\infty} F^n(x,y) = 0\right\}. \end{aligned}$$

Then

$$W = W_C = W_s = \operatorname{graph} \psi_s.$$

Proof. Obviously, graph $\psi_s \subseteq W_C \subseteq W$. Suppose, conversely, that we have an orbit $(x_n, y_n)_{n\in\mathbb{N}}$ in W. Set $Z_n = Z \cap \{\psi : \psi(x_n) = y_n\}$. The sequence Z_n is forward invariant under the preimage graph transform. Therefore ψ_s lies in $\overline{Z_0}$ and $\psi_s(x_0) = y_0$. This yields graph $\psi_s = W_C = W$.

We will now show that $W_C \subseteq W_s$ and then use the obvious inclusion $W_s \subseteq W$ to prove the assertion. Let $(x_\bullet, y_\bullet)$ be an orbit in W_C. We can then estimate $|x_{n+1}| \leq |A||x_n| - \delta(|x_n| + |y_n|) \leq (|A| - 2\delta)|x_n|$. Therefore, $W_C \subseteq W_s$. □

Remark 1.32. The statement that for every map ψ in Z the itererates $F^{n,*}$ converge to the stable manifold ψ_s for $n \to \infty$ is the crucial part of the λ-Lemma or Inclination Lemma, which can be found in e.g. [KH95, Proposition 6.2.23]. Especially the convergence in the $\|\cdot\|_{C^{k-1}}$-norm is remarkable. This statement means geometrically, that the iterates under F^{-1} of any small sheet, which is“sufficiently parallel” to the x-axis, converge to the stable manifold in a smooth way. Since the statement of the full λ-Lemma requires the unstable manifold we will omit it.

1.3.3.2 The Strong Stable Manifold Theorem

In the case when we only have a spectral gap which lies on the stable side and does not contain the unit circle, the stable manifold is again a purely local construction. However, we need slightly different methods in this case. Consider again the setting for the preimage graph transform. Assume that now $|A| < 1$ and $C > |B^{-1}| \geq 1$, i.e. A is contracting while we have bounds on the contraction of B; however, B is not expanding. We can fix $\psi'(0) = 0$ and $\psi(0) = 0$. Then a special choice of norm allows us to make use of the expansion of the preimage in x-direction which ”dilutes” the graph. Define the norm $\|\psi\|_{1,\infty} = \sup_x \frac{|\psi(x)|}{|x|}$. Let ψ_1', $\psi_2' \in \operatorname{dom} F^*$ and $\psi_1 = F^*\psi_1'$, $\psi_2 = F^*\psi_2'$. We can estimate:

$$\begin{aligned}
|x|^{-1}|\psi_1 - \psi_2|(x) \leq& |B^{-1}||x|^{-1}[\psi_1'(Ax + f(x,\psi_1(x))) - \psi_1'(Ax + f(x,\psi_2(x)))\\
&+ \psi_1'(Ax + f(x,\psi_2(x))) - \psi_2'(Ax + f(x,\psi_2(x)))\\
&- g(x,\psi_1(x)) + g(x,\psi_2(x))]\\
\leq& |B^{-1}|[|\psi_1'|_{C^1}|f|_{C^1}\|\psi_1 - \psi_2\|_{1,\infty}\\
&+ \|\psi_1' - \psi_2'\|_{1,\infty}(|A| + |f|_{C^1}) + |g|_{C^1}\|\psi_1 - \psi_2\|_{1,\infty}]
\end{aligned}$$

By solving for $\|\psi_1 - \psi_2\|_{1,\infty}$ we get the estimate

$$\|\psi_1 - \psi_2\|_{1,\infty} \leq \frac{|B^{-1}|(|A| + \|f\|_{C^1})}{1 - |B^{-1}|(\|f\|_{C^1}|\psi_1'|_{C^1} + |g|_{C^1})}\|\psi_1' - \psi_2'\|_{1,\infty}$$

This estimate allows us to repeat basically the same arguments as before as long as we have $|A||B^{-1}| < 1$.

Theorem 1.33 (Strong Stable Manifold Theorem). *Let $F : X \times Y \to X \times Y$ be continuous with*

$$F(x,y) = (Ax + f(x,y), By + g(x,y))$$

where A and B are linear and $|A| < 1$ as well as $|A||B^{-1}| < 1 \leq |B^{-1}|$. Suppose further $f,g \in C^k$ with $f(0,0) = 0$ and $g(0,0) = 0$ and that $|f|_{C^1}, |g_{C^1}| < \delta$ with

$$\delta \leq \frac{1}{4|B^{-1}|}(1 - |A||B^{-1}|)$$

Then there exists a Lipschitz function $\psi_{ss} : B_1^X(0) \to B_1^Y(0)$ which is at least $C^{k-1,1}$ and invariant under the graph transform. We call graph ψ_{ss} *the strong stable manifold.*

Proof. Let the space

$$Z = \{\psi : B_1^X(0) \to B_1^Y(0) : |\psi|_{C^1} < 1, \psi \in C^k, \psi(0) = 0\}$$

By the estimate (1.3.1.4) on the graph transform, we have for $\psi \in Z$:

$$\|F^*\psi\|_{C^1} \leq \frac{|B^{-1}|(|A| + \delta) + \delta}{1 - 2|B^{-1}|\delta} \leq \frac{1 - 3|B^{-1}|\delta}{1 - 2|B^{-1}|\delta} < 1$$

Higher order estimates can be proved similar as for the stable manifold in Theorem 1.29 by induction. Therefore, the space Z is forward invariant under the preimage graph transform and uniform bounds on the C^k-Norm hold.
The contraction estimate (1.3.1.5) for the preimage graph transform yields:

$$\begin{aligned}\|F^*\psi_1 - F^*\psi_2\|_{1,\infty} &\leq \frac{|B^{-1}|(|A| + \delta)}{1 - 2\delta|B^{-1}|}\|\psi_1 - \psi_2\|_\infty \\ &\leq \frac{1 - 3|B^{-1}|\delta}{1 - 2|B^{-1}|\delta}\|\psi_1 - \psi_2\|_\infty\end{aligned}$$

Therefore the nonautonomous contraction mapping principle yields the assertion. □

The strong stable manifold has also a characterization via growth conditions:

Proposition 1.34. *Let $L > 0$, such that $|A| + \delta < L < |B^{-1}|^{-1} - \delta$. Define*

$$W_C = \mathrm{inv}_F^+ \left(B_1^X(0) \times B_1^Y(0) \cap \{(x,y) : |x| \leq |y|\}\right)$$

$$W_L = \left\{(x,y) : \sup_{n \in \mathbb{N}} L^{-n}|F_x^n(x,y)| \leq 1\right\}.$$

Then

$$\operatorname{graph}\psi_{ss} = W_C = W_L.$$

Proof. Obviously, $\operatorname{graph}\psi_{ss} \subseteq W_C$. Suppose, conversely, that we have an orbit $(x_n, y_n)_{n\in\mathbb{N}}$ in W_C. Set $Z_n = Z \cap \{\psi : \psi(x_n) = y_n\}$. The sequence Z_n is forward invariant under the preimage graph transform (we have to consider W_C instead of $\operatorname{inv}_F^+\left(B_1^X(0) \times B_1^Y(0)\right)$ in order to ensure that Z_n is nonempty). Therefore, $\operatorname{graph}\psi_s s = W_C$. In order to see that $W_C \subseteq W_L$, note the estimate $|Ax + f(x,y)| \le (|A| + \delta)|x|$ for (x,y) with $|y| \le |x| \le 1$.

In order to see the reverse inclusion, note that we have $|By + g(x,y)| \ge (|B^{-1}|^{-1} - \delta)|y|$ and $|Ax + f(x,y)| \le (|A| + \delta)|x|$ for (x,y) with $|x| < |y| \le 1$. □

Remark 1.35. The trick used in order to prove the strong stable manifold theorem can be extended to a somewhat more general situation. Using a similar choice of norm $\|\psi\|_{1,C^1} = \sup|x|^{-1}|D\psi|$ we can get uniform bounds on the iterates of the graph transform if $|A|^2|B^{-1}| < 1$. Then we can use the $|\cdot|_{2,\infty}$-Norm defined by $\|\psi\|_{2,\infty} = \sup|x|^{-2}|\psi|$ to get a contraction and therefore a unique invariant manifold, which is again C^k regular.
This method has been carried out in [dlL03]. The result is especially remarkable, since the thus constructed invariant manifold is not tangent to the spectral subspace belonging to a spectral gap; it is even not necessarily tangent to any spectral subspace.

1.3.3.3 The Pseudo-stable Manifold Theorem

The stable and strong stable manifold is local, i.e. it depends only on the behaviour of F in a small neighborhood of 0. Furthermore, it has very good regularity properties: It is almost as regular as the system itself.

If we only have a spectral gap on the unstable side of the unit circle, it is still possible to construct an invariant manifold which contains all points, which do not diverge to quickly – i.e. an invariant manifold tangential to the most stable part of the iteration F. This construction has however more subtle regularity properties, as we shall see.

Theorem 1.36 (Pseudo-stable Manifold Theorem). *Let $F : X \times Y \to X \times Y$ be C^1 with*

$$F(x,y) = (Ax + f(f,y), By + g(x,y))$$

where A and B are linear with

$$|B^{-1}| < 1 \le |A|.$$

Suppose further $f(0,0) = 0$ and $g(0,0) = 0$ and that $|f|_{C^1}, |g|_{C^1} < \delta$ with

$$\delta = \frac{1}{5}(1 - |A||B^{-1}|).$$

Then there exists a Lipschitz function $\psi_{ps} : X \to Y$ *which is invariant under the graph transform. Set*

$$Z = \{\psi : X \to Y : |\psi|_{C^1} < 1, |\psi|_\infty < \frac{\delta}{1 - |B^{-1}|}, \psi \in C^1\}$$

Then every function in Z converges to ψ_{ps} *in the* $\|\cdot\|_\infty$*-Norm.*

Proof. We proof the theorem analogously to the stable manifold theorem. By the estimate (1.3.1.4) on the graph transform, we have for $\psi \in Z$:

$$|F^*\psi|_{C^1} \le \frac{|B^{-1}|((|A| + \delta) + \delta}{1 - 2|B^{-1}|\delta} \le \frac{|B^{-1}||A| + 2\delta}{1 - 2\delta} \le \frac{1 - 3\delta}{1 - 2\delta}$$

as well as

$$|F^*\psi|_\infty \le |B^{-1}|(|\psi|_\infty + \delta).$$

Therefore, the space Z is forward invariant under the preimage graph transform.

The contraction estimate (1.3.1.5) for the preimage graph transform yields:

$$\begin{aligned}\|F^*\psi_1 - F^*\psi_2\|_\infty &\le \frac{|B^{-1}|}{1 - 2\delta|B^{-1}|}\|\psi_1 - \psi_2\|_\infty \\ &\le \frac{1 - 5\delta}{1 - 2\delta}\|\psi_1 - \psi_2\|_\infty\end{aligned}$$

The non-autonomous contraction mapping principle now yields the assertion. □

Regularity for pseudostable manifolds is slightly more subtle and depends on the size of the spectral gap. We will use Hölder-spaces for the regularity results. We will use the following well known interpolation estimate (cf. e.g. [GT98, Chapter 4]) for $R > 0$ and $f \in C^{k,\alpha}(B^n_R(0))$:

$$|f|_{C^k} \le C(\alpha)\varepsilon^{-1}|f|_{C^{k-1}} + \varepsilon^\alpha \lfloor \mathrm{D}^k f \rfloor_\alpha \quad \forall 0 < \varepsilon < \min(1, R) \tag{1.3.3.1}$$

We will also use the fact that the $|\cdot|_\infty$-closure of the $C^{k,\alpha}$-ball $\mathrm{cl}_{C^0} B_1^{C^{k,\alpha}}(0) = B_1^{C^{k,\alpha}}(0)$.

In order to prove apriori Hölder estimates we will use the following general estimates of the chain and product rule, which hold on convex domains whenever the expressions are well-defined:

$$\begin{aligned}\lfloor f \circ g \rfloor_\alpha &\le \lfloor f \rfloor_\alpha |g|^\alpha_{C^1} \\ \lfloor f \circ g \rfloor_\alpha &\le |f|_{C^1} \lfloor g \rfloor_\alpha \\ \lfloor f \cdot g \rfloor_\alpha &\le |f|_\infty \lfloor g \rfloor_\alpha + \lfloor f \rfloor_\alpha |g|_\infty\end{aligned}$$

Theorem 1.37 (Regularity of the Pseudo-stable Manifold). *Suppose that the setting for the pseudostable manifold theorem holds. Suppose furthermore that there is* $0 < \alpha \leq 1$ *such that*

$$(|A| + \delta)^{k+\alpha}|B^{-1}| < 1 - 8\delta$$

and that f *and* g *are* C^{k+1}. *Then the pseudostable manifold is actually Hölder continuous in* $C^{k,\alpha}$.

Proof. We need to check the boundedness of the preimage sequence in the $C^{k,\alpha}$-Norm. Let $\psi \in Z \cap C^{k+1}$. We can estimate for $n+1 \leq k$:

$$\begin{aligned}
\|F^*\psi\|_{C^{n+1}} \leq & \frac{|B^{-1}|(|A|+\delta)}{1-4\delta}\left((|A|+\delta)^n + \delta|B^{-1}|\frac{(|A|+\delta)^{n-1}}{1-4\delta}\right)\|\psi\|_{C^{n+1}} \\
& + C(\|f\|_{C^{n+1}}, \|g\|_{C^{n+1}}, \|\psi\|_{C^{n+1}}) \\
\leq & \frac{1-8\delta}{(1-4\delta)}\left(1+\frac{\delta}{1-4\delta}\right)\|\psi\|_{C^n} + C \\
\leq & \frac{1-8\delta}{(1-4\delta)}\frac{1-3\delta}{1-4\delta}\|\psi\|_{C^n} \\
\leq & \left(1-4\delta\frac{1-2\delta}{(1-4\delta)^2}\right)\|\psi\|_{C^{n+1}} + C(\|f\|_{C^{n+1}}, \|g\|_{C^{n+1}}, \|\psi\|_{C^n})
\end{aligned}$$

Therefore we have uniform bounds on the $\|\cdot\|_{C^k}$-Norm. Using the Hölder versions of the chain and product rule we can estimate

$$\begin{aligned}
\|F^*\psi\|_{C^{k,\alpha}} \leq & \frac{|B^{-1}|}{1-4\delta}\left((|A|+\delta)^{k+\alpha} + \delta|B^{-1}|\frac{(|A|+\delta)^{k+\alpha-1}}{1-4\delta}\right)\|\psi\|_{C^{k,\alpha}} \\
& + C(\|f\|_{C^k}, \|g\|_{C^k}, \|\psi\|_{C^k}) \\
\leq & \frac{1-8\delta}{(1-4\delta)}\left(1+\frac{\delta}{1-4\delta}\right)\|\psi\|_{C^n} + C \\
\leq & \left(1-4\delta\frac{1-2\delta}{(1-4\delta)^2}\right)\|\psi\|_{C^{n+1}} + C(\|f\|_{C^{n+1}}, \|g\|_{C^{n+1}}, \|\psi\|_{C^n})
\end{aligned}$$

The non-autonomous contraction mapping principle and the properties of Hölder spaces now yield the assertion. □

The pseudostable manifold also has a characterization via growth conditions.

Proposition 1.38. *Let* $L > 0$, *such that* $|A| + \delta < L < |B^{-1}|^{-1} - \delta$. *Define*

$$\begin{aligned}
W_C &= \mathrm{inv}_F^+\left(B_1^X(0) \times B_1^Y(0) \cap \{(x,y) : |x| \leq |y|\}\right) \\
W_L &= \left\{(x,y) : \sup_{n\in\mathbb{N}} L^{-n}|F_x^n(x,y)| \leq 1\right\}.
\end{aligned}$$

Then

$$\operatorname{graph}\psi_{ps} = W_C = W_L.$$

Proof. Obviously, $\operatorname{graph}\psi_{ps} \subseteq W_C$. Suppose, conversely, that we have an orbit $(x_n, y_n)_{n\in\mathbb{N}}$ in W_C. Set $Z_n = Z \cap \{\psi : \psi(x_n) = y_n\}$. The sequence Z_n is forward invariant under the preimage graph transform (we have to consider W_C instead of $\operatorname{inv}_F^+\left(B_1^X(0) \times B_1^Y(0)\right)$ in order to ensure that Z_n is nonempty). Therefore, $\operatorname{graph}\psi_s s = W_C$. In order to see that $W_C \subseteq W_L$, note the estimate $|Ax + f(x,y)| \leq (|A| + \delta)|x|$ for (x,y) with $|y| \leq |x| \leq 1$.

In order to see the reverse inclusion, note that we have $|By + g(x,y)| \geq (|B^{-1}|^{-1} - \delta)|y|$ and $|Ax + f(x,y)| \leq (|A| + \delta)|x|$ for (x,y) with $|x| < |y| \leq 1$. □

1.3.4 A note on Unstable Manifolds

Up to now we have only considered stable manifolds. There is a completely analogous statement about unstable manifolds with analogous techniques. Part of the result is a simple corollary of the stable manifold theorem:

Theorem 1.39 (Unstable Manifold Theorem). *Let $F : X \times Y \to X \times Y$ be continuous with*

$$F(x,y) = (Ax + f(f,y), By + g(x,y))$$

where A and B are linear and $|B^{-1}| < 1$ as well as $1 < |A^{-1}| < \infty$. Suppose further that $f, g \in C^k$ with $f(0,0) = 0$ and $g(0,0) = 0$ and $\mathrm{D}f(0,0) = 0$ and $\mathrm{D}g(0,0) = 0$.Then there exists an $\varepsilon > 0$ and a Lipschitz function $\psi_u : B_\varepsilon^Y(0) \to B_\varepsilon^X(0)$ which is at least $C^{k-1,1}$ and has the property that $\operatorname{graph}\psi_u$ *is backward invariant. We call the* $\operatorname{graph}\psi_u$ *the unstable manifold.*

Proof (rough sketch). By the inverse function theorem, F can inverted in a neighborhood of 0. We can then apply the stable manifold theorem.

Remark 1.40. One can similarly construct pseudo-unstable and strong unstable manifolds.

Remark 1.41. This method of proof is unsatisfactory, since it requires that A is invertible.

It is possible to prove the theorem directly by considering the image graph. We will, however, only sketch this approach. If we again assume the situation $F : X \times Y \to X' \times Y'$ and $\psi : Y \to X$, then the image graph transform $F_*\psi = \psi' : Y' \to X'$ is defined by

$$F[\operatorname{graph}\psi] = \operatorname{graph} F_*\psi$$

This can be written as

$$\psi'(\psi_{Y'}F(\psi(y),y)) = \pi_{X'}F(\psi(y),y).$$

We can solve this equation with a contraction mapping principle by considering $Z = \{\Gamma : Y' \to Y\}$ and the iteration $T : Z \to Z$ given by

$$(T\Gamma)(y') = B^{-1}y' - B^{-1}g(\psi(\Gamma(y')),\Gamma(y')).$$

Then we can set with the fixed point Γ of T:

$$\psi'(y') = A\psi(\Gamma(y')) + f(\psi(\Gamma(y')),\Gamma(y'))$$

It can be shown that T is a contraction in order to prove existence and regularity for the image graph transform. Afterwards, all theorems and ideas of proof carry over to the unstable case.

Chapter 2
A Topological Stable Set Theorem

In this chapter, we will introduce a topological generalization of the stable manifold theorem. We will start by giving a motivational example and then introduce some homotopy theory. In Section 2.3, we will introduce the main concepts of this chapter. We will close by applying these concepts to a simple system in Section 2.4.

2.1 Motivation

Consider the following example: Let $Q = [0,1] \times [0,1]$ and let $L = \{(x,y) : x \leq 0\} \subseteq \mathbb{R}^2$ and $R = \{(x,y) : x \geq 1\} \subseteq \mathbb{R}^2$. Consider a continuous map $F : Q \to \mathbb{R}^2$. Assume that $F(Q) \subseteq \operatorname{int} Q \cup L \cup R$.

Assume further that $F(Q \cap L) \subseteq \operatorname{int} L$ and $F(Q \cap R) \subseteq \operatorname{int} R$. For an illustration, see Figure 2.1. We aim for results similiar to the following Proposition:

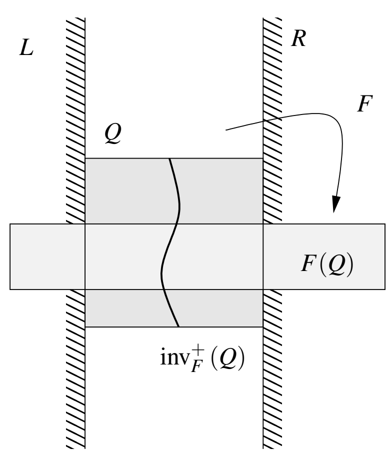

Fig. 2.1 The motivational Example

Proposition 2.1. *The set $Q \setminus \mathrm{inv}_F^+(Q)$ is not path-connected and $Q \cap L$ and $Q \cap R$ lie in different path-connected components of $Q \setminus \mathrm{inv}_F^+(Q)$.*

This Proposition implies a simple corollary, which illustrates the power of such results:

Claim. Let $y \in [0,1]$. Then there exists $x \in [0,1]$ such that $(x,y) \in \mathrm{inv}_F^+(Q)$.

Proof. Consider the path $\gamma(t) = (t,y)$ for $t \in [0,1]$. Since $Q \cap L$ and $Q \cap R$ lie in different components of $Q \setminus \mathrm{inv}_F^+(Q)$, we must have $\gamma([0,1]) \cap \mathrm{inv}_F^+(Q) \neq \emptyset$. □

We will at first state a rough proof idea for Proposition 2.1. The details will be filled out in the remainder of this chapter, i.e. Section 2.2 and Section 2.3.

Proof (Rough Idea for Proposition 2.1).

Definition of Separation. It is useful to crystallize the main property of $\mathrm{inv}_F^+(Q)$ into a definition. Ee define $C \subseteq Q$ to be separating, if $Q \cap L$ and $Q \cap R$ lie in different path-connected components of $Q \setminus C$. We will give a more general Definition in Definition 2.9.

Separating Preimage Property. The next step is to show, that for any separating $C \subseteq Q$ the complete preimage $F^{-1}[C]$ is also separating (note that F is only defined on Q). The idea of proving this is indirect (or "by duality"): For a L-R-connecting continuous path γ, we can also get a L-R-connecting continuous path by considering $F \circ \gamma$. If γ does not intersect $F^{-1}[C]$, then $F \circ \gamma$ does not intersect C. This step will be detailed in Lemma 2.16.

Construction of $\mathrm{inv}_F^+(Q)$. The next step is to use the construction $\mathrm{inv}_F^+(Q) = \bigcap_{n\in\mathbb{N}} F^{-n}[Q]$ in Definition 1.2 and iteratedly apply the Separating Preimage Property. This will yield a construction of $\mathrm{inv}_F^+(Q)$ as the intersection of a descending chain of compact separating sets. More details on this step will be given in Theorem 2.19.

Compactness argument. We can close the proof by using the fact, that the intersection of a descending sequence of compact separating sets is compact and separating. This can be seen by considering the intersection of the descending sequence of nonempty compact sets $\mathrm{Ran}\,\gamma \cup F^{-n}[Q]$. This step will be detailed in Lemma 2.12.

The Proposition 2.1 also has an elementary proof, which however does not generalize as gracefully to higher dimensions. In order to give confidence into Proposition 2.1, we will give this proof, even though the arguments in it will not be revisited later in this work.

Proof (Proposition 2.1). Define

$$\begin{aligned} M_L &= \left\{ p \in Q : \exists n \in \mathbb{N} : F^n(p)_x < 0,\ F^k(p) \in Q\ \forall k \in \{0, \dots n-1\} \right\} \\ M_R &= \left\{ p \in Q : \exists n \in \mathbb{N} : F^n(p)_x > 1,\ F^k(p) \in Q\ \forall k \in \{0, \dots n-1\} \right\}. \end{aligned} \tag{2.1.0.1}$$

Intuitively, M_L (resp. M_R) is the set of initial conditions, for which the forward iterates leave Q first to the left (resp. right) side. The three sets M_L, M_R and $\mathrm{inv}_F^+(Q)$ are pairwise disjoint. Since $F(Q) \subseteq Q \cup L \cup R$, we can see that $Q = \mathrm{inv}_F^+(Q) \cup M_L \cup M_R$. By continuity of F and by the definitions of M_L and M_R, both M_L and M_R are open. Therefore, $M_L \cup M_R = Q \setminus \mathrm{inv}_F^+(Q)$ is not connected and $Q \cap L \subseteq M_L$ and $Q \cap R \subseteq M_R$ lie in different connected components. □

2.2 Topological Prerequisites: Homotopy Theory

Homotopy theory from algebraic topology will provide a convenient language to describe our setting of interest. All definitions can be found in most books on algebraic topology, e.g. [Mun00] or [Oss92].
One of the main ingredients of the argument in the motivation is the topological fact that $[0,1]$ is connected and the usage curves $\gamma : [0,1] \to Q$. The argument holds even if we deform the curve γ, provided that it still connects the left and right boundary of Q. These concepts are formalised by the notion of homotopy.

Definition 2.2. Let X and Y be topological spaces and $f : X \to Y$, $g : X \to Y$ be continuous functions. We define f and g to be *homotopic* if there exists a continuous $h : X \times [0,1] \to Y$ with $h(\cdot,0) = f$ and $h(\cdot,1) = g$. We then write $f \sim g$. We call f *null-homotopic* if there exists a constant function g such that $f \sim g$. We write $[f]$ for the equivalence class of f under $\sim$.

The relation $\sim$ is an equivalence relation. In the language of homotopy our motivational argument yielded the following result:
All maps $c : \{0,1\} \to Q \setminus M_0$, i.e. all pairs of points (c_0, c_1), with c_0 lying in the left boundary of Q and c_1 in the right one are not null-homotopic, i.e. there is no deformation which makes c constant. This means that there exists no continuation $\tilde{c} : [0,1] \to Q \setminus M_0$ with $\tilde{c}_{|0,1} = c$. Therefore, there exists no continuous curve connecting the left and right boundary at all, which does not intersect M_0.

Definition 2.3. Let X be a topological space and for $n \in \mathbb{N}$ let $S^n = \partial B_1^{n+1}(0) \subset \mathbb{R}^{n+1}$. Define $h_n(X) := C(S^n, X)/\sim$ as the set of homotopy classes of continuous functions $\gamma : S^n \to X$. We call h_n *trivial* if it contains only the trivial, i.e. null-homotopic, class. We sometimes call an element $\gamma \in C(S^n, X)$ an n-sphere.

The motivational example was in the case of $n = 0$.

Remark 2.4. Since many authors study homotopy theory from a group theoretic point of view, we remark that it is possible to endow $h_n(X)$ with a group structure by fixing a basepoint $x_0 \in X$ and assuming $n > 0$ and path-connectedness of the space X. This group is called the nth homotopy group or the nth fundamental group. Since we do not use this group structure in this work, we refer to [Oss92, Chapter 3] for details.

One of our main uses for homotopy theory is the study of *induced maps*:

Definition 2.5. Let X and Y be topological spaces, $n \in \mathbb{N}$ and let $F : X \to Y$ be a continuous map. Define the *induced map* $h_n(F) : h_n(X) \to h_n(Y)$ for $\gamma \in C(S^n, X)$ as

$$h_n(F)[\gamma]_{h_n(X)} = [F \circ \gamma]_{h_n(Y)}.$$

Then the following holds:

Proposition 2.6. *1. The induced map is well defined.*
2. Let Z be another topological space and $G : Y \to Z$ continuous. Then $h_n(G \circ F) = h_n(G) \circ h_n(F)$.
3. If $F \sim F'$ then $h_n(F) = h_n(F')$.

The proof follows directly from the definitions. In order to use these definitions, we have to understand how $h_n(X)$ looks like in some important cases. There is a deep result, Theorem 2.8, from algebraic topology which classifies $h_n(S^n)$. A proof can be found in [Oss92, p. 205f, Satz 5.7.10]. We will need polar coordinates in order to state the result:

Definition 2.7. Let $P_n : [0, 2\pi] \times [0, \pi]^{n-1} \to S^n$ be the *polar coordinate map* defined for $n > 1$ by

$$P_n(\phi_1, \dots, \phi_{n-1}, \phi_n) = (\cos(\phi_n) P_{n-1}(\phi_1, \dots, \phi_{n-1}), \sin(\phi_n))$$

and for $n = 1$ by $P_1(\phi) = (\cos(\phi), \sin(\phi))$.

Theorem 2.8. *Consider the mapping $G_k : S^n \to S^n$ defined in polar coordinates for $k \in \mathbb{Z}$ by*

$$G_k : (\phi_1, \phi_2, \dots, \phi_N) \to (k\phi_1, \phi_2, \dots, \phi_n)$$

Then the relation $\mathbb{Z} \to h_n(S^n)$ given by $k \to [G_k]$ is a bijection and only $[G_0]$ is null-homotopic. We write $[k] := [G_k]$ in $h_n(S^n)$.

Now that we have some topological language set up, it is possible to generalize the result of our motivational argument: M_0 separates the left and right boundary in Q.

Definition 2.9 (Separating Sets). Let X be a topological space and $E, M \subset X$ be disjoint subsets of X where $E \neq \emptyset$. Let $n \geq 0$ and $\gamma \in h_n(E)$ not null-homotopic. We say that M *separates* γ in X if γ cannot be contracted in $X \setminus M$. For a more formal definition, let $\iota = \iota_{E \to X \setminus M}$ be the inclusion map. M is a separating subset with respect to E and γ, if $h_n(\iota)\gamma$ is not null-homotopic.

In many cases it is not necessary to study individual $\gamma \in h_n(E)$. We therefore call M separating in codimension n, if all non null-homotopic $\gamma \in h_n(E)$ are mapped to non null-homotopic spheres by $h_n(\iota)$, i.e. if M separates all $\gamma \in h_n(E)$.

Remark 2.10. The fact that M separates $\gamma \in h_n(E)$ is only useful, if the empty set does not separate γ, i.e. γ cannot be contracted in X. However, we include this case into the definition because it makes the theorems in 2.3 more convenient.

A similiar concept is widely used in the calculus of varations. The concept there is called "linking" and will be related to the concept of "separating sets" in Section 2.2.1.

Example 2.11. The statement of the motivational example can be found with $n = 0$. The homotopy classes $h_0(Z)$ of a topological space Z just count path-connected components: If Z has the path-connected components $Z = \cup_{i\in I} Z_i$, then the homotopy classes $h_0(Z)$ can be represented as pairs $(i, j) \in I^2$. Such a 0-sphere (i, j) is null-homotopic if and only if $i = j$ (by the definition of path-connected components).

In the example, consider $X = Q$ and $E = Q \cap (L \cup R)$ and $M = \mathrm{inv}_F^+(Q)$. The set E has two path-connected components, i.e. the left part $Q \cap L$ and the right part $Q \cap R$. The statement that the specific 0-sphere (L, R) is separated by M means literally that the left and the right boundary cannot be connected in Q without intersecting M. The statement that M is separating E in codimension 0 without qualifying any 0-sphere means, that each non-trivial 0-sphere in $h_0(E)$ is separated by M.

Separating sets have good limit properties.

Lemma 2.12 (Separating Limit Lemma).

1. *Let $M \subset M' \subset X \setminus E$ and let $\gamma \in h_n(E)$ be not null-homotopic. If M separates γ, then M' also separates γ.*
2. *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed separating (with respect to γ) subsets of X and $C_n \subset C_{n+1}$ for all $n \in \mathbb{N}$ and assume $C_0 \cap E = \emptyset$. Then $C = \bigcap_{n\in\mathbb{N}} C_n$ is also a closed separating subset of X with respect to γ.*

Proof. The first part follows directly from the definition.
The proof of the second assertion is indirect. We assumed that $\gamma : S^n \to E$ is not null-homotopic. Assume that there exists some continuous $\omega : S^n \times [0,1] \to X \setminus C$ with $\omega(\cdot, 0) = \gamma$ and $\omega(\cdot, 1) = \text{const}$. Since C_n is separating for all $n \in \mathbb{N}$, $K_n = h^{-1}(C_n) \subset S^n \times [0,1]$ is nonempty. By continuity of ω, K_n is compact and $\omega^{-1}(C) = \bigcap_{n\in\mathbb{N}} K_n$ is nonempty, since it is the intersection of a descending sequence of nonempty compact sets. Therefore we have a contradiction to the assumption $\omega : S^n \times [0,1] \to X \setminus C$. □

2.2.1 Relations to Linking

A similiar concept to separating sets (Definition 2.9) is widely used in the calculus of varations. The concept there is called "linking". In [Str08], the following definition of linking sets is given:

Definition 2.13. Let S be a closed subset of a Banach space V, Q a submanifold of V with relative boundary ∂Q, we say S and ∂Q *link* if :

L1 $S \cap \partial Q = \emptyset$

L2 for any map $\phi \in C(V,V)$ such that $\phi_{|\partial Q} = id$ there holds $\phi(Q) \cap S \neq \emptyset$

In order to illustrate the connection and the differences between the definitions of linking and separation, we will give a short example.

Example 2.14. Consider $V = \mathbb{R}^3$. Let $A_i = \{x : x_i \geq 0, x_j = 0 \ \forall \ j \neq i\}$ be the three non-negative half-axes and $A = A_1 \cup A_2 \cup A_3$. Let $Q = \{x : d(x,A) \leq 1\}$ and $\partial Q = \{x : d(x,A) = 1\}$. The set Q looks like a thickened three-armed star stretching to infinity, as illustrated in 2.2. Then the following holds:

1. The set $A_1 \cup A_2$ is separating for some $\gamma \in h_1(\partial Q)$ with respect to some but not all $\gamma \in h_1(\partial Q)$ (separating with repect to $X = Q$, $E = \partial Q$).
2. The set $A_1 \cup A_2$ is linked to ∂Q.

Proof. Set $\gamma^1 : S^1 \to \partial Q$ with $\gamma^1(\xi) = (10, \cos\phi, \sin\xi)$ and analogously $\gamma^2(\xi) = (\sin\xi, 10, \cos\xi)$ and $\gamma^3(\xi) = (\cos\xi, \sin\xi, 10)$. It is obvious by the Figure 2.2 that γ^1 and γ^2 are separated by $A_1 \cup A_2$, while γ^3 is not separated.

Suppose that there is an “unlink”, i.e. some continuous $\phi : V \to V$ with $\phi_{|\partial Q} = id$ and $\phi(Q) \cap A_0 = \emptyset$. Let π be a retraction of $\mathbb{R}^3$ on Q, i.e. $\pi_{|Q} = id$ and $\operatorname{Ran} \pi_{|\mathbb{R}^3 \setminus Q} \subseteq \partial Q$ (e.g. $\pi(x) = x \cdot \sup\{t < 1 : tx \in Q\}$). Now consider $\omega(s,\xi) = \pi(\phi(10, s\cos\xi, s\sin\xi))$. We have $\omega(0,\cdot) = const$ and $\omega(1,\cdot) = \gamma^1$, since $\operatorname{Ran}\gamma^1 \subset \partial Q$ and $\phi_{|\partial Q} = \pi_{|\partial Q} = id$. Since $\operatorname{Ran}\omega \subseteq Q \setminus (A_1 \cup A_2)$, the existence of an “unlink” ϕ would imply that the 1-sphere γ^1 is not separated by $A_1 \cup A_2$.

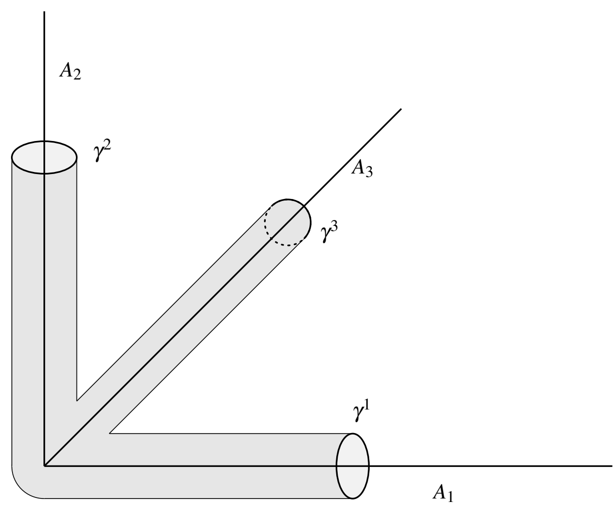

Fig. 2.2 An example for Separation and Linking

We do not use the Definition 2.13 of linking for three reasons:

Firstly, the restriction of using only Banach spaces V as a base space is unnecessary. In this work we will mainly work with topological spaces X, which are subsets of Banach spaces.

Secondly, the restriction of only working with submanifold-boundaries ∂Q is unnecessary. In this work, we will always specify a set E, which plays the role of a boundary. Even while it may be possible to view these sets as relative boundaries, such a viewpoint requires more work and does not give any additional insight.

The third disadvantage is the primary reason for not using the concept of "linking" in this work: We need to track some more details in our applications in dynamical systems, namely the induced maps $h_n(F) : h_n(\partial Q) \to h_n(\partial Q)$. This is not possible with only the concept of "linking" without specifying exactly which elements of $h_n(\partial Q)$ are separated, if any. An alternative wording of this critique would be, that it does not suffice to know for our applications whether two sets are linked. We need to specify how exactly they are linked, i.e. what the obstructions to an "unlink" are.

2.3 The Forward Invariant Separating Set Theorem

With the results of the last section, i.e. the generalizations of the concepts of connectedness and separation, we can generalize the other parts of the construction in 2.1. There are several ways to do this, and the choice of the construction is mainly a matter of personal preference. We will now give a straightforward construction and then relate it in Section 2.3.1 to standard constructions in Conley index theory.

Definition 2.15. Let $X = N \dot\cup E$ and $X' = N' \dot\cup E'$ be topological spaces and $F : X \to X'$ be a continuous map. Suppose that

$$\operatorname{int} F^{-1}[E'] \supset E.$$

Then we call the quadruple (N, E, N', E') a *block pair* for F.

The definition can be illustrated by Figure 2.3. We will need to study the induced map $h_n(F) : h_n(E) \to h_n(E')$.

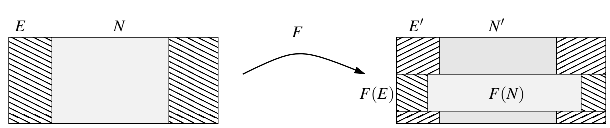

Fig. 2.3 The Block pair construction

The definition of a block is compatible with the definition of separating sets; the following Lemma is actually the justification for these definitions:

Lemma 2.16 (Separating preimage Lemma). *Let $F : X \to X'$ be continuous and $X = N \dot\cup E$ and $X' = N' \dot\cup E'$ be a block for F. Let $\gamma \in h_n(E)$ be not null-homotopic and* *let $\gamma' = h_n(F)\gamma \in h_n(E')$ be not null-homotopic as well. Let $M' \subset N'$ be a separating subset for γ'. Then $F^{-1}[M]$ is separating for γ.*

Proof. Let $\omega : [0,1] \times S^n \to X$ with $[\omega(0,\cdot)] = \gamma$ and $\omega(1,\cdot) = \text{const}$. Then $\omega' = F \circ \omega$ is continuous and $[\omega'(0,\cdot)] = \gamma'$ while $\omega'(1,\cdot) = \text{const}$. Since we assumed that M is separating for γ' and E', there is a $(t,s) \in [0,1] \times S^n$ with $\omega'(t,s) \in M$. Therefore, $\omega(t,s) \in F^{-1}[M]$. □

In many cases it is not necessary to keep track of the homotopy classes in h_n, since it suffices for our purposes to know whether they are null-homotopic. In this case we can use the following simplified corollary:

Corollary 2.17. *Let $F : X \to X'$ be continuous and $X = N \dot\cup E$ and $X' = N' \dot\cup E'$ be a block pair for F. Suppose that for some $n \in \mathbb{N}$ and $M' \subset N'$:*

1. *The sets of homotopy classes $h_n(E)$ and $h_n(E')$ are not trivial.*
2. *The induced map $h_n(F) : h_n(E) \to h_n(E')$ maps non null-homotopic classes on non null-homotopic classes.*
3. *M' is separating E' in codimension n (in the sense of Definition 2.9).*

Then $M = F^{-1}[M']$ is separating E in codimension n.

With the definition of block pairs, we can immediately define a block sequence:

Definition 2.18. Let $X_k = N_k \dot\cup E_k$ for $k \in \mathbb{N}$ be a sequence of topological spaces and $F_k : X_k \to X_{k+1}$ be a sequence of continuous maps. The decomposition is called *block sequence*, if $(N_k, E_k, N_{k+1}, E_{k+1})$ is a block pair for F_k for every $k \in \mathbb{N}$.

We can now iteratedly apply the separating preimage Lemma 2.16 and use the limit property of separating sets (Lemma 2.12) in order to achieve a separating forward invariant set.

Theorem 2.19 (Forward Invariant Separating Set Theorem). *Let $X_k = N_k \dot\cup E_k$ and $F_k : X_k \to X_{k+1}$ for $k \in \mathbb{N}$ be a block sequence. Assume that $\gamma_0 \in h_n(E_0)$ is not null homotopic and assume that for all $k \in \mathbb{N}$ the sphere $\gamma_k \in h_n(E_k)$ with $\gamma_k = h_n(F_{k-1}) \circ \ldots \circ h_n(F_0)\gamma_0$ is not null-homotopic. Then $\mathrm{inv}^+_{F_\bullet}(N_\bullet)_k$ separates γ_k for all $k \in \mathbb{N}$.*

Proof. We will follow the construction of $\mathrm{inv}^+_{F_\bullet}(N_\bullet)$. Set $C_k^1 = \overline{F_k^{-1}[N_{k+1}]}$. By the Definition 2.15 of a block sequence we have $C_k^1 \subseteq N_k$, i.e. $C_k^1 \cap E_k = \emptyset$. By assumption, N_{k+1} separates γ_{k+1} and by the separating preimage Lemma 2.16, C_k^1 separates γ_k.

Define

$$C_k^m = F_k^{-1}[C_{k+1}^{m-1}].$$

By continuity, the separating preimage Lemma and induction over m, $C_k^m \subseteq N_k$ is closed and separates γ_k for all $k, m \in \mathbb{N}$. Furthermore and also by induction over m, the C_k^m are a descending chain, i.e. $C_k^{m+1} \subseteq C_k^m$. Define $C_k^\infty = \bigcap_{m\in\mathbb{N}} C_k^m$. By the limit property of separating sets (Lemma 2.12), we can see that C_k^∞ separates γ_k for all $k \in \mathbb{N}$.

In order to see that $\left(C_k^\infty\right)_{k\in\mathbb{N}} = \mathrm{inv}^+_{F_\bullet}(N_\bullet)$, consider the construction of $\mathrm{inv}^+_{F_\bullet}(N_\bullet)$ in Definition 1.7 as the intersection U_k^∞ of the descending chain defined by $U_k^0 = N_k$ and $U_k^m = F_k^{-1}[U_{k+1}^{m-1}]$. We have $U_k^{m+1} \subseteq C_k^{m+1} \subseteq U_k^m$ and therefore the limits coincide. □

Similarly to the simpler version of the separating preimage Lemma 2.16, i.e. Corollary 2.17, it is often not necessary to track the entire sequence γ_k of homotopy classes. This makes the formulation simpler:

Corollary 2.20. *Let $N_k \dot\cup E_k$ and $F_k : X_k \to X_{k+1}$ for $k \in \mathbb{N}$ be a block sequence. Let $n \in \mathbb{N}$. Suppose that:*

1. *The set of homotopy classes $h_n(E_k)$ is not trivial for $k \in \mathbb{N}$.*
2. *The induced maps $h_n(F_k) : h_n(E_k) \to h_n(E_{k+1})$ map non null-homotopic classes on non null-homotopic classes.*

Then $(\mathrm{inv}^+_{F_\bullet}(N_\bullet))_k$ is a closed separating subset for E_k in codimension n and for all $k \in \mathbb{N}$.

If we study an autonomous system, we can also state a simpler version of the theorem for which we need to check only finitely many conditions.

Corollary 2.21. *Let $X = N \dot\cup E$ and $F : X \to X$ be a block, i.e. let (N,E,N,E) be a block pair in the non-autonomous sense. Suppose that $h_n(E)$ is non-trivial and that $h_n(F) : h_n(E) \to h_n(E)$ maps non null-homotopic classes to non null-homotopic ones. Then $\mathrm{inv}^+_F(N)$ is separating for E in codimension n.*

2.3.1 Relations To Conley Index Theory

We will now discuss the relations of our Definition 2.15 of block pairs to other theories and the rationale of our definition. The Figure 2.4 makes clear a drawback of our choice of definition for block pairs: In most applications, we will be faced with a map $F : N \dot\cup E \to Y$ with $N' \dot\cup E' \subsetneq Y$, i.e. the assumption $F(X) \subseteq X'$ fails. If we are only interested in forward invariant sets $\mathrm{inv}^+_{N_\bullet}(F_\bullet)$, this does not pose an

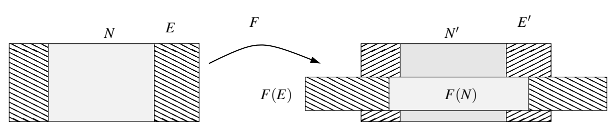

Fig. 2.4 A realistic example for the Block pair construction

insurmountable problem: Points, whose images are not in N' cannot belong to the maximal forward invariant set anyway. This problem does however introduce some technicalities. There are at least three ways of handling this problem:

The first approach is the one used in this work. If F does not satisfy $F(X) \subseteq X'$, then we try to find some continuous map $\pi : \operatorname{Ran} F \to X'$ and consider $\tilde{F} = \pi \circ F$. If we can archieve $\tilde{F}^{-1}[N'] = F^{-1}[N]$ and $\tilde{F}_{|F^{-1}[N]} = F_{|F^{-1}[N]}$, then this will allow us to use the theory developed in this chapter, since we will then get $\operatorname{inv}^+_F(N) = \operatorname{inv}^+_{\tilde{F}}(N)$.

The second approach is used in [Zgl04], where a similiar theory is constructed for the special case $X = \{x \in \mathbb{R}^n : |x|_\infty \leq 1\}$. In this paper, the analogous Definition [Zgl04][Definition 2] to our Definition 2.15 of block pairs includes the case when $F(X) \not\subseteq X'$, but makes many more technical assumptions. This approach has the immediate drawback of making the definition of a block pair immensely unflexible and inapplicable to situations as encountered in Chapter 3. Therefore, we will not go into more details on this approach.

The third approach is used in Conley Index theory, as in e.g. [FR00],[Gid98] and [RW10]. The basic idea is to collapse (identify) the set E'.

We will now illustrate the approach of collapsing E', which is used in Conley Index theory. Afterwards we will discuss the approach which is used in this work.

In [RW10], Richeson uses the following definition of block pairs:

Definition 2.22. Let X and X' be topological spaces and $F : X \to X'$ be continuous. Let $L \subset N \neq \emptyset$ be subsets of X. Let $L' \subset N' \neq \emptyset$ be subsets of X'. They are called a *block pair* (in the sense of Richeson), if L is an open neighborhood of $N \cup F^{-1}[X' \setminus \mathring{N}']$.

With this definition, Richeson studies the homotopy type of the pointed quotient spaces $(N/L, L)$ and $(N'/L', L')$ as well as induced map $\hat{F}$ on the pointed quotient spaces. Our definition is similar if one considers $X = N \setminus L \dot{\cup} L$ and a retraction $\pi : \operatorname{Ran} F \to L$. Then the maps $h_{n+1}(F) : h_{n+1}(N/L, L) \to h_{n+1}(N'/L', L')$ and $h_n(\pi \circ F) : h_n(E) \to h_n(E')$ are comparable. This definition and its application is illustrated in Figure 2.5.

We will now discuss how to apply our Definition 2.15 of block pairs. We will project back some neighborhood $M \supseteq N' \dot{\cup} E'$ with an appropriate projection $\pi : M \to N' \cup E'$, as illustrated in Figure 2.6. Then we can apply our definitions and theory to the composed map $\tilde{F} = \pi \circ F$. However, the induced map $h_n(\pi \circ F) : h_n(E) \to h_n(E')$ may in general depend on the choice of M and π, as illustrated in Figure 2.7.

This ambiguity is the reason I decided to define block pairs for $\tilde{F} = \pi \circ F$ and not for maps $F : N \dot{\cup} E \to Y$. This means that the problem of choosing an appropriate retraction π has to be solved in each specific application of the definition of block pairs. Even while the Conley Index approach of collapsing the exit set is more elegant than our approach, it requires some more topological machinery and is not all that helpful in the applications in this work, where we can give retractions explicitly. Furthermore, the ambiguity does not need to concern us when we restrict our ambitions to proving existence of forward invariant separating sets and do not seek to construct topological invariants of dynamical systems.

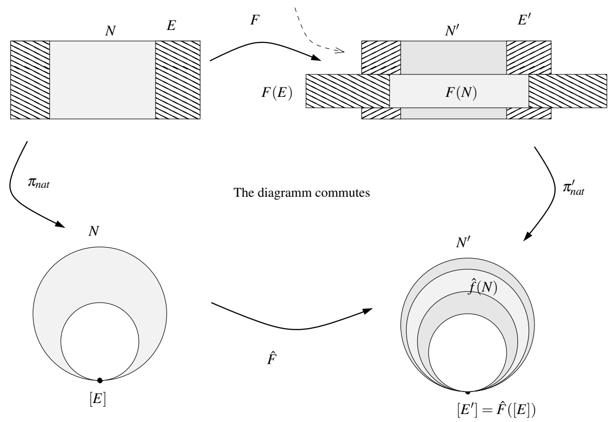

Fig. 2.5 The block pair construction in [RW10]

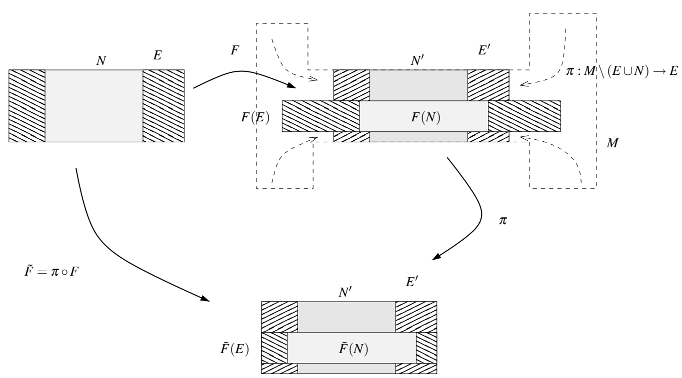

Fig. 2.6 Application of block pairs and retractions

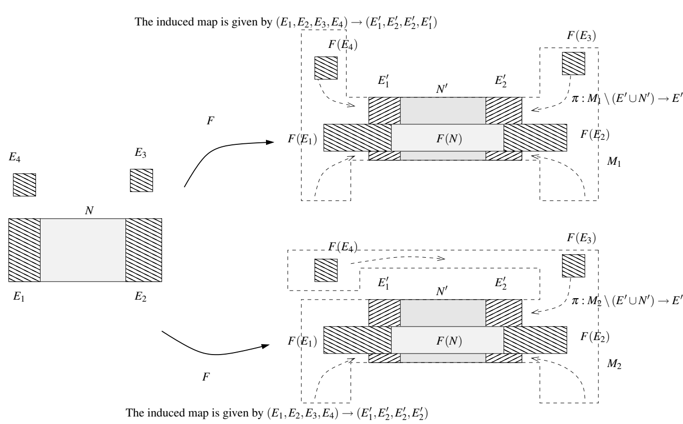

Fig. 2.7 An illustration of ambiguity

Now we will explain the details of what an "appropriate" projection is and in which cases we can get induced maps independently of the chosen projection. We will need some definitions first:

Definition 2.23. Let X be a topological space and $A \subset X$. We call a continuous map $\pi : X \to A$ with $\pi_{|A} = id$ a *retraction* of X to A. We call a continuous map $R : [0,1] \times X \to X$ with $R(0,\cdot) = id$, $R(1,\cdot) \subset A$ and $R(\cdot,\cdot_{|A}) = id$ a *deformation retraction* of X to A. If such maps exist we call A a *retract*, respective a *deformation retract*, of X.

We can contruct a block pair in applications by choosing a "strip" $M \subseteq Y$ with $\operatorname{Ran} F \subseteq M$ and a retraction $\pi : M \to M \cap X'$ such that $\pi_{|M\setminus N'} : M \setminus N' \to E' \cap M$ is also a retraction.

Proposition 2.24. *Let $F : E\dot{\cup}N \to Y$ be continuous and suppose $\operatorname{Ran} F \subseteq M \subseteq Y$. Let $X' = N'\dot{\cup}E' \subset Y$ and $\pi : M \to X'$ be continuous with $\pi_{|X'\cap M} = id$ and $\operatorname{Ran} \pi_{M\setminus X'} \subset E'$. Suppose further that $F(E) \subset M \setminus X'$ and that $X' \cap M$ is relatively closed in M. Then (N,E,N',E') is a block pair for $\tilde{F} = \pi \circ F$.*

Proof. We need to show that $E \subset \operatorname{int} \tilde{F}^{-1}(E')$. Since $F(E) \subset M \setminus X'$ we have $E \subset F^{-1}(M \setminus X')$. Because of the projection we have $F^{-1}(M \setminus X') \subset \tilde{F}^{-1}(E')$. The set $F^{-1}(M \setminus X')$ is however open, since F is continuous and X' is relatively closed in M. $\square$

This approach yields block pairs but has the unsatisfactory property of the resulting induced map $h_n(\tilde{F})$ not being independent of the choice of M and π. If we use deformation retractions instead, the resulting induced map becomes at least partially independent of this choice:

Proposition 2.25. *Let $F : E\dot{\cup}N \to Y$ be continuous and $X' = N'\dot{\cup}E' \subset Y$. Suppose that $\operatorname{Ran} F \subset M \subset \hat{M} \subset Y$ and $R : [0,1] \times M \to M$ and $\hat{R} : [0,1] \times \hat{M}$ are two deformation retractions on $X' \cap M$ and $X' \cap \hat{M}$ respectively. Suppose further that the retractions fulfill $R(1,\cdot) : M \setminus X' \to E' \cap M$ and $\hat{R}(1,\cdot) : \hat{M} \setminus X' \to E' \cap \hat{M}$.*

Then $h_n(R(1,\cdot) \circ F) = h_n(\hat{R}(1,\cdot) \circ F)$ for the induced maps $h_n(\tilde{F}) : h_n(E) \to h_n(E')$.

Proof. Consider the homotopy $f : [0,1] \times E \to E'$ given by

$$f(t,x) = \begin{cases} \hat{R}(1, R(1-2t, F(x))) & \text{for } t \in [0,1/2] \\ \hat{R}(2-2t, F(x)) & \text{for } t \in [1/2,1] \end{cases}$$

By this homotopy we have $R(1,\cdot) \circ F \sim \hat{R}(1,\cdot) \circ F$ and therefore $h_n(R(1,\cdot) \circ F) = h_n(\hat{R}(1,\cdot) \circ F)$. $\square$

In general, however, the induced map $h_n(\pi \circ F) : h_n(E) \to h_n(E')$ may depend on the choice of M and π, even when π is a deformation retraction, as illustrated in Figure 2.7.

2.4 An Application of the Theorem

In this section we will show how to apply the topological theorem to cases similar to the ones covered by the stable manifold theorem.

Proposition 2.26. *Let* $F : \mathbb{R}^m \times \mathbb{R}^{n+1} \to \mathbb{R}^m \times \mathbb{R}^{n+1}$ *with*

$$F(x,y) = (Ax + f(x,y), By + g(x,y))$$

where A *and* B *are linear and* $|A|, |B^{-1}| < \sigma < 1$ *and* f, g *are continuous and bounded. Assume* $\|f\|_\infty < 1 - \sigma$ *and* $\|g\|_\infty < 1 - \sigma$.
Set

$$X = \{(x,y) : x \in \mathbb{R}^m, y \in \mathbb{R}^{n+1}, |y| \le 1, |x| \le 1\}$$

$$E = \{(x,y) : x \in \mathbb{R}^m, y \in \mathbb{R}^{n+1}, |y| = 1, |x| \le 1\}$$

Set $N = X \setminus E$. *Then* $\mathrm{inv}^+_F(N)$ *is separating in* E *in codimension* n.

Proof. We wish to apply the topological theorem and therefore need to set up conditions for it. At first we will set up the block pair for Corollary 2.21 and a projection like the one used in Proposition 2.24. Let $U = \{(x,y) : |x| \le 1\}$. Then $F(X) \subset U$ since $|Ax + f(x,y)| \le \sigma + \|f\|_\infty < 1$. Set

$$\pi(x,y) = (x, \frac{y}{max(1,|y|)})$$

Then $\pi : U \to X$ is continuous and maps $U \setminus X$ to E while $\pi_{|X} = \mathrm{id}$. Now we will show that $F(E) \subset U \setminus X$.
Assume $(x,y) \in E$. Then $|F_y(x,y)| = |By + g(x,y)| \ge \sigma|y| - \|g\|_\infty > 1$. Since F is continuous and X is relatively closed in U, $F^{-1}(U \setminus X)$ is relatively open in X.

Now consider $\tilde{F} = \pi \circ F$. The map $\tilde{F}$ is continuous as the composition of continuous maps. Furthermore $\tilde{F}^{-1}(E) = F^{-1}\left(\pi^{-1}(E)\right) \supset F^{-1}(U \setminus X)$. Therefore $\tilde{F}^{-1}(E)$ contains an open neighborhood of E and $\mathrm{inv}^+_F(N) = \mathrm{inv}^+_{\tilde{F}}(N)$.

We have set up a (constant) block sequence for $\tilde{F}$. We can easily see that $h_n(E) = h_n(B^m_1(0) \times S^n) = h_n(S^n) = \mathbb{Z}$. Now we need to determine the induced map $h_n(F) : h_n(E) \to h_n(E)$. It is given by $h_n(\tilde{F})([k]) = [(\mathrm{sign}\det B)k]$. Then we can apply the topological stable set theorem. This proves that $\mathrm{inv}^+_F(N)$ separates E codimension n.

In order to complete the proof, we still need to establish the fact that $h_n(\tilde{F}) = \mathrm{sign}\det B$. Although this seems to be intuitively clear, a formal proof would be lengthy and does not lie in the scope of this work and will therefore just be sketched. The basic idea is to exploit that $F \sim G$ implies $h_n(F) = h_n(G)$ and homotope away the nonlinearity. Then one can use the fact that $GL(n)$ has only two path-connected components, namely $\{A \in GL(n) : \det A > 0\}$ and $\{A \in GL(n) : \det A < 0\}$. This can be proved by considering the real Jordan form and then homotoping pairs of eigenvalues around the zero in the complex plane. An alternative proof can be found in [Oss92, p. 204, Satz 5.7.4]. □

In order to see that separation is a strong concept, we will give a short corollary of the fact that $\mathrm{inv}_F^+(N)$ separates E in codimension n:

Corollary 2.27. *Let $x_0 \in \mathbb{R}^m$ with $|x_0| < 1$. Then there exists a $\hat{y}$ such that $(x_0, \hat{y}) \in \mathrm{inv}_F^+(N)$.*

Note that we had a similar statement for the stable manifold theorem: In this case $\hat{y} \in \mathbb{R}^n$ is unique and therefore defines a function $\hat{y} = \psi(x_0)$. This function has even better regularity, as shown in Theorem 1.29.

Proof. Consider the map $\omega : S^n \times [0,1] \to X$ given by $\omega(y,t) = (x_0, (1-t)y)$. Then $\omega(\cdot, 0) \in h_n(E)$ is not null-homotopic, while $\omega(\cdot, 1) = \text{const}$. Since $\mathrm{inv}_F^+(N)$ is separating E in codimension n, there exists $(y_0, t_0) \in S^n \times [0,1]$ such that $\omega(y_0, t_0) \in \mathrm{inv}_F^+(N)$. Therefore, there exists $\hat{y} = t_0 y_0$ such that (x_0, y_0) lies in the maximal forward invariant subset of N. $\square$

Chapter 3
Application to Homoclinic Orbits in a 3-Dimensional System

In this section we want to study a 3-dimensional time-continuous system with the flow ϕ^t and a homoclinic orbit, i.e. a trajectory $\gamma(t)$ with $\lim_{t\to\pm\infty}\gamma(t) = x_0$.

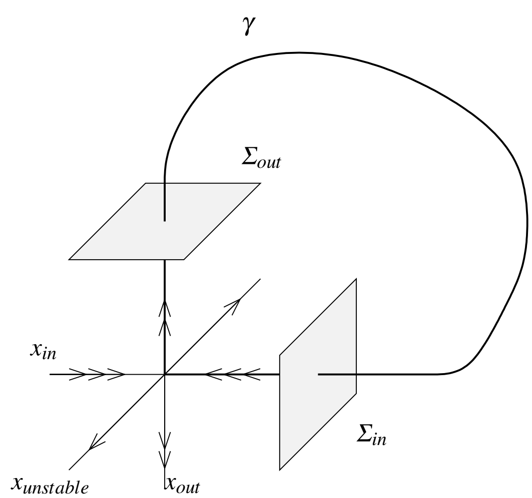

Fig. 3.1 The homoclinic orbit

Assumptions 3.1. More precisely the system we study has the form

$$\begin{aligned}\dot{x_i} &= -\lambda_i x_i + f_i(x_i, x_o, x_u)\\ \dot{x_o} &= \lambda_o x_o + f_o(x_i, x_o, x_u)\\ \dot{x_u} &= \lambda_u x_u + f_u(x_i, x_o, x_u),\end{aligned} \tag{3.0.0.1}$$

where $f = (f_i, f_o, f_u)$ fulfills $f \in C^3$ and $f(0) = 0$ as well as $\mathrm{D}f(0) = 0$. We assume that there is a homoclinic orbit $\gamma(t)$ with $\lim\limits_{t\to\pm\infty}\gamma(t) = 0$, which is tangent to the x_i-

axis for $t \to \infty$ and tangent to the x_o-axis for $t \to -\infty$. Furthermore, the eigenvalues are assumed to fulfill the following inequalities:

$$0 < \lambda_o < \lambda_i \tag{3.0.0.2}$$

$$0 < \lambda_u < \lambda_o \tag{3.0.0.3}$$

Let W_s denote the stable set of γ, i.e.

$$W_s = \{x \in \mathbb{R}^3 : \lim_{t\to\infty} d(\phi^t(x), \gamma) = 0.\} \tag{3.0.0.4}$$

We will see how to apply the topological Theorem 2.19 to prove that W_s is separating in codimension 1 (and therefore especially nontrivial) under some additional geometric assumptions. This chapter will culminate in the following main theorem:

Theorem 3.2. *Suppose we have a system fulfilling the Assumptions 3.1. Then there exists a vector $v(t)$ along γ which determines whether W_s is trivial, i.e. whether there are points which converge to the homoclinic orbit. v is invariant under the flow, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \mathrm{D}F(\gamma(t))v(t)$$

and it fulfills

$$\lim_{t\to\infty} \frac{v(t)}{|v(t)|} = e_o$$

Suppose that $v(t)$ points to the same side of the unstable manifold as γ_{in} for $t \to -\infty$. Then W_s is separating a cusp in codimension 1 (the cusp is defined in (3.2.4.2) *in Section 3.2.4). The stable set is tangent to v in the sense, that it is contained in a cusp around v. Therefore, W_s is especially nontrivial.*

A more precise statement of the assumption on $v(t)$ is this: Let the local unstable manifold W_u in a neighborhood U of 0 *be given by $W_u = \{x \in U : \psi(x) = 0\}$ with $\psi \in C^1$, where* 0 *is a regular value of ψ and $\mathrm{D}_{x_i}\psi(0) > 0$. Then we suppose that there is $T_0 < 0$ such that $\mathrm{D}_{v(t)}\psi(\gamma(t)) > 0$ for all $t < T_0$.*

Remark 3.3. The last assumption (3.0.0.3) is made for technical reasons. Although the relation

$$\lambda_i > \lambda_o > \lambda_u > 0. \tag{3.0.0.5}$$

is needed in our proof, there are strong indications that (3.0.0.2) and $\lambda_o - \lambda_i < \lambda_u$ might be enough. The regularity assumption $f \in C^3$ is made for technical reasons as well. There are indications, that $f \in C^1$ might be enough.

Remark 3.4. We would actually like to construct a stable manifold to the homoclinic orbit. A naive way would be to look at the equilibrium and construct an invariant manifold which contains both the incoming and the exiting part of the homoclinic orbit. However, the standard theorems of Chapter 1 do not provide such a manifold: We can only take intersections of strong or pseudo-stable and unstable manifolds, i.e. get an invariant manifold tangent to the spectral subspace to an interval. Since $-\lambda_i < \lambda_u < \lambda_o$, we cannot get a manifold tangent to the $x_i x_o$-plane with these methods. This is one of the reasons why the results of [dlL03], which have been mentioned in Remark 1.35, are remarkable. However, these results do not help us in this spectral setting, since we are not looking for a subset of the stable manifold.

Even if we were in the situation $-\lambda_i < \lambda_o < \lambda_u$, the existence of orbits converging to the homoclinic would not be obvious. We could construct a pseudostable manifold tangent to the $x_i x_o$-plane– but in view of its non-uniqueness, it may not coincide with the stable set constructed in this chapter.

The general idea to prove that W_s is nontrivial for the described setting is to consider two small sections $\Sigma_{in} = \{x : x_i = \delta_i\}$ and $\Sigma_{out} = \{x : x_o = \delta_o\}$. The homoclinic orbit will return from Σ_{out} to Σ_{in} in finite time, thus yielding a smooth Poincaré map $\Phi_{glob} : U \subset \Sigma_{out} \to \Sigma_{in}$ for some small neighborhood of the point corresponding to the homoclinic orbit. The passage near the equilibrium will yield a Poincaré map $\Phi_{loc} : \Sigma_{in} \to \Sigma_{out}$. However, the domain of this local map will not contain a complete neighborhood of the point corresponding to the homoclinic orbit. We will show that it is continuous in the interior of a small cone, the tip of which is the homoclinic orbit. Because of this lack of smoothness, the stable manifold theorem and the theory of the first chapter are not applicable. This is the reason for using a topological theorem.

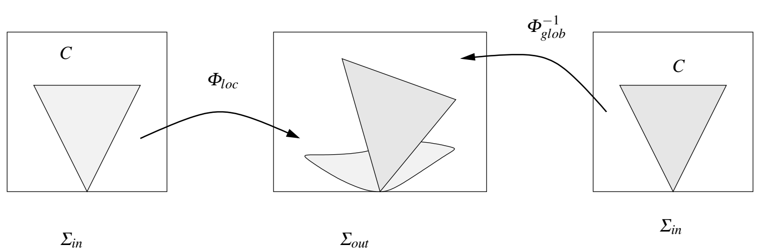

Fig. 3.2 The cone-like block pairs

We will construct a block pair for the local map as depicted in Figure 3.2. The first block will be a cusp-like cone in Σ_{in} with the homoclinic orbit as tip; the second block will be a "pancake"-like cone in Σ_{out} which has a tangency at the homoclinic orbit. Then we will consider the global map Φ_{glob} and modify our construction slightly in order to make the pair a block pair for Φ_{glob}. Since Φ_{glob} is a diffeomorphism this will work similar to the example in Section 2.4.

This will then enable us to apply Theorem 2.19 to obtain the desired separating set.

Remark 3.5. Most proofs in this chapter can be modified to remain valid in the case $\lambda_u = 0$. The necessary modifications will be sketched in Section 3.2.6.

3.1 The Linear Case

Since the local estimates are lengthy in the general case, we will start by assuming that our system is linear near the equilibrium. Once the entire construction is completed, we will relax the required conditions.

We consider the system (3.0.0.1) with the Assumptions 3.1 and a box $B = \{x : |x_i| \leq 1, |x_o| \leq 1, |x_u| \leq 1\}$. We assume that $f = (f_i, f_o, f_u)$ vanishes inside of B, i.e.

$$f(x) = 0 \quad \forall x \in B. \tag{3.1.0.6}$$

Furthermore we assume that γ contains the positive x_i-axis as well as the positive x_o-axis in B.

Remark 3.6. It is in some cases possible to achieve a linear system by a change of coordinates. We do not study this viewpoint in detail becaus it does not generalize to more interesting situation like a line of equilibria (i.e. $\lambda_u = 0$). An overview of linearization theorems can be found e.g. in [BD84] for C^k-maps and in [KH95, Chapter 6.6].

The goal of this section is to prove the following theorem for the linear setting, which is analogous to Theorem 3.2:

Theorem 3.7. *Suppose we have a system fulfilling the Assumptions 3.1 as well as* (3.1.0.6)*. Let γ be parametrized such that $\gamma(0) = e_i$ $\gamma(-T) = e_o$. Let $v(t)$ be the solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \mathrm{D}F(\gamma(t))v(t) \quad v(0) = e_o.$$

Suppose that $v(-T)_i > 0$. Then there is $\varepsilon > 0$ such that W_s is separating the region N_{in} (defined in (3.1.1.2) *in Section 3.1.1) in codimension 1 (see Definition 2.9). Therefore, W_s is especially nontrivial.*

The proof will follow in Section 3.1.3 after we constructed a suitable block sequence as outlined the introduction of this chapter.

3.1.1 Constructing a Block Pair for the Local Poincaré Map

Since the system is linear in the neighborhood of 0, we can get an explicit formula for the local return map. Define

$$\Sigma_{in} = \{x : x_i = 1, 0 < x_o \leq 1, |x_u| \leq 1\}$$

and analogously

$$\Sigma_{out} = \{x : x_o = 1, 0 < x_i \leq 1, |x_u| \leq 1\}$$

The return map for $\Phi_{loc} : \Sigma_{in} \to \Sigma_{out}$ is given by

$$\begin{aligned} T &= -\lambda_o^{-1} \log x_o(0) \\ x_o(T) &= 1 \\ x_i(T) &= x_o(0)^{\frac{\lambda_i}{\lambda_o}} \\ x_u(T) &= x_o(0)^{-\frac{\lambda_u}{\lambda_o}} x_u(0) \end{aligned} \tag{3.1.1.1}$$

The map is well defined and continuous in the cone $C = \{x : x_i = 1, 0 < x_o < 1, |x_u| \leq x_o\}$. We therefore have a superlinear contraction in the $x_i x_o$-direction and a superlinear expansion of the x_u direction.

The next step is the construction of apropriate block pairs. The incoming block will be cusp-like with the following form for $\varepsilon_{in} > 0$ and $\vartheta_{in} \geq 0$:

$$\begin{aligned} N_{in}^{\vartheta_{in},\varepsilon_{in}} &= \{x \in \Sigma_{in} : 0 < x_o \leq \varepsilon_{in}, |x_u| < |x_o|^{1+\vartheta_{in}}\} \\ E_{in}^{\vartheta_{in},\varepsilon_{in}} &= \{x \in \Sigma_{in} : 0 < x_o \leq \varepsilon_{in}, |x_u| = |x_o|^{1+\vartheta_{in}}\}. \end{aligned} \tag{3.1.1.2}$$

The outgoing block will be pancake-like for $\varepsilon_{out} > 0$ and $\vartheta_{out} > 0$:

$$\begin{aligned} N_{out}^{\vartheta_{out},\varepsilon_{out}} &= \{x \in \Sigma_{out} : 0 < x_i \leq \varepsilon_{out}, |x_u| < |x_i|^{1-\vartheta_{out}}\} \\ E_{out}^{\vartheta_{out},\varepsilon_{out}} &= \{x \in \Sigma_{out} : 0 < x_i \leq \varepsilon_{out}, |x_u| = |x_i|^{1-\vartheta_{out}}\}. \end{aligned} \tag{3.1.1.3}$$

In view of the explicit formula, we can check the parameter ranges for which these sets form a block pair for Φ_{loc}. We will consider the projection $\pi_{out} : \{x : x_o = 1, 0 < x_i < \varepsilon_{out}\} \to N_{out} \cup E_{out}$ defined via

$$\pi_{out}(x_i, x_o = 1, x_u) = \begin{cases} (x_i, 1, (\operatorname{sign} x_u) x_i^{1-\vartheta_{out}}) & \text{for } x \notin N_{out} \\ (x_i, 1, x_u) & \text{for } x \in N_{out} \end{cases} \tag{3.1.1.4}$$

Lemma 3.8 (Block pair for the local map). *Suppose that we have $\varepsilon_{out} > \varepsilon_{in}^{\frac{\lambda_i}{\lambda_o}}$ as well as*

$$\lambda_u + \lambda_i(1 + \vartheta_{out}) - \lambda_o(1 + \vartheta_{in}) > 0.$$

Then $\left(N_{in}^{\vartheta_{in},\varepsilon_{in}}, E_{in}^{\vartheta_{in},\varepsilon_{in}}, N_{out}^{\vartheta_{out},\varepsilon_{out}}, E_{out}^{\vartheta_{out},\varepsilon_{out}}\right)$ is a block pair for $\tilde{\Phi}_{loc} = \pi_{out} \circ \Phi_{loc}$. Furthermore, the induced map $h_0(\tilde{\Phi}_{loc}) : h_0(E_{in}) \to h_0(E_{out})$ maps non null-homotopic to non null-homotopic classes (i.e. does not merge any path-connected components). This means the the images of the $\pm x_u > 0$ sides of E_{in} are contained in the $\pm x_u > 0$ sides of E_{out}.

Proof. The proof is done by direct calculation. At first we have to check, that the image of any point in $N_{in} \cup E_{in}$ lies in the strip on which π_{out} is defined. This can be calculated easily.

Secondly, we have to check that the images of the components of E_{in} lie outside of E_{out}, i.e. that for $\pm x_u = x_o^{1+\vartheta_{in}}$ in Σ_{in} the image under Φ_{loc} has $\pm x_u > x_i^{1-\vartheta_{out}}$. Plugging in the explicit formula of Φ_{loc} we get for $\pm x_u = x_o^{1+\vartheta_{in}}$

$$\begin{aligned}\log\left|x_u(T)x_i(T)^{-(1-\vartheta_{out})}\right| &= \log \pm x_u(0) - \frac{\lambda_u}{\lambda_o}\log x_o(0) - (1-\vartheta_{out})\frac{\lambda_i}{\lambda_o}\log x_o(0) \\ &= \left(1+\vartheta_{in} - \frac{\lambda_u}{\lambda_o} - (1-\vartheta_{out})\frac{\lambda_i}{\lambda_o}\right)\log x_o(0) \\ &= -\left(\lambda_u + \lambda_i(1-\vartheta_{out}) - \lambda_o(1+\vartheta_{in})\right)\lambda_o^{-1}\log x_o(0)\end{aligned}$$

Positivity of the last term shows that the proof is done. □

With these calculation we can see that it is possible to chose positive ϑ_{in} and ϑ_{out} which make $(N_{in}, E_{in}, N_{out}, E_{out})$ a block pair. The other parameters ε_{in} and ε_{out} will be chosen later.

3.1.2 Constructing a Block Pair for the Global Poincaré Map

Our aim is now to make $(N_{out}, E_{out}, N_{in}, E_{in})$ a block pair for the global return map $\Phi_{glob} : \Sigma_{out} \to \Sigma_{in}$. By the assumptions on the homoclinic orbit we have $\Phi_{glob}(0,1,0) = (1,0,0)$ and by the implicit function theorem Φ_{glob} is a local diffeomorphism. Since Φ_{glob} is a diffeomorphism we can equivalently prove that $(N_{out}, E_{out}, \Phi_{glob}^{-1}[N_{out}], \Phi_{glob}^{-1}[E_{out}])$ is block pair for a suitable projection (i.e. for $F = id$ and $\tilde{F} = \pi$).

Whether this is the case depends on the vector $\mathrm{D}\Phi_{glob}^{-1}(1,0,0) \cdot e_o$ where e_o is the unit vector in x_o-direction. This is because N_{in} is a cusp around e_o; if $\mathrm{D}\Phi_{glob}^{-1}(1,0,0) \cdot e_o$ points into the inside of N_{out} we can construct a block pair, as depicted in Figure 3.3.

Lemma 3.9 (Block pair for the global map). *Suppose that (N_{out}, E_{out}) and (N_{in}, E_{in}) have been constructed with some $\vartheta_{in} > 0$ and $\vartheta_{out} > 0$. Suppose further that $\left(\mathrm{D}\Phi_{glob}^{-1}(1,0,0) \cdot e_o\right)_i > 0$, i.e. $\partial_{x_o}\Phi_{glob}^{-1}(0,0,1)_i > 0$. Then it is possible to chose $\varepsilon_{out} > 0$ and $\varepsilon_{in} > 0$ small enough that a block pair for Φ_{glob}^{-1} is formed.*

Proof. The easiest way to proof this is "proof by picture" and say that the assertion is obvious by Figure 3.3. This however does not provide us with explicit estimates on the ε and may be unsatisfactory if one is not inclined to visual proofs. Therefore we will now construct ε_{in} explicitly.

Consider

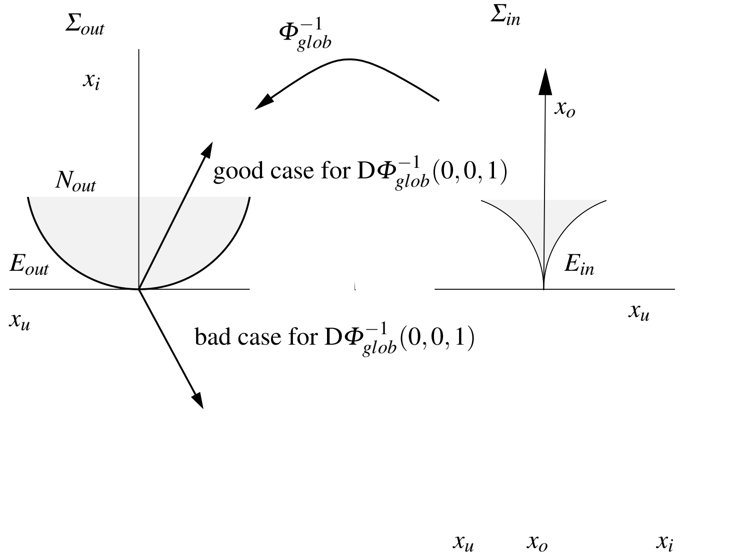

Fig. 3.3 The geometric condition for Block pairs

$$A = \mathrm{D}\Phi^{-1}_{glob}(1,0,0) = \begin{pmatrix} \partial_{x_o}\Phi^{-1}_{glob,u} & \partial_{x_o}\Phi^{-1}_{glob,i} \\ \partial_{x_u}\Phi^{-1}_{glob,i} & \partial_{x_u}\Phi^{-1}_{glob,u} \end{pmatrix} = \begin{pmatrix} a^i_o & a^u_o \\ a^i_u & a^u_u \end{pmatrix}. \tag{3.1.2.1}$$

The required sign condition reads now as $a^i_o > 0$. Suppose that $\varepsilon_{in} > 0$ is small enough that $|\Phi^{-1}_{glob} - A|_{B^{\Sigma_{in}}_{\varepsilon_{in}}(1,0,0),C^1} < \delta$ for some $\delta > 0$, which will be fixed later. The set E_{in} consists of two curves $\gamma_\pm$ with $\gamma_{\pm,u} = \pm\gamma^{1+\vartheta_{in}}_{\pm,o}$. The curves are graphs over the x_o-axis with

$$\pm\frac{\mathrm{d}\gamma_{\pm,u}}{\mathrm{d}\gamma_{\pm,o}} = (1+\vartheta_{in})\gamma^{\vartheta_{in}}_{\pm,o} \le 2\varepsilon^{\vartheta_{in}}_{in}.$$

The preimages of these two bounding curves are again curves $\tilde{\gamma}_\pm$ with

$$\left|\frac{\mathrm{d}\tilde{\gamma}_{\pm,u}}{\mathrm{d}\tilde{\gamma}_{\pm,i}}\right| = \frac{\partial_{x_o}\Phi^{-1}_{glob,u}\mathrm{d}\gamma_{\pm,o} + \partial_{x_u}\Phi^{-1}_{glob,u}\mathrm{d}\gamma_{\pm,u}}{\partial_{x_o}\Phi^{-1}_{glob,i}\mathrm{d}\gamma_{\pm,o} + \partial_{x_u}\Phi^{-1}_{glob,i}\mathrm{d}\gamma_{\pm,u}} \le \frac{|a^u_o| + |a^u_u| 2\varepsilon^{\vartheta_{in}}_{in} + 2\delta}{a^i_o - |a^i_u| 2\varepsilon^{\vartheta_{in}}_{in} - 2\delta}$$

If $0 < \delta < \frac{1}{3}a^i_o$ and ε_{in} are small enough we therefore get bounds on the derivatives and the curves are again graphs over the x_i-axis. Next, we need to construct an appropriate projection of the block $N_{out} \cup E_{out}$ on $\Phi^{-1}_{glob}[E_{in} \cup N_{in}]$ which is the identity on $\Phi^{-1}_{glob}[N_{in}]$ and projects the remainder of the strip on E_{in}. A similar projection as in the previous part works well, if the upper boundary of the cusp lies above the strip, i.e. if

$$(a_o^i - |a_u^i|\varepsilon_{in}^{\vartheta_{in}} - 2\delta)\varepsilon_{in} \geq \varepsilon_{out}$$

The next part –the most important part– is to show that the bounding curves $\tilde{\gamma}_\pm$ do not intersect the bounding curves $\zeta_\pm$ of the E_{out}-block. We can calculate that

$$\pm\frac{\mathrm{d}\zeta_{\pm,u}}{\mathrm{d}\zeta_{\pm,i}} = (1-\vartheta_{out})\zeta_{\pm,o}^{-\vartheta_{out}} \geq \frac{1}{2}\varepsilon_{out}^{-\vartheta_{out}}.$$

The assertion therefore holds when

$$\frac{1}{2}\varepsilon_{out}^{-\vartheta_{out}} \geq \frac{|a_o^u| + |a_u^u|2\varepsilon_{in} + 2\delta}{a_o^i - |a_u^i|2\varepsilon_{in} - 2\delta}.$$

which is certainly true for ε_{out} small enough. □

The assumptions on $\mathrm{D}\Phi_{glob}^{-1}$ of Lemma 3.9 are unsatisfactorily dependent on our choice of the Poincaré-sections. There is another characterization, which was used in the Theorem 3.7.

Proposition 3.10. *Let γ be parametrized such that $\gamma(0) = e_i$ $\gamma(-T) = e_o$. Let $v(t)$ be the solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \mathrm{D}F(\gamma(t))v(t) \quad v(0) = e_o.$$

Then $v_i(T) > 0$ if and only if $\partial_{x_o}\Phi(1,0,0)_i > 0$.

Proof. We have $v(T) = \mathrm{D}\phi^{-T}(1,0,0)\cdot e_o$. By the implicit function theorem,

$$\mathrm{D}\Phi_{glob}^{-1} = \left(\mathrm{D}\phi_{x_i}^{-T}, \mathrm{D}\phi_{x_u}^{-T}\right)^\top.$$

□

3.1.3 Application of the Topological Method

Now we finally prove Theorem 3.7 by combing the two block pairs in Lemma 3.8 and Lemma 3.9 to get a block sequence and applying Theorem 2.19 to obtain a separating stable set.

Proof (Proof of Theorem 3.7). By Proposition 3.10, we can apply the two Lemmas Lemmas 3.8 and 3.9 on block pairs. We just need two show that we can fulfill the requirements on ε_{in} and ε_{out} simultaneously. Therefore we will walk through all the Lemmas again and chose one constant after the other.

Chose $0 < \vartheta_{in} < 1/2$ and $0 < \vartheta_{out} < 1/2$ according to the Lemma on the block pair for the local map. Then chose $0 < \varepsilon_{in}$ small enough that the bounds on the slopes of the bounding curves in the global Lemma 3.9 are finite. Then there are constants $C_1 > 0$ and $C_2 > 0$, such that $\varepsilon_{out} < C_1\varepsilon_{in}$ and $\varepsilon_{out} < C_2$ suffice for Lemma

3.9 to hold. Fix $\varepsilon_{out} = C_1/2\varepsilon_{in}$ and chose ε_{in} small enough that $\varepsilon_{out} < C_2$ and $C_1/2\varepsilon_{in} = \varepsilon_{out} > \varepsilon_{in}^{\lambda_i/\lambda_o}$ in Lemma 3.8 (which is possible since $\lambda_i > \lambda_o$). Therefore, we have constructed a block sequence for Φ_{glob} and Φ_{loc} and can apply the topological Theorem 2.19. □

3.2 The Nonlinear Case

If we want to extend our analysis to the nonlinear case, where f does not vanish outside of a neighborhood of 0, we have one main difficulty: We do not have an explicit formula for Φ_{glob} anymore, as it was used in Section 3.1.1. We therefore have to work harder to get local estimates, which allow for an analogue of the Local Block Lemma 3.8. We will start by using a set of standard changes of coordinates to simplify our system in Section 3.2.1. Then we will show some basic local estimates and argue that these estimates are insufficient for a satisfactory Local Block Lemma in Section 3.2.2. In the Section 3.2.3, we will construct an approximate stable manifold to γ_{in}, which will allow us to a local block Lemma in Section 3.2.4 and thus prove the main Theorem 3.2 in Section 3.2.5. We will close by discussing the necessary changes to our proof in the center case $\lambda_u = 0$ in Section 3.2.6.

3.2.1 Changing the Coordinates

We will now use the invariant manifold theorems of Chapter 1 to change the coordinates of the System 3.1. The goal of this section is to prove the following proposition:

Proposition 3.11. *Suppose we have a dynamical system $\dot{x} = F(x)$, which fulfills the Assumptions 3.1 and suppose that $\varepsilon_g > 0$. Then there is a C^1-diffeomorphism $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ and a bounded C^1-Euler Multiplier $\mu : \mathbb{R}^3 \to \mathbb{R}$ with $\sup_{x\in\mathbb{R}^3}(\mu(x), \mu(x)^{-1}) < \infty$, such that the transformed system $\dot{x} = \tilde{F}(x)$ with $\tilde{F}(x) = \mu(x)\mathrm{D}\psi(x)^{-1}F(\psi(x))$ has the following properties:*

1. *The system fulfills the Assumptions 3.1, except for the regularity assumptions.*
2. *The homoclinic orbit is locally contained in the x_i and x_o axis i.e. has the property that for $B = \{x : |x_i| \le 1, |x_o| \le 1, |x_u| \le 1\}$ the following holds:*

$$\gamma(\mathbb{R}) \cap B = \{x : x_i \in (0,1], x_o = 0, x_u = 0\} \cup \{x : x_i = 0, x_o \in (0,1], x_u = 0\}.$$

3. *The system has locally the following form (i.e. $\tilde{F}_{|B}$ is given by):*

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + g_i^i(x_i, x_o, x_u)x_i \\ \dot{x}_o &= \lambda_o x_o + g_o^o(x_i, x_o, x_u)x_o + g_o^u(x_i, x_o, x_u)x_u \\ \dot{x}_u &= \lambda_u x_u + g_u^{io}(x_i, x_o, x_u)x_i x_o, \end{aligned} \tag{3.2.1.1}$$

4. The following estimates on the nonlinearities hold:

$$\max\left(\|g_i^i\|_{C^2,B}, \|g_o^u\|_{C^2,B}, \|g_o^o\|_{C^2,B}, \|g_u^{io}\|_{C^1,B}\right) \leq \varepsilon_g$$

The change of coordinates only uses standard methods. Readers who are familiar with such changes of coordinates may therefore omit the remainder of this section.

3.2.1.1 Application of Invariant Manifold Theorems

The first step of our transformation is to apply invariant manifold theorems in order to ensure that the homoclinic orbit γ lies locally on the x_i and the x_o half-axes.

The Unstable Manifold.
By the unstable manifold theorem, there exists a neighborhood $U \subset \mathbb{R}^2$ of 0 and a function $\xi^{ou} : U \to \mathbb{R}$ with $\xi^{ou} \in C^k$, such that

$$M_{uo} = \{(x_i, x_o, x_u) : x_i = \xi^{ou}(x_o, x_u), (x_o, x_u) \in U\}$$

is the set of those x, which converge to 0 under the time-reversed flow and whose projections never leave U. The unstable manifold is tangential to the $x_o x_u$-plane, i.e. $\xi^{ou}(0) = 0$ and $D\xi^{ou}(0) = 0$. The unstable manifold contains the backward part of the homoclinic orbit, i.e. for $t < t_0$ we have $\gamma(t) \in M_{ou}$.
Consider the changed coordinates

$$\tilde{x} = \varphi(x) = (x_i - \xi^{ou}(x_o, x_u), x_o, x_u)$$

This change of coordinates is C^3 and the unstable manifold in these coordinates has the simple form

$$M_{uo} = \{\tilde{x} : \tilde{x}_i = 0\}$$

Since the unstable manifold is invariant we can assume for the transformed vectorfield

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = (-\lambda_i \tilde{x}_i, \lambda_o \tilde{x}_o, \lambda_u \tilde{x}_u) + \tilde{f}(\tilde{x})$$

that $\tilde{f}_i(0, \tilde{x}_o, \tilde{x}_u) = 0$. We then rename the variables back and can therefore without loss of generality assume this identity.

The Strong Unstable Manifold.
By the strong unstable manifold theorem, there exists a neighborhood $0 \in U \subset \mathbb{R}$ and a function $\xi^o : U \to \mathbb{R}^2$ with $\xi^o \in C^k$, such that

$$M_o = \{(x_i, x_o, x_u) : x_i = \xi_i^o(x_o), x_u = \xi_u^o(x_o), x_u \in U\}$$

is the set of x, which converge to 0 under the time-reversed flow sufficiently fast and whose projections never leave U. Furthermore, the homoclinic orbit γ lies in

M_o for all sufficiently small times, i.e. for $t \to -\infty$. The strong unstable manifold is tangential to the x_o-axis, i.e. $\xi^o(0) = 0$ and $D\xi^o(0) = 0$. By our first change of coordinates and since the strong unstable manifold is a submanifold of the unstable manifold, we have $\xi_i^o = 0$.
Consider the changed coordinates $\tilde{x} = \varphi(x) = (x_i, x_o, x_u - \xi_u^o(x_o))$. This change of coordinates is C^3 and the strong unstable manifold in these coordinates has the simple form $M_o = \{\tilde{x} : \tilde{x}_i = 0, \tilde{x}_u = 0\}$. Since the strong unstable manifold is invariant we can assume for the transformed vectorfield that $\tilde{f}_u(0, \tilde{x}_o, 0) = 0$. The identity $\tilde{f}_i(0, \tilde{x}_o, \tilde{x}_u) = 0$ is preserved. We then rename the variables back and can therefore without loss of generality assume these identities.

The Stable Manifold.
By the stable manifold theorem, there exists a neighborhood $0 \in U \subset \mathbb{R}$ and a function $\xi^i : U \to \mathbb{R}^2$ with $\xi^i \in C^k$, such that

$$M_i = \{(x_i, x_o, x_u) : x_o = \xi_o^i(x_i), x_u = \xi_u^i(x_i), x_i \in U\}$$

is the set of x, which converge to 0 under the flow and whose projections never leave U. The stable manifold is tangential to the x_i-axis, i.e. $\xi^i(0) = 0$ and $D\xi^i(0) = 0$. The stable manifold contains the forward part of the homoclinic orbit, i.e. for $t > t_0$ we have $\gamma(t) \in M_i$.
Consider the changed coordinates $\tilde{x} = \varphi(x) = (x_i, x_o - \xi_o^i(x_i), x_u - \xi_u^i(x_i))$ This change of coordinates is C^3 and the stable manifold in these coordinates has the simple form $M_i = \{\tilde{x} : \tilde{x}_o = 0, \tilde{x}_u = 0\}$
Since the stable manifold is invariant we can assume for the transformed vectorfield that $\tilde{f}_o(x_i, 0, 0) = 0$ and $\tilde{f}_u(x_i, 0, 0) = 0$.
Since $\xi^i(0) = 0$, the change of coordinates is the identity on the unstable manifold. Therefore the previous identities get preserved.

3.2.1.2 Applying the Taylor Theorem

By the unstable manifold theorem, we can assume w.l.o.g. that $f_i(0, x_o, x_u) = 0$ in a neighborhood U of 0. By the strong unstable manifold theorem we can assume that $f_u(0, x_o, 0) = 0$ in a neighborhood U of 0. By the stable manifold theorem we can assume that $f_o(x_i, 0, 0) = f_u(x_i, 0, 0) = 0$ in a neighborhood U of 0. We will now use a Taylor expansion in order to show that we can write the system in a form similiar to the one given in Proposition 3.11.
By applying the Taylor theorem we can write

$$f_i(x_i, x_o, x_u) = \int_0^1 \partial_{x_i} f_i(\vartheta x_i, x_o, x_u) \mathrm{d}\vartheta \cdot x_i = g_i(x_i, x_o, x_u) x_i$$

Similarly we can write

$$\begin{aligned} f_o(x_i,x_o,x_u) =& \int_0^1 \partial_{x_o} f_o(x_i,\vartheta x_o,\vartheta x_u)\mathrm{d}\vartheta \cdot x_o \\ &+ \int_0^1 \partial_{x_u} f_o(x_i,\vartheta x_o,\vartheta x_u)\mathrm{d}\vartheta \cdot x_u \\ =& g_o^o(x_i,x_o,x_u)x_o + g_o^u(x_i,x_o,x_u)x_u \end{aligned}$$

In order to rewrite f_u we need to apply the Taylor theorem twice. Just as we did for f_o we first project onto the stable manifold:

$$\begin{aligned} f_u(x_i,x_o,x_u) =& \int_0^1 \partial_{x_u} f_u(x_i,\vartheta x_o,\vartheta x_u)\mathrm{d}\vartheta \cdot x_u \\ &+ \int_0^1 \partial_{x_o} f_u(x_i,\vartheta x_o,\vartheta x_u)\mathrm{d}\vartheta \cdot x_o \end{aligned}$$

Then we project onto the strong stable manifold:

$$\begin{aligned} \partial_{x_o} f_u(x_i,\vartheta x_o,\vartheta x_u) =& \partial_{x_o} f_u(0,\vartheta x_o,0) \\ &+ \int_0^1 \partial_{x_i}\partial_{x_o} f_u(\sigma x_i,\vartheta x_o,\sigma\vartheta x_u)\mathrm{d}\sigma \cdot x_i \\ &+ \int_0^1 \partial_{x_u}\partial_{x_o} f_u(\sigma x_i,\vartheta x_o,\sigma\vartheta x_u)\mathrm{d}\sigma \cdot \vartheta x_u \end{aligned}$$

The first term vanishes identically. The other terms can be plugged into the first integral and yield

$$f_u(x_i,x_o,x_u) = g_u^u(x_i,x_o,x_u)x_u + g_u^{io}(x_i,x_o,x_u)x_i x_o.$$

3.2.1.3 Rescaling and Euler Multipliers

We have now shown that we can transform the systen (3.0.0.1) into the following form:

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + g_i^i x_i \\ \dot{x}_o &= \lambda_o x_o + g_o^o x_o + g_o^u x_u \\ \dot{x}_u &= \lambda_u x_u + g_u^u x_u + g_u^{io} x_i x_o, \end{aligned} \tag{3.2.1.2}$$

In order to prove Propositoin 3.11, we will now prove that we can also archieve the estimates on the g and $g_o^o \equiv 0$. At first we will rescale the system in order to get the estimates on $|g|_\infty < \varepsilon_g$.
Consider the change of coordinates $\psi(x) = x/\lambda$ for $\lambda > 0$ and a vectorfield $F(x) = Ax + f(x)$ with $f \in C^N$, $N \geq 2$ and $f(0) = 0$ as well as $\mathrm{D}f(0) = 0$. Then the transformed vectorfield $\tilde{F} = \mathrm{D}\psi^{-1} F \circ \psi$ has the form $\tilde{F} = A\tilde{x} + \tilde{f}(\tilde{x})$ with $\tilde{f}(\tilde{x}) = \lambda f(\frac{x}{\lambda})$. Therefore we get for $2 \leq k \leq N$:

$$|\tilde{f}|_{C^k,B_1(0)} \leq \lambda^{1-k}|f|_{C^k,B_{\lambda^{-1}}(0)}$$

This yields immediately the estimates

$$|\tilde{f}|_{C^1,B_1(0)} \leq \lambda^{-1}|f|_{C^2,B_{\lambda^{-1}}(0)}$$

and

$$|\tilde{f}|_{C^0,B_1(0)} \leq \lambda^{-1}|f|_{C^2,B_{\lambda^{-1}}(0)}.$$

By considering the construction of the g in Section 3.2.1.2, we can see that rescaling with $\lambda > 0$ suffieciently large yields

$$\max\left(\|g_i^i\|_{C^2,B}, \|g_o^u\|_{C^2,B}, \|g_o^o\|_{C^2,B}, \|g_u^u\|_{C^2,B}, \|g_i^{io}\|_{C^1,B}\right) \leq \varepsilon_g$$

The final step is to use an Euler multiplier to archieve $g_u^u \equiv 0$. Suppose we study a system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$. A Lipschitz continuous function $\mu : \mathbb{R}^n \to \mathbb{R}^+$ which is bounded and bounded away from zero is called Euler multiplier. Then the changed system $\dot{x} = \mu(x)f(x)$ has obviously the same trajectories as the original systems and all Poincaré-maps are unchanged. Euler multipliers can be viewed as a space-dependent rescaling of time.
Consider the Euler multiplier

$$\mu(x_i, x_o, x_u) = \frac{\lambda_u}{\lambda_u + g_u^u(x_i, x_o, x_u)}.$$

This Euler multiplier is well-defined when $\varepsilon_g < \lambda_u$. Furthermore, $|1-\mu|_{C^2,B} \leq \varepsilon_g$ for $\lambda > 0$ large enough.
The equations then become after multiplying with μ:

$$\begin{aligned}
\dot{x}_i &= -\lambda_i x_i + \frac{\lambda_u g^i + \lambda_i g_u^u}{\lambda_u + g_u^u} x_i \\
\dot{x}_o &= \lambda_o x_o + \frac{\lambda_u g_o^o - \lambda_o g_u^u}{\lambda_u + g_u^u} x_o + \frac{\lambda_u g_u^o}{\lambda_u + g_u^u} x_u \\
\dot{x}_u &= \lambda_u x_u + \frac{\lambda_u g_{io}^u}{\lambda_u + g_u^u} x_i x_o
\end{aligned}$$

After renaming all functions we see that Proposition 3.11 is proven.

3.2.2 General Estimates

We have now proven, that we can consider without loss of generality a system of the form stated in Proposition 3.11. We will now start with a local analysis, similiar to Section 3.1.1 in the linear case. We will now fix $\varepsilon_g > 0$ small enough, such that :

$$\varepsilon_g < \frac{1}{4}(\lambda_i - \lambda_o) \tag{3.2.2.1}$$

We want to get a well defined map which maps an initial condition in $\Sigma_{in} = \{x : x_i = 1\}$ to a point in $\Sigma_{out} = \{x : x_o = 1\}$. This must fail for initial conditions which result in the trajectory leaving the box $B = \{x : |x_i| \leq 1, |x_o| \leq 1, |x_u| \leq 1\}$ through any other boundary. Therefore we will only consider solutions in the "tent" $C = \{(x_i, x_o, x_u) : |x_u| \leq x_o, 0 < x_o < 1, 0 < x_i < 1\}$. Furthermore, we will make heavy use of the quantity $d = x_i x_o$, which is a good measure for the distance from the homoclinic orbit.

Proposition 3.12. *Let $x \in B$ and let $T_o = \inf\{t > 0 : x(t) \in B\}$ denote the exit time of x. If $|x_u(t) \leq x_o(t)|$, then $|x_u(\tau)| \leq x_o(\tau)$ for all $\tau \in [t, T_o]$. If $d(t) \leq |x_u(t)|$, then $d(\tau) \leq |x_u(\tau)|$ for all $\tau \in [t, T_o]$.*

Proof. Suppose that for some time $|x_u| = x_o$. Then $\mathrm{D}_t|x_u| \leq (\lambda_u + 2\varepsilon_g)\,|x_u|$ and $\frac{\mathrm{d}}{\mathrm{d}t}x_o \geq (\lambda_o - 2\varepsilon_g)\,x_o$. But by our assumptions, $x_u = x_0$ and $\lambda_u + 2\varepsilon_g < \lambda_o - 2\varepsilon_g$. Suppose that for some time $d(t) = |x_u(t)|$. Then $\dot{d}d < 0$ and $\dot{x}_u/x_u > 0$. □

Let T_o denote the time at which an orbit leaves B. By Proposition 3.12, the return map $\Phi_{loc} : C \cap \Sigma_{in} \to \Sigma_{out}$ is well-defined and continuous (it is continuous since the passage time T_o is finite at every point. It is well-defined, since we must leave the box trough Σ_{out}, by virtue of Proposition 3.12) We get inside of C the following trivial bounds:

$$\begin{aligned} \lambda_o - \lambda_i - 3\varepsilon_g \leq \dot{d}/d \quad &\leq \lambda_o - \lambda_i + 3\varepsilon_g \\ \lambda_o - 2\varepsilon_g \leq \dot{x}_o/x_o \quad &\leq \lambda_o + 2\varepsilon_g \\ -\lambda_i - \varepsilon_g \leq \dot{x}_i/x_i \quad &\leq -\lambda_i + \varepsilon_g \end{aligned} \tag{3.2.2.2}$$

Since $x_o(T_o) = 1$ we get

$$\frac{-1}{\lambda_o + 2\varepsilon_g}\log x_o(0) \leq T_o \leq \frac{-1}{\lambda_o - 2\varepsilon_g}\log x_o(0)$$

and therefore

$$x_i(T_o) \leq x_o^{\left(\frac{\lambda_i - \varepsilon_g}{\lambda_o + 2\varepsilon_g}\right)}(0)$$

Therefore we have a superlinear contraction for the local passage in the $x_i x_0$ direction with a power $\frac{\lambda_i - \varepsilon_g}{\lambda_o + 2\varepsilon_g} > 1$.

Remark 3.13. In principle we could now construct a local block-pair, since we can prove estimates on $x_u(T_o)$ for the boundary cases $x_u(0) = \pm x_o(0)$:
By Propositon 3.12, we can assume $|x_u(t)| > d(t)$ for all $t \in [0, T_o]$ when $\pm x_u(t) = x_o(0)$. Then $\frac{\dot{x}_u}{x_u} \geq \lambda_u - 2\varepsilon_g$. Therefore we get

$$\log|x_u(T_o)| \geq \left(1 - \frac{\lambda_u - 2\varepsilon_g}{\lambda_o + 2\varepsilon_g}\right) \log x_o(0).$$

We could immediately use these estimates to construct a block-pair which is cusp-like in Σ_{in} and pancake-like in Σ_{out}. This would allow us to prove the following Theorem:

Claim. Suppose we have a system fulfilling the Assumptions 3.1 in the form given in Proposition 3.11. Suppose that

$$\mathrm{D}\Phi_{glob}^{-1}[C \cap \Sigma_{in}] \subseteq \{x : x_i > 0\}.$$

Then there is a $\varepsilon > 0$, such that W_s is separating the cone-boundaries $\partial^{\pm}C = \{x \in B_{\Sigma_{in}}(\varepsilon)1,0,0 : x_o = \pm x_u\}$ in C. Therefore, W_s is especially nontrivial.

This geometric condition on $\mathrm{D}\Phi_{glob}$ is however extremely unsatisfactory. Further rescaling allows us to construct block pairs for smaller cones C, but will at the same time cost us in terms of $\mathrm{D}\Phi_{glob}$. Therefore we will not prove this claim and will instead do a more intricate local analysis, in order to get a useful geometric condition.

3.2.3 Approximation of the Stable Set

We have changed the coordinates of the System (3.1) to the form given in Proposition 3.11 and proven some preliminary estimates. These estimates do however not suffice to prove our main Theorem 3.2. We will need to construct a block-pair which is cusp-like in Σ_{in}.
If we consider the equation for x_u,

$$\dot{x}_u = \lambda_u x_u + g_u^{io}(x_i, x_o, x_u) x_i x_o$$

then we see that the nonlinear terms are of the same order as the x_u terms near the stable trajectory. Therefore, we can only get control of the x_u dynamics when x_u is of the same order as the distance from the stable manifold. In order to chose arbirarily small cones or get estimates for a cusp, we will need to turn the nonlinear g_u^{io}-term into a higher order term.
The rough idea is that the stable set should intuitively be a C^1 Manifold. Coordinates in this stable manifold would make the term vanish completely. Coordinates which at least have the correct tangent space along the incoming trajectory should transform the linear order part in the x_o-direction away and leave us with only higher order terms. Then we should be able to construct a block pair which is a cusp around the tangent vector v to the stable manifold in the section Σ_{in}. This would allow us to prove the main Theorem 3.2.
There are many different ways to archieve such coordinates. Our approach will be to consider a suitable approximation, which will allow us to immediately find the

stable manifold. Then we will show that the difference in behaviour from the approximation is of higher order.

Remark 3.14. An alternative approach is to consider the linearization along the γ_{in} trajectory. Then we can consider the non-autonomous linear system $\dot{v} = \mathrm{D}F(\gamma(t))v$ and look for solutions with $\lim_{t\to\infty} v_u(t) = 0$. Since the system is linear, there are no regularity problems and the approach generalizes to higher dimensions. The main appeal of this aproach in higher dimensions is that we can use topological methods to achieve some solution with prescribed asymptotics and can then take direct sums of the spanned subspaces (in nonlinear systems we can only take intersections of invariant manifolds and never get any regularity for free). We will revisit this approach in Proposition 3.19.

The main problem in finding a stable manifold is the instability of the x_o-direction; however, the product $d = x_i x_o$ is stable and is in some sense the stable direction our stable set corresponds to. Therefore we will simplify the system by replacing x_o by d and then apply the stable manifold theorem. Let x' denote the approximate vectorfield and $\dot{x}$ denote the original one. A good approximation is given by

$$\begin{aligned} x_i' &= -\lambda_i x_i + g_i^i(x_i,0,x_u)x_i + \partial_{x_o} g_i^i(x_i,0,x_u)d \\ x_u' &= \lambda_u x_u + g_u^{io}(x_i,0,x_u)d \\ d' &= (\lambda_o - \lambda_i)d + (g_i^i(x_i,0,x_u) + g_o^o(x_i,0,x_u))d \\ &\quad + (g_o^u(x_i,0,x_u)x_i + \partial_{x_o} g_o^u(x_i,0,x_u)d)\, x_u \end{aligned} \tag{3.2.3.1}$$

Let $\xi_u = \xi_u(x_i,d)$ be the stable manifold of the approximate system. By the stable manifold theorem, ξ_u is C^1 and invariant. Therefore we get

$$x_u'(x_i,\xi_u(x_i,d),d) = \partial_{x_i}\xi_u(x_i,d)x_i'(x_i,\xi_u(x_i,d),d) + \partial_d \xi_u(x_i,d)d'(x_i,\xi_u(x_i,d),d)$$

Putting the invariance into the original equation yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x_u - \xi_u) &= \dot{x}_u - \partial_{x_i}\xi_u \dot{x}_i - \partial_d \xi_u \dot{d} && (3.2.3.2) \\ &= \dot{x}_u(x_i,x_o,x_u) - x_u'(x_i,\xi_u(x_i,x_i x_o),x_i x_o) && (3.2.3.3) \\ &\quad - \partial_{x_i}\xi_u(x_i,x_i x_o)\cdot\big(\dot{x}_i(x_i,x_o,x_u) - x_i'(x_i,\xi_u(x_i,x_i x_o),x_i x_o)\big) && (3.2.3.4) \\ &\quad - \partial_d \xi_u(x_i,x_i x_o)\cdot\big(\dot{d}(x_i,x_o,x_u) - d'(x_i,\xi_u(x_i,x_i x_o),x_i x_o)\big) && (3.2.3.5) \end{aligned}$$

We will spend the remaining part of this section proving the following fact:

Proposition 3.15. *There exist C^0-Functions $\tilde{g}_u$ and $\tilde{g}_u^{ioo}$ with $|\tilde{g}_u|_{B,C^0} \le 7\varepsilon_g$ and $|\tilde{g}_u^{ioo}|_{B,C^0} \le 4\varepsilon_g$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_u - \xi_u) = (\lambda_u + \tilde{g}_u \cdot x_i)(x_u - \xi_u) + \tilde{g}_u^{ioo} x_i x_o^2. \tag{3.2.3.6}$$

In view of the rescaling in Section 3.2.1.3, we will however assume without loss of generality that $|\tilde{g}_u|_{B,C^0} \le \varepsilon_g$ and that $|\tilde{g}_u^{ioo}|_{B,C^0} \le \varepsilon_g$.

Proof. Each of the three terms of (3.2.3.2) will be handled separately. All three terms can be split into one part which can be absorbed into $x_u - \xi_u$ and another part of order $x_i x_o^2$. We expand the first term (3.2.3.3) by Taylor's formula:

$$\begin{aligned}\dot{x}_u - x_u' =& \lambda_u(x_u - \xi_u(x_i, x_i x_o)) + \left(g_u^{io}(x_i, x_o, x_u) - g_u^{io}(x_i, 0, \xi_u(x_i, x_i x_o))\right) x_i x_o \\ =& \lambda_u(x_u - \xi_u) + \int_0^1 \frac{\partial}{\partial x_o} g_u^{io}(x_i, \vartheta x_o, \vartheta x_u + (1-\vartheta)\xi_u) \mathrm{d}\vartheta \cdot x_o \cdot x_i x_o \\ &+ \int_0^1 \frac{\partial}{\partial x_u} g_u^{io}(x_i, \vartheta x_o, \vartheta x_u + (1-\vartheta)\xi_u) \mathrm{d}\vartheta \cdot (x_u - \xi_u) \cdot x_i x_o \\ =& (\lambda_u + \tilde{g}_{u,u} x_i x_o)(x_u - \xi_u) + \tilde{g}_{u,ioo} x_i x_o^2\end{aligned}$$

The second term (3.2.3.4) can be handled with a mixed-order Taylor's Formula. The first order expansion yields

$$\begin{aligned}\dot{x}_i - x_i' &= \left(g_i^i(x_i, x_o, x_u) - g_i^i(x_i, 0, \xi_u) - \partial_{x_o} g_i^i(x_i, 0, \xi_u) x_o\right) x_i \\ &= \int_0^1 \partial_{x_o} g_i^i(x_i, \vartheta x_o, \vartheta x_u + (1-\vartheta)\xi_u) - \partial_{x_o} g_i^i(x_i, 0, \xi_u) \mathrm{d}\vartheta \cdot x_o \cdot x_i \\ &\quad + \int_0^1 \partial_{x_u} g_i^i(x_i, \vartheta x_o, \vartheta x_u + (1-\vartheta)\xi_u) \mathrm{d}\vartheta \cdot (x_u - \xi_u) \cdot x_i.\end{aligned}$$

A second-order expansion then provides

$$\begin{aligned}\dot{x}_i - x_i' &= \int_0^1 \int_0^1 \partial_{x_o}^2 g_i^i(x_i, \sigma\vartheta x_o, \sigma\vartheta x_u + (1-\sigma\vartheta)\xi_u) \vartheta x_o \mathrm{d}\sigma \mathrm{d}\vartheta \cdot x_o \cdot x_i \\ &\quad + \int_0^1 \int_0^1 \partial_{x_u} \partial_{x_o} g_i^i(x_i, \sigma\vartheta x_o, \sigma\vartheta x_u + (1-\sigma\vartheta)\xi_u) \vartheta (x_u - \xi_u) \mathrm{d}\sigma \mathrm{d}\vartheta \cdot x_o \cdot x_i \\ &\quad + \int_0^1 \partial_{x_u} g_i^i(x_i, \vartheta x_o, \vartheta x_u + (1-\vartheta)\xi_u) \mathrm{d}\vartheta \cdot (x_u - \xi_u) \cdot x_i \\ &= \tilde{g}_{i,u} x_i (x_u - \xi_u) + \tilde{g}_{i,ioo} x_i x_o^2\end{aligned}$$

The third term (3.2.3.5) can be split:

$$\begin{aligned}\dot{d} - d' =& (g_i^i(x_i, x_o, x_u) + g_o^o(x_i, x_o, x_u)) x_i x_o + g_o^u(x_i, x_o, x_u) x_i x_u \\ &- (g_i^i(x_i, 0, \xi_u) + g_o^o(x_i, 0, \xi_u)) x_i x_o + (g_o^u(x_i, 0, \xi_u) + \partial_{x_o} g_o^u(x_i, 0, \xi_u) x_o) x_i \xi_u \\ =& \left(g_i^i(x_i, 0, \xi_u) + g_o^o(x_i, 0, \xi_u) - g_i^i(x_i, x_o, x_u) - g_o^o(x_i, x_o, x_u)\right) x_i x_o \\ &+ (g_o^u(x_i, 0, \xi_u) + \partial_{x_o} g_o^u(x_i, 0, \xi_u) x_o) \cdot x_i \cdot (x_u - \xi_u) \\ &- (g_o^u(x_i, 0, \xi_u) + \partial_{x_o} g_o^u(x_i, 0, \xi_u) x_o - g_o^u(x_i, x_o, x_u)) x_i x_u\end{aligned}$$

The first of these three subterms can be handled analogously to the term (3.2.3.3), i.e. via first-order Taylor's formula. The second subterm can be immediately absorbed into $x_u - \xi_u$. The third subterm allows a similar mixed-order Taylor expansion as the term of (3.2.3.4). We therefore get

$$\dot{d} - d' = \tilde{g}_{d,u} x_i \cdot (x_u - \xi_u) + \tilde{g}_{d,ioo} x_i x_o^2$$

Combining the three terms yields the expression in the proposition. Counting the absorbed terms and using the estimates in Proposition 3.11 yields the estimates on $\tilde{g}_u$ and $\tilde{g}_u^{ioo}$. □

3.2.4 Constructing a Block Pair for the Local Poicaré Map

In this section, we will prove a local block Lemma comparable to Lemma 3.1.1. The stable manifold to the approximate system has given us the coordinates which allow finer estimates. We will split the passage of an orbit into two sections: At first $x_i > x_o$ and we have higher order control over $x_u - \xi_u$. This allows us to control initial conditions in a cusp around ξ_u, which will get blown up to a “pancake” even in the first part of the passage. Afterwards everything is harmless: The cusp has expanded to a pancake, which is larger than a cone and we have already controlled cones with our general estimates in Section 3.2.2.
We will again use a cusp-like incoming block comparable to (3.1.1.2). Define

$$\begin{aligned} N_{in}^{\vartheta_{in},\varepsilon_{in}} &= \{x \in \Sigma_{in} : 0 < x_o \le \varepsilon_{in}, |x_u - \xi_u(1,x_o)| < |x_o|^{1+\vartheta_{in}}\} \\ E_{in}^{\vartheta_{in},\varepsilon_{in}} &= \{x \in \Sigma_{in} : 0 < x_o \le \varepsilon_{in}, |x_u - \xi_u(1,x_o)| = |x_o|^{1+\vartheta_{in}}\} \end{aligned} \tag{3.2.4.1}$$

We will use the same pancake-like outgoing block used in (3.1.1.3), which will be stated here again for convenience:

$$\begin{aligned} N_{out}^{\vartheta_{out},\varepsilon_{out}} &= \{x \in \Sigma_{out} : 0 < x_i \le \varepsilon_{out}, |x_u| < |x_i|^{1-\vartheta_{out}}\} \\ E_{out}^{\vartheta_{out},\varepsilon_{out}} &= \{x \in \Sigma_{out} : 0 < x_i \le \varepsilon_{out}, |x_u| = |x_i|^{1-\vartheta_{out}}\}. \end{aligned} \tag{3.2.4.2}$$

We also use the same projection $\pi_{out} : \{x : x_o = 1, 0 < x_i < \varepsilon_{out}\} \to N_{out} \cup E_{out}$ which was used in (3.1.1.4) and will be stated here again for convenience:

$$\pi_{out}(x_i, x_o = 1, x_u) = \begin{cases} \left(x_i, 1, (\operatorname{sign} x_u) x_i^{1-\vartheta_{out}}\right) & \text{for } x \notin N_{out}^{\vartheta_{out},\varepsilon_{out}} \\ (x_i, 1, x_u) & \text{for } x \in N_{out} \end{cases} \tag{3.2.4.3}$$

Lemma 3.16 (Local Block Pair). *There exist* $0 < \vartheta_{in} < 1/2$ *and* $0 < \vartheta_{out} < 1/2$ *as well as* $\hat{\vartheta} > 0$*, such that* $\left(N_{in}^{\vartheta_{in},\varepsilon_{in}}, E_{in}^{\vartheta_{in},\varepsilon_{in}}, N_{out}^{\vartheta_{out},\varepsilon_{out}}, E_{out}^{\vartheta_{out},\varepsilon_{out},}\right)$ *is a block pair for* $\tilde{\Phi}_{loc} = \pi_{out} \circ \Phi_{loc}$ *for all* $\varepsilon_{in} > 0$ *and* $\varepsilon_{in}^{1+\hat{\vartheta}} < \varepsilon_{out} < 1$*. Furthermore, the induced map* $h_0(\tilde{\Phi}_{loc}) : h_0(E_{in}) \to h_0(E_{out})$ *maps non null-homotopic to non null-homotopic classes (i.e. does not merge any path-connected components). This* *means the the images of the* $\pm(x_u - \xi_u) > 0$ *sides of* E_{in} *are contained in the* $\pm x_u > 0$ *sides of* E_{out}.

Proof. We consider only the "tent" C. This is possible, since $N_{out} \cup E_{out} \subseteq C$ and because of Proposition 3.12. Let $\hat{T}$ be the first time when $x_i = x_o$, i.e. $\hat{T} = \inf\{t > 0 : x_o(t) > x_i(t)\}$. Let T_o denote the exit-time, i.e. $T_o = \inf\{t > 0 : x_o(t) = 1\}$. Our first step will be to prove that the local map Φ_{loc} is continuous in $N_{in} \cup E_{out}$ and $\Phi_{loc}(N_{in} \cup E_{in}) \subseteq \operatorname{dom} \pi_{out}$, when $\varepsilon_{in}^{1+\hat{\vartheta}} < \varepsilon_{out} < 1$. This directly follows from the estimates in Section 3.2.2, if

$$0 < \hat{\vartheta} < \frac{\lambda_i - \lambda_o - 3\varepsilon_g}{\lambda_o + 2\varepsilon_g}.$$

The main step is to prove that the induced map $h_0(\pi_{out} \circ \Phi_{loc}) : h_0(E_{in}) \to h_0(E_{out})$ does indeed map the $\pm(x_u - \xi_u) > 0$-components of E_{in} to the $\pm x_u > 0$-components of E_{out}. We need therefore to consider inital conditions in $E_{in}^{\vartheta,\varepsilon}$ and fine-tune the constants ϑ_{in} and ε_{out}. We will now give sufficient constraints on these constants; the required inequalities follow from the following calculations. We assume that $\vartheta_{in}, \vartheta_{out} \in (0, \frac{1}{2})$ and the following inequality holds:

$$\vartheta_{out} + 2\vartheta_{in} < \frac{\lambda_i + \lambda_u - \lambda_o - \varepsilon_g}{2\lambda_i} \tag{3.2.4.4}$$

For $t < \hat{T}$, i.e. $x_o < x_i$, we can estimate $x_i x_o^2 \leq d^{3/2}$. Assume that $|x_u - \xi_u| \geq d^{1+\vartheta_{in}}$. Then we can estimate:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|x_u - \xi_u| &\geq (\lambda_u - \varepsilon_g)|x_u - \xi_u| - |\tilde{g}_u^{ioo}|_\infty d^{\frac{3}{2}} \\ &\geq \left(\lambda_u - \varepsilon_g - \varepsilon_g d^{\frac{1}{2} - \vartheta_{in}}\right)|x_u - \xi_u| \\ &\geq (\lambda_u - 2\varepsilon_g)\,|x_u - \xi_u|. \end{aligned} \tag{3.2.4.5}$$

By this estimate and the fact that d is monotonically decreasing, the inequality $|x_u - \xi_u| \geq d^{1+\vartheta}$ gets preserved for $t < \hat{T}$ if $d(0) < \varepsilon_{in}$.
We can get lower bounds on $\hat{T}$ by using $\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{x_o}{x_i} \leq \lambda_o + \lambda_i + 3\varepsilon_g$:

$$\hat{T} \geq -\frac{1}{\lambda_o + \lambda_i + 3\varepsilon_g} \log x_o(0).$$

This allows us to get the following estimate for initial conditions in $E_{in}^{\vartheta,\varepsilon}$:

$$\begin{aligned} &\log|x_u(\hat{T}) - \xi_u(\hat{T})| \geq \left(1 + \vartheta_{in} - \frac{\lambda_u - 2\varepsilon_g}{\lambda_o + \lambda_i + 3\varepsilon_g}\right) \log d(0) \\ &\operatorname{sign}(x_u(\hat{T}) - \xi_u(\hat{T})) = \operatorname{sign}(x_u(0) - \xi_u(0)) \end{aligned} \tag{3.2.4.6}$$

By plugging the lower bound on $\hat{T}$ into (3.2.2.2), we can see that:

$$\log d(\hat{T}) \leq \left(1 - \frac{\lambda_o - \lambda_i + 3\varepsilon_g}{\lambda_o + \lambda_i + 3\varepsilon_g}\right) \log d(0).$$

Combining these two estimates yields

$$\log |x_u(\hat{T}) - \xi_u(\hat{T})| \geq \frac{\lambda_o + \lambda_i - \lambda_u + \varepsilon_g + \vartheta_{in}(\lambda_o + \lambda_i + 3\varepsilon_g)}{2\lambda_i} d(\hat{t}).$$

Therefore, we can archieve $\log |x_u(\hat{T}) - \xi_u(\hat{T})| \geq (1 - \vartheta_{out}) d(\hat{T})$, when (3.2.4.4) holds.
We now have to show that the estimates get preserved for $\hat{T} < t < T_o$. Since the inequality $x_u(\hat{T}) \geq d(\hat{T})$ gets preserved for $\hat{T} < t < T_o$, we can estimate $\dot{x}_u / x_u \geq (\lambda_u - \varepsilon_g) > 0$. Therefore, $|x_u|$ is monotonically increasing for $\hat{T} < t < T_o$ and d is monotonically decreasing. This yields

$$\begin{aligned} &\log |x_u(T_o)| \geq (1 - \vartheta') \log d(T_o) \\ &\operatorname{sign} x_u(T_o) = \operatorname{sign}(x_u(0) - \xi_u(0)). \end{aligned} \tag{3.2.4.7}$$

□

3.2.5 Application of the Topological Method

We have now proven a local block Lemma 3.16. We will now modify the global block Lemma 3.9 and then proceed analogously to the linear case in Section 3.1.3 to prove the main Theorem 3.2.

Lemma 3.17 (Block pair for the global map). *Suppose that (N_{out}, E_{out}) and (N_{in}, E_{in}) have been constructed according to* (3.2.4.1) *and* (3.2.4.2). *Suppose further that* $\left(\mathrm{D}\Phi_{glob}^{-1}(1,0,0) \cdot (0,1,\partial_d \xi_u(1,0))\right)_i > 0$. *Then it is possible to choose $\varepsilon_{in} > 0$ small enough that for $\varepsilon_{out} = \varepsilon_{in}^{\hat{\vartheta}}$ a block pair for Φ_{glob}^{-1} is formed for* $\left(N_{in}^{\vartheta_{in},\varepsilon_{in}}, E_{in}^{\vartheta_{in},\varepsilon_{in}}, N_{out}^{\vartheta_{out},\varepsilon_{out}}, E_{out}^{\vartheta_{out},\varepsilon_{out}}\right)$.

Proof. Since $\xi_u \in C^1$, the proof of Lemma 3.9 applies without modification.

We will now prove the following preliminary version of the main Theorem 3.2:

Theorem 3.18. *Suppose we have a system fulfilling the Assumptions 3.1, which is in the form given in Proposition 3.11. Suppose ξ_u has been constructed according to Proposition 3.15. Suppose that*

$$\left(\mathrm{D}\Phi_{glob}^{-1}(1,0,0) \cdot (0,1,\partial_d \xi_u)\right)_i > 0. \tag{3.2.5.1}$$

Set $U^\varepsilon = \{x \in \Sigma_{in} : 0 < x_o < \varepsilon, |x_u| \leq \varepsilon\}$ *and* $\partial^\pm U^\varepsilon = \{x \in \Sigma_{in} : 0 < x_o < \varepsilon, x_u = \pm\varepsilon\}$. *Then there is $\varepsilon > 0$ such that $W_s \cap U^\varepsilon$ is* *separating the boundaries $\partial^{\pm}U^{\varepsilon}$ in U^{ε} in codimension 1. Therefore, W_s is especially nontrivial.*

Proof. The Theorem follows directly by combining Lemma 3.17 and Lemma 3.16 and applying Theorem 2.19. □

In order to prove the main Theorem 3.2, we will now massage the assumptions of Theorem 3.18 in order to make them independent of our construction, similiar as in the linear case in Section 3.1.3. We will at first proof the Remark 3.14 about an alternative characterization of $v = \partial_{x_o}\xi_u(1,0)$:

Proposition 3.19. *Let $v^0 = (0,1,\partial_{x_o}\xi_u(1,0))$ and $v(t)$ be the solution of the linearized equation $\dot{x} = F(x)$ around γ with $\gamma(0) = (1,0,0)$:*

$$\dot{v} = \mathrm{D}F(\gamma(t))v(t) \qquad v(0) = v^0. \tag{3.2.5.2}$$

Then for $t > 0$, we have:

$$v_u(t) = \partial_{x_i}\xi_u \cdot v_i + x_i\partial_d\xi_u \cdot v_o$$

and especially

$$\lim_{t\to\infty}\frac{v(t)}{|v(t)|} = e_o.$$

Proof. We transform the approximate system (3.2.3.1) back in order to compare it to the original system (3.2.1.1). Setting $x_o = d/x_i$ we get for the approximate system $x' = \tilde{F}$:

$$\begin{aligned} x_i' =& -\lambda_i x_i + g_i^i(x_i,0,x_u)x_i + \partial_{x_o}g_i^i(x_i,0,x_u)x_i x_o \\ x_o' =& \lambda_o x_o + g_o^o(x_i,0,x_u)x_o + (g_o^u(x_i,0,x_u) + \partial_{x_o}g_o^u(x_i,0,x_u)x_o)\,x_u - \partial_{x_o}g_i^i(x_i,0,x_u)x_o^2 \\ x_u' =& \lambda_u x_u + g_u^{io}(x_i,0,x_u)x_i x_o. \end{aligned} \tag{3.2.5.3}$$

We can see that $F_{|\gamma} = \tilde{F}_{|\gamma}$ and $\mathrm{D}F_{|\gamma} = \mathrm{D}\tilde{F}_{|\gamma}$. By invariance of the manifold $\{x_u = \xi_u(x_i, x_i x_o)\}$ under the approximate vectorfield $\tilde{F}$, the vector $v(t)$ is tangent to it. The assertion about $\lim_{t\to\infty} v(t)/|v(t)|$ follows easily. □

This allows us to prove the main Theorem:

Proof (of the main Theorem 3.2). The Theorem follows by combining Theorem 3.18 and Proposition 3.19. □

3.2.6 The Center Case: $\lambda_u = 0$

The proofs and statements of this chapter can be modified for the case $\lambda_u = 0$. We will now sketch all necessary modifications.
The change of coordinates needs to be slightly modified. We will need to replace the unstable manifold by a $C^{2,1}$ center-unstable manifold for this change of coordinates. We can do this by changing the system outside of a small neighborhood of 0 in order to achieve the necessary inequalities for the pseudo-stable manifold theorem for the time one map ϕ^1 of the flow and since the spectral gap $-\lambda_i < 0 = \min(\lambda_u, \lambda_o)$ near zero allows any finite regularity of pseudo-stable manifolds. Similarly we need to replace the stable manifold by the strong stable manifold.
Another difficulty arises with the Euler multiplier in Section 3.2.1.3: This change of coordinates is simply not possible anymore, since μ is not bounded. Therefore we will need to keep the g_u^u-term and handle it later.
The approximation of the stable part therefore needs to be modified. We can instead use

$$x_u' = \lambda_u x_u + g_{io}^u(x_i, 0, x_u)d + g_u^u(x_i, 0, x_u)x_u$$

and will therefore get get an additional summand in (3.2.3.6):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x_u - \xi_u) = (\lambda_u + \tilde{g}_u \cdot x_i)(x_u - \xi_u) + \tilde{g}_u^{ioo} x_i x_o^2 \\ + g_u^u(x_i, x_o, x_u)x_u - g_u^u(x_i, 0, \xi_u)\xi_u \end{aligned} \tag{3.2.6.1}$$

We can absorb part of this extra term (3.2.6.1) into $x_u - \xi_u$ and end up with a bad additional term of the form $\tilde{g}_u^{uo} x_o x_u$.
The construction of the local block in Lemma 3.16 and the local estimates therefore need to be changed as well. We consider the time $\hat{T}$ when $x_i^2 = x_o$ for the first time. We can then estimate under the assumtions $x_o \leq x_i^2$ and $|x_u| \leq x_o$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x_u - \xi_u| &\geq (\lambda_u - \varepsilon_g)|x_u - \xi_u| - |\tilde{g}_u^{ioo}|_\infty x_i x_o^2 - |\tilde{g}_u^{uo}|_\infty x_o |x_u| \\ &\geq (\lambda_u - \varepsilon_g)|x_u - \xi_u| - \varepsilon_g d^{\frac{5}{3}} - \varepsilon_g d^{\frac{4}{3}} \\ &\geq \left(\lambda_u - \varepsilon_g - \varepsilon_g d^{\frac{2}{3} - \vartheta_{in}} - \varepsilon_g d^{\frac{1}{3} - \vartheta_{in}}\right)|x_u - \xi_u|\end{aligned}$$

This crucial estimate allows us to proceed as in the case $\lambda_u > 0$ with only minor modifications in the constants.

Chapter 4
Outlook and Discussion

In this work we gave an overview of the theory of stable manifolds and introduced a topological generalization of the stable manifold theorem, which we related to Conley Index theory. We demonstrated the usefulness of this topological generalization in the analysis of heteroclinic sets by studying a three-dimensional toy-model with a homoclinic orbit, where the standard theory of stable manifolds is not applicable due to a lack of smoothness of the Poincaré map. We furthermore showed that the topological technique is especially useful in conjunction with direct calculations, since its application only requires C^0-estimates.
There are currently two lines of future work planned:
Firstly, the procedure demonstrated in the third chapter can be genereralized. It is possible to directly extend the result 3.2 to heteroclinic chains, i.e. sequences of orbits $\gamma_i(t)$ with $\lim_{t\to+\infty}\gamma_i(t) = x_i = \lim_{t\to-\infty}\gamma_{i+1}(t)$. In this case we will require a geometric sign condition similiar to the one in Theorem 3.2 for every orbit γ_i of the heteroclinic chain as well as a growth-condition on $\mathrm{D}F$ and on the spectral gaps at x_i.
Furthermore, it should be possible to generalize the analysis leading to result 3.2 to heteroclinic chains in $\mathbb{R}^n$. In this case geometric conditions similiar to the ones used in Theorem 3.2 will require more subtle local analysis. This is due to the fact that a direct application of the approach used in the 3-dimensional case yields a block pair of the general form

$$\begin{aligned} N_{in} &= \text{cusp} \times \mathbb{R}^{n_s} \\ N_{out} &= \text{halfspace} \times \mathbb{R}^{n_s} \end{aligned}$$

where $n_s + 1$ is the stable dimension of the equilibrium. Then any tilt of the $\mathbb{R}^{n_s}$-spaces introduced by Φ_{glob} will pose a serious problem for our analysis. Future work will seek to overcome this geometric problem in order to formulate a general theorem on separating basins of attraction of heteroclinic chains.
The second line of future work will apply the techniques developed in this work to problems in mathematical cosmology, which actually were the prime motivation for the development of said techniques. In the study of Bianchi IX cosmologies,

there arises an attractor consisting of heteroclinic chains. Up to now only partial progress has been made in determining which of these chains have a nonempty basin of attraction and in understanding these basins of attractions (see e.g. the excellent survey [HU09]). Since the geometry of the Bianchi system prevents any tilt in the $\mathbb{R}^{ns}$-spaces, which will make a general theorem on heteroclinic chains so difficult, we hope to use methods similar to the third chapter of this work, i.e. the topological technique in conjunction with direct calculations, on the Bianchi system. More specifically, we hope to prove existence of separating basins of attraction for a large class of heteroclinic chains occuring in the Bianchi system, thus proving a variant of the Kasner map convergence conjecture [HU09, p.26] for a set of full measure on the Kasner circle.

A corollary of this would be a new proof for some of the conjectures in [HU09, p.23f], which have been recently solved. These are the positive results for the Mixmaster attractor conjecture as given in [HU09] and the ”attractor beliefs” (i) and (ii) which have been recently proved in [LHWG10], as well as the negative result to the attractor belief (iii), which has been recently disproved in [Beg10].

References

BD84. Patrick Bonckaert and Freddy Dumortier. On a linearization theorem of sternberg for germs of diffeomorphisms. *Mathematische Zeitschrift*, 185:115–135, 1984.

Beg10. Francois Beguin. Aperiodic oscillatory asymptotic behavior for some bianchi spacetimes, 2010. arXiv:1004.2984.

dlL03. Rafael de la Llave. Invariant manifolds associated to invariant subspaces without invariant complements: A graph transform approach. *Mathematical Physics Electronical Journal*, 9, 2003.

FR00. J. Franks and D. Richeson. Shift equivalence and the conley index. *Transactions of the AMS*, 352(7):3305–3322, 2000.

Gid98. Marian Gidea. The discrete conley index for noninvariant sets. In *Proceedings of the Topological Methods in Differential Equations and Dynamical Systems Conference*. Universitatis Jagellonicae Acta Mathematica, 1998.

GT98. D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, 1998.

HU09. Mark Heinzle and Claes Uggla. Mixmaster: Facts and beliefs. *Preprint*, 2009.

KH95. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

LHWG10. Stefan Liebscher, Jörg Härterich, Kevin Webster, and Marc Georgi. Ancient dynamics in bianchi models: Approach to periodic cycles, 2010. arXiv:1004.1989.

Mun00. James R. Munkres. *Topology*. Prentice Hall, 2000.

Oss92. Erich Ossa. *Topologie*. Vieweg, 1992.

RS80. M. Reed and B. Simon. *Methods of Modern Mathematical Physics: Functional Analysis*. Academic Press, 1980.

RW10. David Richeson and Jim Wiseman. Symbolic dynamics for nonhyperbolic systems. *Proc. Amer. Math. Soc.*, 2010.

Sel67. George R. Sell. Nonautonomous differential equations and topological dynamics i. the basic theory. *Transactions of the American Mathematical Society*, 127(2):241–262, 1967.

Str08. Michael Struwe. *Variational Methods: Applications To Nonlinear Partial Differential Equations And Hamiltonian Systems*. Springer Verlag, 2008.

Zei95. Eberhard Zeidler. *Applied Functional Analysis: Main Principles and Their Application*. Springer-Verlag, 1995.

Zgl04. Piotr Zgliczynski. Covering relations for multidimensional dynamical systems. *J. Differential Equations*, 202(1):32–58, 2004.